GEOMETRIC BRUHAT ORDER ON (0,1)-MATRICES

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ABSTRACT. The combinatorially and the geometrically defined partial orders on the set of permutations coincide. We extend this result to $(0,1)$-matrices with fixed row and column sums. Namely, the Bruhat order induced by the geometry of a Cherkis bow variety of type A coincides with one of the two combinatorially defined Bruhat orders on the same set.

1. INTRODUCTION

Let $\mathrm{BCT}(r,c)$ denote the set of 01 matrices with row sum vector r and column sum vector c—BCT stands for binary contingency table. Recent advances in enumerative geometry identified a class of smooth varieties $X(r,c)$, called Cherkis bow varieties of type A ("bow varieties"), that are parameterized by pairs of vectors r,c. Bow varieties are endowed with a torus action, and the torus fixed points of $X(r,c)$ are naturally identified with $\mathrm{BCT}(r,c)$. Bow varieties are very general (eg. they contain Nakajima quiver varieties of type A as special cases), they have significance in physics [3], and they have favorable properties for enumerative geometry [14, 2] (eg. bow varieties are closed for "combinatorial 3d mirror symmetry").

For $r = c = (1, 1, \ldots, 1) \in \mathbb{N}^n$ the bow variety is the total space of the cotangent bundle over the full flag variety $\mathrm{Fl}(n)$, and the torus action is the natural $(\mathbb{C}^\times)^n$ action. The fixed point set is $\mathrm{BCT}((1,\ldots,1),(1,\ldots,1))$ which is the set of $n \times n$ permutation matrices. More generally, if only $c = (1, 1, \ldots, 1)$ then the bow variety is the total space of the cotangent bundle of a partial flag variety on $\mathbb{C}^n$, with its natural $(\mathbb{C}^\times)^n$ action. The fixed point set is $\mathrm{BCT}(r,(1,\ldots,1))$ which is indeed in obvious bijection with $S_n/(\times_i S_{r_i})$.

In these Schubert Calculus examples one can use geometry to define a partial order, called *geometric Bruhat order*, on the set of fixed points. Namely, after fixing a reference flag the variety is partitioned into Schubert cells, with each Schubert cell containing one torus fixed point (we can forget the cotangent directions for now). The "containment in the closure" relation then defines a partial order on the set of Schubert cells, and hence also on the set of fixed points.

It turns out that there is an analogous partial order on the torus fixed points of general bow varieties $X(r,c)$. The role of the reference flag is played by a general cocharacter of the torus. The role of Schubert cells will be played by attracting sets with respect to this cocharacter. In fact,

one needs to work with iterated attracting sets (a.k.a. full attracting sets). The "containment in the closure" part of the definition is also modified: we consider which fixed points belong to the closure of which attracting sets. Nevertheless, there is a notion of geometric Bruhat order on $\mathrm{BCT}(r,c)$ for general r,c vectors that generalizes the notion of geometric Bruhat order in Schubert Calculus. This Bruhat order plays an important role in the theory of Stable Envelopes developed by Okounkov and his coauthors [10, 12, 1].

Our paper is about finding a combinatorial characterization of the geometric Bruhat order on $\mathrm{BCT}(r,c)$. In fact, combinatorists already studied "natural" partial orders on 01 matrices with prescribed row and column sums [4]. The surprising outcome of their studies is that there are two different combinatorially natural partial orders, that they named Bruhat order and secondary Bruhat order. We will call them the *combinatorial Bruhat order* and the *combinatorial secondary Bruhat order*. The two orders coincide if $c = (1, 1, \ldots, 1)$ and it is a well-known fact of Schubert Calculus that in this case the geometric Bruhat order is also the same.

In this paper we prove that the geometric Bruhat order on $\mathrm{BCT}(r,c)$ is the same as the combinatorial secondary Bruhat order.

After introducing the necessary combinatorics and geometry, we first define the geometric Bruhat order using the geometry of $X(r,c)$. Then we present a characterization of the partial order using invariant curves on $X(r,c)$. In Sections 4 and 5 we give two proofs of our main theorem. The first one relies on the recent work [8] describing all invariant curves on $X(r,c)$, and the second one relies on a technique, called *geometric resolution*, developed in [2] to prove the so-called 3d mirror symmetry property of bow varieties.

We will use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $k^n = (k, k, \ldots, k) \in \mathbb{N}^n$.

2. Combinatorial Bruhat orders

Let $r \in \mathbb{N}^m$ and $c \in \mathbb{N}^n$ satisfy $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. Define $\mathrm{BCT}(r,c)$ to be the set of $m \times n$ matrices whose entries are all 0 or 1, and whose row sum vector is r and column sum vector is c.

Example 2.1. The set of $n \times n$ permutation matrices is a special case for $r = c = (1, 1, \ldots, 1)$. A more general special case will play a role in this paper: let $r = (r_1, r_2, \ldots, r_m)$, $n = \sum r_i$, $c = (1^n)$. In Algebra this $\mathrm{BCT}(r,c)$ is in bijection with $S_n/(\times_i S_{r_i})$, and in Geometry it parametrizes the torus fixed points (or the Schubert cells) of the partial flag variety $\{V_\bullet : V_1^{r_1} \subset V_2^{r_1+r_2} \subset \ldots \subset V_{m-1}^{r_1+r_2+\ldots+r_{m-1}} \subset \mathbb{C}^n\}$.

Combinatorial properties of $\mathrm{BCT}(r,c)$ have been extensively studied, see [4] for a survey. For example, the Gale-Ryser theorem [16, Ch. 6, Thm. 1.1] provides a simple necessary and sufficient condition for $\mathrm{BCT}(r,c)$ to be non-empty. Another fundamental result is that the graph whose

vertex set is $\mathrm{BCT}(r,c)$ and whose edges are pairs of matrices that only differ in the intersection of two rows and two columns by

$$(1) \qquad L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

is connected [15] (the rows and columns are not necessarily adjacent). In this paper we will be concerned with partial orders on $\mathrm{BCT}(r,c)$.

2.1. The combinatorial Bruhat order. For $M \in \mathrm{BCT}(r,c)$ let $\Sigma_M = (\sigma(M)_{i,j})$ be the $m \times n$ matrix where

$$\sigma(M)_{i,j} = \sum_{k=1}^{i} \sum_{l=1}^{j} m_{kl}.$$

Define the *combinatorial Bruhat order* by $M_1 \leqslant_B M_2$ if $\Sigma_{M_1} \geqslant \Sigma_{M_2}$ entrywise.

2.2. The combinatorial secondary Bruhat order. The *combinatorial secondary Bruhat order* is defined by letting $M_1 \leqslant_{\hat{B}} M_2$ if M_2 can be transformed into M_1 by a sequence of $L_2 \to I_2$ interchanges, see (1). Nota bene, we permit the surgery depicted in (1) in the intersection of two (not necessarily consecutive) rows and two (not necessarily consecutive) columns *only in one direction*: from L_2 to I_2.

Recall that the covering relation of a (finite) partial order $\leqslant$ is defined by: M covers M' provided $M' < M$ and there is no M'' with $M' < M'' < M$. A combinatorial description for the covering relation of $\leqslant_{\hat{B}}$ is proved by Brualdi and Deaett.

Theorem 2.2. [5, Thm. 3.1] *Let* $M = (m_{ij})$ *be a matrix in* $\mathrm{BCT}(r,c)$ *where* $M[\{i,j\},\{k,l\}] = L_2$ *(cf. (1)). Let* M' *be the matrix obtained from* M *by the* $L_2 \to I_2$ *interchange that replaces* $M[\{i,j\},\{k,l\}]$ *with* I_2. *Then* M *covers* M' *in the combinatorial secondary Bruhat order on* $\mathrm{BCT}(r,c)$ *if and only if*

(1) $m_{pk} = m_{pl}$ *for* $i < p < j$,
(2) $m_{iq} = m_{jq}$ *for* $k < q < l$,
(3) $m_{pk} = 0$ *and* $m_{iq} = 0$ *imply* $m_{pq} = 0$ *for* $i < p < j, k < q < l$,
(4) $m_{pk} = 1$ *and* $m_{iq} = 1$ *imply* $m_{pq} = 1$ *for* $i < p < j, k < q < l$.

Example 2.3. For $r = c = (2,1,2)$ the combinatorial Bruhat order and the combinatorial secondary Bruhat order on $\mathrm{BCT}(r,c)$ are the same. The Hasse diagram (only drawing the cover relations) is on the left of Figure 1.

2.3. The two combinatorial Bruhat orders are different. It is clear that $M_1 \leqslant_{\hat{B}} M_2$ implies $M_1 \leqslant_B M_2$. However, the two partial orders are not the same. The counterexample from [5,
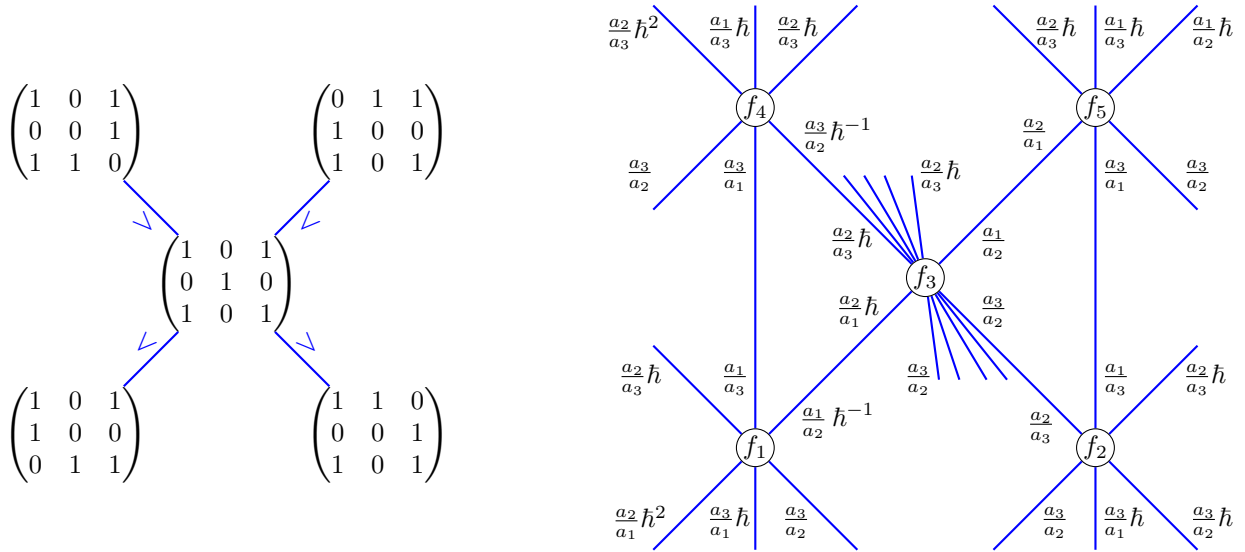
FIGURE 1. *Left*: The Hasse diagram of $\mathrm{BCT}((2,1,2),(2,1,2))$ for either of the two combinatorial Bruhat orders $\leqslant_B$, $\leqslant_{\hat{B}}$. Observe that the two leftmost matrices (as well as the two rightmost matrices) are connected by an interchange (1), but that relation is not a cover relation. *Right*: The fixed points and the invariant curves (together with their weights) of the bow variety $X(\texttt{/2/3/5\textbackslash 3\textbackslash 2\textbackslash})$. The combs of blue lines around the middle indicate two 1-parameter families of invariant curves. The geometric Bruhat order, that is, the "a_{large}/a_{small} flow" in the right picture coincides with the $>$ relation in the left picture. The bijective correspondence between interchanges of type (1) in the left picture and the "bounded" curves in the right picture holds for this example, but not in general.

Sect. 2] is the pair of matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

for which $M_1 \leqslant_B M_2$ but $M_1 \not\leqslant_{\hat{B}} M_2$. This pair is the smallest counterexample the authors are aware of. They live in $\mathrm{BCT}((1,4,5,2,1,3),(3,1,2,5,4,1))$ which is an 89-element set.

A well known fact is that for $r = c = 1^n$, that is, for permutation matrices, the two combinatorial Bruhat orders coincide. However, a complete characterization of pairs of vectors r, c for which the two combinatorial Bruhat orders coincide is not known. For some examples and counterexamples see [7, 18]. For example, it is proved that the orders $\leqslant_B$ and $\leqslant_{\hat{B}}$ coincide for $r = c = (k^n)$ if and only if either $0 \leqslant n \leqslant 5$ or $k \in \{0, 1, 2, n-2, n-1, n\}$ with $n \geqslant 6$. Note that

$|\operatorname{BCT}(3^6, 3^6)| = 297200$. It is also proved that the orders $\leqslant_B$ and $\leqslant_{\hat{B}}$ coincide if either r or c equals either (1^n) or (2^n). In particular, the two orders coincide for the set corresponding to the partial flag variety of Example 2.1.

3. The geometric Bruhat order

3.1. **Bow varieties.** Combinatorial objects like $\mathcal{D} = $ /2\2/2\4/3/3/4\3/2\2\ are called (type A) brane diagrams. The red forward-leaning lines are called NS5 branes, denoted by $\mathcal{Z}$. The blue backward-leaning lines are called D5 branes, denoted by $\mathcal{A}$. The positions between 5-branes are called D3 branes, denoted by $\mathcal{X}$, and the integer sitting there is called its multiplicity or dimension $d_{\mathcal{X}}$.

If all NS5 branes are to the left of all D5 branes, we call the diagram *separated*, if all NS5 branes are to the right of all D5 branes, we call the diagram *co-separated*.

Let $\mathcal{A}^+$, $\mathcal{Z}^+$ be the D3 brane to the right, and $\mathcal{A}^-$, $\mathcal{Z}^-$ be the D3 brane to the left of $\mathcal{A}, \mathcal{Z}$. The charge of 5-branes are defined by

$$ch(\mathcal{Z}) = d_{\mathcal{Z}^+} - d_{\mathcal{Z}^-} + |\{\text{D5 branes left of } \mathcal{Z}\}| \qquad \text{for NS5 branes } \mathcal{Z},$$
$$ch(\mathcal{A}) = d_{\mathcal{A}^-} - d_{\mathcal{A}^+} + |\{\text{NS5 branes right of } \mathcal{A}\}| \qquad \text{for D5 branes } \mathcal{A}.$$

To a brane diagram $\mathcal{D}$ one associates a smooth holomorphic symplectic variety called Cherkis bow variety $X(\mathcal{D})$ with a torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_{\hbar}^{\times}$ action, where $\mathbb{A} = (\mathbb{C}^{\times})^{\text{D5 branes}}$, see [11, 14]. The action has finitely many fixed points. The fixed points are in an explicit bijection with $\operatorname{BCT}(r, c)$ where r is the vector of charges of NS5 branes read from left to right, and c is the vector of D5 brane charges read from left to right. For instance, the vertices of the graph on the left of Figure 1 are in bijection with the torus fixed points of $X($/2/3/5\3\2\$)$, or with the torus fixed points of $X($/2/3\2/3\2\$)$. Spaces with the same charge vectors are in fact isomorphic as we will see in the next section.

Assumption. In the whole paper we assume that $X(\mathcal{D})$ has at least one fixed point. Equivalently, that for the associated charge vectors r, c the set $\operatorname{BCT}(r, c)$ is not empty.

3.2. **Hanany-Witten transition.** The local surgery on a brane diagram a\b/c $\leftrightarrow$ a/a+c−b+1\c (in either direction) is called Hanany-Witten (HW) transition, cf. [9]. The charge vectors form a complete invariant of the HW equivalence class of brane diagrams.

The HW transition induces an explicit isomorphism between the associated bow varieties. Let the D5 brane involved in the HW transition be $\mathcal{A}$ (ie. the blue \ brane that moves over an NS5 brane in the surgery). The HW isomorphism is equivariant for the torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_{\hbar}^{\times}$ under the automorphism of $\mathbb{A}$ obtained by re-scaling $\mathbb{C}_{\mathcal{A}}^{\times}$ by $\hbar$.

Since the $\mathbb{C}_{\hbar}^{\times}$ action will not concern us, we will not distinguish between $X(\mathcal{D})$ varieties associated to HW equivalent brane diagrams, and use the notation $X(r, c)$ for $X(\mathcal{D})$ for any $\mathcal{D}$ with charge vectors r and c. If we wish, we can choose the separated

$$\mathcal{D} = /r_1/r_1 + r_2/r_1 + r_2 + r_3/ \ldots \backslash c_{n-2} + c_{n-1} + c_n \backslash c_{n-1} + c_n \backslash c_n \backslash$$

representative.

3.3. **Geometric Bruhat order.** Let us choose a permutation $\sigma \in S_n$. It induces a cocharacter

$$\lambda_\sigma : \mathbb{C}^\times \to \mathbb{A} \times \mathbb{C}_\hbar^\times, \qquad t \mapsto (t^{\sigma(1)}, t^{\sigma(2)}, \ldots, t^{\sigma(n)}, 1).$$

For a torus fixed point $f \in X(\mathcal{D})$ we define the corresponding attractive manifold

$$\mathrm{Att}_\sigma(f) = \{x \in X(\mathcal{D}) : \lim_{t \to 0} \lambda_\sigma(t) \cdot x = f\}.$$

Remark 3.1. Since the tangent $\mathbb{A}$-weights at the fixed points in $X(\mathcal{D})$ are of the form a_i/a_j, their pullback by λ_σ are nontrivial. As a consequence, the $\mathbb{C}^\times$-action on $X(\mathcal{D})$ is generic, i.e. we have $X(\mathcal{D})^{\mathbb{C}^\times} = X(\mathcal{D})^{\mathbb{A}}$.

For torus fixed points $f, g \in X(\mathcal{D})$ we define

$$f \leqslant g \qquad\qquad \text{if} \qquad\qquad f \in \overline{\mathrm{Att}_\sigma(g)},$$

and its transitive closure will be called the *geometric Bruhat order* and denoted by $\leqslant_G$.

Remark 3.2. This definition of geometric partial order applies to any smooth quasiprojective variety $X \subseteq \mathbb{P}^N$ equipped with the action of a torus $\mathbb{T}$ and does not exploit extra geometric structure carried by bow varieties. In particular, the fact that this is really a partial order, i.e. that $f \leqslant_G g$ and $g \leqslant_G f$ imply $f = g$, follows from smoothness, which in turn ensures that the embedding $X \subseteq \mathbb{P}^N$ can be made $\mathbb{T}$-equivariant [6, Theorem 5.1.25].

Notice that $\leqslant_B$, $\leqslant_{\hat{B}}$, $\leqslant_G$ are partial orders on the same set $\mathrm{BCT}(r,c)$, the last one depending on the choice of the permutation σ.

Remark 3.3. In the $c = 1^n$ special case, which is relevant in Schubert Calculus, the three partial orders are the same (where $\leqslant_G$ is meant with the identity permutation). The coincidence of the first two is proved in [5, 7] and the equality of $\leqslant_B$ and $\leqslant_G$ is the content of [13, Thm.5A].

The main theorem of our paper is the identification of the geometric Bruhat order with one of the combinatorial ones, for arbitrary r, c.

Theorem 3.4. *The geometric Bruhat order for the identity permutation is the same as the secondary combinatorial Bruhat order.*

The theorem implies that the geometric Bruhat order for another permutation σ coincides with an appropriate modification of the combinatorial secondary Bruhat order—one where a σ-permutation of the columns is taken into account. In Sections 4 and 5 we will give two different proofs for Theorem 3.4.

Assumption. In theory the vectors r and c can have 0 components. However, simply erasing those components effects none of our notions or statements. Hence, from now on in the whole paper we assume that r and c have no 0 components.

3.4. **More on the geometric Bruhat order.** Since the union of all attracting sets in a bow variety is closed, cf. [2, Lemma 5.12], the geometric Bruhat order admits the following equivalent characterization: $f \leqslant_G g$ iff there exists a collection of $\mathbb{A}$-fixed points $f = f_0, f_1, \ldots, f_{n-1}, f_n = g$ connected by a chain of attracting $\mathbb{A}$-orbits. Namely, if there exist $\mathbb{A}$-orbits $\mathcal{O}_i \subseteq X(\mathcal{D})$ such that $f_i, f_{i-1} \in \overline{\mathcal{O}_i}$ and

$$\lim_{t \to 0} \lambda_\sigma(t) \cdot x = f_{i-1} \qquad \forall x \in \mathcal{O}_i \,.$$

In general, the dimension of these orbits can be up to $\mathrm{rk}(\mathbb{A}) - 1$[1] However, in Lemma 3.6 we will show that we can restrict ourselves to closures of *one-dimensional* orbits. Before proving this result, let us discuss its consequences. By an easy adaptation of Lemma A.1, it follows that one dimensional orbit closures admit the following evocative description: two distinct points $f, g \in X^{\mathbb{A}}$ belong to the closure $\overline{\mathcal{O}} \subset X(\mathcal{D})$ of a one dimensional $\mathbb{A}$-equivariant orbit $\mathcal{O}$ iff there exists an embedding $\gamma : \mathbb{P}^1 \to \overline{\mathcal{O}}$ whose restriction to $\mathbb{C}^\times = \mathbb{P}^1 \setminus \{0, \infty\}$ is onto $\mathcal{O}$ and such that $\gamma(0) = f$, $\gamma(\infty) = g$. In particular, this implies

$$\lim_{t \to 0} \lambda_\sigma(t) \cdot x = f \qquad \lim_{t \to \infty} \lambda_\sigma(t) \cdot x = g$$

for every $x \in \mathcal{O}$. Hence, a compact one dimensional $\mathbb{A}$-orbit closure can be thought of as a $\mathbb{P}^1 \subseteq X(\mathcal{D})$ preserved by the $\mathbb{A}$-action and whose poles $0, \infty \in \mathbb{P}^1$ are $\mathbb{A}$-fixed. We call such curves $\mathbb{A}$-invariant curves. For convenience, we always parametrize the domain in such a way that $\gamma(\infty) \leqslant_G \gamma(0)$. This in particular implies that the tangent weight at $0 \in \mathbb{P}^1$ of an $\mathbb{A}$-invariant curve is always attractive, i.e. its pullback by λ_σ is of the form $\lambda_\sigma^* \chi(t) = t^k$ for some $k > 0$.

Remark 3.5. Not all one dimensional orbit closures are compact. For example, consider the bow variety $X(/1/2\backslash1\backslash) = \mathrm{T}^* \mathbb{P}^1$. The fibers $T_0^* \mathbb{P}^1$ and $T_\infty^* \mathbb{P}^1$ are one dimensional orbit closures, with boundaries given by the base-points $0, \infty \in \mathbb{P}^1$. A similar situation can be observed in Figure 1, where both compact and non-compact invariant curves are drawn. In particular, notice that non-compact invariant curves do not affect the geometric Bruhat order. This is indeed a general fact, which directly follows from the characterization of the geometric Bruhat order discussed above.

We now characterize the geometric Bruhat order in term of one dimensional attracting curves.

Lemma 3.6. *Let $f, g \in X(\mathcal{D})^{\mathbb{A}}$. Then $f \leqslant_G g$ iff there exist $\mathbb{A}$-fixed points $f = f_0, f_1, \ldots, f_n = g$ and a chain of $\mathbb{A}$-invariant curves $\gamma_i : \mathbb{P}^1 \to X(\mathcal{D})$ such that $\gamma_i(0) = f_{i-1}$ and $\gamma_i(\infty) = f_i$ for all $i = 1, \ldots, n$*

Proof. To save on notation, set $X = X(\mathcal{D})$. Notice that the geometric Bruhat order only depends on $\mathbb{A}$ via the character $\lambda : \mathbb{C}^\times \to \mathbb{A}$. Since the latter is generic in the sense of Remark 3.1, it follows that $f \leqslant_G g$ iff f is connected to g via a chain of a $\mathbb{C}^\times$-invariant curves. As a consequence, it suffices to show that an attractive $\mathbb{C}^\times$-invariant curve $\gamma : \mathbb{P}^1 \to X$ degenerates to a chain of attractive $\mathbb{A}$-invariant curves. As we will see in the proof, the existence of such degeneration requires some amount of compactness. Although bow varieties are non-compact in general, they

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[1] In principle, it could be up to $\mathrm{rk}(\mathbb{A})$, but the one dimensional torus consisting of elements the form $(a, \ldots, a)$ acts trivially on the bow variety.

are defined as GIT quotients of some affine variety with respect to the action of a reductive group. As a consequence, any bow variety X comes with a canonical proper $\mathbb{T}$-equivariant map $\pi : X \to X_0 = \mathrm{Spec}(\Gamma(\mathcal{O}_X))$.This relative compactness will be sufficient for the proof.

Consider the composition $\mathbb{P}^1 \xrightarrow{\gamma} X \xrightarrow{\pi} X_0$. Since X_0 is affine, $\pi \circ \gamma$ is constant. Let $x \in X_0^{\mathbb{A}}$ be its image. Since π is equivariant, the fiber $Z := \pi^{-1}(x)$ is preserved by $\mathbb{A}$ and contains the image of γ. Moreover, since π is proper, Z is complete. As a result, to prove the lemma it suffices to show that the $\mathbb{C}^\times$-invariant curve $\gamma : \mathbb{P}^1 \to Z \subset X$ degenerates to a chain of attractive $\mathbb{A}$-invariant curves inside Z, which is complete.

Let $\chi_1, \ldots \chi_d$ be set of attractive tangent weights of $X^{\mathbb{A}} \subset X$. This set is really finite because the $X(\mathcal{D})^{\mathbb{A}}$ is a finite set and is nonempty unless X is zero dimensional, in which case the lemma is trivial. Set $\mathbb{A}_{d+1} := \mathbb{A}$, $\mathbb{A}_k := \cap_{j=k}^d \mathrm{Ker}(\chi_j) \subseteq \mathbb{A}$ for all $k = 1, \ldots, d$ and choose splittings $\mathbb{A}_k = \mathbb{A}_{k-1} \times \mathbb{A}_k / \mathbb{A}_{k-1}$. Notice that $\mathbb{A}_1 \subset \cdots \subset \mathbb{A}_d \subset \mathbb{A}$, the tori $\mathbb{A}_k / \mathbb{A}_{k-1}$ are one dimensional, and $\mathbb{A}_1$ acts trivially on X. The cocharacter λ and the weight χ factor as

$$
\begin{array}{ccc}
\mathbb{C}^\times \xrightarrow{\quad\lambda\quad} \mathbb{A}_{d+1} & \qquad & \mathbb{A}_{d+1} \xrightarrow{\chi_d} \mathbb{C}^\times \\
\Delta \downarrow \qquad \quad \| & & \downarrow \quad \nearrow \\
\mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\lambda' \times \lambda''} \mathbb{A}_d \times \mathbb{A}_{d+1} / \mathbb{A}_d & & \mathbb{A}_{d+1} / \mathbb{A}_d
\end{array}
$$

Consider the induced action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on $Z \subset X$ and fix a point $t \in \mathbb{P}^1 \backslash \{0, \infty\}$. Since Z is complete, Lemma A.2 (i) provides a map

$$
\Gamma : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p^{-1}} \overline{\mathrm{Graph}(\gamma(t))} \to X
$$

such that γ is recovered as the composition $\mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \mathbb{P}^1 \to X$. Moreover, since λ is generic, the same holds for λ' and λ''. As a consequence, Lemma A.2 (ii)-(iii) implies that $\gamma_d : \{0\} \times \mathbb{P}^1 \to Z \subset X$ is fixed by $\mathbb{A}_d$ and its image is the closure of an $\mathbb{A}_{d+1} / \mathbb{A}_d$ orbit, which must be either constant or one dimensional with tangent weight χ_d. Hence, combining the two diagrams above we conclude that γ_d is either constant or an attractive $\mathbb{A}$-invariant curve (with tangent weight χ_d at $0 \in \mathbb{P}^1$). Similarly, we see that the map $\gamma' : \mathbb{P}^1 \times \{\infty\} \to Z \subset X$ is either constant or an attractive $\mathbb{C}^\times$-invariant curve for the $\mathbb{C}^\times$-action induced by the character $\lambda' : \mathbb{C}^\times \to \mathbb{A}_d$.

Overall, we have shown that we may replace the curve γ with the concatenation of γ_d and γ'. Iterating the argument for all pairs $(\mathbb{A}_k, \mathbb{A}_{k+1})$, we see that we may replace γ by a chain of attracting orbits $\gamma_k : \mathbb{P}^1 \to Z \subset X$ (with tangent weight χ_k at $0 \in \mathbb{P}^1$) connecting f to g[2]. This proves the lemma.

$\square$

Remark 3.7. Since every $\mathbb{A}$-fixed point in a bow variety is also $\mathbb{T}$-fixed, cf. [14, Section 4], the argument above actually shows that the $\mathbb{T}$-invariant curves are sufficient to characterize the geometric Bruhat order as in Lemma 3.6.

For completeness, we give a counterexample to stress that the argument above fails in absence of completeness.

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[2]Notice that in general $n \leqslant d$ because some maps $\gamma_i : \mathbb{P}^1 \to X$ might be constant, so we can neglect them.

Counterexample 3.8. Consider the bow variety $X = \mathrm{T}^*\mathrm{Gr}(2,4)$, and set $Y = \mathrm{Gr}(2,4)$. The affinization map $\pi : X \to X_0$ is the Springer resolution of a nilpotent orbit in $X_0 \subset \mathfrak{gl}_4$ and all the fixed points $X^{\mathbb{A}}$ are contained in the zero fiber $\pi^{-1}(0) = Y$.

The $\mathbb{A}$-fixed points $Y^{\mathbb{A}} = X^{\mathbb{A}}$ are the coordinate planes $f_{ij} = \mathbb{C}_{a_i} \oplus \mathbb{C}_{a_j} \subset \mathbb{C}^4$ for $1 \leqslant i < j \leqslant 4$. Set $Y^0 = Y \backslash \{f_{13}, f_{14}, f_{23}, f_{24}\}$ and let $\lambda_{id} : \mathbb{C}^* \to \mathbb{A}$ be the character $t \mapsto (t, t^2, t^3, t^4)$. Unlike Y, Y_0 is not complete and hence the argument above fails. Indeed, it is easy to check that there exists a $\mathbb{C}^*$-invariant curve connecting f_{12} to f_{34} in Y^0 (and hence $f_{24} \in \overline{\mathrm{Att}_\sigma(f_{12})}$ or vice versa) but no chain of $\mathbb{A}$-invariant curves. On the contrary, this chain of curves exists in X, and consists of two copies of $\mathbb{P}^1$ glued at a pole. In particular, the pole at the intersection of the two $\mathbb{P}^1$ in X must be one of the fixed points $\{f_{13}, f_{14}, f_{23}, f_{24}\}$, which are not contained in Y^0.

4. The proof via combinatorial description of invariant curves

4.1. **Classification of Compact Invariant Curves.** For a bow variety $X(\mathcal{D})$, the $\mathbb{T}$-invariant curves can be read directly from the BCTs of fixed points. First, we restate the classification from [8] for the compact invariant curves.

Definition 4.1. *Let $M \in \mathrm{BCT}(r,c)$. A submatrix $\overline{M} = (m_{pq})_{q=k,l}^{p=i,i+1,\ldots,j}$ with only two columns is called a matched block if $\sum_{p=i}^{j} m_{pk} = \sum_{p=i}^{j} m_{pl}$. We call $\overline{M}$ a minimal matched block if it cannot be partitioned into two matched sub-blocks.*

Definition 4.2. *Let $M \in \mathrm{BCT}(r,c)$. A block swap move ψ on M consists of swapping the columns of a matched block $\overline{M}$ within M. If $\overline{M}$ is minimal, then ψ is called indecomposable.*

Note that if a minimal matched block has top row equal to $(0,1)$ then the bottom row is $(1,0)$, and vice-versa. Moreover, every block swap move uniquely decomposes into simultaneous indecomposable block swap moves with minimal matched blocks separated by $(0,0)$ and $(1,1)$ rows.

Before classifying invariant curves in $X(\mathcal{D})$, it is important to note that bow varieties generally contain pencils of invariant curves, by which we mean a family of invariant curves whose tangent directions at a fixed point span one weight space. For an example, consider $(\mathbb{P}^1)^k$ with the diagonal action by $\mathbb{C}^\times$. In fact, all compact invariant pencils in a bow variety are of this form [8]. Also pencils can occur non-compactly, as in Figure 1, to no effect on $\leqslant_G$.

Theorem 4.3. *Let $\mathcal{D}$ be a separated brane diagram with charge vectors r and c.*
  (1) *The block swap moves in $\mathrm{BCT}(r,c)$ are in bijection with the pencils of compact invariant curves in $X(\mathcal{D})$.*
  (2) *If a block swap move decomposes into k simultaneous indecomposable block swap moves, then the associated pencil has dimension k.*
  (3) *For a block swap move from M to M', let i be the index of the topmost affected row. Let (i, q_0) be the index of the 0 that becomes a 1 and let (i, q_1) be the index of the 1 that becomes a 0. Then the associated pencil has tangent weight $\frac{a_{q_1}}{a_{q_0}} \hbar^{1 + s_{iq_0} - s_{iq_1}}$ in $T_M X(\mathcal{D})$, where $s_{pq} = \sum_{p'=1}^{p} m_{p'q}$.*

Remark 4.4. These block swap moves contain the same data as "butterfly surgeries", which are the natural way to explicitly define these curves and were first described in [14] and [17]. Checking that these are the only compact invariant curves is done in [8].

4.2. **First proof of Theorem 3.4.** With compact invariant curves now classified, we return to the geometric Bruhat order on fixed points with greater clarity. First note that we do not need pencils of dimension greater than 1, as the indecomposable block swap moves are sufficient to generate the geometric order. Now define the relation $\prec_G$ such that $M' \prec_G M$ if there exists an indecomposable block swap move from M to M' with tangent weight $\frac{a_N}{a_n}\hbar^d$ with any $d \in \mathbb{Z}$ and $N > n$. In other words, $\prec_G$ corresponds to the relation on fixed points given by directed invariant curves. Note that $\prec_G$ does generate $\leqslant_G$ by transitive closure, but it is not a covering relation, as triangles may still appear in $\prec_G$, as in Figure 1.

Proof. Let $\lessdot_{\hat{B}}$ be the covering relation for $\leqslant_{\hat{B}}$. We first show that if $M' \lessdot_{\hat{B}} M$, then $M' \prec_G M$. By Theorem 2.2, the $L_2 \to I_2$ interchange on $M[\{i,j\},\{k,l\}]$ yields M', and with condition (1), $m_{pk} = m_{pl}$ for $i < p < j$. This means that $\overline{M} = (m_{pq})_{q=k,l}^{p=i,i+1,...,j}$ is a minimal matched block, as all the middle rows of $\overline{M}$ are either $(0,0)$ or $(1,1)$. By Theorem 4.3, since the $L_2 \to I_2$ interchange has the same effect as the block swap move swapping the columns of $\overline{M}$, there is an invariant curve between M' and M. It has tangent weight $\frac{a_l}{a_k}\hbar^{1+s_{ik}-s_{il}}$. Since $l > k$, we have $M' \prec_G M$.

Next we show if $M' \prec_G M$, then $M' \leqslant_{\hat{B}} M$. Let there be an indecomposable block swap move from M to M' with tangent weight $\frac{a_l}{a_k}\hbar^d$ with $l > k$. Let this block swap move have minimal matched block $\overline{M} = (m_{pq})_{q=k,l}^{p=i,i+1,...,j}$. By the form of its tangent weight, Theorem 4.3 says that the ith row is $(0,1)$. By minimality, the jth row is $(1,0)$. We seek a sequence of $L_2 \to I_2$ interchanges to swap the columns of $\overline{M}$.

We will induct on the number of $(0,1)$ rows in $\overline{M}$. If there is only 1 row of the form $(0,1)$, then it is the top row, and only one $L_2 \to I_2$ interchange is needed. If there are $n+1$ rows of the form $(0,1)$, then find the first row under the ith row that is either $(0,1)$ or $(1,0)$, and call its index t. Note that if row t is $(1,0)$, then minimality is violated, as $(m_{pq})_{q=k,l}^{p=i,i+1,...,t}$ would be a minimal matched subblock. Thus, row t is $(0,1)$ and all rows between i and t are $(0,0)$ or $(1,1)$. Similarly, there is a lowest row t' of the form $(1,0)$ that is above row j, and between t' and j are only $(0,0)$ or $(1,1)$ rows. First, perform the $J_2 \to I_2$ interchange on rows i and j. The part between t and t' is unchanged and has n rows of $(0,1)$. By inductive hypothesis, a sequence of $L_2 \to I_2$ interchanges will do the swap $(m_{pq})_{q=k,l}^{p=t,t+1,...,t'} \to (m_{pq})_{q=l,k}^{p=t,t+1,...,t'}$. Note that this matched block from rows t to t' may not be minimal, in which case break it into minimal matched subblocks and apply the inductive hypothesis multiple times. We now have the total block swap move produced from $L_2 \to I_2$ interchanges, so $M' \leqslant_{\hat{B}} M$.

We have proven if $M' \lessdot_{\hat{B}} M$, then $M' \prec_G M$, so by transitive closure, if $M' \leqslant_{\hat{B}} M$, then $M' \leqslant_G M$. We have proven if $M' \prec_G M$, then $M' \leqslant_{\hat{B}} M$, so by transitive closure, if $M' \leqslant_G M$, then $M' \leqslant_{\hat{B}} M$. $\qquad\square$
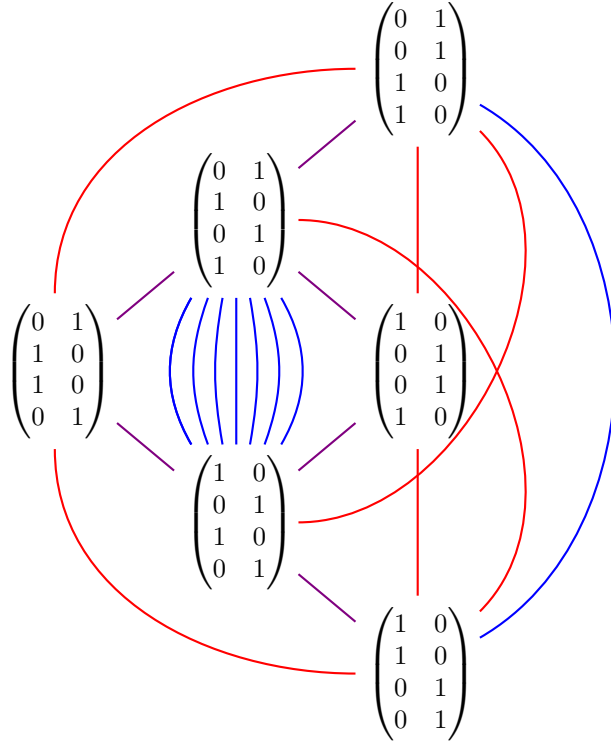
FIGURE 2. A graph with vertex set $\mathrm{BCT}((1,1,1,1),(2,2))$. A blue edge means a compact invariant curve connecting the corresponding fixed points on the bow variety, and the comb of blue edges indicate a pencil of such curves. A red edge indicates an $L_2 \to I_2$ interchange. A purple edge means we have both an invariant curve and an interchange.

Note that Theorem 3.4 is not just a perfect matching of $L_2 \to I_2$ interchanges and invariant curves. Although they do generate the same partial order, they are fundamentally different. We illustrate this phenomenon in Figure 2 for the example $X(/1/2/3/4\backslash 2\backslash)$, whose fixed point set is $\mathrm{BCT}((1,1,1,1),(2,2))$. This bow variety is also a Nakajima quiver variety $\mathcal{N}\left(\begin{smallmatrix} 1 & & 2 & & 1 \\ \bullet & \!\!\!-\!\!\!- & \bullet & \!\!\!-\!\!\!- & \bullet \\ & & | & & \\ & & \square\, 2 & & \end{smallmatrix} \right)$, cf. [14, Sect. 6] and is the "3d mirror dual" of $\mathrm{T}^*\mathrm{Gr}(2,4)$, cf. [2].

5. THE PROOF VIA BRANE-RESOLUTION

5.1. **Column resolutions for BCT tables.** Let $r \in \mathbb{N}^m$ and $c \in \mathbb{N}^n$ satisfying $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. For any $k = 1, \ldots n$, let $\tilde{c} \in \mathbb{N}^{n+1}$ be the vector obtained from c by replacing an entry $c_k \geqslant 2$ with a pair of consecutive entries $c'_k, c''_k \in \mathbb{N}_{>0}$ such that $c'_k + c''_k = c_k$. We call this operation a resolution of the charge vector c. Let $\mathrm{BCT}(r, \tilde{c}, c) \subset \mathrm{BCT}(r, \tilde{c})$ the subset consisting of those $\widetilde{M} = (\tilde{m}_{ij}) \in \mathrm{BCT}(r, \tilde{c})$ such that $\tilde{m}_{ik} + \tilde{m}_{ik+1} \in \{0, 1\}$ for all $i = 1, \ldots, m$. There is a well defined

map

$$q : \mathrm{BCT}(r, \tilde{c}, c) \to \mathrm{BCT}(r, c), \qquad \widetilde{M} \to q(\widetilde{M})$$

sending a matrix $\widetilde{M} = (\tilde{m}_{ij})$ to

$$q(\widetilde{M})_{ij} = \begin{cases} \tilde{m}_{ij} & \text{if } j < k \\ \tilde{m}_{ik} + \tilde{m}_{ik+1} & \text{if } j = k \\ \tilde{m}_{ij+1} & \text{if } j > k \end{cases}$$

Thus, the map q merges the k-th and $(k+1)$-th columns. Because of the defining condition of $\mathrm{BCT}(r, \tilde{c}, c)$, this operation really produces a BCT table for $\mathrm{BCT}(r, c)$.

The fiber $q^{-1}(M)$ is in one to one correspondence with the $S_{c_k}/(S_{c'_k} \times S_{c''_k})$ ways of producing a BCT table $\widetilde{M} \in \mathrm{BCT}(r, c)$ by replacing the k-th column $M_{\bullet k}$ of M with two adjacent columns that sum to $M_{\bullet k}$. This observation motivates the following definition:

Definition 5.1. *Let $M \in \mathrm{BCT}(r, c)$. The elements $\widetilde{M} \in q^{-1}(M) \subset \mathrm{BCT}(r, \tilde{c}, c)$ are called resolutions of the k-th column of M.*

Example 5.2. *Let $M \in \mathrm{BCT}(r, c)$ be of the form*

$$M = \begin{pmatrix} \heartsuit & \begin{vmatrix} 1 \\ 0 \\ 1 \\ 1 \end{vmatrix} & \blacklozenge \end{pmatrix}.$$

The central column is the k-th column and the placeholders $\heartsuit$ and $\blacklozenge$ are put in place of the other columns. Notice that $c_k = 3$. Let $\tilde{c}$ be the resolution of c such that $c'_k = 2$ and $c''_k = 1$. Then the set $q^{-1}(M) \subset \mathrm{BCT}(r, \tilde{c})$ consists of the three column resolutions

$$\widetilde{M}_1 = \begin{pmatrix} \heartsuit & \begin{vmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} & \blacklozenge \end{pmatrix} \qquad \widetilde{M}_2 = \begin{pmatrix} \heartsuit & \begin{vmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{vmatrix} & \blacklozenge \end{pmatrix} \qquad \widetilde{M}_3 = \begin{pmatrix} \heartsuit & \begin{vmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{vmatrix} & \blacklozenge \end{pmatrix}.$$

The following lemma, which directly follows from the definitions, shows that the BCT resolutions introduced above are compatible with the secondary Bruhat order.

Lemma 5.3. *Let $M_1, M_2 \in \mathrm{BCT}(r, c)$. Then $M_1 \geqslant_{\hat{B}} M_2$ if and only if for every resolution $\widetilde{M}_1 \in q^{-1}(M_1) \subset \mathrm{BCT}(r, \tilde{c}, c)$ there exists $\widetilde{M}_2 \in q^{-1}(M_2)$ such that $\widetilde{M}_1 \geqslant_{\hat{B}} \widetilde{M}_2$.*

5.2. **D5 resolutions.** Throughout this section, we assume that all bow varieties are separated, i.e. their brane diagrams are of the form $\mathcal{D} = /d_1/d_2/\ldots/d_m = h_n\backslash\ldots\backslash h_2\backslash h_1\backslash$. The combinatorics of the dimension vectors implies that

$$d_k = \sum_{i=1}^{k} r_i \qquad h_l = \sum_{i=n-l+1}^{n} c_i$$

for all $k = 1 \ldots, m$ and $l = 1, \ldots, n$. In particular, constraint $d_m = h_n$ is exactly the one entering in the definition of the set $\mathrm{BCT}(r, c)$.

Let $\mathcal{D}(r, c)$ be a separated brane diagram with charges $r \in \mathbb{N}^m$ and $c \in \mathbb{N}^n$ and let $\mathcal{D}(r, \tilde{c})$ be the diagram obtained by replacing the charge vector c with a resolution $\tilde{c}$ of its k-th entry. In other words, the diagram $\mathcal{D}(r, \tilde{c})$ is obtained via the local surgery

$$c'_k + c''_k \qquad\qquad c'_k \quad c''_k$$

$$\longrightarrow$$

We call this local modification of the brane diagram a D5 brane resolution.

Let $X(r, c)$ and $X(r, \tilde{c})$ be the bow varieties associated with $\mathcal{D}(r, c)$ and $\mathcal{D}(r, \tilde{c})$ respectively. Notice that, according to our conventions, $X(r, c)$ is acted on by $\mathbb{A} = (\mathbb{C}^\times)^n$ while $X(r, \tilde{c})$ is acted on by $\widetilde{\mathbb{A}} = (\mathbb{C}^\times)^{n+1}$. Throughout this section, we regard $\mathbb{A}$ as a subtorus of $\widetilde{\mathbb{A}}$ via the embedding

$$\varphi : \mathbb{A} \hookrightarrow \widetilde{\mathbb{A}} \qquad (a_1, \ldots, a_k, \ldots, a_n) \mapsto (a_1, \ldots, a_k, a_k, \ldots, a_n)$$

and consider its induced action on $X(r, \tilde{c})$.

Theorem 5.4. [2, Section 6.1] *There exists an $\mathbb{A}$-equivariant embedding $j : X(r, c) \hookrightarrow X(r, \tilde{c})$. Moreover, for any fixed point $f \in X(r, c)^{\mathbb{A}}$ the $\mathbb{A}$-fixed component $F \subseteq X(r, \tilde{c})^{\mathbb{A}}$ such that $j(f) \in F$ is isomorphic to the bow variety with brane diagram*

$$/1/2/3/\ldots/c'_k + c''_k \backslash c''_k \backslash$$

This theorem encodes all the data entering in the combinatorics described in the previous section. Indeed, we have:

(i) the $\mathbb{A}$-fixed points in $X(r, c)$ are in one to one correspondence with the set $\mathrm{BCT}(r, c)$,

(ii) the $\widetilde{\mathbb{A}}$-fixed points in $X(r, \tilde{c})$ are in one to one correspondence with the set $\mathrm{BCT}(r, \tilde{c})$,

(iii) for any $M \in \mathrm{BCT}(r, c)$ corresponding to some $f \in X(r, c)^{\mathbb{A}}$, the set $F^{\widetilde{\mathbb{A}}}$ is in one to one correspondence with the fiber $q^{-1}(M) \subset \mathrm{BCT}(r, \tilde{c})$.

5.3. **Maximal resolutions.** For any $d \in \mathbb{N}$ such that $d = \sum_{k=1}^{m} r_k$, let $\mathrm{Fl}(r, d)$ be the variety parametrizing quotients[3]

$$\mathbb{C}^d = V_m \twoheadrightarrow V_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow V_1 \twoheadrightarrow V_0 = 0$$

where $\dim(V_k) = d_k = \sum_{i=1}^{k} r_k$. In particular, for $1^d = (1, 1, \ldots, 1) \in \mathbb{N}^d$ we get the full flag variety $\mathrm{Fl}(1^d, d)$. The maximal torus $\widetilde{\mathbb{A}} \subset \mathrm{GL}(d)$ naturally acts on $\mathrm{Fl}(r, d)$ and hence also on its cotangent bundle $\mathrm{T}^*\mathrm{Fl}(r, d)$. For a given $c \in \mathbb{N}^n$ such that $\sum_{i=1}^{m} c_i = \sum_{i=1}^{m} r_i$, let $\mathbb{A} = (\mathbb{C}^\times)^n \subset \widetilde{\mathbb{A}}$ be the subtorus acting on V_m with weight decomposition $V_m = \mathbb{C}^{d_m} = a_1 \mathbb{C}^{c_1} \oplus \cdots \oplus a_n \mathbb{C}^{c_n}$.

Applying Theorem 5.4 c_i times for every D5 brane in $X(r, c)$, we get

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[3]The choice of considering quotients rather than inclusions is due to our choice of stability condition on bow varieties, see [14, 2].

Corollary 5.5. *There exists an $\mathbb{A}$-equivariant embedding $X(r,c) \hookrightarrow \mathrm{T}^*\mathrm{Fl}(r,d)$. Moreover, for any fixed point $f \in X(r,c)^{\mathbb{A}}$ the $\mathbb{A}$-fixed component $F \subseteq \mathrm{T}^*\mathrm{Fl}(r,d)^{\mathbb{A}}$ such that $j(f) \in F$ is the n-fold fiber product*

$$(2) \qquad F = \mathrm{T}^*\mathrm{Fl}(1^{c_1}, c_1) \times \cdots \times \mathrm{T}^*\mathrm{Fl}(1^{c_n}, c_n).$$

Definition 5.6. *Let $f \in X(r,c)^{\mathbb{A}}$ be a fixed point corresponding to an element $M \in \mathrm{BCT}(r,c)$. The associated fixed points $F^{\widetilde{\mathbb{A}}} \subset X^{\widetilde{\mathbb{A}}}$ are called maximal resolutions of f. Similarly, the BCT tables corresponding to the fixed points $F^{\widetilde{\mathbb{A}}} \subset X^{\widetilde{\mathbb{A}}}$ are called maximal resolutions of M.*

The combinatorial interpretation of this corollary is a straightforward generalization of the points (i)-(iii) above. Each term in the n-fold fiber product (2) corresponds to a column of M, and the $c_i!$ fixed points in $\mathrm{T}^*\mathrm{Fl}(1^{c_i}, c_i)$ correspond to all possible ways of "resolving" the i-th column of M with c_i columns with exactly one nonzero entry. Hence, altogether, the maximal resolutions $\tilde{f} \in F^{\widetilde{\mathbb{A}}}$ correspond to the $\prod_{i=1}^{n} c_i!$ ways of "maximally resolving" M. The resulting matrices $\widetilde{M} \in \mathrm{BCT}(r, 1^d)$ are exactly the fixed points in $\mathrm{T}^*\mathrm{Fl}(r,d)$.

Example 5.7. *Consider the bow variety $X(r,c) = $ /1/2/4\2\. Then, $r = (1,1,2)$ and $c = (2,2)$, and two fixed points in $X(r,c)^{\mathbb{A}}$ are described by the BCTs*

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Each BCT admits four maximal resolutions:

$$M_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \quad M_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \quad M_3 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad M_4 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$N_1 = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad N_2 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad N_3 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad N_4 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The case when X has exactly two D5 branes, and hence its BCTs have two rows, will be the most relevant for us. As a consequence, we invite the reader to keep in mind the example above.

5.4. **Second proof of Theorem 3.4.** Theorem 4.3 provides an explicit description of all invariant curves in a bow variety $X(r,c)$ in terms of certain combinatorial operations for BCT tables and, hence, opens the way to our first proof of Theorem 3.4. However, the results introduced in the previous section suggest an alternative approach to its proof. Indeed, instead of classifying all the invariant curves, we can exploit Corollary 5.5 to control them in terms of the invariant curves in a partial flag variety $\mathrm{Fl}(r,d)$. As we will see, this will be the key idea behind our second proof of the main theorem.

We begin with two technical lemmas. The first one characterizes the secondary Bruhat order for BCT tables with two columns, and the second one provides a geometric analog of Lemma 5.3.

Lemma 5.8. *Let M, M' be two BCTs with n rows and exactly two columns. Then $M' \leqslant_{\hat{B}} M$ if and only if $M' = M + Z$ where $Z \in \mathrm{Mat}(n, 2)$ such that*

  (i) *each row of Z is either $(0, 0)$, $(1, -1)$ or $(-1, 1)$.*
  (ii) *$\#\{(1, -1)$ among the first k rows of $Z\} \geqslant \#\{(-1, 1)$ among the first k rows of $Z\}$ for all $k \leqslant n$.*
  (iii) *$\#\{(1, -1)$ in $Z\} = \#\{(-1, 1)$ in $Z\}$.*

We invite the reader to test this result in the situation described in Figure 2.

Proof. For a given matrix $A \in \mathrm{Mat}(2, 2)$, let $A^{ij} \in \mathrm{Mat}(n, 2)$ be the matrix all whose rows are $(0, 0)$ except from the i-th and the j-th, which we set to be equal to the first and the second row of A, respectively. Note that the matrices L_2 and I_2 involved in the transition defining the secondary Bruhat order are related by

$$L_2 + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = I_2.$$

As a consequence, if $M' \leqslant_{\hat{B}} M$ then $M' = M + \sum_k {}_+ Z$, where

(3) $$Z = \sum_{k=1}^{K} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^{i_k j_k}$$

for some $i_k, j_k \leqslant n$ and $K \in \mathbb{N}$. As a consequence, the rows of Z are of the form $n_1(1, -1) + n_2(-1, 1)$ for some $n_1, n_2 \in \mathbb{N}$. But since the rows of M and M' are either $(0, 0)$, $(1, 0)$, $(0, 1)$ or $(1, 1)$, the condition $M' = M + \sum_k Z$ forces (i). Conditions (ii) and (iii) are then immediate consequences of equation (3).

Conversely, given a Z satisfying (i)-(iii) such that $M' = M + Z$, let K be the number of $(1, -1)$-type (equivalently $(-1, 1)$-type by (iii)) rows in Z. For $k = 1, \ldots, K$, let i_k (resp. j_k) be the column where the k-th row of type $(1, -1)$ (resp. of type $(-1, 1)$) appears. Then, Z is of the form (3) and each term in the summation corresponds to a $L_2 \to I_2$ interchange, hence $M' \leqslant_{\hat{B}} M$. $\qquad\square$

Lemma 5.9. *Let $A = A_1 \times A_2$ be a torus acting on a complete variety X and let F, G be two connected components for the action of the subtorus $A_1 \subset A$. Let $\sigma : \mathbb{C}^\times \to A$ be a generic cocharacter and denote by $\sigma_1 : \mathbb{C}^\times \to A_1$ the composition of σ with the projection $A \to A_1$. Then $\overline{\mathrm{Att}_{\sigma_1}(F)} \cap G \neq 0$ iff there exist A-fixed points $\tilde{f} \in F^A$, $\tilde{g} \in G^A$ such that $\overline{\mathrm{Att}_\sigma(f)} \cap g \neq 0$.*

Proof. The proof is analog to Lemma 3.6. Assume first that $\overline{\mathrm{Att}_\sigma(F)} \cap G \neq 0$, or, equivalently, that there exists a point $x \in X$ and an orbit $\mathbb{C}^\times \ni t \mapsto \sigma_1(t) \cdot x$ such that $\lim_{t \to 0} \sigma_1(t) \cdot x \in F$ and $\lim_{t \to \infty} \sigma_1(t) \cdot x \in G$. The cocharacter $\sigma : \mathbb{C}^\times \to A$ splits as

$$\begin{array}{ccc}
\mathbb{C}^\times & \xrightarrow{\ \sigma\ } & A_1 \times A_2 \\
{\scriptstyle \Delta} \downarrow & \nearrow & \\
\mathbb{C}^\times \times \mathbb{C}^\times & {\scriptstyle \sigma_1 \times \sigma_2} &
\end{array}$$

with $\sigma_1 \in \mathfrak{C}$. Since σ is generic for the action of A, the cocharacters σ_1 and σ_2 are generic for the actions of A_1 and A_2, respectively. Consider the action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on X induced by $\sigma_1 \times \sigma_2$ and the map

$$(4) \qquad \Gamma : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p^{-1}} \overline{\mathrm{Graph}(x)} \to X$$

introduced in Lemma A.2. The closure of the orbit $\mathbb{C}^\times \ni t \mapsto \sigma_1(t) \cdot x$ is contained in the image of (4), and in particular the limit points $\lim_{t \to 0} \sigma_1(t) \cdot x$ and $\lim_{t \to \infty} \sigma_1(t) \cdot x$ are contained in the images of $\{0\} \times \mathbb{P}^1$ and $\{\infty\} \times \mathbb{P}^1$ respectively. Since the cocharacter σ_1 is generic, by the second point of Lemma (4) these limit points are fixed by A_1. Then, by connectedness, it follows that

$$(5) \qquad \Gamma(\{0\} \times \mathbb{P}^1) \subseteq F \qquad \Gamma(\{\infty\} \times \mathbb{P}^1) \subseteq G.$$

Now consider the orbit $\sigma(t) \cdot x$. It can be seen as the image of the composition

$$\mathbb{C}^\times \xrightarrow{\Delta} \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p^{-1}} \overline{\mathrm{Graph}(x)} \to X$$

Its limit point $\lim_{t \to 0} \sigma(t) \cdot x$ coincides with $\Gamma(0 \times 0)$, hence (5) implies that $f := \lim_{t \to 0} \sigma(t) \cdot x \in F$. By the third point of Lemma A.2, the point f is fixed by the action of $\mathbb{C}^\times \times \mathbb{C}^\times$ induced by $\sigma = \sigma_1 \times \sigma_2$. Since the cocharacters are generic, the limit point f is even fixed by the whole $A = A_1 \times A_2$. Analogously, one can argue that $g := \lim_{t \to \infty} \sigma(t) \cdot x$ is equal to $\Gamma(\infty \times \infty)$, belongs to G, and is A-fixed. Overall, we have shown that $\lim_{t \to 0} \sigma(t) \cdot x = f \in F^A$ and $\lim_{t \to 0} \sigma(t) \cdot x = g \in G^A$, which implies that $\overline{\mathrm{Att}_\sigma(f)} \cap g \neq 0$.

The opposite direction is analogue. In this case one assumes the existence of an orbit $\sigma(t) \cdot x$ with limit points at f and g and shows that the orbit $\sigma_1(t) \cdot x$ connects F to G. Details are left to the reader. As before, the closure of $\sigma(t) \cdot x$ corresponds to the diagonal $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, while $\sigma_1(t) \cdot x$ corresponds to $\mathbb{P}^1 \times \{1\} \hookrightarrow \mathbb{P}^1 \times \mathbb{C}^\times \subset \mathbb{P}^1 \times \mathbb{P}^1$. $\qquad \square$

We are now ready to give an alternative proof of our main theorem.

Second proof of Theorem 3.4. The {combinatorial} $\implies$ {geometric} direction is the easiest part of the proof. Indeed, an attracting invariant curve connecting two fixed points such that $M \prec_{\hat{B}} M'$ can be explicitly constructed following [17, Section 3.4.1]. Therefore, we only prove the opposite direction.

For convenience, we denote by M_X a BCT table defining a fixed point in the bow variety X. It suffices to show that if there exists an invariant curve $\gamma : \mathbb{P}^1 \to X$ with attractive tangent weight $\chi \in \mathrm{Char}(\mathbb{A})$ at 0 and such that $\gamma(0) = M_X$ and $\gamma(\infty) = M_X'$, then $M_X' < M_X$.

First, we reduce ourselves to the case where there are only two D5 branes. This corresponds to the case where the BCT table has only two columns. By [14, Section 4.6], the tangent weight χ of γ at 0 must be of the form a_i/a_j for some $i \neq j \in \{1, \dots, n\}$. Let $B \subset \mathbb{A}$ be the sub-torus given by the equation $a_i = a_j$. Then, the curve γ is contained in some connected component $Y \subset X^B$. Moreover, its endpoints M_X and M_X' correspond to two $\mathbb{A}/B$-fixed points in Y. By [2, Theorem 3.1], every fixed component in X^B is isomorphic to a bow variety with exactly two branes (these two branes correspond to the i-th and j-th D5 branes in X). Let M_Y and M_Y' be the BCT tables describing the fixed points $\gamma(0), \gamma(\infty) \in Y^{\mathbb{A}/B}$. The matrix M_Y (resp. M_Y') is simply obtained from M_X (resp. M_X') by erasing all but the i-th and j-th columns. In particular, if we show

that $M'_Y <_{\hat{B}} M_Y$, then $M'_X <_{\hat{B}} M_X$ follows. Moreover, since by construction γ is contained in $Y \subset X$, it follows that $M'_Y < M_Y$. Overall, we have shown that

$$M'_X <_G M_X \implies M'_Y <_G M_Y \qquad M'_Y \leqslant_{\hat{B}} M_Y \implies M'_X \leqslant_{\hat{B}} M_X.$$

Hence, it suffices to prove that $M'_Y <_G M_Y \implies M'_Y <_{\hat{B}} M_Y$. In words, we have reduced ourselves to the case of a bow variety with only two D5 branes (equivalently, to the case of BCTs with only two columns). This concludes the first step of the proof.

In the second step, we exploit the results of the previous section to constrain the BCT table of a fixed point M'_Y satisfying $M'_Y <_G M_Y$. To ease the notation, we drop the subscripts in the fixed points since there is no more risk of confusion. By Corollary 2, there is an $\mathbb{A}$-equivariant embedding $Y \hookrightarrow \mathrm{T}^*\mathrm{Fl}$, where Fl is a partial flag variety. Let $F \subseteq \mathrm{T}^*\mathrm{Fl}^{\mathbb{A}}$ (resp. $G \subseteq \mathrm{T}^*\mathrm{Fl}^{\mathbb{A}}$) be the $\mathbb{A}$-fixed component containing M (resp. M'). Let F_0 (resp. G_0) be the intersection of F (resp. G) with the zero section $\mathrm{Fl} \subset \mathrm{T}^*\mathrm{Fl}$. They are both isomorphic to the fiber products of two full flag varieties. Since by hypothesis M and M' are connected by a chain of invariant curves, the same is true for F and G and hence for F_0 and G_0 via the projection $\mathrm{T}^*\mathrm{Fl} \to \mathrm{Fl}$. Therefore, we deduce that $\overline{\mathrm{Att}_\sigma(F_0)} \cap G_0 \neq \varnothing$. Let $\widetilde{\mathbb{A}} \supset \mathbb{A}$ be the maximal torus acting on Fl from the framing.

Choose a splitting $\widetilde{\mathbb{A}} = \mathbb{A} \times \widetilde{\mathbb{A}}/\mathbb{A}$ and a cocharacter $\lambda_{\tilde{\sigma}} : \mathbb{C}^\times \to \widetilde{\mathbb{A}}$ such that its projection to $\mathbb{A}$ is $\lambda_\sigma : \mathbb{C}^\times \to \mathbb{A}$. By Lemma 5.9 below, it follows that that there exist $\widetilde{\mathbb{A}}$-fixed points $\widetilde{M}, \widetilde{M'} \in \mathrm{Fl}^{\widetilde{\mathbb{A}}}$ such that $\overline{\mathrm{Att}_{\tilde{\sigma}}(\widetilde{M})} \cap \widetilde{M'} \neq \varnothing$. Notice that $\widetilde{M}$ (resp. $\widetilde{M'}$) is a maximal resolution of M (resp. M') in the sense of Definition 5.6. Since for flag varieties either combinatorial Bruhat orders coincide with the geometric order—see Remark 3.3—we deduce that $\widetilde{M'}$ is obtained from $\widetilde{M}$ by a sequence of $L_2 \to I_2$ interchanges. If one of these interchanges occurs among the columns resolving a single column of $\widetilde{M}$, they leave M unchanged. If instead they involve columns of $\widetilde{M}$ resolving both columns of M, they have the effect of adding to M a matrix of the form

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Hence, we deduce that $M' = M + Z$, where

$$Z = \sum_k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^{i_k j_k}$$

for some indices $1 \leqslant i_k, j_k \leqslant m$. But then the same argument used in the proof of Lemma 5.8 shows that the assumptions of the latter are satisfied. Hence the $M <_{\hat{B}} M'$ and the proof follows. $\square$

Appendix A.

In this appendix, we record some results about orbits of torus actions on complete algebraic varieties. In the first lemma, which is classical, we recall the characterization of $\mathbb{C}^\times$-oribits closures in terms of morphisms from $\mathbb{P}^1$. Then, we extend the result to rank two tori. Although

this second lemma is also probably well known, we could not find a reference for it. Hence, we provide a proof.

Lemma A.1 (Lemma 2.4.1 in [6])**.** *Let X be a complete variety and let $\mathbb{C}^\times$ be one dimensional torus acting on X. Fix $x \in X$ and let $\overline{\mathrm{Graph}(x)}$ be the closure in $\mathbb{P}^1 \times X$ of $\mathrm{Graph}(x) = \{(z, z \cdot x) \in \mathbb{C}^\times \times X \mid z \in \mathbb{C}^\times\}$. Then*

    (1) *The projection $p : \overline{\mathrm{Graph}(x)} \to \mathbb{P}^1$ is an isomoprhism.*

    (2) *The image of the points $0, \infty \in \mathbb{P}^1$ under the composition $\mathbb{P}^1 \xrightarrow{p^{-1}} \overline{\mathrm{Graph}(x)} \to X$ are fixed points for the $\mathbb{C}^\times$-action.*

With this well known result in mind, we extend the previous result to rank two tori. In principle, we could admit an arbitrary rank, but this level of generality suffices for our scopes.

Lemma A.2. *Let X be a complete variety and let $\mathbb{C}^\times \times \mathbb{C}^\times$ act on X via $m : \mathbb{C}^\times \times \mathbb{C}^\times \times X \to X$. Fix $x \in X$ and let $\overline{\mathrm{Graph}(x)}$ be the closure in $\mathbb{P}^1 \times \mathbb{P}^1 \times X$ of*

$$\{(z, w, m(z, w, x)) \in \mathbb{C}^\times \times \mathbb{C}^\times \times X \mid (z, w) \in \mathbb{C}^\times \times \mathbb{C}^\times\}.$$

Then

    (1) *The projection $p : \overline{\mathrm{Graph}(x)} \to \mathbb{P}^1 \times \mathbb{P}^1$ is an isomorphism.*

    (2) *The images of the curves $\{0\} \times \mathbb{P}^1$ and $\{\infty\} \times \mathbb{P}^1$ under the composition $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p^{-1}} \overline{\mathrm{Graph}(x)} \to X$ are fixed by the action of the one dimensional torus $\mathbb{C}^\times \times \{1\} \subset \mathbb{C}^\times \times \mathbb{C}^\times$. Similarly, the images of $\mathbb{P}^1 \times \{0\}$ and $\mathbb{P}^1 \times \{\infty\}$ are fixed under the action of the one dimensional torus $\{1\} \times \mathbb{C}^\times \subset \mathbb{C}^\times \times \mathbb{C}^\times$.*

    (3) *The images of the "corners" $0 \times 0, 0 \times \infty, \infty \times 0, \infty \times \infty \in \mathbb{P}^1 \times \mathbb{P}^1$ are fixed under the action of $\mathbb{C}^\times \times \mathbb{C}^\times$.*

Proof. Since $\overline{\mathrm{Graph}(x)}$ is connected and $\mathbb{P}^1 \times \mathbb{P}^1$ is normal, it suffices to show that $\pi : \overline{\mathrm{Graph}(x)} \to \mathbb{P}^1 \times \mathbb{P}^1$ is bijective on $\mathbb{C}$-points. The proof then essentially follows by iteration of Lemma A.1. Fix $z \in \mathbb{C}^\times \subset \mathbb{P}^1$ and consider the following pullback diagram

$$
\begin{array}{ccc}
\overline{\mathrm{Graph}(x)}_z & \longrightarrow & \overline{\mathrm{Graph}(x)} \\
\downarrow & & \downarrow \\
\{z\} \times \mathbb{P}^1 \times X & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \times X
\end{array}
$$

By construction, $\overline{\mathrm{Graph}(x)}_z$ is closed in $\{z\} \times \mathbb{P}^1 \times X$. Indeed, it coincides with the closure of the orbit $\{(w, m(z, w, x)) \in \mathbb{C}^\times \times X \mid w \in \mathbb{C}^\times\}$. By the previous Lemma A.1, we deduce that the composition $\overline{\mathrm{Graph}(x)}_z \to \{z\} \times \mathbb{P}^1 \times X \to \{z\} \times \mathbb{P}^1$ is an isomorphism for all $z \in \mathbb{C}^\times$. As a consequence, it follows that the restriction of p on the preimage of $\mathbb{C}^\times \times \mathbb{P}^1$ is a bijection. Swapping the order of the two $\mathbb{C}^\times$, one similarly shows that the restriction of p to the preimage of $\mathbb{P}^1 \times \mathbb{C}^\times$ is also a bijection. Altogether, we deduce that the restriction of p to

$\mathbb{P}^1 \times \mathbb{P}^1 \backslash \{0 \times 0, 0 \times \infty, \infty \times 0, \infty \times \infty\}$ is bijective. Now consider the pullbacks

$$
\begin{array}{ccc}
\overline{\mathrm{Graph}(x)}_\Delta & \hookrightarrow & \overline{\mathrm{Graph}(x)} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times X & \xrightarrow{\Delta \times id} & \mathbb{P}^1 \times \mathbb{P}^1 \times X
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathrm{Graph}(x)}_\nabla & \hookrightarrow & \overline{\mathrm{Graph}(x)} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times X & \xrightarrow{\nabla \times id} & \mathbb{P}^1 \times \mathbb{P}^1 \times X
\end{array}
$$

along the diagonal $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and the antidiagonal $\nabla : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ respectively (here ∇ is the composition of Δ with the inversion $w \to 1/w$ on the second factor). The variety $\overline{\mathrm{Graph}(x)}_\Delta$ corresponds to the image of the orbit $\{(z, z, m(z, z, x)) \in \mathbb{C}^\times \times X \mid z \in \mathbb{C}^\times\}$. Likewise, the variety $\overline{\mathrm{Graph}(x)}_\nabla$ corresponds to the image of the orbit $\{(z, 1/z, m(z, 1/z, x)) \in \mathbb{C}^\times \times X \mid z \in \mathbb{C}^\times\}$. Again, by Lemma A.1 it follows that the restrictions of p to the preimages of Δ and ∇ are bijections. Since these restrictions clealry agree on their common intersection with the restriction of p on the preimage of $\mathbb{P} \times \mathbb{P} \backslash \{0 \times 0, 0 \times \infty, \infty \times 0, \infty \times \infty\}$, we conclude that $p : \overline{\mathrm{Graph}(x)} \to \mathbb{P}^1 \times \mathbb{P}^1$ is bijective, as desired. This completes the proof of the first point.

We now prove the second point. By construction, the image of $\{0\} \times \mathbb{C}^\times$ is fixed by $\mathbb{C}^\times \times \{1\}$. Since fixed loci are closed, also the preimage of $\{0\} \times \mathbb{P}^1$ is closed. This proves the claim for the first curve. The other cases are analog.

As for the third point, simply notice that by point (2) each of the "diagonal" points $0 \times 0, 0 \times \infty, \infty \times 0, \infty \times \infty \in \mathbb{P}^1 \times \mathbb{P}^1$ lie at the intersection of two curves. One of these curves is fixed by $\mathbb{C}^\times \times \{1\}$, while the other is fixed by $\{1\} \times \mathbb{C}^\times$, hence their intersection if fixed by the whole torus $\mathbb{C}^\times \times \mathbb{C}^\times$.

$\square$

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