GRADUATE TOPOLOGY COLORING BOOK

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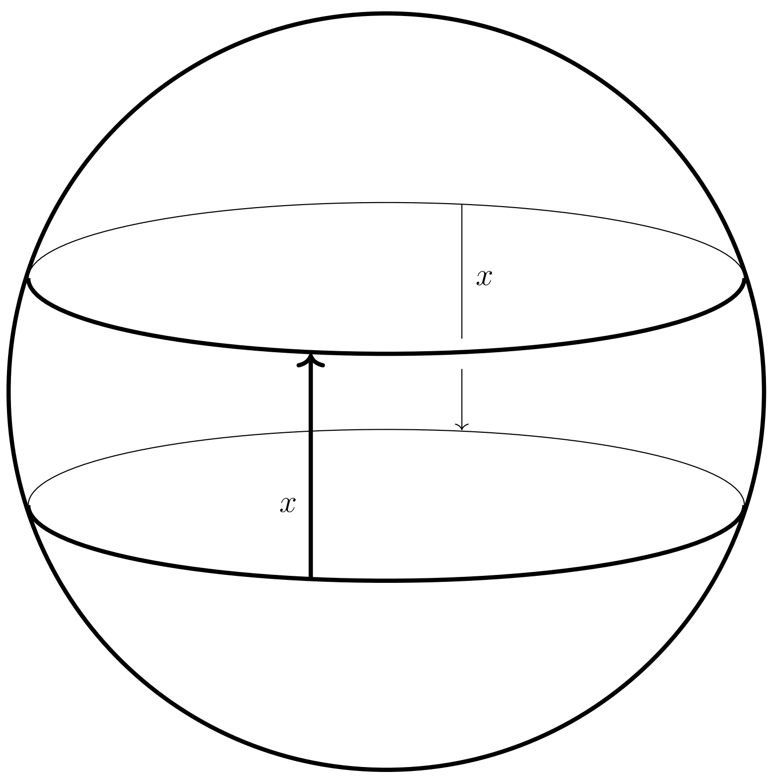
Abstract. This series of lectures contains the material for the class Math 681, Graduate Topology, as it was taught in Fall 2021—a.k.a. the δ semester—at the University of North Carolina at Chapel Hill. It is called a Coloring Book, because numerous arguments, indicated by the sign ♠, that were presented in the class are *not* typed in. In fact, those arguments are deliberately left out of this text: reading those arguments would have no educational value for the reader. Figuring out those arguments ("coloring between the contours") does. Hence, whenever the reader meets a ♠ sign, they should stop and fill in the missing proof. Such "coloring" of this Coloring Book is an essential part of learning the subject.



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Contents

1. Point-set topology

1.1. **Topological space, basis.** We denote the powerset (the set of subsets) of a set X by 2^X.

Definition 1.1. *The ordered pair* $(X, \mathcal{T})$ *is a* topological space *(X is the base set, $\mathcal{T}$ is the topology on X), if X is an arbitrary set, and $\mathcal{T} \subset 2^X$ satisfies*

- $\emptyset, X \in \mathcal{T}$,
- *the union of arbitrary many sets from $\mathcal{T}$ is in $\mathcal{T}$,*
- *the intersection of finitely many sets from $\mathcal{T}$ is in $\mathcal{T}$, (equivalently, the intersection of two elements of $\mathcal{T}$ belongs to $\mathcal{T}$).*

Elements of $\mathcal{T}$ we call *open* sets. If the complement $A^c = X - A$ is open we call A a *closed* set. Find a few (all) topologies on small sets (eg. $|X| = 1, 2, 3$) ♠.

Example 1.2. $(X, 2^X)$ *is a topological space, we call it the* discrete *topological space.*

Example 1.3. $(X, \{\emptyset, X\})$ *is a topological space, we call it the* antidiscrete *topological space.*

Example 1.4. *For a given X let $U \in \mathcal{T}$ if U^c is finite or if $U = \emptyset$. Then $(X, \mathcal{T})$ is a topological space* ♠*. We call it the* finite complement topological space *(or the* cofinite *topological space).*

Example 1.5. *For a given X let $U \in \mathcal{T}$ if U^c is countable (meaning: finite or countably infinite) or if $U = \emptyset$. Then $(X, \mathcal{T})$ is a topological space* ♠*. We call it the* countable complement topological space.

The last two examples can be generalized (if you know the theory of large infinities). Let $\aleph$ be an infinite cardinality. Then $U \in \mathcal{T}$ if and only if $U = \emptyset$ or $|U^c| < \aleph$ defines a topology ♠.

We say $(X, \mathcal{T}')$ is a finer topological space than $(X, \mathcal{T})$ if $\mathcal{T}' \supset \mathcal{T}$. In this case the latter is coarser than the first one. The discrete and the antidiscrete topologies are the finest and the coarsest topologies on X.

Definition 1.6. *Let $(X, \mathcal{T})$ be a topological space. The set $\mathcal{B} \subset \mathcal{T}$ is called a* basis *of $(X, \mathcal{T})$ if every element of $\mathcal{T}$ can be written as a union of some of the elements of $\mathcal{T}$.*

For example, $\mathcal{T}$ is a basis of $\mathcal{T}$. Usually there are bases of $\mathcal{T}$ that are "much smaller" than $\mathcal{T}$. The set $\mathcal{B} = \{\{x\} : x \in X\}$ (the set of "singletons") is a basis of the discrete topology ♠.

The basis $\mathcal{B}$ determines $\mathcal{T}$ ♠.

Theorem 1.7. *Let X be a set. The set $\mathcal{B} \subset 2^X$ is a basis of a topology on X if and only if $\mathcal{B}$ satisfies the "basis properties":*

- *the union of elements of $\mathcal{B}$ is X,*
- *the intersection of any two elements in $\mathcal{B}$ can be written as a union of some of the elements of $\mathcal{B}$.*

Proof. ♠ □

This last notion *basis* gives us an economical way of defining topologies. We do not need to define $\mathcal{T}$, just define $\mathcal{B}$ (but make sure it satisfies the basis properties).

Example 1.8. *Let $X = \mathbb{R}$ and $\mathcal{B} = \{$open intervals$\}$. This $\mathcal{B}$ satisfies the basis properties ♠, hence it determines a topology. We call it the* Euclidean topology *on $\mathbb{R}$ and denote it by $\mathbb{R}_{\mathcal{E}}$.*

In $\mathbb{R}_{\mathcal{E}}$ a set $U \subset \mathbb{R}$ is open if for all $x \in U$ there exists a positive number ε such that $B(x,\varepsilon) := \{y \in \mathbb{R} : |y-x| < \varepsilon\} \subset U$ ♠. That is, $\mathbb{R}_{\mathcal{E}}$ is the topological space studied in Real Analysis.

Example 1.9. *Let $X = \mathbb{R}$ and $\mathcal{B} = \{[a,b) : a < b\}$. This $\mathcal{B}$ satisfies the basis properties ♠, hence it determines a topology. We call it the* Sorgenfrey line *or the* lower limit topology *on $\mathbb{R}$ and denote it by $\mathbb{R}_{\mathrm{ll}}$.*

1.2. **Order topology.**

Definition 1.10. *$(X,\leq)$ is a totally ordered set if $\leq$ is irreflexive, transitive, anti-symmetric, and satisfies the trichotomy law ($\forall a,b \in X : a \leq b$ or $b \leq a$). We write $a < b$ for $(a \leq b, a \neq b)$.*

A total order on X restricts to a total order on a subset $Y \subset X$ ♠. In a totally ordered set intervals $(a,b), [a,b), (a,b], [a,b]$ and rays $[a,\infty)$, (a,∞), $(-\infty,a)$, $(-\infty,a]$ are defined for $a,b \in X$.
For the totally ordered set $(X,\leq)$ the set

$$\mathcal{B} = \{\emptyset, (a,b), (a,\infty), (-\infty,a), X \ : \ a,b \in X\}$$

satisfies the basis axioms ♠. The generated topology is called the order topology on X.
Totally ordered spaces are, for example, $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c, \mathbb{Q}\cap[0,1], \mathbb{Q}\cap[0,1), [0,1)\cup\{2\}$ with the standard $\leq$; $\mathbb{R}^2$ with the lexicographic order. Hence we have order topology on them.

1.3. **Product topology.**

Definition 1.11. *Let $(X,\mathcal{T})$, $(Y,\mathcal{S})$ be topological spaces. The set $\mathcal{B} = \{U \times V : U \in \mathcal{T}, V \in \mathcal{S}\}$ satisfies the basis axioms ♠. The generated topology is called the product topology on $X \times Y$.*

We have $\mathbb{R}_{\mathcal{E}} \times \mathbb{R}_{\mathcal{E}} =$ the Euclidean topology on $\mathbb{R}^2$, that is, a set $\subset \mathbb{R}^2$ is open if there exists an $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$ ♠.
Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases of $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ respectively. The set $\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ is also a basis of the same product topology ♠.

1.4. **Subspace topology.**

Definition 1.12. *Let $(X,\mathcal{T})$ be a topological space, and $Y \subset X$. The set $\{U \cap Y : U \in T\}$ is a topology on Y ♠. It is called the subspace topology on Y inherited from X.*

For example in $[0,1]$ with topology inherited from $\mathbb{R}_{\mathcal{E}}$ the set $[1,1/3)$ is open ♠. Interesting examples include $\mathbb{Z} \subset \mathbb{R}$, $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q}^c \subset \mathbb{R}$ with the topology inherited from $\mathbb{R}_{\mathcal{E}}$. If $K \subset \mathbb{R}^3$ is a knot, the topological space $\mathbb{R}^3 - K \subset \mathbb{R}^3$ is an important topological space used to study the knot.
Let $\mathcal{B}$ be a basis of $\mathcal{T}$. The set $\mathcal{B}' = \{U \cap Y : U \in \mathcal{B}\}$ is a basis of the subspace topology ♠.

One may ask if some of our operations commute or not.

- If $(X,\mathcal{T})$, $(Y,\mathcal{S})$ are topological spaces, $A\subset X$, $B\subset Y$ we can consider two topologies on $A\times B$. First, the subspace topology from the product topology $X\times Y$. Second, the product topology of the subspace topologies on A and B. Are they necessarily the same or not? ♠
- If $(X,\leq)$ is a totally ordered set and $A\subset X$ then on A we can consider two topologies. First, the subspace topology from the order topology on X. Second, the order topology defined by $(A,\leq|_A)$. Are they necessarily the same or not? ♠

1.5. **Metric topology.**

Definition 1.13. *(X,d) is a metric space if X is a set and $d : X\times X\to\mathbb{R}_{\geq 0}$ is a function satisfying*

- $d(x,y)=d(y,x)$,
- $d(x,y)=0\Leftrightarrow x=y$,
- *(triangle inequality)* $d(x,y)+d(y,z)\geq d(x,z)$.

In a metric space we define $B(x,r)=\{y\in X : d(x,y)<r\}$.

The set $\mathcal{B}=\{B(x,r) : x\in X, r\in\mathbb{R}_{>0}\}$ satisfies the basis axioms ♠. [This lemma will be useful: For $y\in B(x,r)$ there is an $s>0$ such that $B(y,s)\subset B(x,r)$ ♠.] The induced topology on X is called the metric topology. Different metrics may induce the same topology ♠.
The Euclidean topology $\mathbb{R}^n_{\mathcal{E}}$ is induced from the metric $d(x,y)=\sqrt{\sum_i(x_i-y_i)^2}$ on $\mathbb{R}^n$ ♠.
The function $d(x,y)=\begin{cases}1 & x\neq y\\ 0 & x=y\end{cases}$ is a metric on any set ♠. It induces the discrete topology.
The restriction of a metric to a subset A of X is a metric. The topology induced by the restricted metric is the same as the subspace topology inherited from (X,d) ♠.

1.6. **On open and closed sets.** The collection of closed sets includes $\emptyset$, X; finite union and arbitrary intersection of closed sets is closed ♠. The collection of closed sets satisfying these properties as axioms could be an alternative definition of topological space ♠.

Let X be a topological space, and $A\subset Y\subset X$. Then A is a closed set in the subspace topology on Y, if and only if there is a set Z closed in X with $A=Z\cap Y$. ♠

Let Y be open in the topological space X, $U\subset Y$. Then U is open in Y if and only if it is open in X ♠.

Let Y be closed in the topological space X, $V\subset Y$. Then V is closed in Y if and only if it is closed in X ♠.

The projection map $\pi : X\times Y\to X$ is an *open map*, that is, U open implies $\pi(U)$ open ♠.
The same projection is not necessarily a *closed map.* ♠

1.7. **Interior, closure.**

Definition 1.14. *Let A be a subset in a topological space. Define*

$$\operatorname{int} A=\cup\{U : U \textit{ is an open set}, U\subset A\}\qquad \overline{A}=\cap\{V : V \textit{ is a closed set}, V\supset A\}.$$

Clearly $\operatorname{int} A \subset A \subset \overline{A}$.
If $U \subset A$ is an open set then $U \subset \operatorname{int} A$. If $V \supset A$ is a closed set then $V \supset \overline{A}$. That is, $\operatorname{int} A$ is the largest open set contained in A, and $\overline{A}$ is the smallest closed set containing A ♠.
Let X be a topological space, Y a subspace, and $A \subset Y$. Then the closure of A *in the topological space* Y is equal to $\overline{A} \cap Y$ ♠. That is, (using obvious temporary notation) we have

$$\overline{A}^Y = \overline{A}^X \cap Y.$$

The analogous statement for $\operatorname{int} A$ is not true ♠.
An open set containing $x \in X$ is called a neighborhood of x.

Proposition 1.15 (Pointwise characterization of interior). *We have $x \in \operatorname{int} A$ if and only if there is a neighborhood of x which is a subset of A ♠.*

Proposition 1.16 (Pointwise characterization of closure). *We have $x \in \overline{A}$ if and only if for every neighborhood U of x the intersection $U \cap A$ is not empty ♠.*

1.8. Limit points.

Definition 1.17. *Let A be a subset of the topological space X. The point $x \in X$ is called a limit point of A (notation $x \in A'$) if for every neighborhood U of x the set $(U \cap A) - \{x\}$ is not empty.*

Clearly $x \in A' \Leftrightarrow x \in \overline{A - \{x\}}$ ♠.

Proposition 1.18. *We have $\overline{A} = A \cup A'$ ♠.*

The following conditions are equivalent: (i) A is closed, (ii)$A = \overline{A}$, (iii) A cointains all of its limit points ♠.

1.9. Boundary.

Definition 1.19. *Define the boundary of the set A in a topological space to be $\partial A = \overline{A} \cap \overline{(A^c)}$.*

The boundary ∂A is a closed set ♠.
A point x belongs to ∂A if and only if every neighborhood of x intersects both A and A^c ♠.

1.10. Continuous functions.

Definition 1.20. *The map $f : X \to Y$ between topological spaces is* continuous *if the preimage of an open set in Y is open in X.*

The following are equivalent characterizations of continuity ♠.

- Let $\mathcal{B}$ be a basis in Y. We have $U \in \mathcal{B} \Rightarrow f^{-1}(U)$ is open in X.
- The preimage of a closed set in Y is closed in X.
- $f(\overline{A}) \subset \overline{f(A)}$.
- (pointwise characterization) $\forall x \in X$ and for all neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V$.

The identity map $X \to X$ is continuous ♠.
The constant map $X \to Y$ is continuous ♠.
If X is discrete then $f : X \to Y$ is continuous ♠.
If Y is antidiscrete then $f : X \to Y$ is continuous ♠.
The composition of continuous functions is continuous ♠.
The identity map $(X, \mathcal{T}) \to (X, \mathcal{S})$ is continuous if and only if $\mathcal{T}$ is a finer topology than $\mathcal{S}$ ♠.

Definition 1.21. *A bijection $f : X \to Y$ between topological spaces for which both f and f^{-1} are continuous is called a homeomorphism. A map $f : X \to Y$ for which the induced map $X \to f(X)$ is a homoeomorphism ($f(X) \subset Y$ with the subspace topology) is called an embedding.*

The map $(0, \infty) \to \mathbb{R}^2$ (with Euclidean topologies) defined by $x \mapsto (x, \sin(1/x))$ is an embedding.
The map $[0, 2\pi) \to \{x \in \mathbb{R}^2 : \|x\| = 1\}$ (with Euclidean topologies) defined by $x \mapsto (\cos x, \sin x)$ is *not* a homeomorphism ♠.
If there exists an $X \to Y$ homeomorphism then we call X and Y homeomorphic. This is an equivalence "relation" ♠.

1.11. Continuity vs product space, continuity vs subspace.

The projection map $\pi_X : X \times Y \to X$ is continuous ♠.

Proposition 1.22. *The map $X \to Y \times Z$ is continuous if and only if both $\pi_Y \circ f$ and $\pi_Z \circ f$ are continuous ♠.*

Let $A \subset X$ be a subspace. The inclusion map $i : A \subset X$ is continuous ♠.

Proposition 1.23. *Let $X = \cup_a U_\alpha$ where U_α is open in X for all α. The map $f : X \to Y$ is continuous if and only if $f|_{U_\alpha}$ is continuous for all α.*

Proof. Use ♠

- $f^{-1}(V) = f^{-1}(V) \cap X = f^{-1} \cap (\cup_\alpha U_\alpha) = \cup_\alpha (f^{-1}(V) \cap U_\alpha)$, and
- if $U \subset X$ is open then for $A \subset U \subset X$ we have A open in $U \Leftrightarrow A$ open if X.

□

Proposition 1.24. *(Pasting lemma) Let $X = \cup_{i=1}^n A_i$ where A_i is closed in X for all i (finite union). The map $f : X \to Y$ is continuous if and only if $f|_{A_i}$ is continuous for all i ♠.*

Proof. Use the "preimage of closed is closed" definition of continuity, and an appropriate modification of the proof above ♠. □

1.12. Separation axioms.

Definition 1.25. *The topological space $(X, \mathcal{T})$ is called*

- *T_0 if for any $x \neq y \in X$ there is a $U \in \mathcal{T}$ that contains one of x, y but not the other;*
- *T_1 if for any $x \neq y \in X$ there is a $U \in \mathcal{T}$ such that $x \in U, y \notin U$;*
- *T_2 (Hausdorff) if for any $x \neq y \in X$ there are disjoint $U, V \in \mathcal{T}$ such that $x \in U, y \in V$;*
- *T_3 (regular) if T_1 and for any $x \notin A \subset X$, A closed there are disjoint $U, V \in \mathcal{T}$ such that $x \in U, A \subset V$;*

- T_4 *(normal) if* T_1 *and for any* $A, B \subset X$ *disjoint closed sets there are disjoint* $U, V \in \mathcal{T}$ *such that* $A \subset U, B \subset V$.

The space X is T_1 if and only if singletons are closed ♠. Equivalently, if finite sets are closed.
We have $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ ♠.
The antidiscrete topology is not T_0.
The space $(\{1,2\}, \{\emptyset, \{1\}, \{1,2\}\})$ is T_0 but not T_1. This example has two generalizations:

(1) The "distinguished point topology": Fix $x \in X$. Let $U \in \mathcal{T}$ if $x \in U$.
(2) The "avoid a point topology": Fix $x \in X$. Let $U \in \mathcal{T}$ if $x \notin U$.

Both of these are T_0 but not T_1 ♠.
If X has infinitely many points then the finite complement topology on X is T_1 but not T_2.

Definition 1.26 (Ravioli)**.** *Let* $X = (\mathbb{R} - \{0\}) \cup \{0', 0''\}$. *Define a topology on* X *via the following basis. The intervals* (a, b) *with either* $a, b < 0$ *or* $a, b > 0$ *belong to* $\mathcal{B}$. *Also, for* $a < 0, b > 0$ *the set* $((a, b) - \{0\}) \cup \{0'\}$ *and the set* $((a, b) - \{0\}) \cup \{0''\}$ *belong to* $\mathcal{B}$. *This* $\mathcal{B}$ *satisfies the basis properties* ♠. *The defined topology we will call the ravioli space.*

The ravioli is T_1 but not T_2 ♠.

Proposition 1.27. *The property*

- *for all* $x \in X$ *and neighborhood* U *of* x *there exist a neighborhood* V *of* x *with* $\overline{V} \subset U$

is equivalent to the T_3 *property.*

Proposition 1.28. *The order topology is* T_2. *The metric topology is* T_2 ♠.

Proposition 1.29. *A subspace of a* T_2 *space is* T_2. *Product of* T_2 *spaces is* T_2 ♠.

In fact the analogous proposition holds for T_3 but not for T_4—we do not prove these statements here.

Proposition 1.30. *If* $A \subset X$, X *is* T_1 *then* $x \in A'$ *if and only if every neighborhood of* x *contains infinitely many points of* A ♠.

Definition 1.31. *A sequence is a map* $a : \mathbb{N} \to X$. *We use the standard notation* a_n. *A point* $b \in X$ *is a limit of* a_n *if for every neighborhood* U *of* b *there is a threshold* N *such that* $n \geq N$ *implies* $a_n \in U$.

In an antidiscrete space every point is the limit of every sequence.

Proposition 1.32. *In* T_2 *space the limit of a sequence, if exists, is unique* ♠.

1.13. **Countability axioms.**

Definition 1.33. *A neighborhood basis of a point* x *in a topological space* X *is a collection* U_α *of neighborhoods of* x *such that for all neighborhood* V *of* x *there is an* α *with* $U_\alpha \subset V$.

Definition 1.34. *The topological space* $(X, \mathcal{T})$ *is called*

- M_1 *(first countable) if every point* $x \in X$ *has a countable neighborhood basis;*

- *M_2 (second countable) if $(X, \mathcal{T})$ has a countable basis.*

Without loss of generality we may assume that a countable neighborhood basis is 'nested': $U_1 \supset U_2 \supset U_3 \supset \ldots$. ♠.
We have $M_2 \Rightarrow M_1$ ♠. The Euclidean line is M_2 (think of rational endpoint open intervals) ♠.

Proposition 1.35. *The Sorgenfrey line is M_1 but not M_2.*

Proof. That it is M_1 at x is proved by $\{[x, r) : r \in \mathbb{Q}, r > x\}$ ♠. Now let $\mathcal{B}$ be a basis of $\mathbb{R}_{ll}$. For $x \in \mathbb{R}$ there is a neighborhood of x contained in $[x, x+1)$ ♠. Choose one such U_x for each $x \in \mathbb{R}$. The chosen U_x's are all different, because $\inf U_x = x$. Hence the cardinality of $\mathcal{B}$ is at least the cardinality of $\mathbb{R}$. □

The space $\mathbb{R}$ with the countable complement topology is not M_1 ♠.

Proposition 1.36. *If X is M_1 and $A \subset X$, then the following are equivalent*

- *$x \in \overline{A}$;*
- *there is a sequence a_n in A whose limit is x.*

Proof. ♠ □

Proposition 1.37. *If X is M_1 then the following are equivalent*

- *$f : X \to Y$ is continuous;*
- *(f is sequentially continuous, that is) $\lim x_n = a \ \Rightarrow \lim f(x_n) = f(a)$.*

Proof. The proof from Real Analysis applies. One direction is convenient to prove by contradiction ♠. □

1.14. **Quotient space.**

Definition 1.38. *Let $(X, \mathcal{T})$ be a topological space, A a set, and $q : X \to A$ a surjection. We define a topology on A by setting U open in A if $q^{-1}(U)$ is open in X.*

The defined collection of open sets in A is indeed a topology ♠. The quotient topology on A is the finest topology among those for which q is continuous ♠.
The surjection q is often given by an equivalence relation $\sim$ on X. Namely, $\sim$ determines a surjection $X \to X/\sim$ to the set of equivalent classes. Thus, we defined a topology of $X/\sim$.

Example 1.39. *Let $X = [0, 1] \cup [2, 3]$ and define $1 \sim 2$ (and the obvious other relations that are forced by the fact that $\sim$ is an equivalence relation). Then the quotient space is homeomorphic with $[0, 2]$ ♠.*

Example 1.40. *Let $X = [0, 1]$, and define $0 \sim 1$ (and the obvious other relations that are forced by the fact that $\sim$ is an equivalence relation). Then the quotient space is homeomorphic with S^1 ♠.*

Here and in the whole course $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ with the Euclidean topology.

Definition 1.41. *Let G be a group, X a topological space. The map*

$$G \times X \to X, \qquad \text{written as} \qquad (g, x) \mapsto g \cdot x$$

is a (continuous) left group action if

- $1 \cdot x = x$ *(for all x);*
- $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ *(for all g_1, g_2, x);*
- $x \mapsto g \cdot x$ *is a continuous map $X \to X$ (for all g).*

The map in the last requirement is actually a homeomorphism ♠.
For a group action we define the relation $x \sim y$ if there is a g such that $g \cdot x = y$. It is an equivalence relation ♠. An equivalence class is called an orbit. Hence the quotient space construction defines a topology on the space of orbits.

Example 1.42. *The multiplicative group $S^1 \subset \mathbb{R}^2 = \mathbb{C}$ acts on $\mathbb{R}^2 = \mathbb{C}$ by multiplication. The space of orbits is homeomorphic to $[0, \infty)$ ♠.*

Example 1.43. *The additive group $\mathbb{R}$ acts on $\mathbb{R}^2$ by $u \cdot (x, y) = (x + uy, y)$. As a set, the space of orbits can be identified with the union of the axes in $\mathbb{R}^2$. Describe the obtained topology ♠.*

The last example produced a rather "ugly" topological space. Other pathological examples of quotient spaces include

- $X = \mathbb{R}$, $x \sim y$ if $y - x \in \mathbb{Q}$.
- $X = \mathbb{R}$, $x \sim y$ if they are both rational, or if $x = y$.

Example 1.44. Let $X = [0, 1] \times \mathbb{Z}_+$, and define $(0, a) \sim (0, b)$ for all a, b (and the obvious other relations that are forced by the fact that $\sim$ is an equivalence relation). Let Y be the quotient space. Let Z be the union of the segments in the plane connecting $(0, 0)$ with $(1/n, 1)$ for all $n \in \mathbb{Z}_+$, with the Euclidean topology. It is tempting to think that Y is homeomorphic with Z. It is not ♠.

Let $q : X \to Y$ be a quotient map. A map $\tilde{f} : Y \to Z$ determines a map $f : X \to Z$ making $f = \tilde{f} \circ q$ true. Conversely, a map $f : X \to Z$ that is constant on the equivalence classes determine a map $\tilde{f} : Y \to Z$ making $f = \tilde{f} \circ q$ true ♠.

Proposition 1.45. *The map f is continuous if and only if $\tilde{f}$ is continuous ♠.*

1.15. **Connectedness.**

Definition 1.46. *Let X be a topological space. The pair (U, V) is called a separation of X, if $X = U \cup V$, both are non-empty, and are disjoint. If X has no separation, it is called connected. A subset A in a topological space is connected if it is a connected topological space with the subspace topology.*

In a separation U is both open and closed, and $U \neq \emptyset, U \neq X$.
Examples: $\mathbb{R} - \{0\}$ is not connected. The space $\{(x, 1/x) : x > 0\} \cup y$-axis is not connected. The space $\mathbb{Q}$ is not connected. The space $\{(x, \sin 1/x) : x > 0\} \cup y$-axis is connected ♠.

Lemma 1.47. *Let $X = U \cup V$ be a separation, and $A \subset X$ a connected set. Then $A \subset U$ or $A \subset V$ ♠.*

Lemma 1.48. *Let $X = \cup_\alpha U_\alpha$, with each U_α connected. If there is a point $p \in \cap_\alpha U_\alpha$, then X is connected ♠.*

Proposition 1.49. *Let A be a connected subset of X, and $A \subset B \subset \overline{A}$. Then B is connected.*

Proof. Let $B = U \cup V$ be a separation of the topological space B. Since Lemma 1.47 we can assume $A \subset U$. Then we have $\overline{A}^B \subset \overline{U}^B$. The left hand side is B, and the right hand side is U, hence $B \subset U$, which is a contradiction ♠. □

Proposition 1.50. *Let $f : X \to Y$ be continuous, and A a connected subset of X. Then $f(A)$ is connected.*

Proof. Let $\tilde{f} : A \to f(A)$ be induced by f. It is also continuous ♠. If $\tilde{f}(A) = U \cup V$ was a separation of $\tilde{f}(A)$ then $\tilde{f}^{-1}(U)$, $\tilde{f}^{-1}(V)$ would be a separation of A. □

Proposition 1.51. *The product of two connected spaces is connected ♠.*

Proposition 1.52. *For $A \subset \mathbb{R}$ (with Euclidean topology) the following are equivalent:*

(1) *A is connected;*
(2) *A is convex ($x, y \in A$ and $x < z < y$ imply $z \in A$);*
(3) *A is an interval (arbitrary open/closed on each side, finite, half infinite, or infinite).*

Proof. (1) ⇒ (2) is proved indirectly by $((-\infty, z) \cap A) \cup ((z, \infty) \cap A)$.
(2) ⇒ (3): First show that $(\inf A, \sup A) \subset A$ ♠.
(3) ⇒ (1) [Heine's theorem] Let $u \in U, v \in V$, $u < v$ for a separation of an interval. Consider $x = \sup U$ and check the cases $x \in U$, $x \in V$ ♠.

□

Corollary 1.53 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b]) \supset [f(a), f(b)]$ ♠.*

Definition 1.54. *Let X be a topological space. Define $x \sim y$ if there is an open set $A \subset X$ with $x, y \in A$. Its is an equivalence relation on X (use Lemma 1.48 ♠). The equivalence classes are called the connected components of X.*

Determine the connected components of the examples after Definition 1.46 ♠.

Proposition 1.55. *Connected components are*

- *connected ♠;*
- *closed (♠ use Proposition 1.49).*

If X has finitely many connected components then connected components are open ♠.

1.16. **Path connectedness.**

Definition 1.56. *For $x, y \in X$ define $x \sim y$ if there is a path ($\gamma : [0,1] \to X$ continuous) connecting them (ie. $\gamma(0) = x$, $\gamma(1) = y$).*

This is an equivalence relation ♠. The equivalence classes are called path components. If X is one path component, it is called path-connected. If $A \subset X$ is path-connected, it must be contained in a path component of X.
Instructive example: $\{(x, \sin(1/x)) : x > 0\} \cup y$-axis—this is a connected but not path-connected space ♠.
Path components must be contained in connected components ♠. In general path components are neither open nor closed.

1.17. **Compactness.**

Definition 1.57. *A collection of open sets U_α in X such that $\cup U_\alpha = X$ is called an open cover(ing) of X. A space X is called compact, if every open covering has a finite sub-cover.*

$\mathbb{R}$ is not, $(0,1]$ is not, $\{1/n\}$ is not, $\{1/n\} \cup \{0\}$ is compact.
If A is a subspace of X, then A being compact is equivalent with this property: If U_α are open in X and $\cup U_\alpha \supset A$, then finitely many of them also cover A ♠.

Proposition 1.58. *A closed subset of a compact space is compact ♠.*

Proposition 1.59. *A compact subspace of a Hausdorff space is closed.*

Proof. $A \subset X$. We will show that A^c is open, using the pointwise criterion of openness. Let $x \in A^c$. Since X is T_2, for all $a \in A$ there exists disjoint neighborhoods U_a and V_a of a and x respectively. Finitely many of the U_a's cover A. The intersection of the corresponding V_a's is a neighborhood of x, disjoint from A ♠. □

Proposition 1.60. *Continuous image of compact space is compact ♠.*

Corollary 1.61. *If $f : X \to Y$ is a continuous bijection, X compact, Y Hausdorff, then f is a homeomorphism.*

Proof. For $A \subset X$ closed, the set $(f^{-1})^{-1}(A)$ is closed, using the propositions above ♠. □

Proposition 1.62. *If X and Y are compact then $X \times Y$ is compact.*

Proof. Let U_α be an open cover of $X \times Y$. Let $x \in X$. The fiber $\{x\} \times Y$ is compact, hence finitely many of the U_α's cover it. Let V_x be the union of these U_α's. Claim ("tube lemma"): there exists a neighborhood W_x of x such that $W_x \times Y \subset V_x$ ♠. From the collection W_x finitely many cover X. Each $W_x \times Y$ is covered by a finite collection of U_α's, so $X \times Y$ is covered by a finite collection of U_α's. □

Proposition 1.63. *The closed interval $[0,1]$ is compact.*

Proof. Let U_α be an open cover of $[0,1]$, and assume that no finite subcollection covers $[0,1]$. Consider $[0,1/2]$ and $[1/2,1]$. At least one of them is not covered by a finite subcollection of U_α's, choose that one, and continue halving it, always shoosing a half which is not covered by a finite subcollection of U_α's. Let x be in the intersection of these intervals (completeness of $\mathbb{R}$). Since x is contained by one of the U_α's, there is an ε such that the ε-neighborhood of x is in a U_α. However, after a while the halving intervals are contained in this ε-neighborhood, which is a contradiction. $\square$

Corollary 1.64. *A set $A \subset \mathbb{R}^n$ compact if and only if it is bounded and closed* ♠.

Corollary 1.65 (Extreme Value Theorem)**.** *If X is comapct then the continuous function $f : X \to \mathbb{R}$ attains its maximum (minimum)* ♠.

1.18. **Topological groups — sketch.**

Definition 1.66. *A T_1 topological space G that is also a group is called a topological group if the group operations $G \times G \to G$ (multiplication) and $G \to G$ (inverse) are continuous.*

We get an equivalent notion if we just require that the map $f : G \times G \to G$, $(x,y) \mapsto xy^{-1}$ is continuous ♠.
A topological group is necessarily T_2 (hint: consider $f^{-1}(1)$) ♠.

Proposition 1.67. *An open subgroup of a topological group is also closed* ♠.

Proposition 1.68. *The closure of a subgroup of a topological group is also a subgroup* ♠.

Important examples of topological groups include $(\mathbb{Z},+)$, $(\mathbb{R},+)$, $(\mathbb{R}-\{0\},*)$, $(S^1,*)$, $(\mathrm{GL}_n(\mathbb{R}),*)$, $(O(n),*)$, (S^3, quaternionic multiplication).

1.19. **Metrization theorems — sketch.**

Theorem 1.69 (Urysohn metrization theorem — not proved in this course)**.** *A T_3, M_2 topological space is metrizable (homeomorphic with a metric space).*

Remark 1.70.

- T_3, M_2 implies T_4.
- Somewhere in the proof we need to "create" some functions out of thin air (out of just spaces). This is done using **Urysohn's lemma** [not proved in this course]: If A, B are disjoint closed subsets of a T_4 topological space X then there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$, $f^{-1}(1) = B$. It is instructive to glance at the proof of this lemma (in the textbook or online) to see how the T_4 property is used to "create a function", and hence to prove Urysohn's lemma.
- The Sorgenfrey line is T_3, it is not M_2, and it is not metrizable ♠.

1.20. **Infinite products — sketch.** Let X_α be topological spaces for all $\alpha \in A$. Define

$$\prod_{\alpha\in A} X_\alpha = \{f : A \bigcup_{\alpha\in A} : f(\alpha) \in X_\alpha\}$$

as a set. We define two topologies on this set:

- [basis of *box topology*] $\prod_{\alpha\in A} U_\alpha$ where $U_\alpha \subset X_\alpha$ open;
- [basis of *product topology*] $\prod_{\alpha\in A} U_\alpha$ where $U_\alpha \subset X_\alpha$ open, $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$.

Both of these collections satisfy the axioms of a basis ♠.

Proposition 1.71. *A map $f : A \to \prod_{\alpha\in A} X_\alpha$ (with the product topology) is continuous if and only if all the component functions are continuous ♠.*

The map $f : \mathbb{R} \to \prod_{n\in\mathbb{N}} \mathbb{R}$, $t \mapsto (t, t, t, \ldots)$ has continuous coordinate functions, but is not continuous if the codomain is equipped with the box topology. (♠ Hint: consider the preimage of the set $(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots$.)

Theorem 1.72 (Tychonoff's theorem)**.** *If X_α is compact for all $\alpha \in A$ then $\prod_{\alpha\in A} X_\alpha$ (with the product topology) is compact [not proved in this course].*

1.21. Locally compact spaces, Alexandrov compactification — sketch.

Definition 1.73. *The space X is called locally compact at $x \in X$ if there exists a neighborhood of x which is contained in a compact set. The space is locally compact if it is locally compact at all $x \in X$.*

Proposition 1.74. *If X is T_2 then the following are equivalent ♠*

- *X is locally compact;*
- *For all $x \in X$ and all neighborhood U of x there is a compact set A and another neighborhood V of x, such that $x \in V \subset A \subset U$.*

Definition 1.75. *The space Y is an Alexandrov compactification (a.k.a. 1-point compactification) of the space X if*

- *$X \subset Y$ (with subspace topology), $Y - X$ is one point;*
- *Y is compact T_2.*

Theorem 1.76. *The space X has an Alexandrov compactification if and only if it is locally compact and T_2. In this case the Alexandrov compactification is unique.*

Proof. Let X be locally compact T_2. The construction of Y is as follows: let $Y = X \cup \{\infty\}$, with the topology defined by the open sets

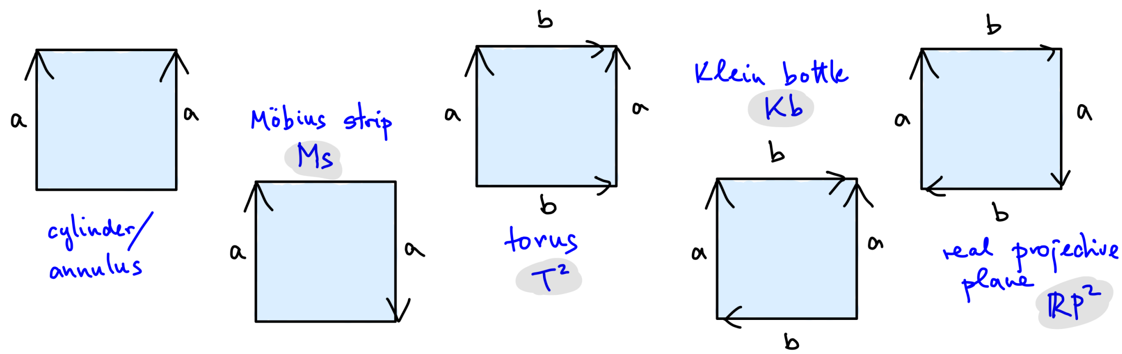
(1) $U \subset X$ open;
(2) $U \cup \{\infty\}$, where $U \subset X$ is open, $X - U$ is compact.

Verifying that this is an Alexandrov compactification, as well as other parts of the proof are straightforward. ♠ □

Example 1.77. The stereographic projection $S^2 - \{\text{North Pole}\} \to \mathbb{R}^2$ shows that the Alexandrov compactification of $\mathbb{R}^2$ is S^2 ♠. In fact this works in every dimension for $\mathbb{R}^n$ and S^n ♠. The Alexandrov compactification of a compact T_2 space X is $X \cup \{\infty\}$ (discrete union) ♠. Find a familiar space that is homeomorphic to the Alexandrov compactification of $\mathbb{Z}$ ♠.

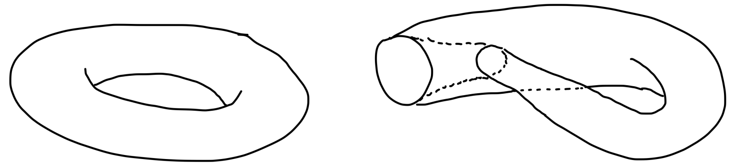
2. Surfaces

2.1. **Getting familiar with some spaces.** The pictures below define quotient spaces (with intuitive notation of gluing), they will be called (i) cylinder or annulus, (ii) Möbius strip Ms, (iii) torus T^2, (iv) Klein bottle Kb, (v) real projective plane $\mathbb{RP}^2$.



In these spaces some points x have a neighborhood homeomorphic to $\mathbb{R}^2$, and some points x have a neighborhood homeomorphic to the half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ in such a way that the point x is mapped to $(0,0)$. Points of the second kind will be called edge-points, or boundary points. (This is not the "boundary of a set" notion we learned before.) For each space find these boundary points ♠.

The cylinder can be realized as a subspace of $\mathbb{R}^2$ (c.f. the name annulus ♠). The Ms and the torus (see picture on the left) can be realized as a subspace of $\mathbb{R}^3$ ♠. The Klein bottle can be realized as a subspace of $\mathbb{R}^4$, starting from the usual picture (on the right) below ♠.



The picture on the right is not an *embedding* of the Kb into $\mathbb{R}^3$, because it has self intersections. Similar pictures are called immersions. The space $\mathbb{RP}^2$ can also be immersed in $\mathbb{R}^3$ (search for pictures online) and can be embedded in $\mathbb{R}^4$.

The following are all homeomorphic to $\mathbb{RP}^2$ ♠

- $D^2/\sim$, where $x \sim -x$ on the boundary of D^2. ($D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ is the 2-disc.)
- $S^2/\sim$, where $x \sim -x$.

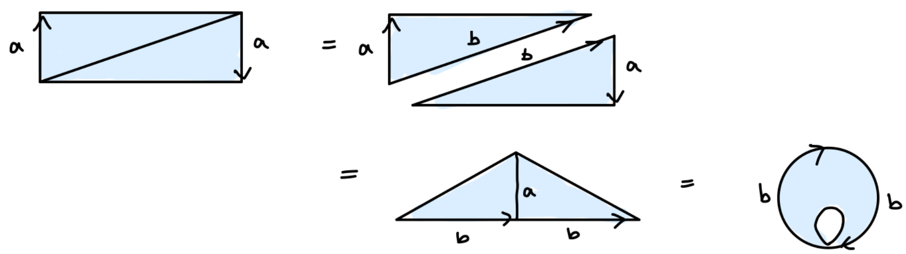
As a set {1-dimensional linear subspaces of $\mathbb{R}^3$} is in bijection with $\mathbb{RP}^2$ ♠. In this disguise $\mathbb{RP}^2$ is also called the $\mathrm{Gr}_1(\mathbb{R}^3)$ Grassmannian.

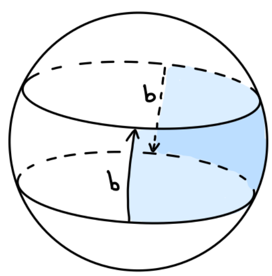
Define $\mathbb{R}\mathrm{P}^{\mathrm{n}} = S^n/\sim$, where $x \sim -x$. As a set $\mathbb{R}\mathrm{P}^{\mathrm{n}}$ is in bijection with $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ ♠. In geometry (but not in this course) Grassmannians are endowed with extra structures besides their topology.

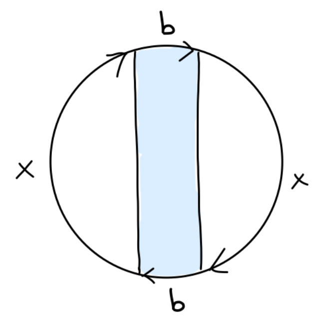
Proposition 2.1.

- *Gluing together two copies of D^2 along a homeomorphism of their boundaries results S^2.*
- *Gluing together a D^2 and a Ms along a homeomorphism of their boundaries results $\mathbb{R}\mathrm{P}^2$.*
- *Gluing together two copies of Ms along a homeomorphism of their boundaries results* Kb.

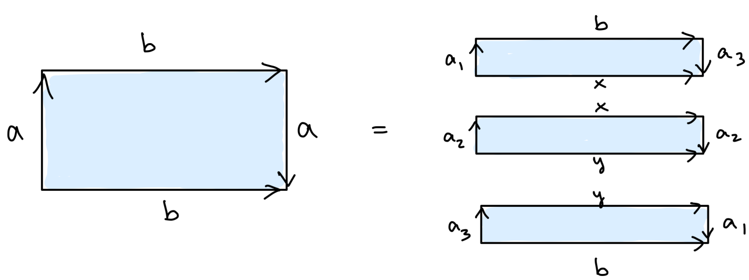
Three different proofs of the second statement are illustrated below



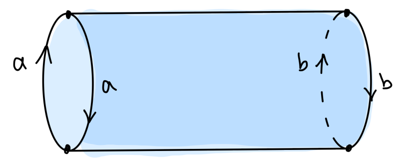




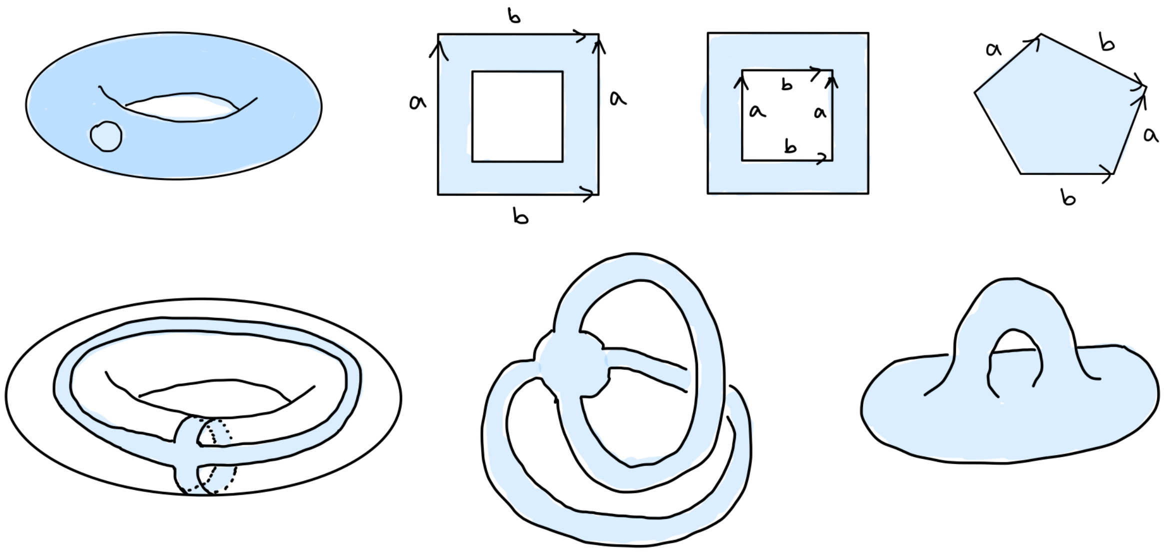
Finish the proof of the third statement started in the figure below. (♠ Hint: the next step is realizing the b gluing.)



We may use the proposition above to identify some other quotient spaces. Eg. the space below is the Kb ♠.



Proposition 2.2. *The spaces below are all the "punctured torus' (ie. the torus with an open disc removed)* ♠.



Find analogous pictures of the punctured Klein bottle ♠.

2.2. **Surfaces.**

Definition 2.3. *The T_2, M_2 topological space X is called a (topological n-) manifold, if every $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$.*

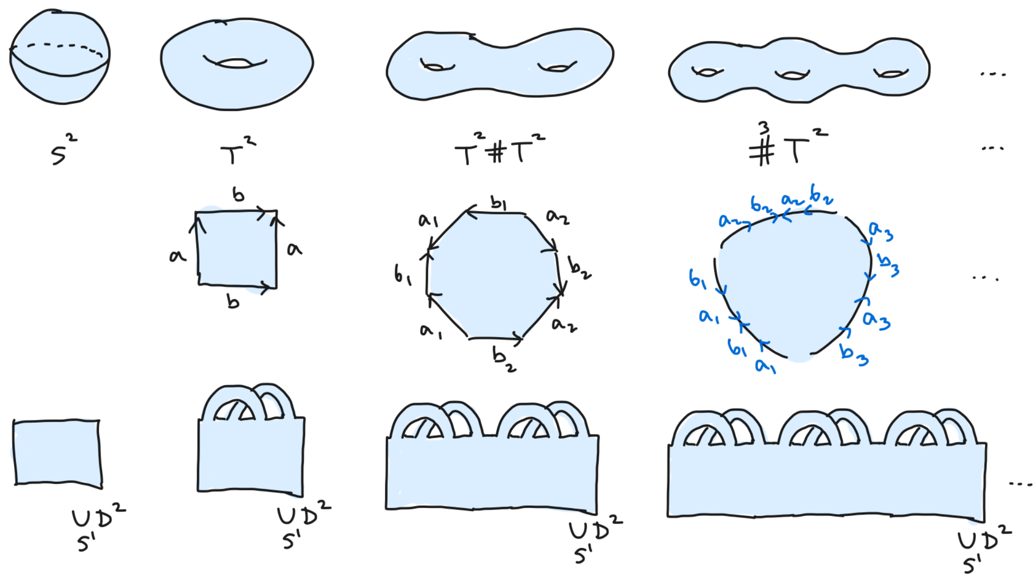
1-manifolds are also called curves, 2-manifolds are called surfaces.

Remark 2.4. The T_2 condition is there to exclude spaces like the ravioli. The M_2 condition is there to exclude spaces like an uncountable disjoint union of $\mathbb{R}^2$s. A very interesting topological space, named Alexandroff line (or "long line", not covered in this course) also satisfies all, but the M_2 property.

Remark 2.5. There are other versions of manifolds, namely smooth manifolds, and PL (piecewise linear) manifolds. We will not meet them in this course.

The spaces $\mathbb{R}^n$, S^n are n-manifolds. An open subset of $\mathbb{R}^n$ is an n-manifold. The cylinder $S^1 \times \mathbb{R}^1$ is a surface. The annulus or the Ms are not surfaces, because points of the boundary circles do not satisfy the requirement. If we remove those boundary circles, the remaining spaces are surfaces.

2.2.1. *The first list.* Consider the infinite sequence of spaces in the figure.

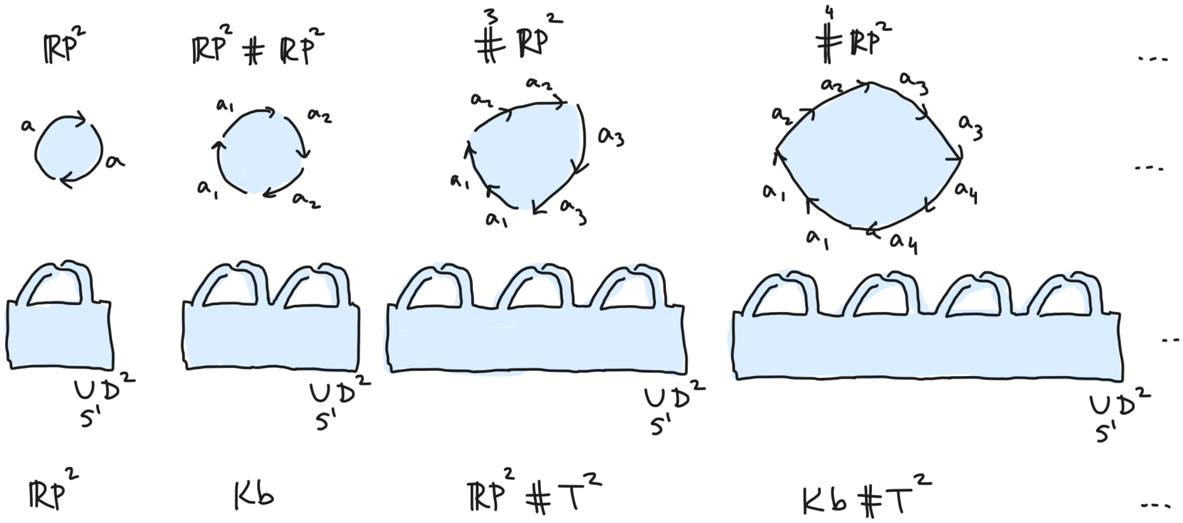


The figures in the middle line will be called the plane models with gluing scheme $aba^{-1}b^{-1}$, $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$, etc.
In the figure we used the following notations

- If A and B both has a distinguished circle (typically a boundary circle) as a subspace, then $A \cup_{S^1} B$ is obtained by gluing those two circles together via a homeomorphism. Whenever we use this notation it will be true that the choice of the homeomorphism $S^1 \leftrightarrow S^1$ does not matter.
- If A and B are surfaces then $A\#B$ is obtained by removing a (small) open disc, and then gluing the resulting edge circles together by a homeomorphism (again, does not matter which homeomorphism).

Proposition 2.6. *The three sequences of spaces (the three lines) above are the same sequence of spaces.* ♠

2.2.2. *The second list.* Consider the infinite sequence of spaces in the figure.



The figures in the second line will be called the plane models with gluing scheme a^2, $(a_1)^2(a_2)^2$, $(a_1)^2(a_2)^2(a_3)^2$, etc.
The bottom line is

$$\mathbb{RP}^2, \mathrm{Kb}, \mathbb{RP}^2 \# T^2, \mathrm{Kb} \# T^2, \mathbb{RP}^2 \# T^2 \# T^2, \mathrm{Kb} \# T^2 \# T^2, \mathbb{RP}^2 \# T^2 \# T^2 \# T^2, \ldots$$

Proposition 2.7. *The four sequences of spaces (the four lines) above are the same sequence of spaces.*

Proof. The following lemma is useful in the proof: If A is a surface which contains a Ms then $A \# T^2 \sim A \# \mathrm{Kb}$. ♠ □

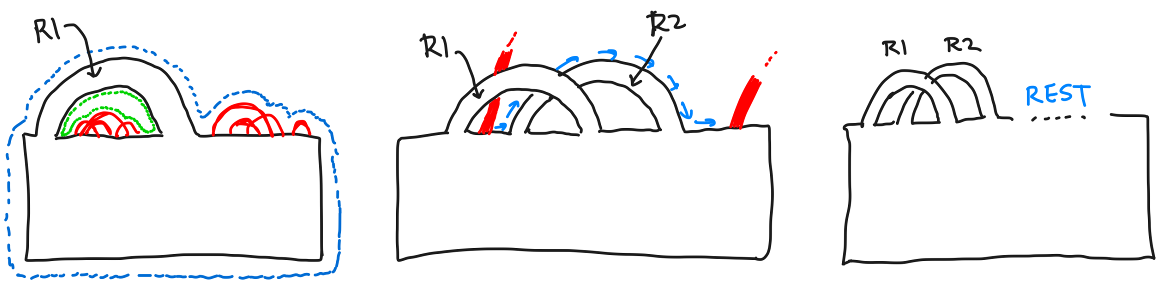
2.3. **The Classification Theorem.**

Theorem 2.8 (Classification of Closed Surfaces)**.** *The union of the lists in Section 2.2.1 and Section 2.2.2 is a complete, repetition-free list of compact connected surfaces.*

Proof. First we prove that every compact connected surface is one from the lists.
Let M be a compact connected surface. It is possible to remove an open disc from M such that what remains is a disc with a finite number of ribbons (some twisted, some not) attached to it. This fact we do not prove here—see Remark 2.9 below.

Case 1: None of the ribbons are twisted. Consider one of the ribbons $R1$. There must be another ribbon $R2$ that is "linked" with this one. [Linked: one if its foot is below the ribbon $R1$ and the other foot is outside.] This holds because otherwise the boundary would have more than one components—see the picture on the left, the blue and green doted lines ♠. Consider the ribbons $R1$ and $R2$ together. It is possible to "slide" the feet of all other ribbons away from $R1$ and $R2$—see how the foot of the red ribbon slides along the blue arrows in the middle figure

♠. We arrive at the picture on the right, and by induction we arrive at one of the pictures in the third row of the Figure in Section 2.2.1.



Case 2: If there is a twisted ribbon $R1$**.** We can slide the foot of all other ribbons away from $R1$—see the figure below, on the left ♠. Iterating this procedure we arrive at a picture on the right ♠. The rest must not have a twisted ribbon anymore. Hence the rest—according to Case 1—is m pairs of linked untwisted ribbons. We find that M is

$$(\#^n \mathbb{R}\mathrm{P}^2)\#(\#^m T^2),$$

which is the same (cf. the lemma in the proof of Proposition 2.7) as $\#^{n+2m} \mathbb{R}\mathrm{P}^2$.

We still need to prove that the surfaces on our list are pairwise not homeomorphic. We will come back to it later. □

Remark 2.9. Our first step in the proof created a combinatorial structure on the surface. That step can be carried out by proving the existence of a so-called "differentiable structure" on a topological surface, then using a statement from "Morse theory". We will not cover this argument in this course.

Remark 2.10. There are other proofs of the Classification Theorem. Their first step is always giving some kind of combinatorial structure to M. An example of such a combinatorial structure is triangulation. Proving the existence of a triangulation is not easier than our first step above. A proof assuming the existence of triangulations is available in the textbook.

2.4. **Euler characteristic.** An n-simplex is the convex hull of $n+1$ points in $\mathbb{R}^n$, eg. an interval, a triangle, a tetrahedron. An n-simplex has i-faces for $i = 0, 1, \ldots, n$.
A topological space which is the union of finitely many simplices, such that any two either are disjoint, or intersect in a full face of each, is a simplicial complex.

Definition 2.11. *The Euler characteristic of a simplicial complex M is*

$$\chi(M) = \sum_{n=0}^{\infty} (-1)^n |\{n\text{-}dimensional\ simplices\}|$$

Theorem 2.12 (proved later, c.f. Definition-Theorem 4.21)**.** *The Euler characteristic does not depend on the simplicial structure, it only depends on the homeomorphism type of the space.*

We have $\chi(S^2) = 2$ ♠, $\chi(T^2) = 0$ ♠, $\chi(\mathbb{RP}^2 = 1)$ ♠.

Proposition 2.13. *For compact connected surfaces A and B we have* ♠

$$\chi(A\#B) = \chi(A) + \chi(B) - 2.$$

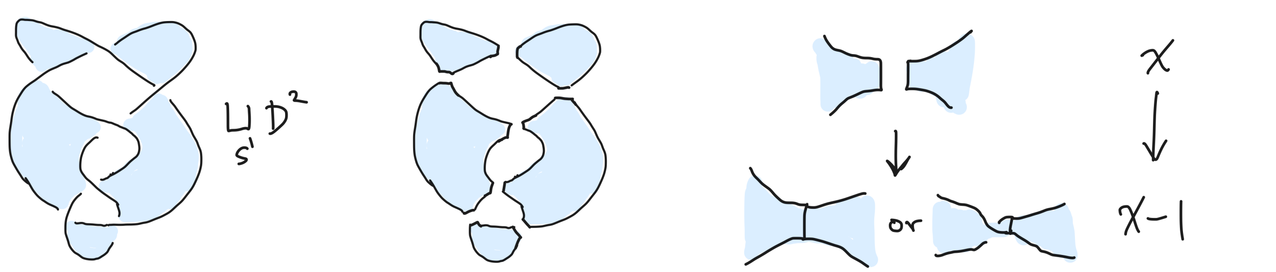
We have ♠

$$\chi(\#^g T^2) = 2 - 2g, \qquad\qquad \chi(\#^g\, \mathbb{RP}^2) = 2 - g.$$

Define a surface non-orientable (orientable) if it contains (does not contain) a Ms as a subspace. Those on our first list are orientable, those on the second list are non-orientable ♠.
The pair (orientability of M, $\chi(M)$) proves that the surfaces in the Classification Theorem are pairwise non-homeomorphic ♠. This completes the proof of Theorem 2.8.

Another way of proving that the list is a repetition free list will be given after Corollary 3.28.

2.5. **Identifying surfaces.** Consider the surface on the left of the figure below. We will identify this surface as one from the Classification Theorem.



The Euler characteristic of the space in the middle figure is 5 ♠. At the operation showed on the right the Euler characteristic decreases by 1 ♠. Hence the Euler characteristic of the surface on the left is

$$5(\text{middle figure}) - 7(\text{joins}) + 1(\text{adding the disc}) = -1.$$

Hence the surface is $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

3. Homotopy, fundamental group, covering spaces

3.1. Homotopy.

Definition 3.1. *The continuous maps $f, g : X \to Y$ are homotopic (we write $f \simeq g$), if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$.*

Any two maps $X \to \mathbb{R}^n$ are homotopic (♠ use the linear structure of $\mathbb{R}^n$). The reader probably "feels" that the identity map of S^1 is not homotopic to the constant map (the map $S^1 \to S^1$ that maps all points to one point). We will prove this fact later.
Homotopy is an equivalence relation on the set of maps $X \to Y$ ♠.

Definition 3.2. *The topological spaces X and Y are homotopy equivalent (we write $X \simeq Y$) if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that*

$$g \circ f \simeq \mathrm{id}_X, \qquad f \circ g \simeq \mathrm{id}_Y .$$

We have $\mathbb{R}^n \simeq \{0\}$ ♠. We have the annulus $\simeq S^1 \simeq$ Ms ♠.
Homotopy equivalence is an equivalence "relation" on topological spaces ♠.

Theorem 3.3 (proved later, cf. Definition-Theorem 4.21)**.** *If $X \simeq Y$ then $\chi(X) = \chi(Y)$.*

The torus minus an open disc is homotopy equivalent to the figure 8 ♠. Derive from this that $\chi(T^2) = 0$ ♠.
We have $D^{n+1} \simeq$ pt ♠, hence $\chi(D^{n+1}) = 1$. Use this to derive that $\chi(S^n) = 1 + (-1)^n$ (hint: consider the $n+1$-simplex as a simplicial structure on D^{n+1}, then take away the interior and see what happens to χ ♠).

3.2. CW complexes.

Consider the following procedure of building a topological space.

- Start with a finite number of points, with the discrete topology, call it X^0 (the 0-skeleton).
- Consider a finite disjoint union of 1-discs (intervals), and a map from their boundary S^0's to X^0. Let us glue these 1-discs to X^0 along the map, and call the resulting space X^1 (the 1-skeleton).
- Consider a finite union of 2-discs, and a map from their boundary S^1's to X^1. Let us glue these 2-discs to X^1 along the map, and call the resulting space X^2.
- etc.

Definition 3.4. *If a space X is obtained by such a (finite) procedure, then we call it (together with the procedure) a finite CW complex. The interiors of the k-discs survive as subspaces of X ♠, we call them k-cells.*

A triangulation of a space (ie. simplicial complex structure on a space) can be considered a CW complex structure ♠.
The plane models we considered for the compact surfaces can be considered as CW complex structures on them, with exactly one 2-cell (and, in fact, exactly one 0-cell too)♠.
The sphere S^n has a CW structure with one 0-cell and one n-cell (nothing else) ♠.
The space $\mathbb{RP}^n$ has a CW structure with exactly one k-cell for all $k = 0, 1, \dots, n$ ♠. It is worth thinking over what the gluing map $S^k \to X^k$ of each cell is ♠.

Theorem 3.5 (proved later, cf. Definition-Theorem 4.21)**.** *The Euler characteristic of a finite CW complex can be calculated by* $\sum_{n=0}^{\infty}(-1)^n|\{n\text{-cells}\}|$.

We say that a space is contractible if it is homotopy equivalent to a one-point space. Eg. $\mathbb{R}^n$, a star-shaped region of $\mathbb{R}^n$, or a tree are contractible.
If $A \subset X$ then by X/A we mean the quotient space where all points of A are declared equivalent, that is, A is collapsed to a point. Eg. $D^n/\partial D^n = S^{n-1}$.

Theorem 3.6 (proved in the HW assignments)**.** *Let A be a sub-complex of the finite CW complex X, and let A be contractible. Then X is homotopy equivalent to X/A.*

3.3. **Fundamental group.** A continuous function $\alpha : [0,1] \to X$ is called a path in X, "from $\alpha(0)$ to $\alpha(1)$". If $\alpha(0) = \alpha(1)$, it is a loop. We denote $\overline{\alpha}(t) = \alpha(1-t)$.
A path homotopy between paths α and β, both from x_0 to x_1, is a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$, $H(0,s) = x_0$, $H(1,s) = x_1$. Draw a picture of $[0,1] \times [0,1]$ indicating which part is mapped where ♠. In notation $\alpha \simeq_p \beta$. Path homotopy is an equivalence relation ♠.

We fix a basepoint x_0 in X, and, as a set, define $\pi_1(X, x_0)$ to be the set of path-homotopy equivalence classes of loops in X based on x_0.
We define a product $*$ on $\pi_1(X, x_0)$ by $[\alpha] * [\beta] = [\alpha * \beta]$ where the $*$-product of paths f and g with $f(1) = g(0)$ is defined by

$$(f * g)(t) = \begin{cases} f(2t) & \text{if } t \leq 1/2, \\ g(2t-1) & \text{if } t \geq 1/2. \end{cases}$$

Proposition 3.7. *The $*$ product*

- *is well defined ♠;*
- *is associative ♠;*
- *has a neutral element, the constant x_0 map ♠;*
- *has a two-sided inverse ♠.*

That is, $\pi(X, x_0)$ is endowed with a group structure, it is called the fundamental group of X, x_0.

Proposition 3.8. *Let β be a path from x_0 to x_1. The assignment $[\alpha] \mapsto [\overline{\beta} * \alpha * \beta]$ is a well defined map $\pi_1(X, x_0) \to \pi_1(X, x_1)$ ♠. It is a group isomorphism ♠.*

Hence, for path-connected spaces X the group $\pi_1(X)$ is well defined as a group isomorphism type. If the path-connected X satisfies $\pi_1(X) = 0$ (the one-element, "trivial" group), we call it simply connected.
A subset $X \subset \mathbb{R}^n$ is called star-shaped, if for all $x \in X$ the segment $[0, x]$ is part of X. Then we have $\pi_1(X) = 0$, use the linear structure of $\mathbb{R}^n$ ♠.
It should be intuitive to "feel" that $\pi_1(S^1) = \mathbb{Z}$ ♠, we will prove this later. Find a loop in $\mathbb{RP}^2$ that 'feels' to have order 2 in $\pi_1(\mathbb{RP}^2)$ ♠.

Proposition 3.9. $\pi_1(S^2) = 0$.

First attempt of the proof: let α be a loop on $x_0 \in S^2$. If $p \notin im(\alpha)$ then we may use a stereographic projection $F : S^2 - \{p\} \to \mathbb{R}^2$ to view α as a loop in $\mathbb{R}^2$. Here it is path homotopic to the constant loop. The F^{-1} image of the path homotopy is a path homotopy in S^2 between α and the constant x_0 loop. ♠

The described "first attempt" only works if α misses at least one point of S^2, so it is not a complete proof yet.

Lemma 3.10 (Lebesgue number lemma)**.** *If X is a compact metric space, covered by open sets U_κ, then there is a (Lebesgue-) number δ such that if $A \subset X$ is part of a δ-radius ball then there is a κ with $A \subset U_\kappa$.* [Find the proof of this lemma in a textbook or online, then close your source and write it down for yourself. ♠]

Let α be a loop in S^2. Let us cover S^2 with small discs. Consider α^{-1} of this cover: we obtain an open cover of $[0, 1]$. Let δ be a Lebesgue number of this cover, and let us subdivide $[0, 1]$ to intervals shorter than δ. On each subinterval change α by a path-homotopy to a "nicer" path, eg. smooth one (or a segment in a fixed homeomorphism between the disc and $\mathbb{R}^2$). Thus we replaced α with a path-homotopic α_0 which is piece-wice nice. The image of such an α_0 has measure 0 (a fact from Analysis), hence it cannot cover S^2. Now out first attempt can be applied. □

The above proof extends to prove that $\pi_1(S^n) = 0$ for all $n \geq 2$ ♠. But it does not apply to $n = 1$ (why? ♠).

3.4. **Functor.** Let $h : X, x_0 \to Y, y_0$ be continuous (ie. $h : X \to Y$ with $h(x_0) = y_0$). The map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $[\alpha] \mapsto [h \circ \alpha]$ is well defined ♠, and is a group homomorphism ♠.
Observe that $\mathrm{id}_* = \mathrm{id}$ ♠, $(h \circ g)_* = h_* \circ g_*$ ♠.

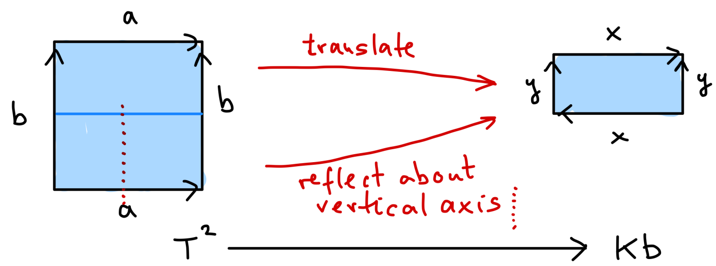
3.5. **Covering spaces.** The continuous map $p : E \to B$ is called a covering space, if all $x \in B$ has a neighborhood U such that

- $p^{-1}(U) = \cup V_\alpha$, the V_α are open in E,
- $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for all α.

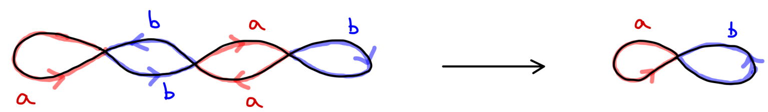
Vocabulary: U is a trivializing neighborhood, E is the total space, B is the base space, p is the projection of the covering.
For a covering the set $B_\kappa = \{b \in B : |p^{-1}(b)| = \kappa\}$ is open ♠. Hence, if B is connected, the cardinality $|p^{-1}(b)|$ is constant ♠, we call it the number of sheets in the covering.
The identity map $X \to X$ is a covering space. If D is a discrete topological space, then the projection map $B \times D \to B$ is a covering map ♠.
The following are covering maps:

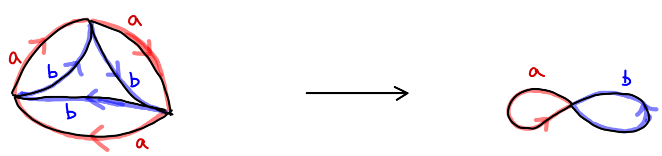
- $S^1 \to S^1$, $z \mapsto z^2$. Draw a picture, where the first S^1 is like the boundary circle of a Ms ♠.
- $S^1 \to S^1$, $z \mapsto z^n$, for $n = 1, 2, 3, \ldots$. Draw pictures ♠.
- The defining quotient map $q : S^n \to \mathbb{RP}^n$ ♠.
- The quotient map $\mathbb{R}^2 \to T^2$ where T^2 is presented by the equivalence relation $(x, y) \sim (x + \text{integer}, y + \text{integer})$.

- The picture below defines a covering map $T^2 \to \mathrm{Kb}$ ♠.

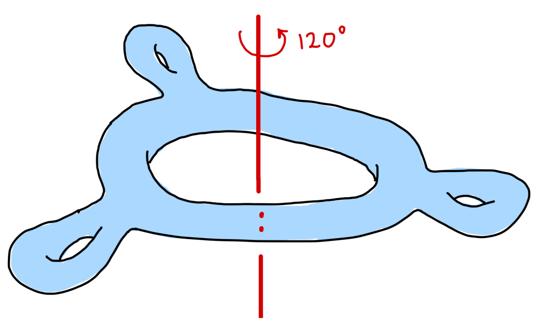


- The pictures below define two covering maps to the figure-8 space ♠. The labels here do not mean gluing, they indicate the map.





- The quotient map by the $\mathbb{Z}_3$-action on $\#^4 T^2$ indicated in the picture below is a covering map ♠.



3.6. **Lifting maps, lifting correspondence, $\pi_1(\mathbb{R}\mathrm{P}^n)$, $\pi_1(S^1)$.** For $p : E \to B$ covering and $f : X \to B$ we call $\tilde{f} : X \to E$ a lifting of f if $f = p \circ \tilde{f}$, that is, if the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

is commutative.

We say $p : E, e_0 \to B, b_0$ is a covering, if $p : E \to B$ be a covering, and $b_0 \in B, e_0 \in E$ such that $p(e_0) = b_0$.

Lemma 3.11 (path lifting)**.** *Let $p : E, e_0 \to B, b_0$ be a covering. Let $f : [0,1] \to B$ be a path with $f(0) = b_0$. Then there exists a unique lifting $\tilde{f}$ of f with $\tilde{f}(0) = e_0$.*

Proof. If a lifting exists and is unique on $[0,t]$, then—due to local triviality of the covering map—a lifting exists uniquely on $[0, t+\varepsilon]$ ♠. Use this idea, precomposed with a Lebesgue number lemma argument to go from the interval $[0,0]$ to the interval $[0,1]$ in finitely many steps ♠. □

Lemma 3.12 (homotopy lifting)**.** *Let $p : E, e_0 \to B, b_0$ be a covering. Let $F : [0,1] \times [0,1] \to B$ be a map with $F((0,0)) = b_0$. Then there exists a unique lifting $\tilde{F}$ of F with $\tilde{F}((0,0)) = e_0$ ♠. Moreover, if F is a path-homotopy then $\tilde{F}$ is also a path-homotopy ♠.*

Corollary 3.13. *Let f, g be path-homotopic paths in B, both from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be their liftings starting at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$ (and, in fact, $\tilde{f}$ and $\tilde{g}$ are path-homotopic) ♠.*

Definition 3.14 (lifting correspondence map)**.** *Let $p : E, e_0 \to B, b_0$ be a covering. Define the lifting correspondence map between sets*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0), \qquad\qquad [f] \mapsto \tilde{f}(1).$$

The lifting correspondence map ϕ is well defined ♠. Work out examples of ϕ-images of elements in $\pi_1(B)$ for all the examples in Section 3.5 ♠.

Theorem 3.15.

- *If E is path-connected then ϕ is surjective ♠.*
- *If $\pi_1(E) = 0$ then ϕ is injective.*

Proof. The following lemma is useful to prove ♠ the second statement: *Any two paths from x to y in a simply connected space are path-homotopic ♠.* □

We have $\pi_1(\mathbb{RP}^{\mathrm{n}}) = \mathbb{Z}_2$ ♠ for $n \geq 2$. We have that $\pi_1(S^1)$ is a countably infinite group ♠.

Theorem 3.16. *The map ϕ for the covering $\mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is also a group homomorphism. Hence $\pi_1(S^1) = \mathbb{Z}$.*

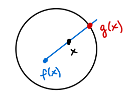
Proof. For $[f], [g] \in \pi_1(S^1, b_0)$ prove that $\tilde{f} * (\tilde{g} + \tilde{f}(1))$ works for $\widetilde{f * g}$ ♠. □

Proposition 3.17. *Let $[f] \in \pi_1(S^1)$ be represented by an* odd *map, that is, assume $f(-x) = -f(x)$. Then $\phi(f) \in \mathbb{Z}$ is an odd number.*

Proof. A idea similar to that in the proof of Theorem 3.16 works: first lift the restriction to a half-circle, then show that a shift of that works for the lift of the other half-circle ♠. □

3.7. Applications of $\pi_1(S^1) = \mathbb{Z}$.

Theorem 3.18 (2d Brouwer fixed point theorem)**.** *Every $f : D^2 \to D^2$ continuous map has a fixed point (ie. $\exists x \in D^2$ such that $f(x) = x$).*

Proof. If f has no fixed point then the map g defined in the picture 

is continuous, and makes the diagram

$$\begin{tikzcd} D^2 \arrow[r, "g"] & S^1 \\ S^1 \arrow[u, hook, "j"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commutative. Applying the π_1 functor (Section 3.4) we obtain a commutative diagram $\spadesuit$

$$\begin{tikzcd} 0 \arrow[r, "g_*"] & \mathbb{Z} \\ \mathbb{Z} \arrow[u, "j_*"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

which is a contradiction. □

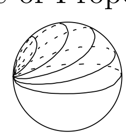
The last step of the proof generalizes. Call a subset A of a topological space X a *retract* of X, if there exists a map (the retraction) $r : X \to A$ such that $r|_A = \mathrm{id}_A$.

Proposition 3.19. *If A is a retract of X then $j_* : \pi_1(A) \to \pi_1(X)$ is an injection. (Here j is the embedding $A \subset X$.) Moreover, r_* is surjective, where r is the retraction $\spadesuit$.*

Theorem 3.20 (Borsuk-Ulam)**.** *For every continuous map $S^2 \to \mathbb{R}^2$ there is a point $p \in S^2$ where $f(p) = f(-p)$.*

Proof. Assuming that there is no such point we can define

$$g : S^2 \to S^1, \qquad g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

The map $h := g|_{S^1} : S^1 \to S^1$ is *odd* in the sense of Proposition 3.17, hence $[h] \in \pi_1(S^1)$ is an odd number $\spadesuit$. On the other hand the picture  shows that $[h] = 0$ $\spadesuit$, proving the needed contradiction. □

Other notable applications of $\pi_1(S^1) = \mathbb{Z}$ include

- **The inside-pointing vector field lemma:** Let $D^2 \to \mathbb{R}^2$ be nowhere 0. Then there is a point $p \in S^1$ such that $X(p)$ is a negative multiple of p.
- **The Fundamental Theorem of Algebra:** Every complex coefficient polynomial of positive degree has a complex root.

- **The hairy ball theorem (a.k.a. Porcupine theorem):** Every continuous tangent vector field on S^2 has a 0.
- **The ham and cheese sandwich theorem:** Let B, H, C be "nice" subsets of $\mathbb{R}^3$ (bread, ham, cheese). Then there is a plane in $\mathbb{R}^3$ that cuts all three of them to two equal volume parts.
- **The Lusternik-Schnirelmann category theorem for $\mathbb{R}\mathrm{P}^2$:** Let A_1, A_2, A_3 be closed subsets of S^2 whose union is S^2. Then at least one of them contains a pair of antipodal points.

These statements will be explored in the homework assignments.

3.8. Fundamental group of a product.

Proposition 3.21. *We have $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. The map $\alpha \mapsto (p_*(\alpha), q_*(\alpha))$ is a homomorphism from the left hand side to the right hand side ♠, where p, q are the projections. It is injective and surjective ♠. □

We have $\pi_1(T^2) = \mathbb{Z}^2$ ♠.

3.9. Homotopy invariance.

Theorem 3.22. *Let $F : X \to Y$ be a homotopy equivalence, with $F(x_0) = y_0$. Then F_* is an isomorphism.*

The proof depends on two arguments. The first one is a lemma.

Lemma 3.23. *Let $f, g : X \to Y$, $f(x_0) = g(x_0) = y_0$. Assume that f and g are homotopic in such a way that during the homotopy x_0 is mapped to y_0. Then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Proof. Let α be a loop on x_0, and let H be the homotopy assumed in the lemma. Then $(\alpha \times \mathrm{id}_{[0,1]}) \circ H$ proves that $f_*([\alpha]) = g_*([\alpha])$ ♠. □

If we could disregard base-points, we would have

$$\begin{tikzcd} X \arrow[r, bend left, "F"] & Y \arrow[l, bend left, "G"] \end{tikzcd} \qquad \text{with} \qquad G \circ F \simeq \mathrm{id}_X,\ F \circ G \simeq \mathrm{id}_Y\,.$$

Applying the π_1 functor, and using the lemma, we would get

$$\begin{tikzcd} \pi_1(X) \arrow[r, bend left, "F_*"] & \pi_1(Y) \arrow[l, bend left, "G_*"] \end{tikzcd} \qquad \text{with} \qquad (G \circ F)_* = (\mathrm{id}_X)_*,\ (F \circ G)_* = (\mathrm{id}_Y)_*.$$

Then using $(F \circ G)_* = F_* \circ G_*$, $(G \circ F)_* = G_* \circ F_*$, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X)}$, $(\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y)}$ from Section 3.4 would prove that $\pi_1(X) \cong \pi_1(Y)$ ♠. The second argument needed to prove Theorem 3.22 is a modification of what we just described, but taking into account the base points ♠.

We have $\pi_1(\mathbb{R}^n - \mathrm{pt}) = \pi_1(S^{n-1})$ ♠. We have $\pi_1(\mathrm{Ms}) = \pi_1(S^1)$ ♠.

3.10. **Free groups and finitely presented groups.** Consider words in the letters

$$x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$$

up to the equivalence that consecutive $x_i x_i^{-1}$ can be deleted, as well as consecutive $x_i^{-1}x_i$ can be deleted. Call the empty word 1. These words (equivalence classes) form a group for concatenation ♠. We call this the free group on n letters, we denote it by F_n or $F(x_1, \ldots, x_n)$ or $\langle x_1, \ldots, x_n \mid \rangle$.

Let $r_1, \ldots, r_m$ be words representing elements in F_n ("the relations"). Let N denote the normal subgroup of F_n generated by these elements. Define $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ to be the quotient group F_n/N.

Remark 3.24. The concept "generated normal subgroup" is not an easy one. Hence it is very difficult to say anything about $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ in general. Even the question whether such group is the trivial group or not is undecidable—in a precise mathematical sense—in general.

We have $F_1 = \langle x | \rangle = \mathbb{Z}$, F_2 is not commutative, $\langle x_1, x_2 | x_1 x_2 x_1^{-1} x_2^{-1} \rangle = \mathbb{Z}^2$, $\langle x | xxxx \rangle = \mathbb{Z}_4$ ♠.

We will use obvious notation: eg. $x^4 = xxxx$, and we will write equalities instead of words r_i, eg. $x_1 x_2 = x_2 x_1$ instead of $x_1 x_2 x_1^{-1} x_2^{-1}$ ♠.

Define the free product $G * H$ of $G = \langle x_1, x_2, \ldots, x_a \mid r_1, r_2, \ldots, r_b \rangle$ and $H = \langle y_1, y_2, \ldots, y_c \mid s_1, s_2, \ldots, s_d \rangle$ to be $\langle x_1, x_2, \ldots, x_a,\ y_1, y_2, \ldots, y_c \mid r_1, r_2, \ldots, r_b,\ s_1, s_2, \ldots, s_d \rangle$. This concept does not depend on the chosen presentations ♠.

3.11. **Seifert-van Kampen theorem.** Let $X = U \cup V$, $x_0 \in U \cap V$. Assume that

- U, V are open,
- $U, V, U \cap V$ are path-connected;

and that

- $\pi_1(U, x_0) = \langle x_1, x_2, \ldots, x_a \mid r_1, r_2, \ldots, r_b \rangle$,
- $\pi_1(V, x_0) = \langle y_1, y_2, \ldots, y_c \mid s_1, s_2, \ldots, s_d \rangle$,
- $\pi_1(U \cap V, x_0) = \langle z_1, z_2, \ldots, z_d \mid * \rangle$,

Theorem 3.25 (SvK). *Let $i : U \subset X$, $j : V \subset X$ be the inclusions. We have*

$$\pi_1(X) = \langle x_1, x_2, \ldots, x_a,\ y_1, y_2, \ldots, y_c \mid r_1, r_2, \ldots, r_c,\ s_1, s_2, \ldots, s_d \\ i_*(z_1) = j_*(z_1), i_*(z_2) = j_*(z_2), \ldots, i_*(z_d) = j_*(z_d) \rangle$$

Some explanations are in order. Namely, the element $i_*(z_k)$ of $\pi_1(U)$ can be represented by a word in the letters $x_1, \ldots, x_a$. By $i_*(z_k)$ above we mean such a word (the statement does not depend on which representative we choose). Similarly, $j_*(z_k)$ means a (chosen) word in the letters $y_1, \ldots, y_c$.
The proof of the SvK theorem is based on the so-called "universality property" of the concept of "generated normal subgroup", as well as numerous applications of the Lebesgue number lemma. We will not give the details of the proof in this course.

Let $X \vee Y$ be the one-point union of the spaces X and Y, that is, the quotient space of $X \cup Y$ obtained by declaring one point of X equivalent to one point of Y. In practice the points chosen in X and Y do not matter, hence we do not indicate them in the notation.

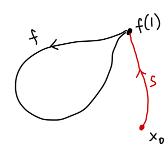
We have $\pi_1(S^2) = 0$ ♠ (we already had a proof using the Lebesgue number lemma, now we have another proof using the SvK theorem—which also depends on the Lebesgue number lemma). We have $\pi_1(S^1 \vee S^1) = F_2$ ♠. We have $\pi_1(S^1 \vee S^2) = \mathbb{Z}$ ♠. We have $\pi_1(S^1 \vee T^2) = \mathbb{Z} * \mathbb{Z}^2$ ♠.

Glue together (identify by a homeomorphism) the longitude of the torus with the longitude of a Klein bottle. The resulting space have ♠

$$\pi_1 = \langle x, y, a, b \mid xyx^{-1}y^{-1}, abab^{-1}, x = b\rangle = \langle x, y, a \mid xyx^{-1}y^{-1}, axax^{-1}\rangle.$$

3.12. π_1 **of CW complexes.** The fundamental group of a connected graph (1-skeleton of a CW complex) is a free group ♠. (Hint: collapse a spanning tree.)

Assume X^1 is connected, and $\pi_1(X^1) = \langle x_1, x_2, \ldots, x_a | \ \rangle$. Let $f_i : S^1 \to X^1$, $i = 1, \ldots, n$ be the attaching maps of the 2-skeleton of a CW complex. To a loop f_i we associate a word $w(f_i)$ in the x-letters as follows: let s be a path connecting the base point x_0 with $f_i(1)$, then the loop $s * f_i * \overline{s}$ is a loop on x_0 hence it is represented by a word—$w(f_i)$.



The choice of s will not matter, hence we do not indicate it in notation.

Theorem 3.26. *We have* $\pi_1(X^2) = \langle x_1, x_2, \ldots, x_a \mid w(f_1), \ldots, w(f_n)\rangle$.

Theorem 3.27. *Attaching 3- or higher dimensional cells do not change* π_1.

Both theorems intuitively follow from the SvK theorem, by attaching the cells one by one ♠. The (quite tedious) details are left to the reader.

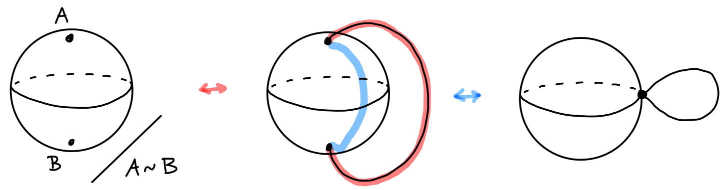
Corollary 3.28. *We have* ♠

$$\pi_1(\#^g T^2) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}\rangle,$$
$$\pi_1(\#^g \mathbb{R}\mathrm{P}^2) = \langle a_1, a_2, \ldots, a_g \mid a_1^2a_2^2 \cdots a_g^2\rangle.$$

Think over what the abelianizations of these groups are ♠, and conclude that the surfaces in the Classification Theorem 2.8 are indeed pairwise non-homeomorphic (even pairwise non homotopy equivalent) ♠.

Consider a triangle with gluing scheme aaa. Its fundamental group is $\mathbb{Z}_3$ ♠.

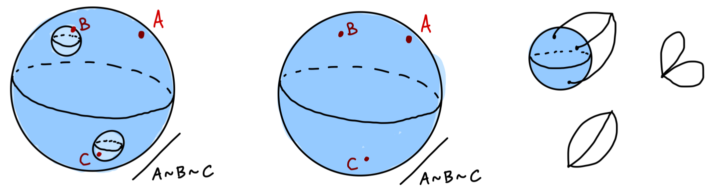
Identify two points A and B of S^2. Use the picture



and Theorem 3.6 to find that $\pi_1 = \mathbb{Z}$.

Let A_1, A_2 be two points on a torus. Let B_1, B_2 be two points on a Klein bottle. Let C_1, C_2 be two points on a $\mathbb{RP}^2$. Let X be the union with $A_1 \sim B_2$, $B_1 \sim C_2$, $C_1 \sim A_2$. Find π_1 by tricks similar to the one above ♠.

Cut out two small open 3-balls from a larger closed 3-ball, and identify 3 points on the resulting space as in the picture on the left. Use the pictures



to find that $\pi_1 = F_2$.

3.13. **Galois correspondence between covering spaces and fundamental groups.** The covering spaces $p : E \to B$ of this lecture will be assumed to have extra properties:

- both E and B are path-connected;
- both E and B are locally path-connected (for every point x and every neighborhood U of x there is a path-connected neighborhood of x contained in U);
- B is semilocally simply connected (every point b has a neighborhood U such that $i_* : \pi_1(U, b) \to \pi_1(B, b)$ is the trivial homomorphism).

Lemma 3.29. *The map $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.*

Proof. Let f be a loop on e_0 and H a path homotopy connecting $p \circ f$ with the constant loop b_0. The unique lift of H (Lemma 3.12) is a path homotopy between f and the constant e_0 loop ♠. □

Let $H_p := p_*(\pi_1(E, e_0))$ be the subgroup associated with the covering p. It is a subgroup of $\pi_1(B, b_0)$ and is isomorphic to $\pi_1(E, e_0)$.

Remark 3.30. Note that the subgroup H_p depends on the choice of e_0 (not indicated in notation). For a different choice of base point in $p^{-1}(b_0)$ the subgroup H_p gets conjugated ♠. Hence, H_p is a concrete subgroup if the base point e_0 is fixed, or is only defined up to conjugation if the base point is not fixed. It should be clear from context which meaning we use.

Lemma 3.31. *The index $[\pi_1(B, b_0) : H_p]$ is equal to the number of sheets of p.*

Proof. The lifting correspondence map induces a bijection between the right cosets of H_p in $\pi_1(B, x_0)$ and $p^{-1}(b_0)$ ♠. □

Verify the statements of the two lemmas above for all of the examples of Section 3.5. ♠♠♠♠♠♠♠ In one of the examples you will see that F_2 has an index 3 subgroup isomorphic to F_4. Name the generators of this subgroup in terms of a, b in $F_2 = F(a, b)$ ♠.

Definition 3.32. *Let $p : E \to B$ and $p' : E' \to B$ be covering spaces of the same base B. We define them equivalent if there is a homeomorphism $f : E \to E'$ with $p = f \circ p'$.*

Theorem 3.33 (Part 1 of Galois correspondence)**.** *There is a bijection between*

- *covering spaces over B, up to equivalence, and*
- *subgroups of $\pi_1(B)$ up to conjugation.*

The subgroup associated with a covering space $p : E \to B$ is H_p (c.f. Remark 3.30).

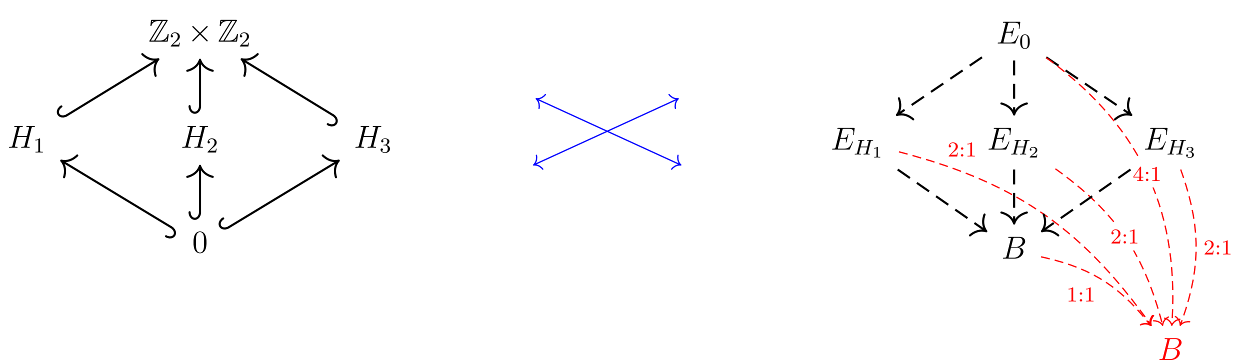
In particular, the identity map $B \to B$ is the covering space associated with the "subgroup" $\pi_1(B) \subset \pi_1(B)$ ♠.
The covering space $p : E \to B$ associated with the trivial subgroup $0 \subset \pi_1(B)$ is called the universal covering. That is, a covering is universal if E is simply connected ♠.

Theorem 3.34 (Part 2 of Galois correspondence)**.** *Let $p : E \to B$ and $q : F \to B$ be covering spaces. There exists a covering $r : E \to F$ with $q \circ r = p$ if and only if $H_p \subset H_q$.*

Note that H_p and H_q are only defined up to conjugation, hence $H_p \subset H_q$ *really* means that there are representative subgroups satisfying the condition. There are pairs of subgroups in Example 3.36 illustrating this phenomenon ♠.

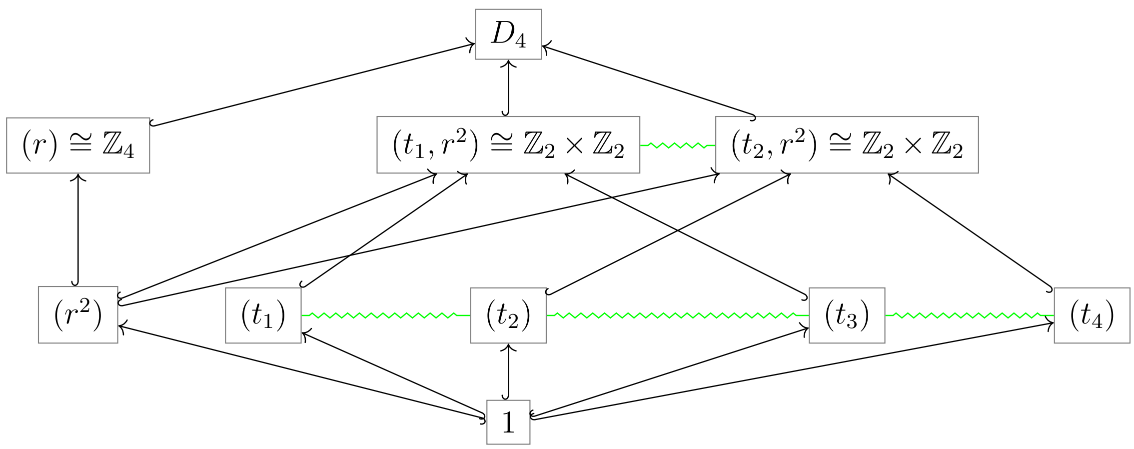
In particular, the total space of the universal covering covers the total space of all other coverings. Hence that name "universal."

Example 3.35. Let $B = \mathbb{RP}^2 \times \mathbb{RP}^2$, and hence $\pi_1(B) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Its subgroups are: itself, the trivial subgroup 0, and three order 2 subgroups H_1, H_2, H_3 ♠. According to Theorem 3.33 they correspond to covering maps. Let the total spaces of these covering maps be denoted by $B, E_0, E_{H_1}, E_{H_2}, E_{H_3}$, respectively. The content of Theorem 3.34 is illustrated in the diagram below (all dashed arrows, black or red, and their compositions indicate covering maps).



Note the "upside down" correspondence between the topological objects and the algebraic objects. Find explicit descriptions of all the spaces E_* ♠, and all the dashed arrows ♠. In this example the fundamental group of B is Abelian, hence the "up to conjugacy" part of the Galois correspondence is trivial.

Example 3.36. Consider $D_4 = \langle r, t \mid r^4 = 1, t^2 = 1, trt = r^3 \rangle$ (the dihedral group of order 8) and define $t_1 = t, t_2 = r^{-1}tr$, $t_3 = r^{-2}tr^2$, $t_4 = r^{-3}tr^3$. The hierarchy of its subgroups is ♠



where ( ) means "generated subgroup", and we connected conjugate subgroups by green squiggly lines. Therefore the hierarchy of covering spaces over a space B with $\pi_1(B) = D_4$ (find such a space ♠) is ♠

$$\begin{tikzcd}
 & E_1 \arrow[dl, dashed, "2:1"] \arrow[dr, dashed, "2:1"] & \\
E_{(r^2)} \arrow[d, dashed, "2:1"] \arrow[drr, dashed, "2:1"] & & E_{(t_i)} \arrow[d, dashed, "2:1"] \\
E_{(r)} \arrow[dr, dashed, "2:1"] & & E_{(t_i, r^2)} \arrow[dl, dashed, "2:1"] \\
 & E_{D_4}. &
\end{tikzcd}$$

The proof of Theorems 3.33 and 3.34 are in the textbook. One of the key steps is the construction of the universal covering over (path-connected, locally path-connected, semilocally simply connected) B. Most of the rest of the proof depends on the following lemma (which is important on it own right too).

Lemma 3.37 (General Lifting Lemma)**.** *Consider a map f into the base of a covering map, as in the picture on the left*

$$\begin{tikzcd} & E, e_0 \arrow[d, "p"] \\ Y, y_0 \arrow[r, "f"] & B, b_0 \end{tikzcd} \qquad \begin{tikzcd} & E, e_0 \arrow[d, "p"] \\ Y, y_0 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B, b_0. \end{tikzcd}$$

There is a lifting $\tilde{f} : Y, y_0 \to E, e_0$ (that is, a map $\tilde{f}$ making the diagram on the right commutative) if and only if

(1) $$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Proof. If $\tilde{f}$ exists then the diagram of group homomorphisms

$$\begin{tikzcd} & \pi_1(E, e_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(B, b_0) \end{tikzcd}$$

is commutative; which proves (1) ♠. To prove the reverse direction, let $y \in Y$, and choose a path α in Y connecting y_0 to y. Denote the unique lift of $f \circ \alpha : [0, 1] \to B$, starting at e_0, by $\tilde{\alpha} : [0, 1] \to E$. Define $\tilde{f}(y) = \tilde{\alpha}(1)$. The key part of the proof is to show that this definition does not depend on the choice of α. Namely, let β be another path connecting y_0 to y, and consider $\tilde{\beta}$. Use the condition (1) to argue that $\tilde{\alpha}(1) = \tilde{\beta}(1)$ ♠. Once $\tilde{f}$ is well defined as above, its continuity and the fact that it is a covering map follow from the construction ♠. □

Let X be the space obtained from a solid triangle by the gluing scheme xxx. To illustrate the power of the General Lifting Lemma 3.37 prove that every map from X to the torus is homotopic to the constant map ♠.

Definition 3.38. *Let $p : E \to B$ be a covering map. Call a homeomorphism $f : E \to E$ a covering transformation (deck transformation) if the diagram*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

is commutative.

Covering transformations of p form a group ♠, denote it by $\mathcal{C}(p)$. Identify this group for all the examples of Section 3.5 ♠♠♠♠♠♠♠.

Theorem 3.39 (Part 3 of Galois correspondence)**.** *For a covering map $p : E \to B$ we have*

$$\mathcal{C}(p) \cong N(H_p)/H_p.$$

Here $N(H_p)$ is the normalizer of H_p in $\pi_1(B)$, that is, the largest subgroup N such that $H_p \leq N \leq \pi_1(B)$ and H_p is a normal subgroup of N ♠. For example, if the group $\pi_1(B)$ is Abelian (or more generally, if H_p is a normal subgroup) then ♠

$$\mathcal{C}(p) \cong \pi_1(B)/H_p.$$

Verify Theorem 3.39 for all the examples of Section 3.5 ♠♠♠♠♠♠♠. In particular, you will see two index 3 subgroups of F_2, both isomorphic with F_4; one of them is a normal subgroup of F_2, the other one is its own normalizer ♠.

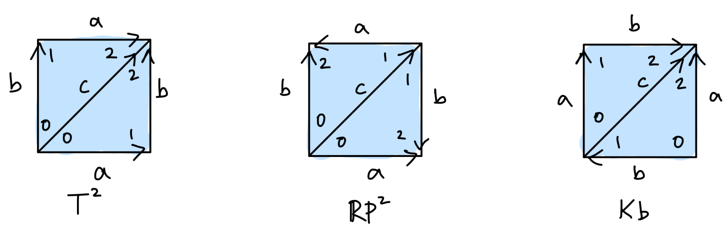
4. Homology

4.1. **Simplicial homology.** Let Δ^n be the convex hull of the elementary basis vectors $v_0, v_1, \ldots, v_n$ of $\mathbb{R}^{n+1}$. It is called the standard n-simplex, and we will mean the total order $v_0 < v_1 < \ldots < v_n$ be part of this concept. To emphasize this convention we use the notation $\Delta^n = [v_0, v_1, \ldots, v_n]$. A subsimplex $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}]$ can be identified with the standard k-simplex $[v_0, v_1, \ldots, v_k]$ by mapping v_{i_k} to v_k and extending this correspondence linearly. In other words the identification is linear *that keeps the total order of the vertices* ♠.
The $n-1$ subsimplices will be called faces. Let $\partial\Delta^n := \cup$faces, $\Delta^{n\circ} := \Delta^n - \partial\Delta^n$.

Definition 4.1. *A finite Δ-complex structure on a topological space X is a finite collection of maps $\sigma_\alpha : \Delta^n \to X$ (n depends on α) such that*

- *the restriction $\sigma_\alpha|_{\Delta^{n\circ}}$ is injective, and every $x \in X$ is covered by exactly one such restriction ("X is the union of the open simplices");*
- *σ_α restricted to a face is one of the σ_β's (recall the identification of a face with a standard $n-1$ simplex);*
- *$U \subset X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open for all α.*

The pictures below indicate Δ-compex structures on T^2, $\mathbb{RP}^2$, Kb.



Decode what the decorations $0, 1, 2$ mean ♠. The arrows got a new meaning too ♠. Verify that these are indeed Δ-complexes ♠.
Find a few different Δ-complex structures on S^1 ♠, on the triangle with gluing scheme xxx ♠.
Find a Δ-complex structure on S^n with exactly two n-simplices ♠.

Let us fix an Abelian group $(G, +)$. For the first reading you may focus on the special case $G = \mathbb{Z}$.

Definition 4.2. *Let X be endowed with a Δ-complex structure.*

- *Formal G-linear combinations of those σ_α's whose domain is an n-simplex will be called* n-chains.
- *n-chains form an abelian group for addition ♠ denote it by $C_n(X;G)$ (precisely speaking we should include the Δ-complex structure on X in the notation; we omit that, but keep it in mind).*

Definition 4.3. *Define the* boundary map $\partial_n : C_n(X;G) \to C_{n-1}(X;G)$ *by linearly extending*

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}. \tag{2}$$

The hat ˆ denotes a missing index.

Lemma 4.4 (Main Equation of Mathematics)**.** *We have $\partial^2 = 0$.*

In fact what we really mean is that the composition $\partial_{n-1} \circ \partial_n$ as a map $C_n(X;G) \to C_{n-2}(X;G)$ is the zero-map, for all n (prove it ♠). Similar economic notation is common in homology theory. We obtain the sequence of Abelian groups and homomorphisms

$$\dots \xrightarrow{\partial_4} C_3(X;G) \xrightarrow{\partial_3} C_2(X;G) \xrightarrow{\partial_2} C_1(X;G) \xrightarrow{\partial_1} C_0(X;G) \xrightarrow{\partial_0} 0.$$

Such sequences, if $\partial^2 = 0$, are called (algebraic) complexes.
Observe that $\partial^2 = 0$ is equivalent to $\operatorname{im}\partial \subset \ker\partial$ ♠. Let $Z_n(X;G) = \ker\partial_n$ be the group of cycles. Let $B_n(X;G) = \operatorname{im}\partial_{n+1}$ be the group of boundaries.

Definition 4.5. *Define the n'th homology group with G coefficients $H_n(X;G)$ of the Δ-complex X to be $Z_n(X;G)/B_n(X;G)$.*

If $G = \mathbb{Z}$ then we do not write it, ie. $C_n(X) = C_n(X,\mathbb{Z})$, $Z_n(X) = Z_n(X,G)$, $B_n(X) = B_n(X,G)$, $H_n(X) = H_n(X,G)$.
For the Δ-complex structures of $T^2, \mathbb{R}\mathrm{P}^2, \mathrm{Kb}$ above we obtain ♠ $\cdots$ ♠

$H_0(T^2) = \mathbb{Z}$	$H_1(T^2) = \mathbb{Z}^2$	$H_2(T^2) = \mathbb{Z}$
$H_0(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}$	$H_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$	$H_2(\mathbb{R}\mathrm{P}^2) = 0$
$H_0(\mathrm{Kb}) = \mathbb{Z}$	$H_1(\mathrm{Kb}) = \mathbb{Z} \oplus \mathbb{Z}_2$	$H_2(\mathrm{Kb}) = 0$
$H_0(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$	$H_1(T^2;\mathbb{Z}_2) = \mathbb{Z}_2^2$	$H_2(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$
$H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$	$H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$	$H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$
$H_0(\mathrm{Kb};\mathbb{Z}_2) = \mathbb{Z}_2$	$H_1(\mathrm{Kb};\mathbb{Z}_2) = \mathbb{Z}_2^2$	$H_2(\mathrm{Kb};\mathbb{Z}_2) = \mathbb{Z}_2$
$H_0(T^2;\mathbb{R}) = \mathbb{R}$	$H_1(T^2;\mathbb{R}) = \mathbb{R}^2$	$H_2(T^2;\mathbb{R}) = \mathbb{R}$
$H_0(\mathbb{R}\mathrm{P}^2;\mathbb{R}) = \mathbb{R}$	$H_1(\mathbb{R}\mathrm{P}^2;\mathbb{R}) = 0$	$H_2(\mathbb{R}\mathrm{P}^2;\mathbb{R}) = 0$
$H_0(\mathrm{Kb};\mathbb{R}) = \mathbb{R}$	$H_1(\mathrm{Kb};\mathbb{R}) = \mathbb{R}$	$H_2(\mathrm{Kb};\mathbb{R}) = 0.$

One possible way of calculating $H_1(\mathrm{Kb})$ is

$$\begin{aligned} &\mathbb{Z}^3\langle a,b,c\rangle/\langle b-c+a, c-a+b\rangle = \mathbb{Z}^3\langle a',b',c'\rangle/\langle b'-c', b'+c'\rangle = \\ &\qquad \mathbb{Z}\langle a'\rangle \oplus \mathbb{Z}^2\langle b',c'\rangle/\langle b'-c', b'-c'\rangle = \mathbb{Z}\langle a'\rangle \oplus \mathbb{Z}^2\langle b'',c''\rangle/\langle b'', b''+2c''\rangle = \\ &\qquad\qquad \mathbb{Z}\langle a'\rangle \oplus \mathbb{Z}\langle c''\rangle/\langle 2c''\rangle = \mathbb{Z} \oplus \mathbb{Z}_2\,. \end{aligned}$$

The point is that when we changed basis, eg.

$$\begin{aligned} a' &= a & & \\ b' &= b & b'' &= b' - c' \\ c' &= c - a & c'' &= c', \end{aligned}$$

we needed to make sure that our transformations were invertible *over the integers* ♠. Recall that an $n \times n$ integer matrix is invertible over the integers if and only if its determinant is ± 1 ♠. For example, the $b'' = b' - c', c'' = b' + c'$ transformation would not be invertible over the integers (so it is *not* a legal change of basis over $\mathbb{Z}$), but it is invertible over $\mathbb{R}$ (so it *is* a legal change of basis if $G = \mathbb{R}$) ♠.

Experiment with other coefficient groups, eg. $G = \mathbb{Z}_3, \mathbb{Z}_4$ ♠. Is there much difference between the coefficient groups $G = \mathbb{Q}, \mathbb{R}, \mathbb{C}$? ♠

4.2. **Singular homology.** Let us fix a topological space X, and an Abelian group G.

Definition 4.6. *A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$. Singular n-chains are formal G-linear combinations of singular n-simplices. They form a group $C_n(X;G)$. (Now the notation is fair, because* this *$C_n(X;G)$ does not depend on any other structure of X, only its topological space structure.) Define the boundary $\partial = \partial_n$ of a singular n-chain by the same formula* (2)*, yielding a homomorphism $C_n(X;G) \to C_{n-1}(X;G)$.*

We have $\partial^2 = 0$ ♠. Hence we have the group $Z_n(X;G) = \ker \partial_n$ of (singular) cycles, the group $B_n(X;G) = \operatorname{im} \partial_{n-1}$ of (singular) boundaries, and we have the singular homology groups $H_n(X;G) = Z_n(X;G)/B_n(X;G)$.

Advantage: the singular groups $H_n(X;G)$ only depend on the topological space. Disadvantage: the groups $C_n(X;G), Z_n(X;G), B_n(X;G)$ are typically huge, we can just hope that the hugeness of $Z_n(X;G)$ and $B_n(X;G)$ cancel in the definition of $H_n(X;G)$ to something manageable—we will see that this is often true.

Proposition 4.7. *We have*

$$H_n(\mathrm{pt};G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proof. For the one-point space we can list all the singular simplices, and calculate $H_n(\mathrm{pt})$ from the definition ♠. □

Proposition 4.8. *If X_α are the path components of X then $H_n(X;G) = \oplus_\alpha H_n(X_\alpha;G)$* ♠.

Proof. The standard simplices are path-connected, hence their continuous images are contained in a path component. Hence $C_n(X;G)$ has the $\oplus_\alpha$ structure, and the ∂ maps respect it ♠. □

Proposition 4.9. *For a path-connected space X we have $H_0(X;G) = G$.*

Proof. Consider the map $\epsilon : C_0(X) \to G$, $\sum n_i p_i \mapsto \sum n_i$ ♠. It is surjective ♠, and its kernel equals the image of ∂_1 ♠. □

Corollary 4.10. *We have* $H_0(X;G)=G^{\{\text{path components of } X\}}$.

Replacing the standard chain complex $\ldots\to C_2(X;G)\to C_1(X;G)\to C_0(X;G)\to 0$ with

$$\ldots \xrightarrow{\partial} C_2(X;G) \xrightarrow{\partial} C_1(X;G) \xrightarrow{\partial} C_0(X;G) \xrightarrow{\epsilon} G$$

we still have a complex ♠, and its ker / im groups are called the *reduced homology groups* of X; denoted by $\tilde{H}_n(X;G)$. Reduced homology is only different from ordinary homology in degree 0, where it is one G less ♠. The content of the two concepts are the same, but some theorems are more convenient to phrase in terms of $\tilde{H}$.

4.3. **Functor.** Let $f:X\to Y$ be continuous. "Composing with f" induces a homomorphism $f_\#: C_n(X;G)\to C_n(Y;G)$ ♠ (more precisely $f_{n\#}$). This map commutes with the ∂'s ♠, giving a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_4"] & C_3(X;G) \arrow[r, "\partial_3"] \arrow[d, "f_\#"] & C_2(X;G) \arrow[r, "\partial_2"] \arrow[d, "f_\#"] & C_1(X;G) \arrow[r, "\partial_1"] \arrow[d, "f_\#"] & C_0(X;G) \arrow[d, "f_\#"] \\
\ldots \arrow[r, "\partial_4"] & C_3(Y;G) \arrow[r, "\partial_3"] & C_2(Y;G) \arrow[r, "\partial_2"] & C_1(Y;G) \arrow[r, "\partial_1"] & C_0(Y;G).
\end{tikzcd}$$

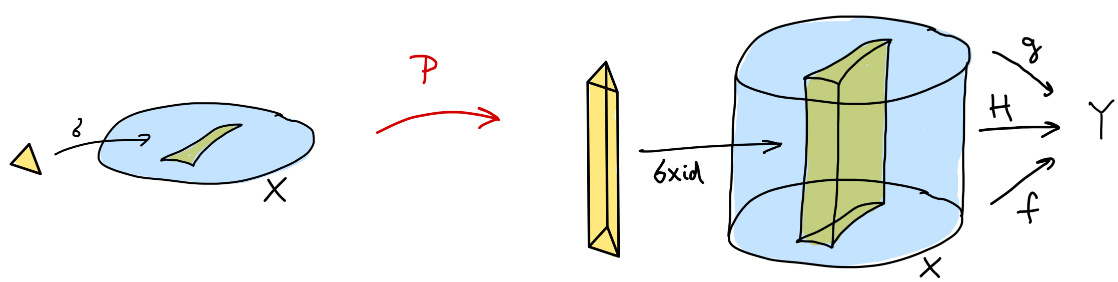
Such a commutative "ladder diagram" induces a homomorphism between the respective $\ker\partial/\operatorname{im}\partial$ groups ♠. Therefore $f:X\to Y$ induces a homomorphism $f_*: H_n(X;G)\to H_n(Y;G)$ (more precisely f_{n*}).

Theorem 4.11. *We have* $(f\circ g)_* = f_*\circ g_*$, $\mathrm{id}_*=\mathrm{id}$ ♠.

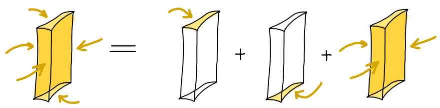
4.4. **Homology is homotopy invariant.** Let the maps $f,g:X\to Y$ be homotopic, that is, we assume the existence of a map $H: X\times[0,1]\to Y$ such that $H|_{X\times\{0\}}=f$, $H|_{X\times\{1\}}=g$. Consider the following "prism" construction

$$\sigma:\Delta^n\to X \qquad \xrightarrow{P} \qquad H\circ\left(\sigma\times \mathrm{id}_{[0,1]}\right)\in C_{n+1}(Y;G),$$

c.f. the picture below.



For this to be precise we would need to fix a subdivision of the prism $\Delta^n\times[0,1]$ into $(n+1)$-simplices and consider their sum with the right plus/minus signs—that is, in such a way that in the boundary of this chain the inside faces cancel. We will skip this detail.

From the picture  we see that

$$\begin{array}{ccccccc}
\text{boundary} & & \text{top lid} & & \text{bottom lid} & & P(\text{boundary} \\
\text{of } P(\sigma) & = & \text{of } P(\sigma) & + & \text{of } P(\sigma) & + & \text{of } \sigma). \\
 & & \| & & \| & & \\
 & & g_\#(\sigma) & & f_\#(\sigma) & &
\end{array}$$

If we worked out the details (namely, the signs), we would obtain $\partial(P(\sigma)) = g_\#(\sigma) - f_\#(\sigma) - P(\partial(\sigma))$, and in effect

$$g_\# - f_\# = \partial \circ P + P \circ \partial \tag{3}$$

for the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "g_\# - f_\#"] \arrow[dl, "P"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "g_\# - f_\#"] \arrow[dl, "P"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "g_\# - f_\#"] \arrow[dl, "P"] & \dots \arrow[dl, "P"] \\
\dots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \dots .
\end{tikzcd}$$

Chasing elements around this diagram—and using (3)—implies that the map induced by $g_\# - f_\#$ is the zero map on the ker / im groups ♠. Thus we obtained

Theorem 4.12. *If $f \simeq g : X \to Y$ then $f_* = g_* : H_n(X; G) \to H_n(Y; G)$.*

Corollary 4.13. *If $X \simeq Y$ (homotopy equivalent spaces) then $H_n(X; G) \cong H_n(Y; G)$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be the homotopy equivalence. Apply the $H_n(-; G)$ functor to $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$, and use Theorems 4.11, 4.12 ♠. □

We have $\tilde{H}_n(\mathbb{R}^m; G) = 0$, $\tilde{H}_n(D^m; G) = 0$, $\tilde{H}_n(\mathbb{R}^{m+1} - \mathrm{pt}; G) = \tilde{H}_n(S^m; G)$.

In the rest of the semester we will present some properties of $H_n(-; G)$ without proofs. The goal is that the reader has enough tools to calculate homology groups efficiently. The proofs and further development of the theory will be given in the follow-up course.

4.5. **Long exact sequences: pair and Mayer-Vietoris.** Let $A \subset X$ be a non-empty closed subset, and we also assume that it is the deformation retract of a neighborhood of it. For example X is a finite CW complex and A is a subcomplex.

Theorem 4.14 (Long exact sequence of a pair). *The (infinitely) long sequence of Abelian groups and group homomorphisms*

$$\begin{tikzcd}
& \dots \arrow[r, "q_*"] & \tilde{H}_{n+1}(X/A;G) \arrow[dll, "\partial"'] \\
\tilde{H}_n(A;G) \arrow[r, "i_*"'] & \tilde{H}_n(X;G) \arrow[r, "q_*"] & \tilde{H}_n(X/A;G) \arrow[dll, "\partial"'] \\
\tilde{H}_{n-1}(A;G) \arrow[r, "i_*"'] & \tilde{H}_{n-1}(X;G) \arrow[r, "q_*"] & \tilde{H}_{n-1}(X/A;G) \arrow[dll, "\partial"'] \\
\tilde{H}_{n-2}(A;G) \arrow[r, "i_*"'] & \dots &
\end{tikzcd}$$

is exact, where $i : A \subset X$ is the inclusion, and $q : X \to X/A$ is the quotient map.

Some explanations: We have not defined the map $\partial : H_{k+1}(X/A) \to H_k(A)$ (it is *not* the boundary map between chain groups, it is just traditionally denoted the same way, sorry). Hence, at this point the theorem reads *"there exist ∂ maps making the sequence exact."* In fact, ∂ is defined "in the proof", so we could have started with that definition and claim that the sequence is exact with that definition. However, in practice the definition of ∂ is rarely needed.

A sequence is exact (by definition) if at every term we have im = ker (of the appropriate maps). In particular the sequence $0 \to G_1 \to G_2$ is exact iff the map $G_1 \to G_2$ is injective, the sequence $G_1 \to G_2 \to 0$ is exact iff the map $G_1 \to G_2$ is surjective, the sequence $0 \to G_1 \to G_2 \to 0$ is exact iff the map $G_1 \to G_2$ is an isomorphism ♠.

Let $m \geq 1$. Apply the theorem for $X = D^m, A = \partial D^m = S^{m-1}$ to obtain that $\tilde{H}_n(S^m;G) = \tilde{H}_{n-1}(S^{m-1};G)$ ♠. Use our knowledge about the homology of the two point space S^0 to conclude that ♠

$$\tilde{H}_n(S^m;G) = \begin{cases} G & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

We have an immediate application.

Theorem 4.15 (n dimensional Brouwer fixed point theorem). *Every $f : D^n \to D^n$ continuous map has a fixed point. ($\exists x \in D^n$ such that $f(x) = x$.)*

We proved the 2d version, Theorem 3.18, using the following properties of the π_1 functor: $\pi_1(S^1) \neq 0$, $\pi_1(D^2) = 0$. Now we have infinitely many functors $H_m(-;G)$, one for each m and each G. Replace π_1 with one of these so that the proof of Theorem 3.18 generalizes to n dimensions ♠.

Theorem 4.16 (Mayer-Vietoris)**.** *Let $A, B \subset X$ such that* $\operatorname{int} A \cup \operatorname{int} B = X$*. Then the (infinitely) long sequence of Abelian groups and group homomorphisms*

$$\begin{tikzcd}
 & \cdots \arrow[r, "i_*-j_*"] & H_{n+1}(X;G) \arrow[dll, "\partial"'] \\
H_n(A\cap B;G) \arrow[r, "(k_*,l_*)"'] & H_n(A;G)\oplus H_n(B;G) \arrow[r, "i_*-j_*"] & H_n(X;G) \arrow[dll, "\partial"'] \\
H_{n-1}(A\cap B;G) \arrow[r, "(k_*,l_*)"'] & H_{n-1}(A;G)\oplus H_{n-1}(B;G) \arrow[r, "i_*-j_*"] & H_{n-1}(X;G) \arrow[dll, "\partial"'] \\
H_{n-2}(A\cap B;G) \arrow[r, "(k_*,l_*)"'] & \cdots &
\end{tikzcd}$$

is exact. Here $i : A \subset X, j : B \subset X$, $k : A \cap B \subset A$, $l : A \cap B \subset B$. The same statement holds for reduced homology $\tilde{H}$.

Word by word the same explanations are in order as the paragraph after Theorem 4.14.

Let $X \vee Y$ mean the "1-point union" of X and Y (gluing a point of X to a point of Y). For example $S^1 \vee S^1$ is the figure-8 space. We have ♠♠

$$H_n(S^1 \vee S^1; G) = \begin{cases} G & \text{if } n = 0 \\ G \oplus G & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{H}_n(\vee_\alpha X_\alpha; G) = \oplus_\alpha \tilde{H}_n(X_\alpha; G),$$

for path-connected X_α if the special point in each X_α is the deformation retract of a neighborhood of it. We have ♠

$$H_n(T^2; G) = \begin{cases} G & \text{if } n = 0 \\ G \oplus G & \text{if } n = 1 \\ G & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.16 (but also Theorem 4.14) intuitively claims that homology can be computed from pieces. This phenomenon can be developed further to obtain the following theorem.

Theorem 4.17. *Let the topological space X be endowed with a Δ-complex structure. Then we have two homology concepts defined for it: the simplicial homology using the (combinatorial) Δ-complex structure, and the singular homology using only the topological structure. These two homology groups coincide.*

One way of looking at this theorem is that even though the definition of singular homology uses infinitely generated Abelian groups (the chain groups), if X has a combinatorial structure then there is another definition that avoids those huge groups. Another advantage of the theorem is

Corollary 4.18. *If the underlying topological space of two Δ-complexes are the same, then their simplicial homologies are the same.*

It is remarkable that while this corollary sounds fully combinatorial, its proof goes through the non-combinatorial concept of singular homology.

There is a further important theorem along the line of "combinatorial homology vs singular homology".

Theorem 4.19 (Cellular homology)**.** *If the topological space X is endowed with a finite CW complex structure then $H_*(X;G)$ can be calculated from an algebraic complex*

$$\ldots \xrightarrow{\partial} G^{(n+1)\text{-}cells} \xrightarrow{\partial} G^{n\text{-}cells} \xrightarrow{\partial} G^{(n-1)\text{-}cells} \xrightarrow{\partial} \ldots$$

by the $\ker\partial/\operatorname{im}\partial$ *definition.*

The full theorem on cellular homology includes the description of the ∂ maps, in terms of a notion called "degree". We will not define degree or the ∂ maps here. The cellular homology theorem is still powerful. For example, use this theorem to *trivially* calculate $\tilde{H}_n(S^m;G)$ ♠.

4.6. **Euler characteristic, the right approach.** Recall the notion of $\operatorname{rk} A$ for a finitely generated Abelian group, as $\dim_{\mathbb{R}}(A\otimes_{\mathbb{R}}\mathbb{R})$ ♠. We have $A=\mathbb{Z}^{\operatorname{rk} A}\oplus$ torsion ♠. If $0\to A\to B\to C\to 0$ is a short exact sequence then $\operatorname{rk} B=\operatorname{rk} A+\operatorname{rk} C$ ♠.

Lemma 4.20. *Assume* $\ldots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots$ *is a complex of Abelian groups for which* $\sum_n(-1)^n\operatorname{rk} C_n$ *makes sense. Let* Z_n, B_n, H_n *denote the groups of cycles, boundaries, and homologies of this complex. Then we have*

$$\sum_n(-1)^n\operatorname{rk} C_n=\sum_n(-1)^n\operatorname{rk} H_n.$$

Proof. We have the $0\to Z_n\to C_n\to B_{n-1}\to 0$ short exact sequences ♠, as well as the $0\to B_n\to Z_n\to H_n\to 0$ short exact sequences ♠. These have consequences on the ranks of the groups involved (see the paragraph above the lemma), and we get $\operatorname{rk} C_n=\operatorname{rk} B_{n-1}+\operatorname{rk} B_n+\operatorname{rk} H_n$ ♠. Adding these equations with alternating signs proves the lemma ♠. □

Putting together Corollary 4.18, Theorem 4.19, and Lemma 4.20 now we have the *right* definition of Euler characteristic, together with its crucial properties.

Definition-Theorem 4.21. *For a topological space X, if the number*

$$\sum_n(-1)^n\operatorname{rk} H_n(X;\mathbb{Z})$$

is defined (that is, if this is a finite sum of finite numbers), then we call it the Euler characteristic $\chi(X)$ of the space (cf. Theorem 2.12). The notion $\chi(X)$ is invariant under homotopy equivalence *(c.f. Theorem 3.3)* ♠. *If the space is endowed with a Δ-complex structure, then $\chi(X)$ can also be calculated as* ♠ *(c.f. Definition 2.11)*

$$\sum_n (-1)^n |\{n\text{-}simplices\}|.$$

If the space is endowed with a CW complex structure, then $\chi(X)$ can also be calculated as ♠ *(c.f. Theorem 3.5)*

$$\sum_n (-1)^n |\{n\text{-}cells\}|.$$

Remark 4.22. Our definition of Euler characteristic can be rephrased $\sum_n (-1)^n \dim_{\mathbb{R}} H_n(X;\mathbb{R})$ ♠. For a field $\mathbb{F}$ the groups $H_n(X;\mathbb{F})$ turn out to be $\mathbb{F}$ vector spaces ♠, and the sum

$$\sum_n (-1)^n \dim_{\mathbb{F}} H_n(X;\mathbb{F})$$

turns out to be the *same* Euler characteristic. Verify this for $\mathbb{F} = \mathbb{Z}_2$ in the examples of Section 4.1 ♠. The dimensions $b_{n,\mathbb{F}}(X) = \dim_{\mathbb{F}} H_n(X;\mathbb{F})$ are called the $\mathbb{F}$-Betti numbers; for $\mathbb{F} = \mathbb{R}$ just Betti numbers $b_n(X)$.

4.7. $H_1(-;\mathbb{Z})$ **vs** $\pi_1(X)$**.**

Theorem 4.23. *Let X be path-connected. Then $H_1(X;\mathbb{Z}) \cong \pi_1(X)^{Ab}$.*

By G^{Ab} we denote the Abelianization of the group G. It is the "largest" (in a precise sense) commutative quotient of G. It is equal to the quotient $G/[G,G]$ where $[G,G]$ is the normal subgroup generated by the commutators $g_1g_2g_1^{-1}g_2^{-1}$ for all $g_1, g_2 \in G$. If $G = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots r_m\rangle$ then

$$\langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_m, \quad x_ix_jx_i^{-1}x_j^{-1} \text{ (for all } i \neq j)\rangle$$

is a presentation of G^{Ab}. Use this theorem to reprove $H_1(T^2;\mathbb{Z}) = \mathbb{Z}^2$ ♠. Find $H_1(T^2\#T^2;\mathbb{Z})$ ♠.

Homology, and its friend, cohomology, are essential tools in half of mathematics. The fields where homology appears include, but are not restricted to, algebraic topology, manifold theory, algebraic geometry, complex analysis, graph theory, dynamical systems, group theory, Lie algebras, statistics. In this course we only introduced the basic concepts. The natural next step is to take an algebraic topology course.
