RESIDUES, GROTHENDIECK POLYNOMIALS AND K-THEORETIC THOM POLYNOMIALS

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Abstract. Grothendieck polynomials were introduced by Lascoux and Schützenberger, and play an important role in K-theoretic Schubert calculus. In this paper, we give a new definition of double stable Grothendieck polynomials based on an iterated residue operation. We illustrate the power of our definition by calculating the Grothendieck expansion of K-theoretic Thom polynomials of $\mathcal{A}_2$ singularities. We present this expansion in two versions: one displays its stabilization property, while the other displays its expected finiteness property.

1. Introduction

From the point of view of enumerative geometry, a very important invariant of a subvariety X in a smooth variety M is its cohomological fundamental class $[X \subset M] \in H^{\operatorname{codim}(X\subset M)}(M)$, obtained from the homology fundamental class by Poincaré duality. A key technique to approach this invariant is to pass to the local equivariant version. Let G be a reductive group, J be a vector space endowed with a G-action, and η be a G-invariant subvariety of J. Then the G-equivariant fundamental class

$$[\eta \subset J] \in H_G^{\operatorname{codim}(\eta\subset J)}(J) = H_G^{\operatorname{codim}(\eta\subset J)}(\mathrm{pt})$$

is an invariant polynomial. The study of this invariant is also known as *degeneracy locus theory* (see eg. [FP, BF, MS, BSz]).

We encounter this setup, for example, in a branch of Schubert calculus where J is a representation vector space of a quiver and the fundamental class is called a *quiver polynomial*, see e.g. [BF, KMS, B6]. Another instance is *global singularity theory*, where J is the vector space of germs (or jets) of maps acted upon by reparametrization groups, and the fundamental class is called the *Thom polynomial* [T, Ri, BSz] of the singularity.

In this paper we will be concerned with the notion of *G-equivariant K-theoretic fundamental class* $[\eta \subset J] \in K_G(J) = K_G(\mathrm{pt})$ of an invariant subvariety η of a G-representation J. In fact, there are at least two inequivalent notions that may be called "K-theoretic fundamental class":

- the class of the structure sheaf of η,
- the push-forward of the class of the structure sheaf of a resolution.

These two notions coincide if η has rational singularities, but not in general (cf. Section 5). The non-ambiguous notion of cohomological fundamental class can be recovered from either of these two K-theoretic fundamental class notions via a limiting procedure. For a review of these two notions, as well as a third notion of K-theoretic fundamental class, see [F, Section 2].

K-theoretic fundamental classes have been computed and studied in numerous situations, in particular, in Schubert calculus, for quivers, and for matroids, (see [B6, BFi] and references therein). At the risk of oversimplification, we can say that the vector spaces J in these situations are direct sums of spaces of linear maps: $\mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b)$, and the group G linearly reparametrizes the vector spaces $\mathbb{C}^a$ and $\mathbb{C}^b$. Linear maps may be thought of as local 1-jets of general maps between manifolds. In this paper, we leave the realm of 1-jets, and our principal object of study, the variety $\mathcal{A}_2$ defined below, is in a 2-jet space of the form $\mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b) \oplus \mathrm{Hom}(S^2\mathbb{C}^a, \mathbb{C}^b)$. A key novel aspect of this variety is that it has non-rational singularities.

It is customary to express $\mathrm{GL}_n(\mathbb{C})$-equivariant cohomological fundamental classes in terms of Schur polynomials in the Chern roots of $\mathrm{GL}(n)$. One reason is for this is that Schur polynomials are themselves cohomological fundamental classes of some basic varieties: the (matrix) Schubert varieties. A key feature of these Schur-expansions is that their coefficients are often positive. More generally, when G is a product of general linear groups, the cohomological fundamental classes are still often expressible as positive linear combinations of some flavors of Schur polynomials, and the coefficients are often related to interesting objects in combinatorics or algebra. See [BF, KMS, BR] for examples of this phenomenon for quiver polynomials, and [PW] for Thom polynomials.

In K-theory, the role of Schur polynomials is played by Grothendieck polynomials, as they are the K-theoretic fundamental classes of Schubert varieties. The expectation is that K-theoretic fundamental classes, when G is a product general linear groups, may be expressed as linear combinations of some flavors of Grothendieck polynomials with coefficients that have *alternating signs* (see for example [B5, M, B6]). In Section 8, we will show that this expectation holds for $\mathcal{A}_2$, in fact, for two different Grothendieck polynomial expansions.

In the process of developing these expansions, we needed a new formula for (double stable) Grothendieck polynomials. This lead us to a novel residue calculus for double stable Grothendieck polynomials, which we present in §4. We briefly describe this formula below, and then present our main result.

1.1. **Grothendieck polynomials, Grothendieck expansions.** In Section 2, we recall the original definition of double stable Grothendieck polynomials [FK1, FK2]. This involves first introducing *ordinary Grothendieck polynomials* $\mathfrak{G}_w$, indexed by permutations and defined by a recursion involving divided differences. Geometrically, the polynomials $\mathfrak{G}_w$ represent torus-equivariant K-theoretic fundamental classes of Schubert varieties in full flag varieties [LS]. Next, *double stable Grothendieck polynomials* $G_\lambda(\alpha;\beta)$ parametrized by partitions are defined by a limiting procedure from ordinary Grothendieck polynomials, where α and β are sequences of variables.

Finally, applying to these latter polynomials a set of certain *straightening laws*, one defines double stable Grothendieck polynomials $G_I(\alpha;\beta)$ parametrized by arbitrary integer sequences. Another approach to double stable Grothendieck polynomials parametrized by partitions uses the combinatorics of set-valued tableaux [B4].

In §4.1, we propose a new formula for the most general integer-sequence parametrized double stable Grothendieck polynomials:

$$(1)\quad G_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l)=$$
$$\operatorname*{Res}_{z_1=0,\infty}\dots\operatorname*{Res}_{z_r=0,\infty}\left(\prod_{j=1}^r(1-z_j)^{I_j-j}\prod_{i>j}\left(1-\frac{z_i}{z_j}\right)\prod_{j=1}^r\frac{\prod_{i=1}^l(1-z_j\beta_i)}{\prod_{i=1}^k(1-z_j\alpha_i)(1-z_j)^{l-k}}\prod_{j=1}^r\frac{dz_j}{z_j}\right).$$

This formula is analogous to the useful residue formula

$$(2)\quad s_I(\bar\alpha_1,\dots,\bar\alpha_k;\bar\beta_1,\dots,\bar\beta_l)=$$
$$(-1)^r\operatorname*{Res}_{z_1=\infty}\dots\operatorname*{Res}_{z_r=\infty}\left(\prod_{j=1}^r z_j^{I_j}\prod_{j>i}\left(1-\frac{z_i}{z_j}\right)\prod_{j=1}^r\frac{\prod_{i=1}^l(1+\bar\beta_i/z_j)}{\prod_{i=1}^k(1+\bar\alpha_i/z_j)}\cdot\prod_{j=1}^r\frac{dz_j}{z_j}\right)$$

for the double stable Schur polynomials (see e.g. [FR3, Lemma 6.1]). Note that in the case of Schur polynomials, the residues are taken only at infinity, while for Grothendieck polynomials, one takes the sum of the residues at 0 and infinity.

Let us explain how our formula (1) helps us to find Grothendieck expansions of certain functions of α_i, β_i. Assume that a function $T=T(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_l)$ is presented in the form

$$(3)\qquad T=\operatorname*{Res}_{z_1=0,\infty}\dots\operatorname*{Res}_{z_r=0,\infty}\left(F\cdot\prod_{i>j}\left(1-\frac{z_i}{z_j}\right)\prod_{j=1}^r\frac{\prod_{i=1}^l(1-z_j\beta_i)}{\prod_{i=1}^k(1-z_j\alpha_i)(1-z_j)^{l-k}}\prod_{j=1}^r\frac{dz_j}{z_j}\right),$$

for some polynomial function $F(z_1,\dots,z_r)$ of the form $F=\sum_I c_I\prod_{j=1}^r(1-z_j)^{I_j-j}$. Then, since the transformation

$$(4)\qquad F\mapsto\operatorname*{Res}_{z_1=0,\infty}\dots\operatorname*{Res}_{z_r=0,\infty}(F\cdot\text{fixed kernel function}(z,\alpha,\beta))$$

is linear in F, we conclude that the function T has Grothendieck expansion $\sum_I c_Ig_I$. We note that the Grothendieck polynomials g_I, which appear in this expansion, are *not* linearly independent. Now, unfortunately, it is too much to ask that we obtain Thom polynomials in the form of (3) with *polynomial* F — in practice, F is often a rational function. Yet, this consideration indicates that the Grothendieck expansion of (3) for general F should be related to a Laurent expansion of F at $z_i=1$. We will give an idea below how to perform this calculation. A detailed analysis will be carried out in Sections 4.3, 8.3, 8.4.

1.2. **K-theoretic Thom polynomials of singularities.** The general reference for singularities of maps is [AVGL]. For a positive integer N, denote by $R^N(\mathbb{C}^a)$ the algebra of N-jets of functions on $\mathbb{C}^a$ at 0; this is the ring of polynomials in a variables modulo monomials of degree at least $N+1$. Let $J^N(\mathbb{C}^a,\mathbb{C}^b)$ be the space of N-jets of maps $(\mathbb{C}^a,0)\to(\mathbb{C}^b,0)$ vanishing at 0. An element of $J^N(\mathbb{C}^a,\mathbb{C}^b)=R^N(\mathbb{C}^a)^b$ is given by a b-tuple of jets from the maximal ideal of $R^N(\mathbb{C}^a)$. A *singularity* η is an algebraic subvariety of $J^N(\mathbb{C}^a,\mathbb{C}^b)$ invariant under the group of formal holomorphic reparametrizations of $(\mathbb{C}^a,0)$ and $(\mathbb{C}^b,0)$ (cf. e.g. [BSz]).

An important set of examples of singularities, called *contact singularities*, is obtained as follows. A reparametrization invariant of N-jets of functions is the *local algebra*, defined for $h = (h_1(x_1,\dots,x_a),\dots,h_b(x_1,\dots,x_a)) \in J^N(\mathbb{C}^a,\mathbb{C}^b)$ as the ideal quotient $R^N(\mathbb{C}^a)/(h_1,\dots,h_b)$. Then for a fixed finite-dimensional local commutative algebra Q and nonnegative integers $a \le b$, we can define the singularity $\eta_Q^{a\to b}$ as the Zariski closure of the set

$$\{g \in J^N(\mathbb{C}^a,\mathbb{C}^b) :\ \text{the local algebra of } g \text{ is isomorphic to } Q\}.$$

(We will omit the dimensions a and b from the notation when this causes no confusion.)

Denote the group of linear reparametrizations $\mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$ by $\mathrm{GL}[a \to b]$, and observe that the space $J^N(\mathbb{C}^a,\mathbb{C}^b)$ is equivariantly contractible, hence we have the identification with the symmetric polynomials:

$$H^*_{\mathrm{GL}[a\to b]}(J^N(\mathbb{C}^a,\mathbb{C}^b)) = H^*_{\mathrm{GL}[a\to b]}(\mathrm{pt}) = \mathbb{Z}[\bar\alpha_1,\dots,\bar\alpha_a,\bar\beta_1,\dots,\bar\beta_b]^{S_a\times S_b},$$
$$K_{\mathrm{GL}[a\to b]}(J^N(\mathbb{C}^a,\mathbb{C}^b)) = K_{\mathrm{GL}[a\to b]}(\mathrm{pt}) = \mathbb{Z}[\alpha_1^{\pm1},\dots,\alpha_a^{\pm1},\beta_1^{\pm1},\dots,\beta_b^{\pm1}]^{S_a\times S_b},$$

where S_m is the permutation group on m elements, and $\bar\alpha_i$ and $\bar\beta_j$ are the cohomological, while α_i, and β_j are the K-theoretic Chern roots of the standard representation of $\mathrm{GL}_a(\mathbb{C})$ and $\mathrm{GL}_b(\mathbb{C})$, correspondingly.

In §5, we recall the definition of the equivariant Poincaré dual class $[X]$ of an invariant algebraic subvariety X in a vector space V acted upon by a Lie group. Using this notion, we define the *Thom polynomial* of the singularity η as the equivariant Poincaré dual

$$\mathrm{Tp}_\eta^{a\to b} = [\eta] \in H^*_{\mathrm{GL}[a\to b]}(J^N(\mathbb{C}^a,\mathbb{C}^b)).$$

The analogous K-theoretic notion

$$\mathrm{KTp}_\eta^{a\to b} = [\eta]^K \in K_{\mathrm{GL}[a\to b]}(J^N(\mathbb{C}^a,\mathbb{C}^b))$$

is, in fact, problematic when η has non-rational singularities, and we will discuss its definition in detail in Section 5 as well.

To simplify our notation, we will denote the Thom polynomial of the contact singularity η_Q as Tp_Q (and KTp_Q) when this causes no confusion. Consider the example of $Q = \mathcal{A}_2 = \mathbb{C}[x]/(x^3)$. We will write formulas for $\mathrm{Tp}_{\mathcal{A}_2}$ in terms of Schur functions $s_\lambda = s_\lambda(\bar\alpha_1,\dots,\bar\alpha_a,\bar\beta_1,\dots,\bar\beta_b)$ defined in (2), or equivalently, by the more standard definition $s_\lambda = \det(c_{\lambda(i)+j-i})$ with

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{\prod_{i=1}^b (1+\bar\beta_i t)}{\prod_{i=1}^a (1+\bar\alpha_i t)}.$$

The general formula due to Ronga [R] is as follows:

$$\mathrm{Tp}_{\mathcal{A}_2}^{a\to a+l} = \sum_{i=0}^{l+1} 2^i s_{l+1+i,l+1-i}. \tag{5}$$

Here are the first few cases:

$$\mathrm{Tp}_{\mathcal{A}_2}^{a\to a} = s_{1,1} + 2s_{2,0}, \qquad \mathrm{Tp}_{\mathcal{A}_2}^{a\to a+1} = s_{2,2} + 2s_{3,1} + 4s_{4,0}, \qquad \mathrm{Tp}_{\mathcal{A}_2}^{a\to a+2} = s_{3,3} + 2s_{4,2} + 4s_{5,1} + 8s_{6,0}.$$

Formula (5) illustrates three key features of cohomological Thom polynomials of contact singularities:

- (stability) The Thom polynomial $\mathrm{Tp}_{\mathcal{A}_2}^{a\to b}$ only depends on the *relative dimension* $b-a$ (denoted by l), not on a and b individually.
- (l-stability) We obtain $\mathrm{Tp}_{\mathcal{A}_2}^{a\to a+l}$ from $\mathrm{Tp}_{\mathcal{A}_2}^{a\to a+l+1}$ by replacing each Schur polynomial $s_{a,b}$ by $s_{a-1,b-1}$ (note that $s_{a,-1}=0$). The general statement of this property for arbitrary Q may be found in [FR1, Theorems 2.1, 4.1].
- (positivity) The coefficients of Schur expansions of Thom polynomials of contact singularities are non-negative [PW].

In §8 we calculate the K-theoretic Thom polynomials $\mathrm{KTp}_{\mathcal{A}_2}^{a\to b}$ for all $a\leq b$, and in §10 we comment on the case of higher singularities.

We will, in fact, obtain two formulas:

- the first (cf. Theorem 8.4), which we call *minimal*, is the unique representation in the basis of the Grothendieck polynomials indexed by partitions. This expression is uniquely defined but it is not l-stable.
- the second (cf. Theorem 8.2), which we will call *formal stable*, is a special representation as a formal infinite sum of Grothendieck polynomials indexed by integer sequences (furnished with a summation procedure), which *has the l-stability property* analogous to the l-stability of cohomological Thom polynomials, see Remark 8.3. This is a new phenomenon in K-theory.

Let us give a visual presentation of the relation between the two Grothendieck expansions of $\mathrm{KTp}_{\mathcal{A}_2}$. Consider the rational function

$$f(x_1,x_2)=\frac{1}{1-z_2/z_1^2}\bigg|_{z_1=1-x_1,z_2=1-x_2}=\frac{1-2x_1+x_1^2}{x_2-2x_1+x_1^2}.$$

The coefficients of its $|x_1| < |x_2|$ Laurent expansion are naturally arranged in the infinite grid as follows:

	1	x_1	x_1^2	x_1^3	x_1^4	x_1^5	x_1^6	x_1^7	x_1^8
x_2^{-1}	1	-2	1						
x_2^{-2}		2	-5	4	-1				
x_2^{-3}			$\boxed{4}$	-12	13	-6	1		
x_2^{-4}				8	-28	38	-25	8	-1
x_2^{-5}					16	-64	104	$\dots$	$\dots$
x_2^{-6}						32	$\dots$	$\dots$	$\dots.$

In the *formal stable version* of Grothendieck expansion of $\mathrm{KTp}_{\mathcal{A}_2}^{a\to a+l}$ (Theorem 8.2), these numbers are exactly the coefficients of the corresponding Grothendieck polynomials, for *any* l, with an appropriate shift. To obtain a finite expression, we sum these Grothendieck polynomials first in the vertical direction, and, as will we show, all but finitely many of these partal sums will vanish, giving a correct finite expression for $\mathrm{KTp}_{\mathcal{A}_2}$. For example, two of the vanishing terms in the formally stable expansion of $\mathrm{KTp}_{\mathcal{A}_2}^{a\to a}$ are

$$\begin{aligned} 4G_{4,0} - 12G_{4,-1} + 8G_{4,-2} &= 4G_4 - 12G_4 + 8G_4 &= 0, \\ -G_{5,0} + 13G_{5,-1} - 28G_{5,-2} + 16G_{5,-3} &= -G_5 + 13G_5 - 28G_5 + 16G_5 &= 0, \end{aligned}$$

corresponding to the x_1^3 and x_1^4 columns above.

To obtain the *minimal version* of our formula, Theorem 8.4, the coefficients of $\mathrm{KTp}_{\mathcal{A}_2,a,a+l}$ for different l's are obtained by different procedures from this grid of integers. For example, for $l = 1$ we "sweep up" all numbers from *below* the third row *to* the third row. That is, replace the $(3, k)$ entry with the sum of entries (r, k) for $r \geq 3$ and then delete the rows from the 4th one down. This sweeping is illustrated by the framed entries in the picture. In the resulting table we get the numbers (reading along the diagonals) $1, 2, 4$; $-2, -5, -12 + 8 = -4$; $1, 4, 13 - 28 + 16 = 1$; -1, and then *infinitely many 0's.* These are exactly the coefficients in the *minimal* Grothendieck expansion of $\mathrm{KTp}_{\mathcal{A}_2}^{a\to a+1}$, cf. (6). To get $\mathrm{KTp}_{\mathcal{A}_2}^{a\to a+2}$ we need to "sweep" the same table below the 4th row, for $l = 3$ we sweep from the 5th row, etc. The exact statement of this sweeping procedure is given in Theorem 8.4.

As a result, we obtain the following minimal expansions:

$$
\begin{aligned}
\mathrm{KTp}_{\mathcal{A}_2}^{a\to a} &= \big(G_{1,1}+2G_2\big)-\big(2G_{2,1}+G_3\big)+G_{3,1}\\
\mathrm{KTp}_{\mathcal{A}_2}^{a\to a+1} &= \big(G_{2,2}+2G_{3,1}+4G_4\big)-\big(2G_{3,2}+5G_{4,1}+4G_5\big)+\big(G_{4,2}+4G_{5,1}+G_6\big)-G_{6,1}\\
\mathrm{KTp}_{\mathcal{A}_2}^{a\to a+2} &= \big(G_{3,3}+2G_{4,2}+4G_{5,1}+8G_6\big)-\big(2G_{4,3}+5G_{5,2}+12G_{6,1}+12G_7\big)\\
&\quad+\big(G_{5,3}+4G_{6,2}+13G_{7,1}+6G_8\big)-\big(G_{7,2}+6G_{8,1}+G_9\big)+G_{9,1}.
\end{aligned}
\tag{6}
$$

It is remarkable that the third key feature, the positivity of cohomological Thom polynomials, extends to a rule of alternating signs for *both* of our expansions. This result will be proved in §9.

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2. Combinatorial definition of Grothendieck polynomials

In this section we will review the traditional definition of various versions of Grothendieck polynomials. We follow the references [LS, FK1, FK2, B1, B4, B6]. Our goal in Sections 2-4 is to replace these traditional definitions with the residue description of Definition 4.2. The reader not interested in the traditional definitions can take Definition 4.2 to be the definition of double stable Grothendieck polynomials and jump to Section 5.

We will use standard notations of algebraic combinatorics. A permutation $w \in S_n$ will be represented by the sequence $[w(1), w(2), \dots, w(n)]$. The length of a permutation $\ell(w)$ is the cardinality of the set $\{i < j : w(i) > w(j)\}$. We will identify S_n with its image under the natural embedding $S_n = \{w \in S_{n+1} | \, w(n+1) = n+1\}$.

2.1. Double Grothendieck polynomials.

Double Grothendieck polynomials. Double Grothendieck polynomials (in variables x_i, y_j) were introduced by Lascoux and Schutzenberger [LS]. In the present paper, following e.g. [B1], we perform the rational substitutions $x_i = 1 - 1/\alpha_i$ and $y_i = 1 - \beta_i$ in those polynomials, and denote the resulting rational functions by $\mathfrak{G}_w(\alpha, \beta)$. To keep the terminology simple, we will continue calling these functions "Grothendieck polynomials".

The functions $\mathfrak{G}_w(\alpha, \beta)$ are defined by the following recursion:

- For the longest permutation $w_0 = [n, n-1, \dots, 1] \in S_n$, let
$$\mathfrak{G}_{w_0} = \prod_{i+j\le n}\left(1 - \frac{\beta_i}{\alpha_j}\right).$$
- Let s_i be the ith elementary transposition. If $\ell(ws_i) = \ell(w) + 1$ then
$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{ws_i}),$$
where the isobaric divided difference operator π_i is defined by

$$\begin{aligned}\pi_i(f) &= \frac{\alpha_i f(\dots,\alpha_i,\alpha_{i+1},\dots) - \alpha_{i+1} f(\dots,\alpha_{i+1},\alpha_i,\dots)}{\alpha_i - \alpha_{i+1}} \\ &= \frac{f(\dots,\alpha_i,\alpha_{i+1},\dots)}{1-\alpha_{i+1}/\alpha_i} + \frac{f(\dots,\alpha_{i+1},\alpha_i,\dots)}{1-\alpha_i/\alpha_{i+1}}.\end{aligned}$$

For example, here is the list of double Grothendieck polynomials for all $w \in S_3$

$$\mathfrak{G}_{321} = \left(1-\frac{\beta_1}{\alpha_1}\right)\left(1-\frac{\beta_2}{\alpha_1}\right)\left(1-\frac{\beta_1}{\alpha_2}\right) \qquad \mathfrak{G}_{231} = \left(1-\frac{\beta_1}{\alpha_1}\right)\left(1-\frac{\beta_1}{\alpha_2}\right)$$

$$\mathfrak{G}_{312} = \left(1-\frac{\beta_1}{\alpha_1}\right)\left(1-\frac{\beta_2}{\alpha_1}\right) \qquad \mathfrak{G}_{213} = 1-\frac{\beta_1}{\alpha_1} \qquad \mathfrak{G}_{132} = 1-\frac{\beta_1\beta_2}{\alpha_1\alpha_2} \qquad \mathfrak{G}_{123} = 1.$$

2.2. **Stable versions.** For a permutation $w \in S_n$ let $1^m \times w \in S_{m+n}$ be the permutation that is the identity on $\{1,\dots,m\}$ and maps $j \mapsto w(j-m)+m$ for $j > m$. The double stable Grothendieck polynomial $G_w(\alpha,\beta)$ is defined to be

$$G_w = \lim_{m\to\infty} \mathfrak{G}_{1^m\times w}\,. \tag{7}$$

For example, $G_{21} = 1-\frac{\beta_1\beta_2\beta_3\cdots}{\alpha_1\alpha_2\alpha_3\cdots}$. The precise definition of this limit may be found in [B1]: roughly, rewritten in the x and y variables mentioned above, each coefficient of $\mathfrak{G}_{1^m\times w}$ stabilizes with m, and hence the limit is defined as a formal power series in x_i, y_j with the stabilized coefficients.

2.3. **Truncated versions.** One usually considers specializations of double stable Grothendieck polynomials of the type

$$G_w^{k,l}(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l) = G_w(\alpha_1,\dots,\alpha_k,1,1,\dots;\beta_1,\dots,\beta_l,1,1,\dots). \tag{8}$$

In fact, $G_w^{k,l}$ may be obtained by substituting $\alpha_i = 1, i > k$, $\beta_i = 1, i > l$ in $\mathfrak{G}_{1^m\times w}$ for $m \gg k,l$. This way the truncated versions (8) may be calculated without the $\lim_{m\to\infty}$ of (7).

Below, we will drop the superscripts k,l whenever they may be determined from the number of α and β variables.

In the case $l = 0$, we will simply write $G_w(\alpha_1,\dots,\alpha_k)$.

2.4. **Stable Grothendieck polynomials parametrized by partitions.** As usual, a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1,\dots,\lambda_r)$ will be called a *partition.* We will identify two partitions if they differ by a sequence of 0's, and we define $L(\lambda)$, the *length* of a partition λ to be the largest i for which $\lambda_i > 0$. The *Grassmannian permutation associated to a partition* λ *with descent in position* p is the permutation

$$w_\lambda(i) = \begin{cases} w_\lambda(i) = i + \lambda_{p+1-i} \text{ for } i \le p, \text{ and} \\ w_\lambda(i) < w_\lambda(i+1) \text{ unless } i = p. \end{cases}$$

Note that necessarily $p \ge L(\lambda)$.

We define the *double stable Grothendieck polynomial* G_λ *of the partition* λ as $G_{w_\lambda}(\alpha;\beta)$. It is easy to show that this definition does not depend on the choice of p above.

2.5. **Stable Grothendieck polynomials parametrized by integer sequences.** The notion G_λ (with λ a partition) is extended to G_I where $I \in \mathbb{Z}^r$ is any finite integer sequence—by repeated applications of the *straightening laws* [B3, Sect. 3]

$$G_{I,p,q,J} = \sum_{k=p+1}^{q} G_{I,q,k,J} - \sum_{k=p+1}^{q-1} G_{I,q-1,k,J} \qquad \text{if } p < q, \tag{9}$$

$$G_{I,p} = G_{I,0} = G_I \qquad \text{if } p < 0. \tag{10}$$

3. Properties of Grothendieck polynomials

We will need the following three properties of Grothendieck polynomials.

Proposition 3.1. [FK2], [B6, (2)] *The polynomial $G_w(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l)$ is $S_k\times S_l$-supersymmetric, i.e. it is symmetric in the α_i and the β_j variables separately, and satisfies*

$$G_w(\alpha_1,\dots,\alpha_{k-1},t;\beta_1,\dots,\beta_{l-1},t) = G_w(\alpha_1,\dots,\alpha_{k-1};\beta_1,\dots,\beta_{l-1}).$$

In particular, the left hand side of this equality does not depend on t.

The next statement is an easy application of the Fomin-Kirillov formulas [FK1], and also follows directly from the set-valued tableau description in [B1].

Proposition 3.2. *Let $\lambda = (\lambda_1,\dots,\lambda_r)$ be a partition with $\lambda_r > 0$ and let $0 < k < r$. Then $G_\lambda(\alpha_1,\dots,\alpha_k) = 0$.*

Proposition 3.3. *Let $\lambda = (\lambda_1,\dots,\lambda_r)$ be a partition with $\lambda_r \geq 0$. We have*

$$G_\lambda(\alpha_1,\dots,\alpha_r) = \sum_{\sigma\in S_r} \frac{\prod_{i=1}^r \left(1-1/\alpha_{\sigma(i)}\right)^{\lambda_i+r-i}}{\prod_{i>j}\left(1-\alpha_{\sigma(i)}/\alpha_{\sigma(j)}\right)}.$$

Proof. Consider the permutation

$$\bar{w}_\lambda = \lambda_1+r, \lambda_2+r-1, \dots, \lambda_r+1, i_1, \dots, i_s$$

where $i_j < i_{j+1}$ for all j, and s is sufficiently large to make this a permutation. The permutation $\bar{w}_\lambda$ is a so-called dominant permutation. For dominant permutations the recursive definition of Section 2.1 can be solved explicitly ([LS], or see the diagrammatic description in [FK1]), and we obtain

$$\mathfrak{G}_{\bar{w}_\lambda}(\alpha_1,\dots,\alpha_r) = \prod_{i=1}^r \left(1-\frac{1}{\alpha_i}\right)^{\lambda_i+r-i}.$$

Observe that $\bar{w}_\lambda \cdot w_0 = w_\lambda$, where w_0 is the longest permutation of $1,\dots,r$. Hence

$$\mathfrak{G}_\lambda(\alpha_1,\dots,\alpha_r) = \mathfrak{G}_{w_\lambda}(\alpha_1,\dots,\alpha_r) = \pi_{w_0(r)}\left(\prod_{i=1}^r\left(1-\frac{1}{\alpha_i}\right)^{\lambda_i+r-i}\right), \tag{11}$$

where

$$\pi_{w_0(r)}(f) = (\pi_1\pi_2 \dots \pi_{r-1})(\pi_1\pi_2 \dots \pi_{r-2}) \dots (\pi_1)(f) = \sum_{\sigma \in S_r} \sigma\left(\frac{f}{\prod_{i>j}(1-\alpha_i/\alpha_j)}\right),$$

where $\sigma \in S_r$ acts by permuting the variables α_i. The right-hand side of (11) is equal to the right-hand side of the displayed formula in the Proposition. If the number of α variables is at least the length of the partition, then $\mathfrak{G}_\lambda(\alpha) = G_\lambda(\alpha)$, which concludes our proof. □

Note that Proposition 3.3 may be used whenever the number of α variables is larger than the length of the partition, because we can append 0's to the end of λ to make the condition satisfied.

4. Grothendieck polynomials in residue form

In this section we introduce a residue calculus for Grothendieck polynomials and show how this new formalism helps to understand some of their properties.

Let z be a complex variable, and introduce the notation

$$\operatorname*{Res}_{z=0,\infty} f(z)\,dz = \operatorname*{Res}_{z=0} f(z)\,dz + \operatorname*{Res}_{z=\infty} f(z)\,dz.$$

The following property of $\operatorname{Res}_{z=0,\infty}$ is straightforward.

Lemma 4.1. *Let $0 \le a \le s-r-2$ and let*

$$f(z) = z^a \cdot \frac{\prod_{i=1}^r (z-x_i)}{\prod_{i=1}^s (z-y_i)}$$

for non-zero complex numbers x_i, y_i. Then $\operatorname{Res}_{z=0,\infty} f(z)\,dz = 0$. □

4.1. **Residue form of double stable Grothendieck polynomials.** Let $z_1, \dots, z_r$ be complex variables. For nonnegative integers k, l, define the differential form

$$M_{k,l}(z_1,\dots,z_r) = \prod_{j=1}^r \frac{\prod_{i=1}^l (1-z_j\beta_i)}{\prod_{i=1}^k (1-z_j\alpha_i)(1-z_j)^{l-k}} \cdot \prod_{j=1}^r \frac{dz_j}{z_j}. \tag{12}$$

When it causes no confusion, we will omit the indices k an l, and denote the vector $(z_1, \dots z_r)$ by z: thus we will write $M(z)$ for $M_{k,l}(z_1,\dots,z_r)$.

Definition 4.2. *For an integer sequence $I \in \mathbb{Z}^r$, define the g-polynomial as*

$$g_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l) = \operatorname*{Res}_{z_1=0,\infty} \dots \operatorname*{Res}_{z_r=0,\infty} \left(\prod_{j=1}^r (1-z_j)^{I_j-j} \prod_{i>j}\left(1-\frac{z_i}{z_j}\right) M_{k,l}(z_1,\dots,z_r)\right). \tag{13}$$

Remark 4.3. In general, iterated residue formulas are sensitive to the order in which one takes the residues Res_{z_i}—see for example [BSz, K1, K2, FR2]—due to factors of the type $z_i - z_j$ in the denominator. However, the denominators in (13) are linear factors each depending on a single variable, and hence the order in this case does not matter.

The following is evident from Definition 4.2.

Lemma 4.4. *We have*

$$g_I(\alpha_1,\dots,\alpha_k,1;\beta_1,\dots,\beta_l)=g_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l,1)=g_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l). \tag{14}$$

The function $g_\lambda(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l)$ is supersymmetric: it is symmetric in the α_i and the β_j variables separately, and we have

$$g_\lambda(\alpha_1,\dots,\alpha_{k-1},t;\beta_1,\dots,\beta_{l-1},t)=g_\lambda(\alpha_1,\dots,\alpha_{k-1};\beta_1,\dots,\beta_{l-1}).$$

In particular, the left hand side does not depend on t. □

Theorem 4.5. *For any integer sequence I, and nonnegative integers k,l, we have*

$$G_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l)=g_I(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_l).$$

First we prove two lemmas.

Lemma 4.6. *Let I and J be integer sequences. Then we have*

$$g_{I,p,q,J}=\sum_{k=p+1}^{q} g_{I,q,k,J}-\sum_{k=p+1}^{q-1} g_{I,q-1,k,J} \qquad \textit{if } p<q, \tag{15}$$

and

$$g_{I,p}=g_I \qquad \textit{if } p\le 0. \tag{16}$$

Proof. For simplicity of notation, we assume that $I=J=\emptyset$. The general case is treated similarly. For $p<q$ consider

$$g_{p,q}-g_{q-1,q}=\operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\left((1-z_1)^{p-1}(1-z_2)^{q-2}-(1-z_1)^{q-2}(1-z_2)^{q-2}\right)\left(1-\frac{z_2}{z_1}\right)\cdot M(z_1,z_2).$$

Applying the identities

$$\left(1-\frac{z_2}{z_1}\right)=-\frac{z_2}{z_1}\left(1-\frac{z_1}{z_2}\right)$$

and

$$\left((1-z_2)^{q-2}(1-z_1)^{p-1}-(1-z_2)^{q-2}(1-z_1)^{q-2}\right)\left(-\frac{z_2}{z_1}\right)=$$
$$\sum_{k=p+1}^{q-1}(1-z_2)^{q-1}(1-z_1)^{k-2}-\sum_{k=p+1}^{q-1}(1-z_2)^{q-2}(1-z_1)^{k-2},$$

we obtain that $g_{p,q}-g_{q-1,q}$ equals

$$\operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\left(\sum_{k=p+1}^{q-1}(1-z_2)^{q-1}(1-z_1)^{k-2}-\sum_{k=p+1}^{q-1}(1-z_2)^{q-2}(1-z_1)^{k-2}\right)\left(1-\frac{z_1}{z_2}\right)\cdot M(z_1,z_2).$$

Using the definition of g with the role of z_1 and z_2 switched, we obtain

$$g_{p,q}-g_{q-1,q}=\sum_{k=p+1}^{q-1} g_{q,k}-\sum_{k=p+1}^{q-1} g_{q-1,k}.$$

This is equivalent to (15) up to the easy equality

$$g_{q-1,q} = g_{q,q},$$

whose proof we leave to the reader.

Formula (16) immediately follows from the fact that, for $p \leq 0$,

$$\operatorname*{Res}_{z_r=0}(1-z_r)^{p-r}\prod_{i=1}^{r-1}\left(1-\frac{z_r}{z_i}\right)M_{k,l}(z_r) = 1,$$

while the residue of this expression at $z_r = \infty$ vanishes. □

Lemma 4.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition with $\lambda_r > 0$. Then for $k < r$, we have $g_\lambda(\alpha_1, \ldots, \alpha_k) = 0$.*

Proof. First note that, according to Lemma 4.4, the case $k = r-1$ implies the case $k < r$.

Now assume $k = r-1$, and introduce the temporary notation γ for the differential form in (13). Assume that the values of the α_is are all different.

We calculate the first residue $\operatorname{Res}_{z_r=0,\infty}\gamma$, taking into account Remark 4.3 and applying the 1-variable Residue Theorem. The exponent $I_r - r + k - l$ of the factor $(1 - z_r\beta_i)$ is nonnegative, since $I_r = \lambda_r > 0$, $k = r-1$, and $l = 0$, and hence there is no pole at $z_r = 1$. The remaining poles are thus the points $z_r = 1/\alpha_i$, $i = 1, \ldots, k$, and each of these poles is simple. The residue at the simple pole $z_r = 1/\alpha_i$, up to a factor of $-\alpha_i$ is obtained by omitting the factor $(1 - \alpha_i z_r)$ in the denominator, and then substituting into the remainder $z_r = 1/\alpha_i$. Continuing the application of residues in (13), we obtain a sum over all choices of indices $1 \leq i_j \leq k$, $j = 1, \ldots r$, of terms of the following form

$$\prod_{j=1}^{r}(1-\alpha_{i_j})^{\epsilon}\prod_{m>j}\left(1-\frac{\alpha_{i_m}}{\alpha_{i_j}}\right)\widetilde{M},$$

where $\epsilon \geq 0$ and $\widetilde{M}$ is some rational expression in the α's. The relevant factor in the product is the second one, which vanishes as long as $i_m = i_j$ for some $1 \leq j < m \leq r$. As $k < r$, this is certainly the case, and this completes the proof. □

Now we are ready to prove Theorem 4.5.

Proof. Since both g and G are supersymmetric (Proposition 3.1 and Lemma 4.4), it is sufficient to prove $G_\lambda = g_\lambda$ for the $\beta_1 = \beta_2 = \ldots = 1$ substitution. For that substitution, both g_λ and G_λ vanish if the number of α's is less then the length of λ (see Proposition 3.2 and Lemma 4.7).

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and consider formula (13) for $k = r, l = 0$. We will apply the Residue Theorem for each residue $\operatorname{Res}_{z_i=0,\infty}$, i.e. we replace $\operatorname{Res}_{z_i=0,\infty}$ by $-\sum_p \operatorname{Res}_{z_i=p}$ where the sum runs over all poles different from 0 and ∞. We claim that the only such poles are at $z_i = 1/\alpha_j$. Indeed the substitution $\beta_i = 1$ makes the exponent of $(1 - z_i)$ in the formula equal to $\lambda_i - i + r$, which is nonnegative.

The only nonzero finite residues hence correspond to permutations $\sigma \in S_r$: $z_i = 1/\alpha_{\sigma(i)}$. Straightforward calculation shows that the $-\operatorname{Res}_{z_i=1/\alpha_{\sigma(i)}}$ operation yields the term corresponding to $\sigma \in S_r$ in Proposition 3.3. This proves the theorem. □

4.2. **Consequences of the $g = G$ theorem.** Grothendieck polynomials have a rich algebraic structure and they display beautiful finiteness and alternating-sign properties. We believe that the residue form for the stable Grothendieck polynomials above sheds light on many of those properties. We will illustrate this in Section 8 in a so-far unexplored situation—the Thom polynomials of singularities. Here we will just sketch a simple example showing how the multiplication structure of Grothendieck polynomials is encoded in their residue form.

4.3. **Multiplication.** Consider the concrete example of calculating the g-expansion of the product $g_2 \cdot g_2$ (here "2" in the subscript is a length 1 partition). We have

$$g_2 \cdot g_2 = \operatorname*{Res}_{z=0,\infty}(1-z)M(z) \cdot \operatorname*{Res}_{u=0,\infty}(1-u)M(u) = \operatorname*{Res}_{z,u=0,\infty}(1-z)(1-u)M(z,u) =$$

$$\operatorname*{Res}_{z,u=0,\infty}\left((1-z)(1-u)\frac{1}{1-\frac{u}{z}}\left(1-\frac{u}{z}\right)M(z,u)\right) =$$

$$\operatorname*{Res}_{z,u=0,\infty}\left((1-z)(1-u)\left(\sum_{i=0}^{2}\frac{(1-z)^i}{(1-u)^{i+1}} - \sum_{i=1}^{2}\frac{(1-z)^i}{(1-u)^i} + \frac{u(1-z)^3}{(z-u)(1-u)^3}\right)\left(1-\frac{u}{z}\right)M(z,u)\right).$$

The term involving $u(1-z)^3/((z-u)(1-u)^3)$ has u-residue 0, because of Lemma 4.1. Hence we further obtain

$$g_2 \cdot g_2 = \operatorname*{Res}_{z,u=0,\infty}\left(\left(\sum_{i=0}^{2}\frac{(1-z)^{i+1}}{(1-u)^i} - \sum_{i=1}^{2}\frac{(1-z)^{i+1}}{(1-u)^{i-1}}\right)\left(1-\frac{u}{z}\right)M(z,u)\right)$$

$$= g_{2,2} + g_{3,1} + g_{4,0} - g_{3,2} - g_{4,1}.$$

In general the calculation of products of arbitrary Grothendieck polynomials is similar, see [AR]. Namely, to find an explicit expression for $g_I \cdot g_J$ as sums of Grothendieck polynomials, one considers

$$\prod_i (1-z_i)^{I_i - i} \prod_j (1-u)^{J_j - j} \prod_{i,j} \frac{1}{1-\frac{u_j}{z_i}},$$

and replaces $1/(1-u_j/z_i)$ with an appropriate initial sum of its Laurent series at $z_i = u_j = 1$. The initial sum needs to be chosen in such a way that the remainder multiplied by $\prod(1-z_i)^{I_i-i}\prod(1-u)^{J_j-j}$ has 0 residue.

Remark 4.8. The example above can be generalized to show that the product of two Grothendieck polynomials (parametrized by integer sequences) is a *finite sum* of Grothendieck polynomials parametrized by integer sequences with coefficients with *alternating signs*, see [AR]. Proving the much more difficult analogous statement for Grothendieck polynomials parametrized by *partitions* [B2] needs extra considerations. We will perform a similar analysis for Thom polynomials in Section 9.

5. Fundamental class in cohomology and K-theory

5.1. **The cohomology fundamental class.** Let X be a subvariety of codimension d in a smooth projective variety M. Then X has a well-defined fundamental class $[X] \in H^{2d}(M, \mathbb{Q})$, satisfying

$$\int_X \iota^*\omega = \int_M [X] \cdot \omega, \tag{17}$$

where $\iota : X \to M$ is the embedding, and $\omega \in H^*(M, \mathbb{Q})$ is arbitrary, cf. [GH].

There is a natural extension of this notion to the equivariant setting, which plays a fundamental role in enumerative geometry. Let V be a complex vector space acted upon by a complex torus T. Then a T-invariant affine subvariety X has a fundamental class $[X]_T \in H^{2d}_T(V) = H^{2d}_T(\mathrm{pt})$, $d = \mathrm{codim}(X)$, which satisfies the equivariant version of (17):

$$\int_X \iota^*\omega = \int_V [X]_T \cdot \omega,$$

where ω is any equivariantly closed, compactly supported form on V.

There is a number of definitions of this notion (cf. [BSz, §3] for a discussion); below we recall one due to Joseph [J]. We begin with introducing some necessary notation.

- Let $\exp : \mathrm{Lie}(T) \to T$ be the exponential map; the pull-back of a function from $f : T \to \mathbb{C}$ to $\mathrm{Lie}(T)$ via this map will be denoted by $\exp^* f$.
- For a character $\alpha \in \mathrm{Hom}(T, \mathbb{C}^*)$, we will write $\bar{\alpha}$ for the corresponding weight in the weight lattice $\mathcal{W}_T \subset \mathrm{Lie(T)}^\vee$. We will thus have the following equality of functions on $\mathrm{Lie}(T)$:
$$\exp^* \alpha = e^{\bar{\alpha}},$$
where factor of $2\pi i$ is considered to be absorbed in the definition of the exponential, and will be ignored in what follows.
- Fix a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_r : T \to \mathbb{C}^*$ of $\mathrm{Hom}(T, \mathbb{C}^*)$. We then have
$$H^*_T(V) = H^*_T(\mathrm{pt}) = \mathbb{Z}[\bar{\beta}_1, \dots, \bar{\beta}_r].$$
- Let x_j, $j = 1, \dots N$ be a set of coordinates on V, corresponding to a basis of eigenvectors of the T action, and denote by $\eta_j \in \mathrm{Hom}(T, \mathbb{C}^*)$, $j = 1, \dots N$, the corresponding characters: for $t \in T$, we have $t \cdot x_j = \eta_j(t)^{-1} x_j$. For what follows, it is convenient to make the following

  Assumption 5.1. *All the weight vectors of the vector space V lie in an open half-space of the weight lattice $\mathcal{W}_T \subset \mathrm{Lie(T)}^\vee$, i.e. there exists an element $Z \in \mathrm{Lie}(T)$ such that we have*
  $$\langle \bar{\eta}_j, Z \rangle > 0, \quad j = 1, \dots N.$$

  One can carry out the constructions of the theory without this assumption as well, but this is more technical, and this case is sufficient for our purposes.

Recall that for a finite-dimensional representation W of T with a diagonal basis

$$W = \oplus_{i=1}^m \mathbb{C}\, w_i,\ t \cdot w_i = \alpha_i(t) \cdot w_i,\ \text{we have } \operatorname{Tr}[t \,|\, W] = \sum_{i=1}^m \alpha_i,\ \text{for } t \in T.$$

This function on T is called the *character of* W.

Now let $X \subset V$ be a T-invariant subvariety, and denote by RX the ring of algebraic functions on X. The character

$$\chi_X(t) = \operatorname{Tr}[t \,|\, RX], \quad t \in T$$

of RX considered as a T-representation is only a formal series since RX is infinite-dimensional whenever the dimension of X is positive. Under Assumption 5.1, however, this series converges in a domain in T, and $\chi_X(t)$ makes sense as a rational function on T.

For example, $RV = \mathbb{C}[x_1, \dots, x_N]$ is the ring of polynomial functions on V, and we have

$$\chi_V = \prod_{j=1}^N \frac{1}{1 - \eta_j^{-1}}, \tag{18}$$

as can be seen by expanding this function in an appropriate domain in T.

The following theorem is a consequence of the Hilbert's syzygy theorem (cf. [MS, Chapter 8]).

Theorem 5.2. *Let $X \subset V$ be a T-invariant subvariety of codimension d. Then χ_X is a function on T defined whenever χ_V is defined (cf. (18)), and has the form of a finite integral linear combination of T-characters multiplied by χ_V:*

$$\chi_X = \chi_V \cdot \sum_{j=1}^M a_j \theta_j, \quad \text{where } a_j \in \mathbb{Z},\ \theta_j \in \operatorname{Hom}(T, \mathbb{C}^*). \tag{19}$$

Moreover, expanding the function $\exp^*(\chi_X/\chi_V) = \sum_{j=1}^M a_j \bar\theta_j$ *on* $\operatorname{Lie}(T)$ *around the origin, we obtain a power series with lowest degree terms in degree d:*

$$\sum_{j=1}^M a_j \exp \bar\theta_j = \frac{1}{d!} \sum_{j=1}^M a_j \bar\theta_j^d + \rho_{d+1} \quad \text{with } \rho_{d+1} \in \mathfrak{m}^{d+1}, \tag{20}$$

where $\mathfrak{m}$ *is the maximal ideal of analytic functions vanishing at the origin in* $\operatorname{Lie}(T)$*.*

The last part of the theorem states that, after the expansion, the terms up to degree $d - 1$ cancel.

Definition 5.3. *Let $X \subset V$ be a T-invariant subvariety of codimension d. We define the T-*equivariant fundamental class *of X in V as the degree-d (leading) term on the right hand side of* (20) *interpreted as an element of $H_T^*(V)$:*

$$[X]_T = (-1)^d \sum_{j=1}^M a_j \bar\theta_j^d.$$

Example 5.4. Let $V = \mathbb{C}^2$ be endowed with a diagonal action of $T = \mathbb{C}^*$ with weight 1 on each of the two coordinate functions x and y, and let $X = \{xy = 0\}$. Then X is T-invariant, and there is a short exact sequence of RV-modules

$$0 \to RV[-2] \to RV \to RX \to 0,$$

where $RV[-2]$ stands for the free module of rank 1, generated by a single element of degree 2, whose image is the function xy. This implies

$$\chi_V = \frac{1}{(1-\beta^{-1})^2}, \quad \text{and} \quad \chi_X = \frac{1-\beta^{-2}}{(1-\beta^{-1})^2} = \frac{1+\beta^{-1}}{1-\beta^{-1}}.$$

Now we substitute $\beta = e^{\bar\beta}$, and we see that modulo $\bar\beta^3$, we have $\chi_X/\chi_V = 1 - \beta^{-2} = 2\bar\beta$, and hence $[X]_T = 2\bar\beta$.

5.2. **Equivariant K-theoretic fundamental classes.** It is not immediately obvious what one should take as the appropriate definition of the equivariant fundamental class in K-theory.

In our setup, we have

$$K_T(\mathrm{pt}) = \mathbb{Z}\,\mathrm{Hom}(T, \mathbb{C}^*) = \mathbb{Z}[\beta_1^{\pm 1}, \beta_2^{\pm 1}, \dots, \beta_r^{\pm 1}],$$

and thus for a T-invariant $X \subset V$, it would seem natural to define as this fundamental class the linear combination of torus characters χ_X/χ_V in (19), which naturally lies in this space.[1] This invariant is very difficult to calculate, however (cf. [K] for a more detailed discussion), and, in fact, there are some alternatives.

Proposition 5.5. *Let $X \subset V$ be a T-invariant subvariety in the vector space V satisfying Assumption 5.1. Then the cohomology groups of the structure sheaf $H^i(Y, \mathcal{O}_Y)$ for a smooth T-equivariant resolution $\pi : Y \to X$ are independent of the choice of Y, and thus are invariants of X. In particular,*

$$\widetilde{\chi}_X(\tau) \overset{\mathrm{def}}{=} \sum_{i=0}^{\dim Y} (-1)^i \mathrm{Tr}\left[\tau \mid H^i(Y, \mathcal{O}_Y)\right] \tag{21}$$

is an invariant of X, which coincides with χ_X if X has only rational singularities. Moreover, χ_X/χ_V and $\widetilde{\chi}_X/\chi_V$ have the same leading term in the sense of (19) *and* (20) *in Theorem 5.2.*

These statements are fairly standard—see for example [MS, H]—hence we only give a *sketch of the proof* to emphasize the key ideas involved. First we recall that for two smooth resolutions $Y_1 \to X \leftarrow Y_2$, there exists a resolution $Y \to X$ which dominates Y_1, Y_2. This fact reduces the theorem to the case when both X and Y are smooth and π is birational. In this case, the first statement may be found in [H, Chapter III].

The statement on rational singularities is essentially a tautology: for an affine variety X, having rational singularities means precisely that for any smooth resolution $Y \to X$, we have $H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

[1] This polynomial is called the K-polynomial in [MS] for this reason.

Finally, note that the cohomology groups $H^i(Y,\mathcal{O}_Y)$ are the sections over X of the derived push-forward sheaves $R^i\pi_*\mathcal{O}_Y$. Applying the flat base change for the smooth locus in X, we see that for $i>1$, these sheaves are supported on the singular locus of X, which is of higher codimension than X itself. For such a sheaf then, the corresponding leading term will be of higher degree than d, the codimension of X (see [MS]), and this completes the proof. $\square$

Definition 5.6. *Let X be a T-invariant subvariety of the vector space V endowed with a T-action and satisfying Assumption 5.1. Then we define the K-theoretic fundamental class $[X]_T^K$ of X in V as the character $\widetilde{\chi}_X/\chi_V$, where $\widetilde{\chi}_X$ is given by the formula* (21).

Now let us revisit Example 5.4. Denote by Y the normalization of X, which is the union of two nonintersecting lines. Then $H^0(Y,\mathcal{O}_Y)$ is two copies of a polynomial ring in one variable, and $H^0(X,\mathcal{O}_X)\subset H^0(Y,\mathcal{O}_Y)$ is the subset of those pairs of polynomials whose constant terms coincide. We have

$$\widetilde{\chi}_X=\chi_Y=\frac{2}{1-\beta^{-1}},\quad \chi_V=\frac{1}{(1-\beta^{-1})^2},\quad \text{and hence}\quad [X]_T^K=\frac{\widetilde{\chi}_X}{\chi_V}=2(1-\beta^{-1}).$$

It is instructive to verify directly the last statement of Proposition 5.5 even in this simple case. When we used χ_X instead of $\widetilde{\chi}_X$, we obtained a different answer:

$$\frac{\chi_X}{\chi_V}=\frac{(1+\beta^{-1})/(1-\beta^{-1})}{1/(1-\beta^{-1})^2}=1-\beta^{-2}.$$

Yet, after substituting $\beta=e^{\bar\beta}$, we see that, modulo $(\bar\beta^3)$ we have the equality:

$$\widetilde{\chi}_X/\chi_V=\chi_X/\chi_V=2\bar\beta \qquad \mathrm{mod}\ (\bar\beta^3),$$

recovering the cohomological fundamental class of Example 5.4.

Remark 5.7. For a holomorphic map between complex manifolds $g:M^a\to P^b$, one can consider the η-singularity points

$$\eta(g)=\{x\in M : \text{the } N\text{-jet of } g \text{ at } x \text{ belongs to } \eta\}.$$

Thom's principle on cohomological Thom polynomials states that if g satisfies certain transversality properties then

$$[\eta(g)]=\mathrm{Tp}_\eta^{a\to b}(\text{Chern roots of } TM, \text{Chern roots of } g^*(TP)).$$

This powerful statement relies on the fact that the notion of "cohomological fundamental class" is *consistent with pullback morphisms*. The way we set up the notion of K-theoretic fundamental class in Definition 5.6 is *not* consistent with pullback morphisms (rather, it is consistent with push-forward morphisms), hence Thom's principle does not hold for our K-theoretic Thom polynomials. The interesting project of studying another version of K-theoretic fundamental class of singularities—one for which Thom's principle holds—is started in [K].

We end this section with an observation addressing the situation when the group G acting on V is a general reductive group with maximal torus T. For a reductive group G, we have

$K_G(\mathrm{pt}) = K_T(\mathrm{pt})^W$ (the Weyl-invariant part). For a G-invariant $X \subset V$, the class $[X]^K_T$ will be in this Weyl-invariant part, and hence we can define $[X]^K_G = [X]^K_T$.

In the rest of the paper, if the group that acts is obvious, we will drop the subscript and use the notation $[X] = [X]_G$, $[X]^K = [X]^K_G$ for the cohomological and K-theoretic fundamental class.

6. Singularities and their Thom polynomials

Recall the notion of contact singularities and their Thom polynomials from §1.2. Let us see a few examples.

Example 6.1.

- The simplest case is $Q = \mathbb{C}$, also known as the $\mathcal{A}_0$-algebra. In this case, we have
$$\eta^{a\to b}_{\mathcal{A}_0} = J_N(\mathbb{C}^a, \mathbb{C}^b),$$
which is essentially the inverse function theorem.
- When the algebra Q is $\mathcal{A}_1 = \mathbb{C}[x]/(x^2)$, the set $\eta^{a\to b}_{\mathcal{A}_1}$ is the set of singular map-jets, i.e. those whose derivative at 0 is not injective.
- For $r > 0$, consider $Q = \mathbb{C}[x_1, \dots, x_r]/(x_1, \dots, x_r)^2$. In this case, $\eta^{a\to b}_Q$ is the set of those map-jets whose linear part has corank at least r (also known as the Σ^r singularity).
- The contact singularities corresponding to the algebra $Q = \mathcal{A}_r = \mathbb{C}[x]/(x^{r+1})$ are called *Morin singularities.* A generic element of $\eta^{2\to 2}_{\mathcal{A}_2}$ may be represented as $(x, y) \mapsto (x^3+xy, y)$; it is called the cusp singularity.

6.1. **The model.** By a *model* for a singularity $\eta \subset J(\mathbb{C}^a, \mathbb{C}^b)$, we mean a $\mathrm{GL}(\mathbb{C}^a) \times \mathrm{GL}(\mathbb{C}^b)$-equivariant commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[rr, bend left, "\rho"] \arrow[dr, "\pi"'] & M \times J(\mathbb{C}^a, \mathbb{C}^b) \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & J(\mathbb{C}^a, \mathbb{C}^b) \arrow[d] \\
& M \arrow[r, "p_M"] & pt,
\end{tikzcd}$$

where

- M is a smooth compact manifold,
- $\pi : X \to M$ is a subbundle of the trivial bundle $\pi_1 : M \times J(\mathbb{C}^a, \mathbb{C}^b) \to M$,
- $\rho = \pi_2 \circ i$ is birational to η,
- and p_M is the map from M to a point pt.

Let ν be the quotient bundle of $\pi_1 : M \times J(\mathbb{C}^a, \mathbb{C}^b) \to M$ by $X \to M$. It follows that for such a model for the singularity η one has
$$\mathrm{Tp}_\eta = p_{M*}(e(\nu)),$$
where e stands for the (equivariant) Euler class. Indeed, we have

$$\mathrm{Tp}_\eta = \rho_*(1) = \pi_{2*}(i_*(1)) = \pi_{2*}(e(\nu)) = p_{M*}(e(\nu)). \tag{22}$$

The advantage of our definition of K-theoretic fundamental class in Section 5 is that the argument (22) goes through without change to the K-theoretic setting, and we have

$$\mathrm{KTp}_\eta = p_{M!}(e(\nu)),$$

where e is now the K-theoretic (equivariant) Euler class, and $p_{M!}$ is the K-theoretic push-forward map.

6.2. **Integration in K-theory using residues.** In what follows we will use residue calculus for the push-forward map in K-theory.

Let the torus T act on the smooth variety X with finitely many fixed points. Let W be a rank-d equivariant vector bundle over X, and let $\omega_1, \dots, \omega_w$ be its Chern roots (i.e. virtual line bundles whose sum is W). Let $p : \mathrm{Gr}(r, W) \to X$ be the Grassmannization of W, that is an equivariant bundle whose fiber over $x \in X$ is the Grassmannian $\mathrm{Gr}(r, W_x)$ of dimension r linear subspaces of the fiber W_x of W over x. Let S be the tautological subbundle over $\mathrm{Gr}(r, W)$, and let $\sigma_1, \dots, \sigma_r$ be its Chern roots. A symmetric Laurent polynomial $g(\sigma_1, \dots, \sigma_r)$ is hence an element of $K_T(\mathrm{Gr}(r, W))$.

Lemma 6.2. *We have*

$$p_!(g(\sigma_1, \dots, \sigma_r)) = \operatorname*{Res}_{z_1=0,\infty} \dots \operatorname*{Res}_{z_r=0,\infty} \left(\prod_{i>j} \left(1 - \frac{z_i}{z_j}\right) \frac{g(z_1, \dots, z_r)}{\prod_{i=1}^r \prod_{j=1}^w \left(1 - \frac{z_i}{\omega_j}\right)} \prod_{i=1}^r \frac{dz_i}{z_i} \right). \tag{23}$$

Proof. Consider first the special case when X is a point. Then the equivariant localization formula for the push-forward map is

$$p_!(f(\sigma_1, \dots, \sigma_r)) = \sum_I \frac{f(\omega_{I_1}, \dots, \omega_{I_r})}{\prod_{i\in I} \prod_{j \in \bar{I}} \left(1 - \frac{\omega_i}{\omega_j}\right)},$$

where the summation is over r-element subsets I of $\{1, \dots, n\}$, and $\bar{I}$ is the complement of I. Applying the Residue Theorem for the right hand side of (23), for $z_1, z_2, \dots$ gives the same expression. This proves the lemma when X is a point.

The general case is shown applying this special case to W restricted to fixed points. □

When G is a connected algebraic group G, Lemma 6.2 may be applied to the maximal torus $T \subset G$, and since $K_G(X)$ is the Weyl-invariant part of $K_T(X)$, formula (23) holds without change.

7. Σ^r singularities

In this short section we illustrate the residue technique to calculate the K theoretic Thom polynomial of singularities that are defined by the behavior of their first derivatives. The obtained results are not new, but our proof will serve as a sample to the more involved calculations of the next section.

7.1. **The model for Σ^r.** The obvious model for the

$$\Sigma^r = \Sigma^r(\mathbb{C}^a, \mathbb{C}^b) = \{g \in J^1(\mathbb{C}^a, \mathbb{C}^b) : \dim\ker g \geq r\}$$

singularity is $M = \mathrm{Gr}(r, \mathbb{C}^a)$, and

$$X = \{(V, g) \in \mathrm{Gr}(r, \mathbb{C}^a) \times J^1(\mathbb{C}^a, \mathbb{C}^b) : g|_V = 0\}.$$

Let the tautological rank r bundle over $\mathrm{Gr}(r, \mathbb{C}^a)$ be S. The bundle $\pi : X \to \mathrm{Gr}(r, \mathbb{C}^a)$ can be identified with $J^1(\mathbb{C}^a/S, \mathbb{C}^b)$, hence the normal bundle is $\nu = J^1(S, \mathbb{C}^b)$. Thus $\mathrm{KTp}_{\Sigma^r} = p_!(e(J^1(S, \mathbb{C}^b)))$ for the map $p : \mathrm{Gr}(r, \mathbb{C}^a) \to \mathrm{pt}$.

Theorem 7.1. *We have*

$$\mathrm{KTp}_{\Sigma^r} = \operatorname*{Res}_{z_1=0,\infty} \dots \operatorname*{Res}_{z_r=0,\infty} \left(\prod_{i>j} \left(1 - \frac{z_i}{z_j}\right) \prod_{i=1}^{r} \frac{\prod_{j=1}^{b} \left(1 - \frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a} \left(1 - \frac{z_i}{\alpha_j}\right)} \prod_i \frac{dz_i}{z_i} \right). \tag{24}$$

Proof. We have

$$\mathrm{KTp}_{\Sigma^r} = p_!(e(J^1(S, \mathbb{C}^b))) = p_! \left(\prod_{i=1}^{r} \prod_{j=1}^{b} \left(1 - \frac{\sigma_i}{\beta_j}\right) \right),$$

and applying Lemma 6.2 proves the Theorem. $\square$

Comparing expression (24) with the residue formula for Grothendieck polynomials (Definition 4.2), we obtain

$$\mathrm{KTp}_{\Sigma^r} = G_{(r+l)^r}(\alpha_1^{-1}, \dots, \alpha_a^{-1}; \beta_1^{-1}, \dots, \beta_b^{-1}).$$

This result is known in Schubert calculus [LS] as the K-theoretic Giambelli-Thom-Porteous formula.

8. $\mathcal{A}_2$ singularities

8.1. **The model for $\mathcal{A}_2$.** Consider the tautological exact sequence $S \to \mathbb{C}^a \to Q$ over $\mathrm{Gr}(1, \mathbb{C}^a)$. Let $M = \mathrm{Gr}(1, S^{\otimes 2} \oplus Q)$ be the projectivization of the vector bundle $S^{\otimes 2} \oplus Q$ over $\mathrm{Gr}(1, \mathbb{C}^a)$, and denote the tautological line bundle over M by D.

According to [BSz, K2] there is a model for the

$$\eta_{\mathcal{A}_2}^{a \to b} = \overline{\{g \in J^2(\mathbb{C}^a, \mathbb{C}^b) : Q_g \cong \mathbb{C}[x]/(x^3)]\}}$$

singularity with this M, and normal bundle $\nu = \mathrm{Hom}(S \oplus D, \mathbb{C}^b)$.

8.2. **Residue formula for $\mathrm{KTp}_{\mathcal{A}_2}$.**

Theorem 8.1. *We have*

$$\mathrm{KTp}_{\mathcal{A}_2}^{a \to b} = \operatorname*{Res}_{z_1=0,\infty} \operatorname*{Res}_{z_2=0,\infty} \left(\frac{1 - \frac{z_2}{z_1}}{1 - \frac{z_2}{z_1^2}} \prod_{i=1}^{2} \frac{\prod_{j=1}^{b} \left(1 - \frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a} \left(1 - \frac{z_i}{\alpha_j}\right)} \frac{dz_2 dz_1}{z_2 z_1} \right).$$

Note that the order of taking residues is important here: first we take residues with respect to z_2, then with respect to z_1.

Proof. We know that $\mathrm{KTp}_{\mathcal{A}_2} = p_{M!}(e(\mathrm{Hom}(D \oplus S, \mathbb{C}^b)))$. Let the Chern roots of the bundle Q be $\omega_1, \dots, \omega_{a-1}$, and let the class of S be σ, and the class of D be τ. We have

$$e(\nu) = \prod_{j=1}^{b}\left(1 - \frac{\sigma}{\beta_j}\right)\prod_{j=1}^{b}\left(1 - \frac{\tau}{\beta_j}\right).$$

Pushing forward this class to $\mathrm{Gr}(1, \mathbb{C}^a)$, using Lemma 6.2 we get

$$\operatorname*{Res}_{z_2=0,\infty}\left(\frac{\prod_j\left(1-\frac{\sigma}{\beta_j}\right)\prod_j\left(1-\frac{z_2}{\beta_j}\right)}{\left(1-\frac{z_2}{\sigma^2}\right)\prod_j\left(1-\frac{z_2}{\omega_j}\right)}\frac{dz_2}{z_2}\right).$$

Using the fact that $S \to \mathbb{C}^a \to Q$ is an exact sequence, this is further equal to

$$\operatorname*{Res}_{z_2=0,\infty}\left(\frac{\prod_j\left(1-\frac{\sigma}{\beta_j}\right)\prod_j\left(1-\frac{z_2}{\beta_j}\right)\left(1-\frac{z_2}{\sigma}\right)}{\left(1-\frac{z_2}{\sigma^2}\right)\prod_j\left(1-\frac{z_2}{\alpha_j}\right)}\frac{dz_2}{z_2}\right).$$

Pushing this class further from $\mathrm{Gr}(1, \mathbb{C}^a)$ to a point, using Lemma 6.2, we obtain

$$\operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\left(\frac{\prod_j\left(1-\frac{z_1}{\beta_j}\right)\prod_j\left(1-\frac{z_2}{\beta_j}\right)\left(1-\frac{z_2}{z_1}\right)}{\left(1-\frac{z_2}{z_1^2}\right)\prod_j\left(1-\frac{z_2}{\alpha_j}\right)\prod_j\left(1-\frac{z_1}{\alpha_j}\right)}\frac{dz_2}{z_2}\frac{dz_1}{z_1}\right),$$

which is what we wanted to prove. $\square$

8.3. $\mathrm{KTp}_{\mathcal{A}_2}$ **in terms of Grothendieck polynomials—the stable expansion.** Let

$$\frac{1}{1 - z_2/z_1^2} = \sum_{r,s} d_{r,s}(1-z_1)^r(1-z_2)^s$$

be the Laurent expansion of the named rational function on the $|1 - z_1| < |1 - z_2|$ region. Equivalently, after substituting $x_1 = 1 - z_1, x_2 = 1 - z_2$, let

$$\frac{1 - 2x_1 + x_1^2}{x_2 - 2x_1 + x_1^2} = \sum_{r,s} d_{r,s}x_1^r x_2^s$$

be the Laurent expansion of the named rational function on the $|x_1| < |x_2|$ region. Based on the calculation

$$\begin{aligned}(25)\qquad \frac{1}{x_2 - 2x_1 + x_1^2} &= \frac{1}{x_2}\cdot\frac{1}{1-(2x_1 - x_1^2)/x_2} = \sum_{k=1}^{\infty}\frac{1}{x_2^k}(2x_1 - x_1^2)^{k-1} \\ &= \sum_{k=1}^{\infty}\sum_{r=k-1}^{2k-2}(-1)^{r-k+1}2^{2k-2-r}\binom{k-1}{2k-2-r}x_1^r x_2^{-k},\end{aligned}$$

we have that

$$d_{r,s} = (-1)^{r+s+1}\left(2^{-2s-2-r}\binom{-s-1}{-2s-r-2} + 2^{-2s-r}\binom{-s-1}{-2s-r-1} + 2^{-2s-r}\binom{-s-1}{-2s-r}\right)$$

for $r = 0, 1, \dots, s = -r-1, \dots, -\lfloor r/2 \rfloor$. In particular, *the sign of* $d_{r,s}$ *is* $(-1)^{r+s+1}$.

For the values of $d_{r,s}$ for small (absolute value) r, s see the table in §1.2.

Theorem 8.2 (Grothendieck expansion of $\mathrm{KTp}_{\mathcal{A}_2}$: the stable version)**.** *Let* $l = b - a$*, and* $N > 2l + 2$*. Then*

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \sum_{r=0}^{N} \sum_{s=-r-1}^{-\lfloor \frac{r}{2} \rfloor} d_{r,s} G_{r+l+1,s+l+2}(\alpha_1^{-1}, \dots, \alpha_a^{-1}; \beta_1^{-1}, \dots, \beta_b^{-1}). \tag{26}$$

Note that for a given r, the set of non-zero $d_{r,s}$ coefficients are exactly those between $s = -r-1$ and $s = -\lfloor r/2 \rfloor$, hence, in the summation above, s runs through all its relevant values.

Remark 8.3. Since N may be arbitrarily large in (26), it is tempting to phrase Theorem 8.2 informally as

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \sum_{r,s} d_{r,s} G_{r+l+1,s+l+2}(\alpha_1^{-1}, \dots, \alpha_a^{-1}; \beta_1^{-1}, \dots, \beta_b^{-1}). \tag{27}$$

This series does not converge, however.

Proof. The finite expansion of $1/(1 - z_2/z_1^2)$ with respect to z_1, around $z_1 = 1$, with remainder term is

$$\frac{1}{1 - z_2/z_1^2} = \sum_{r=0}^{N} \left(\sum_{s} d_{r,s} (1 - z_2)^s \right) (1 - z_1)^r + R_N(z_1, z_2) \tag{28}$$

where the s-summation is finite. A quick calculation shows that the remainder term may be expressed as

$$R_N(z_1, z_2) = -\left(\frac{1 - z_1}{1 - z_2}\right)^{N+1} \frac{z_1 q_N(z_2) + p_N(z_2)}{1 - z_1^2/z_2}. \tag{29}$$

where

$$p_N(z) = \sum_{i=0}^{\lfloor \frac{N+1}{2} \rfloor} \binom{N+1}{2i} z^i, \qquad q_N(z) = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2i+1} z^i.$$

According to Theorem 8.1, we have the following expression for $\mathrm{KTp}_{\mathcal{A}_2}$:

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \operatorname*{Res}_{z_1=0,\infty} \operatorname*{Res}_{z_2=0,\infty} \Bigg((1 - z_1)^l (1 - z_2)^l \frac{1}{1 - z_2/z_1^2} \times$$

$$\times \left(1 - \frac{z_2}{z_1}\right) \prod_{i=1}^{2} \frac{\prod_{j=1}^{b} \left(1 - \frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a} \left(1 - \frac{z_i}{\alpha_j}\right) (1 - z_i)^l} \frac{dz_2 dz_1}{z_2 z_1} \Bigg).$$

Substituting (28), we obtain

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\left(\sum_{r=0}^{N}\left(\sum_s d_{r,s}(1-z_2)^{s+l}\right)(1-z_1)^{r+l}\times\right.$$
$$\left.\times\left(1-\frac{z_2}{z_1}\right)\prod_{i=1}^{2}\frac{\prod_{j=1}^{b}\left(1-\frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a}\left(1-\frac{z_i}{\alpha_j}\right)(1-z_i)^l}\frac{dz_2dz_1}{z_2z_1}\right)+$$
$$\operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\left(R_N(z_1,z_2)\left(1-\frac{z_2}{z_1}\right)\prod_{i=1}^{2}\frac{\prod_{j=1}^{b}\left(1-\frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a}\left(1-\frac{z_i}{\alpha_j}\right)}\frac{dz_2dz_1}{z_2z_1}\right).$$

According to the residue expression for Grothendieck polynomials (Definition 4.2) the first term equals

$$\sum_{r=0}^{N}\sum_s d_{r,s}G_{r+l+1,s+l+2}(\alpha_1^{-1},\dots,\alpha_a^{-1},\beta_1^{-1},\dots,\beta_b^{-1}),$$

and we claim that the second term vanishes for large N. Indeed, using the form (29) of the remainder term $R_N(z_1,z_2)$, we can see that for large N, the rational form

$$R_N(z_1,z_2)\left(1-\frac{z_2}{z_1}\right)\prod_{i=1}^{2}\frac{\prod_{j=1}^{b}\left(1-\frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a}\left(1-\frac{z_i}{\alpha_j}\right)}\frac{dz_2dz_1}{z_2z_1} \tag{30}$$

satisfies the conditions of Lemma 4.1 in z_2. This means that already applying the first residue operation $\operatorname{Res}_{z_2=0,\infty}$ results in 0. This completes the proof. □

8.4. $\mathrm{KTp}_{\mathcal{A}_2}$ in terms of Grothendieck polynomials – the minimal expansion.

Theorem 8.4 (Grothendieck expansion of $\mathrm{KTp}_{\mathcal{A}_2}$, the minimal version)**.** *We have the following expression for* $\mathrm{KTp}^{a\to b}_{\mathcal{A}_2}$ *in Grothendieck polynomials indexed by partitions:*

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \sum_{r=0}^{2l+2}\sum_{s=-l-2}^{-\lfloor\frac{r}{2}\rfloor} D_{r,s,l}\cdot G_{r+l+1,s+l+2}(\alpha_1^{-1},\dots,\alpha_a^{-1};\beta_1^{-1},\dots,\beta_b^{-1}),$$

where $l=b-a$*, and*

$$D_{r,s,l} = \begin{cases} d_{r,s} & \text{if } s>-l-2\\ \sum_{t=-r-1}^{-l-2} d_{r,t} = \sum_{t=-\infty}^{-l-2} d_{r,t} & \text{if } s=-l-2.\end{cases}$$

Proof. It follows from Theorem 8.2 that for large N

$$\mathrm{KTp}^{a\to b}_{\mathcal{A}_2} = \sum_{r=0}^{N}\sum_{s=-r-1}^{-\lfloor\frac{r}{2}\rfloor} d_{r,s}G_{r+l+1,s+l+2}. \tag{31}$$

For notational simplicity we omit the arguments $\alpha_i^{-1}, \beta_i^{-1}$ of the Grothendieck polynomials. Consider the sum

$$\sum_{s=-r-1}^{-\lfloor r/2\rfloor} d_{r,s} G_{r+l+1,s+l+2}$$

for a given r. In it, the occurring Grothendieck polynomials have the same first index $r+l+1$, but varying second index $s+l+2$. Notice that if $r>2l+2$ then all $s+l+2$ indexes are non-positive. Indeed, if $r>2l+2$ then $s\le -\lfloor r/2\rfloor < -\lfloor (2l+2)/2\rfloor = -l-1$ and hence $s+l+2<1$. Then using the straightening law $G_{I,0}=G_{I,-1}=G_{I,-2}=\dots$ (see (10) or Lemma 4.6) we have that

$$\sum_{s=-r-1}^{-\lfloor r/2\rfloor} d_{r,s} G_{r+l+1,s+l+2} = \left(\sum_{s=-r-1}^{-\lfloor r/2\rfloor} d_{r,s}\right) G_{r+l+1,0}. \tag{32}$$

Plugging in $z_2=0$ into $1/(1-z_2/z_1^2)$ results 1, hence for $r>0$ we have $\sum_{s=-r-1}^{\lfloor r/2\rfloor} d_{r,s}=0$, and in turn, the expression (32) is 0. This proves that in (31) the number N can be chosen to be as small as $2l+2$. The same statement may be obtained from a careful analysis of the vanishing of the residues of (30).

Now let $r\le 2l+2$. Using the same straightening law of Grothendieck polynomials we obtain

$$\sum_{s=-r-1}^{-\lfloor r/2\rfloor} d_{r,s} G_{r+l+1,s+l+2} = \underbrace{\left(\sum_{s=-r-1}^{-l-2} d_{r,s}\right)}_{D_{r,s,l}} G_{r+l+1,0} + \sum_{s=-l-1}^{-\lfloor r/2\rfloor} d_{r,s} G_{r+l+1,s+l+2},$$

completing the proof. □

Remark 8.5. The expansion in Theorem 8.4 is minimal in the sense that each occurring Grothendieck polynomial is parametrized by a partition (with non-negative components), and hence can not be simplified by the straightening laws (9)-(10) (or Lemma 4.6).

9. Alternating signs

The coefficients of the Grothendieck polynomials in both the *stable* and the *minimal* Grothendieck polynomial expansions of $\mathrm{KTp}_{\mathcal{A}_2}$ have alternating signs.

Theorem 9.1. *The coefficient of $G_{a,b}(\alpha_1^{-1},\dots,\alpha_e^{-1};\beta_1^{-1},\dots,\beta_b^{-1})$ in both the expansion of Theorem 8.2 and the expansion of Theorem 8.4 has sign $(-1)^{a+b}$.*

Proof. The statement for the expansion in Theorem 8.2 is equivalent to $d_{r,s}$ having sign $(-1)^{r+s+1}$, which follows from the explicit formula for $d_{r,s}$ in Section 8.3.

The statement for the expansion in Theorem 8.4 is equivalent to $D_{r,s,l}$ having sign $(-1)^{r+s+1}$ for any l. For this we need to additionally prove that

$$\text{the sign of } \sum_{t=-\infty}^{-l-2} d_{r,t} \text{ is } (-1)^{r+s+1} \tag{33}$$

for any l.

To prove (33) consider $f=(1-2x_1+x_1^2)/(x_2-2x_1+x_1^2)=\sum_{r,s} d_{r,s}x_1^r x_2^s$ (as before, $|x_1|<|x_2|$), and let $g=(-1+f)/(1-x_2)$. On the one hand $g=1/(x_2-2x_1+x_1^2)$ (from the explicit form of f). On the other hand

$$g=\left(-1+\sum_{r,s} d_{r,s}x_1^r x_2^r\right)(1+x_2+x_2^2+\dots)=\sum_{r,s}\left(\sum_{t=-\infty}^{s} d_{r,t}\right)x_1^r x_2^s.$$

Here we used that $d_{0,-1}=1$ and $d_{0,s}=0$ for all $s\neq -1$.

Comparing the two forms of g we find that statement (33) is equivalent to the the property that the coefficient of $x_1^r x_2^s$ in the expansion of $1/(x_2-2x_1+x_1^2)$ has sign $(-1)^{r+s+1}$. This latter claim follows from the calculation (25). $\square$

10. Remarks on higher singularities

For singularities higher than $\mathcal{A}_2$, it is difficult to carry out our program. There are no practical models for $\mathcal{A}_d$-singularities for $d\geq 7$, but even in the case of $\mathcal{A}_3$, where the model is very simple ([BSz, K2]), the combinatorial problems we face are rather complicated. A proof analogous to that of Theorem 8.1 in this case yields the following statement.

Theorem 10.1. *We have*

$$\mathrm{KTp}_{\mathcal{A}_3}^{a\to b}=\operatorname*{Res}_{z_1=0,\infty}\operatorname*{Res}_{z_2=0,\infty}\operatorname*{Res}_{z_3=0,\infty}\left(\frac{\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{z_3}{z_1}\right)\left(1-\frac{z_3}{z_2}\right)}{\left(1-\frac{z_2}{z_1^2}\right)\left(1-\frac{z_3}{z_1^2}\right)\left(1-\frac{z_3}{z_1z_2}\right)}\prod_{i=1}^{3}\frac{\prod_{j=1}^{b}\left(1-\frac{z_i}{\beta_j}\right)}{\prod_{j=1}^{a}\left(1-\frac{z_i}{\alpha_j}\right)}\frac{dz_3dz_2dz_1}{z_3z_2z_1}\right).$$

$\square$

This formula suggests that to obtain the Grothendieck expansion of $\mathrm{KTp}_{\mathcal{A}_3}$, we ought to consider the expansion

$$\frac{1}{(1-z_2/z_1^2)(1-z_3/z_1^2)(1-z_3/z_1z_2)}=\sum_{r,s,t} d_{r,s,t}(1-z_1)^r(1-z_2)^s(z-z_3)^t,$$

valid in the region $|1-z_1|<|1-z_2|<|1-z_3|$, and then find an appropriate way to resum the series

$$\sum_{r,s,t} d_{r,s,t}G_{r+l+1,s+l+2,t+l+3}(\alpha_1^{-1},\dots,\alpha_a^{-1};\beta_1^{-1},\dots,\beta_b^{-1}). \tag{34}$$

to obtain finite expressions. The concrete form of the resummation procedure and the resulting finite expression is not clear at the moment.

It seems even more difficult to find the analogue of Theorem 8.4 (the minimal Grothendieck expansion) for $\mathcal{A}_3$. To achieve the Grothendieck expansion of Theorem 8.4 from that of Theorem 8.2 we needed to work only with *one* of the straightening laws, namely (10). However, to "straighten" the partitions in (34) one is forced to use the other straightening law, namely (9), and this seems much more complex. It would be interesting to develop the residue calculus or another analytic tool which replaces the combinatorics of (9).

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