# Euclidean and Non-Euclidean Geometries

Richárd RIMÁNYI

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# Chapter 1

# Introduction

Yes, there are hundreds of Geometry textbooks written and published. What is the reason for this one then?

The present lecture notes is written to accompany the course math551, *Euclidean and Non-Euclidean Geometries*, at UNC Chapel Hill in the early 2000s. The students in this course come from high school and undergraduate education focusing on calculus. They have the skills to work with functions and vectors, but typically have completed, no, or little, dedicated geometry classes. Hence, I chose a vector based description of Euclidean geometry, and a model based description of Hyperbolic geometry. Of course, there are still hundreds of excellent Geometry textbooks with the same focus. I decided to write one that contains exactly as much material as we cover in one semester, this way there is no awkward skipping over sections and chapters, and the students are presented a compact set of material they need to learn and practice.

Geometry is the cradle of modern mathematics, with countless faces and interfaces. A student who takes this class early in their math education meets glimpses of other subjects that they will study in future courses. Meeting those subjects early, in action, serves as a motivation. Here is a list of such interfaces.

- *Calculus.* Trigonometric, exponential, and hyperbolic trigonometric functions, in action.

- *Linear algebra.* Linear independence and span vs collinearity. Wedge product, determinant. Bilinear forms, dot product, and Minkowski product. Sylvester's Law of Inertia.

- *Algebraic geometry.* Desargues and Pappus theorems are degenerate cases of natural

Algebraic Geometry theorems. Menelaus and Ceva theorems are manifestations of matroid realization problems.

- *Transformation Groups.* Groups of translations, rotations in Euclidean geometry, $SO(3)$, and the Lorenz group $SO(2,1)$.

- *Real Analysis* Dyadic rationals dense among reals.

- *Analysis, differential geometry.* Differentiation and integration (as area considerations on $S^2$, $\mathbb{H}^2$) all around. Tangent vectors as derivatives of curves. Geodesics. Length of curves. Gauss-Bonnet theorems in three geometries.

- *Complex numbers.* The very last Exercise of the book.

- *Topology.* Projective plane.

- *Number theory.* The cyclotomic polynomials.

- *Special Relativity.* Our model of $\mathbb{H}^2$ lives in the space of Special Relativity. The Lorenz boosts are used to show that $\mathbb{H}^2$ is homogeneous.

The fascinating *philosophical,* and *historical* aspects of Geometries are integral parts of the course. They are not part of the text though—the instructor is encouraged to add their own perspective, and the students are encouraged to do their own research on those faces of the subject.

# Chapter 2

# Affine and Euclidean Geometry

## 2.1   Points and vectors

First we recall coordinate plane geometry from Calculus. The set $\mathbb{R}^2$ will be called the plane. Elements of $\mathbb{R}^2$, that is ordered pairs $(x, y)$ of real numbers, are called points.

Consider directed segments (also called "arrows") between points of the plane. We allow the start point and the end point of an arrow to coincide. Arrows up to translation are called (plane) vectors. That is, the arrow from $A = (1, 3)$ to $B = (5, 6)$ represents the same vector as the arrow from $C = (-4, -4)$ to $D = (0, -1)$. We write $\overrightarrow{AB} = \overrightarrow{CD}$. The vector $\overrightarrow{AA}$ is called the zero-vector and denoted by $\mathbf{0}$.
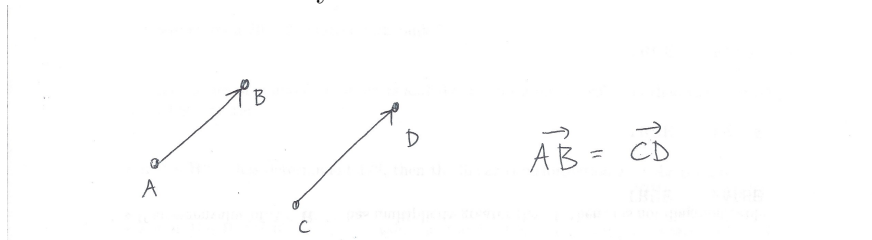


We can represent a vector by an ordered pair of real numbers as well: the vector $\overrightarrow{AB}$ where $A = (a_1, a_2)$ and $B = (b_1, b_2)$ will be represented by $\langle b_1 - a_1, b_2 - a_2 \rangle$. This is a fair definition, because if $\overrightarrow{AB} = \overrightarrow{CD}$ then $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$. The vector $\overrightarrow{AB}$ of the paragraph above is $\langle 4, 3 \rangle$.

## 2.2   Linear operations on vectors

The sum and difference of two vectors are defined geometrically in Figure 2.1. In this context real numbers will also be called scalars. A scalar multiple of a vector is defined in Figure 2.2.
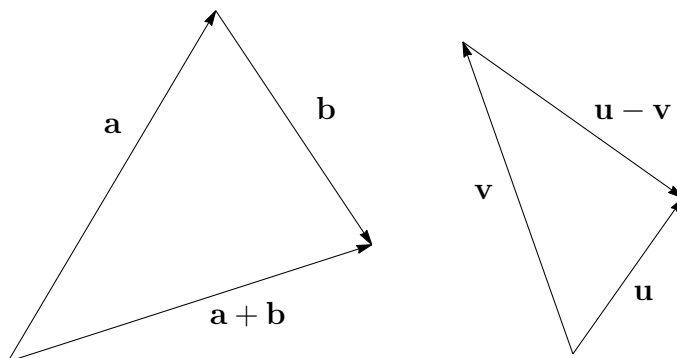


Figure 2.1: Sum and difference of plane vectors
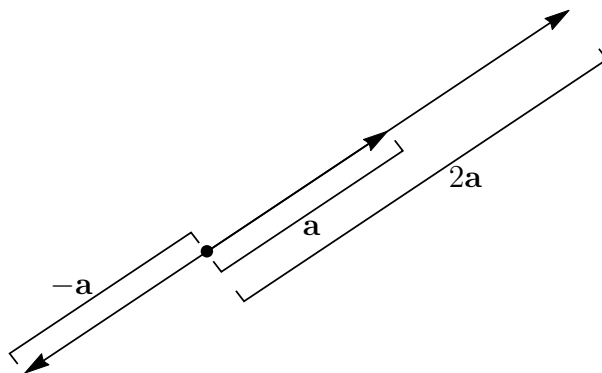


Figure 2.2: Scalar multiple

The operations above (addition, subtraction, multiplication by a scalar) are called the *linear* operations on vectors. The geometric definitions above translate to the following algebraic expressions.

- $\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$

- $\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$

- $\lambda \cdot \langle a_1, a_2 \rangle = \langle \lambda a_1, \lambda a_2 \rangle$

**Proposition 2.2.1** (Vector space "axioms"). *The linear operations on vectors satisfy the following properties.*

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

- $\mathbf{a} + \mathbf{0} = \mathbf{a}$

- $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

- $\lambda \cdot (\mathbf{a} + \mathbf{b}) = \lambda \cdot \mathbf{a} + \lambda \cdot \mathbf{b}$

- $(\lambda + \mu) \cdot \mathbf{a} = \lambda \cdot \mathbf{a} + \mu \cdot \mathbf{a}$

- $\lambda \cdot (\mu \cdot \mathbf{a}) = (\lambda \mu) \cdot \mathbf{a}$

- $1 \cdot \mathbf{a} = \mathbf{a}$

*Proof.* The properties follow from the algebraic expressions for the linear operations. $\square$

**Proposition 2.2.2** (2-dimensionality). *Let $\mathbf{a}$ and $\mathbf{b}$ be non-parallel vectors (algebraically $a_1 b_2 - a_2 b_1 \neq 0$). For a vector $\mathbf{c}$ there are unique real numbers $\lambda, \mu$ such that $\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$.*

The proof of this proposition is left to the readers (see the problems below).

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**Problems**

**2.2.1.** Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be plane vectors. Prove that they are parallel (that is, at least one of them is a scalar multiple of the other) if and only if $a_1 b_2 - a_2 b_1 = 0$.

**2.2.2.** Let $\mathbf{a}, \mathbf{b}$ be not parallel, and let $\mathbf{c} \in \mathbb{R}^2$ be arbitrary. Prove that there are unique real numbers $\lambda, \mu$ such that $\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$.

**2.2.3.** Let $\mathbf{a}, \mathbf{b}$ be not parallel. Prove that $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ are also not parallel.

**2.2.4.** For which real numbers $k_{1,1}, k_{1,2}, k_{2,1}, k_{2,2}$ is it true that if $\mathbf{a}, \mathbf{b}$ are not parallel then $k_{1,1}\mathbf{a} + k_{1,2}\mathbf{b}$ and $k_{2,1}\mathbf{a} + k_{2,2}\mathbf{b}$ are also not parallel.

## 2.3  Convention on identifying points with vectors

To a point $A \in \mathbb{R}^2$ we can associate its "position vector" $\overrightarrow{OA}$ where $O = (0,0)$ is the origin. To a vector $\mathbf{v}$ we can associate a point $P$ by considering an arrow $\overrightarrow{OP}$ representing $\mathbf{v}$.

The above two associations are inverses of each other, they define a one-to-one correspondence between points and vectors. Algebraically this one-to-one correspondence is $(a, b) \leftrightarrow \langle a, b \rangle$.

Throughout this text we will build in this identification in our notation, without further explanation. For example, if $A$ is a point, and we write $5A$ then we really mean either the vector $5\overrightarrow{OA}$ or its endpoint. Or, if we say $A/2 + B/2$ is the midpoint of the segment $AB$ then here is how to read it precisely: the midpoint of the segment $AB$ is the endpoint of the vector $\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$.

## 2.4  Algebraic conditions expressing collinearity

The word "collinear" is a shorthand expression for "on the same line". The word "concurrent" is a shorthand expression for "intersecting in one point".

**Proposition 2.4.1.** *Let $A$ and $B$ be two different points. Point $C$ is on the line through $A$ and $B$ if and only if there is a real number $t$ such that*

$$C = (1 - t)A + tB. \tag{2.1}$$

*Moreover, $t$ and $C$ uniquely determine each other (i.e. for any $C$ on the $AB$ line there is a* unique *real number $t$ satisfying* (2.1)*, and for any real number $t$ there is a* unique *point $C$ on the $AB$ line satisfying (2.1).)*

*Proof.* To obtain the position vector of a point $C$ on the line $AB$ we need to add the vector $\overrightarrow{A}$ and a multiple of $\overrightarrow{AB}$, see Figure 2.3.

Conversely, the end point of any such vector is obviously on the line $AB$. Observe that $\overrightarrow{A} + t \cdot \overrightarrow{AB} = (1 - t)A + tB$, which proves the proposition.

The proof that $C$ and $t$ determine each other is left as an exercise.  □

If $A = B$ then for any $t$ the point $(1 - t)A + tB$ obviously coincides with $A$ and $B$.

A useful rephrasing of Proposition 2.4.1 is that if $A \neq B$ then $C$ is on their line if and only if there exist numbers $x$ and $y$ such that

$$C = xA + yB, \qquad x + y = 1.$$
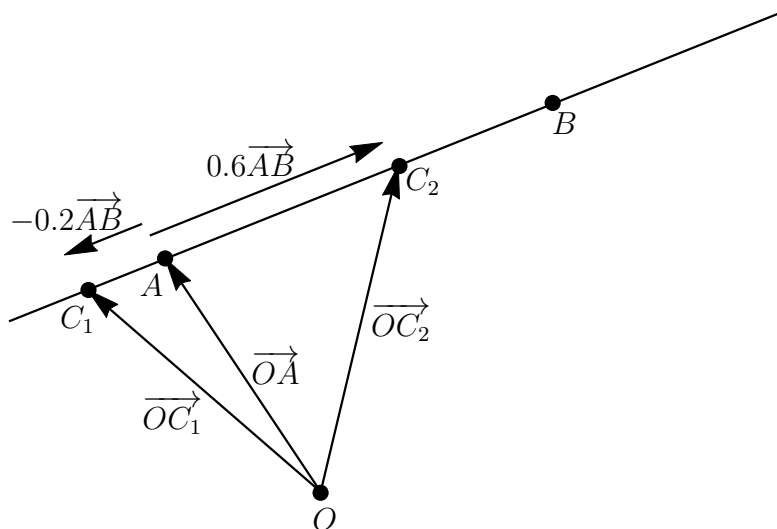
Figure 2.3: $C_1$ and $C_2$ on the line through $A$ and $B$

**Proposition 2.4.2.** *The points $A, B, C$ are collinear if and only if there exist real numbers $x, y, z$ not all 0, such that*

$$xA + yB + zC = \mathbf{0}, \qquad x + y + z = 0.$$

*Proof.* Suppose $A, B, C$ are collinear.

If $A$ and $B$ are different points, then $C$ is on their line. According to Proposition 2.4.1 then there is a $t$ such that $C = (1 - t)A + tB$. After rearrangement we obtain $(1 - t)A + tB + (-1)C = 0$ and hence $1 - t, t, -1$ serve as $x, y, z$.

If $A = B$ then $x = 1, y = -1, z = 0$ satisfy the requirements.

To prove the opposite direction let us now assume that $xA + yB + zC = \mathbf{0}$, $x + y + z = 0$, and not all $x, y, z$ are 0. Let us pick one non-zero among $x, y, z$. Without loss of generality we may assume that it is $z$. Rearrangement gives $C = (-x/z)A + (-y/z)B$. The condition $x+y+z = 0$ translates to $(-x/z)+(-y/z) = 1$. If $A$ and $B$ are different then Proposition 2.4.1 implies that $C$ is on the $AB$ line. If $A$ and $B$ coincide then the remark after Proposition 2.4.1 implies that all $A, B, C$ are the same point, so they are collinear.     □

An important logical consequence of Proposition 2.4.2 is the following Corollary.

**Corollary 2.4.3.** *If $A, B, C$ are not collinear (i.e. they form a triangle), and $x, y, z$ are real numbers with*

$$xA + yB + zC = \mathbf{0} \qquad and \qquad x + y + z = 0,$$

*then $x = y = z = 0$.*     □

**Problems**

**2.4.1.** In the formula $C = (1-t)A + tB$, trace the position of $C$ on the line as $t$ varies from $-\infty$ to $\infty$.

**2.4.2.** Prove that in Proposition 2.4.1 the point $C$ and the real number $t$ uniquely determine each other.

**2.4.3.** In $\triangle ABC$ let $U$ be the midpoint of $AB$ and let $V$ be the midpoint of $AC$. Prove that $2\overrightarrow{UV} = \overrightarrow{BC}$.

## 2.5   The ratio $(\overrightarrow{AC} : \overrightarrow{CB})$ for some collinear points

Let $A$ and $B$ be different points, and let $C$ be on their line. Let us assume that $C \neq B$. We will write that $(\overrightarrow{AC} : \overrightarrow{CB}) = \lambda$ if $\overrightarrow{AC} = \lambda \overrightarrow{CB}$. Such a $\lambda$ exists (and is unique) since $\overrightarrow{AC}$ and $\overrightarrow{CB}$ are collinear vectors with $\overrightarrow{CB} \neq \mathbf{0}$.

**Lemma 2.5.1.** *Let $A, B, C$ be collinear, $A \neq B \neq C$, and write $C = xA + yB$ with $x + y = 1$ (cf. Proposition 2.4.1). Then we have $(\overrightarrow{AC} : \overrightarrow{CB}) = y/x$.*

*Proof.* From $C = xA + yB$ we obtain $\overrightarrow{AC} = y\overrightarrow{AB}$ and $\overrightarrow{CB} = x\overrightarrow{AB}$ (verify!). Also, since $B \neq C$ we know that $x \neq 0$. Thus we have $\overrightarrow{AC} = y\overrightarrow{AB} = y\frac{1}{x}\overrightarrow{CB}$, what we wanted to prove. $\qquad\square$

*Remark* 2.5.2. If we extended the notion of $(\overrightarrow{AC} : \overrightarrow{CB})$ to the case of $C = B$, by defining $(\overrightarrow{AB} : \overrightarrow{BB}) = \infty$, then the last proposition would say $1/0 = \infty$, which agrees with our intuition.

In fact we can interpret the ratio $(\overrightarrow{AC} : \overrightarrow{CB})$ without mentioning vectors. It is the ratio of the length of the segment $AC$ over the length of the segment $CB$, *with a sign convention.* The sign convention is that if $C$ is in between $A$ and $B$, then $(\overrightarrow{AC} : \overrightarrow{CB})$ is positive, and if $C$ is outside of the segment $AB$ then $(\overrightarrow{AC} : \overrightarrow{CB})$ is negative.

**Proposition 2.5.3.** *Let $A \neq B$ be fixed. The ratio $(\overrightarrow{AC} : \overrightarrow{CB})$ uniquely determines $C$.*

The proof is left as an exercise.

*Remark* 2.5.4. We may be sloppy in notation and decide to write $\overrightarrow{AC}/\overrightarrow{CB}$ instead of $(\overrightarrow{AC} : \overrightarrow{CB})$, but we must be careful that this ratio is only defined in the very special situation where $A, B, C$ are collinear (and some coincidences do not happen). In general there is no such operation where we divide a plane vector by another plane vector!

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**Problems**

**2.5.1.** Prove Proposition 2.5.3.

**2.5.2.** For three (distinct) points $A, B, C$ on a line prove that

$$(\overrightarrow{AB} : \overrightarrow{BC}) \cdot (\overrightarrow{BC} : \overrightarrow{CA}) \cdot (\overrightarrow{CA} : \overrightarrow{AB}) = 1.$$

**2.5.3.** For four (distinct) points $A, B, C, D$ on a line prove that

$$(\overrightarrow{AD} : \overrightarrow{DB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CD} : \overrightarrow{DA}) = -1$$

## 2.6   First applications

A quadrilateral $ABCD$ is a parallelogram if $\overrightarrow{AB} = \overrightarrow{DC}$. This condition can be phrased as $B - A = C - D$, or rearranged to $D - A = C - B$, which means $\overrightarrow{AD} = \overrightarrow{BC}$ also holds.

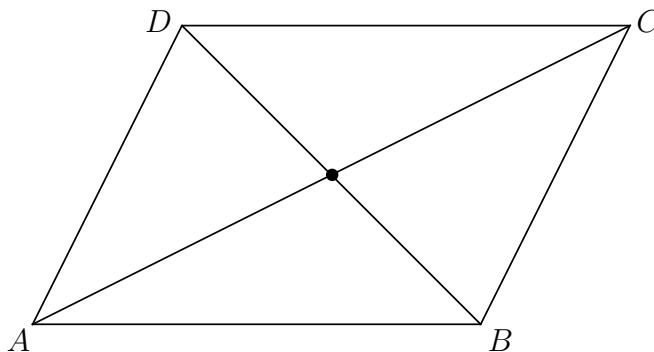**Proposition 2.6.1.** *The diagonals of a parallelogram bisect each other.*



Figure 2.4: The diagonals of a parallelograph bisect each other.

*Proof.* Since $ABCD$ is a parallelogram, we have $B - A = C - D$, or equivalently $A + C = B + D$. Consider the point $P = (A + C)/2 = (B + D)/2$. The first defining expression implies that $P$ is the midpoint of $A$ and $C$. The second expression implies that $P$ is the midpoint of $B$ and $D$. Since they agree, $P$ is the intersection of $AC$ and $BD$, and it bisects both diagonals.                                                                                  $\square$

The median of a triangle is a segment connecting a vertex to the midpoint of the opposite side. A triangle has three medians.

**Proposition 2.6.2.** *The medians of a triangle are concurrent. Moreover they divide each other by 2:1.*
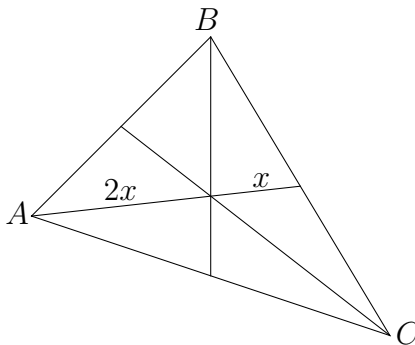


Figure 2.5: The medians of a triangle are concurrent.

*Proof.* Let $ABC$ be a triangle. Consider the point $P = (A + B + C)/3$, and its equivalent expressions

$$P = \frac{2}{3} \cdot \frac{A + B}{2} + \frac{1}{3} \cdot C = \frac{2}{3} \cdot \frac{B + C}{2} + \frac{1}{3} \cdot A = \frac{2}{3} \cdot \frac{C + A}{2} + \frac{1}{3} \cdot B.$$

The first expression claims that $P$ is on the segment connecting the midpoint of $A$ and $B$ with $C$, that is, on the median corresponding to $C$. The second expression claims that $P$ is on the median corresponding to $A$, and the third expression claims that $P$ is on the median corresponding to $B$. Since they are all equal, there is a point, namely $P$, that in on all three medians; and we proved that the medians are concurrent.
A byproduct of the above argument is that the intersection $P$ of the three medians is expressed as 2/3 of the midpoint of a side plus 1/3 the opposite vertex. According to Section 2.5 this proves that $P$ cuts the median $2 : 1$.                                                        $\square$

*Remark* 2.6.3. In the above two propositions we needed to make arguments about the intersections of certain lines. In our proofs we used a trick: we did not "compute" the intersections, but rather we "named" a point and then proved that this point is on the lines, and hence this point must be the intersection. You will find this trick useful when solving exercises.

**PROJECT 1.** *Invent and prove the* $3D, 4D, \ldots$ *versions of Proposition 2.6.2.*

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## Problems

**2.6.1.** For the points $O, A, A', B, B'$ assume that $\overrightarrow{OA'} = \lambda \overrightarrow{OA}$, $\overrightarrow{OB'} = \lambda \overrightarrow{OB}$, see Figure 2.6(a). Prove that $\overrightarrow{A'B'} = \lambda \overrightarrow{AB}$.

**2.6.2.** Assume that in Figure 2.6(b) the lines $x$, $y$ are parallel. Prove that there exists a real number $\lambda$ such that $\overrightarrow{OA'} = \lambda \overrightarrow{OA}$, $\overrightarrow{OB'} = \lambda \overrightarrow{OB}$.

**2.6.3.** Assume $\overrightarrow{O_1A'} = \lambda \overrightarrow{O_1A}$, $\overrightarrow{O_1B'} = \lambda \overrightarrow{O_1B}$ and $\overrightarrow{O_2B'} = \mu \overrightarrow{O_2A}$, $\overrightarrow{O_2A'} = \mu \overrightarrow{O_2B}$, see the Figure 2.6(c). Prove that $\lambda = -\mu$.
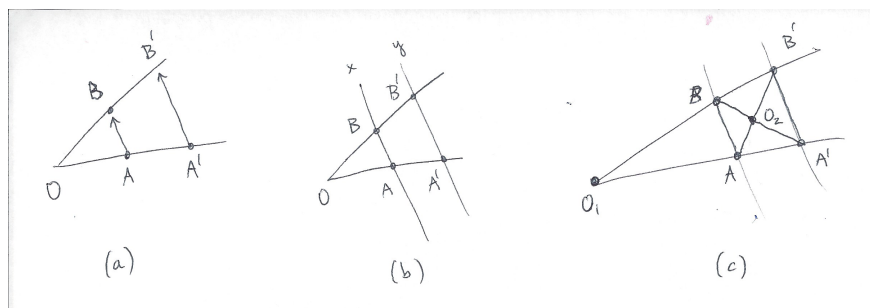


Figure 2.6: Figures for Problems 1.6.1, 1.6.2, 1.6.3

**2.6.4.** In a quadrilateral let $U$ and $V$ be the midpoints of two opposite sides. Prove that the segment $UV$ and the segment connecting the midpoints of the diagonals bisect each other.

**2.6.5.** Let $S$ be the centroid of the $\triangle ABC$. Calculate $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}$.

**2.6.6.** Let $R$ be an arbitrary point in the plane and $ABCD$ a parallelogram. Prove that $\overrightarrow{RA} + \overrightarrow{RC} = \overrightarrow{RB} + \overrightarrow{RD}$.

**2.6.7.** Let $ABCD$ four point of the plane, and let $U$ be the midpoint of $AB$, and $V$ the midpoint of $CD$. Express the vector $\overrightarrow{UV}$ in terms of the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

**2.6.8.** In the $ABCD$ quadrilateral let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{DC} = \mathbf{b}$. Let the points $A, X_1, X_2, X_3, D$ divide the $AD$ side into four equal parts. Let the points $B, Y_1, Y_2, Y_3, C$ divide the $BC$ segment into four equal parts. Express $\overrightarrow{X_1Y_1}$, $\overrightarrow{X_2Y_2}$, $\overrightarrow{X_3Y_3}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

**2.6.9.** Let $ABCDA'B'C'D'$ be a cube ($ABCD$ is a square and $A'B'C'D'$ is a translated copy of $ABCD$). Let $\mathbf{a} = \overrightarrow{AB}, \mathbf{b} = \overrightarrow{AD}, \mathbf{c} = \overrightarrow{AA'}$. Let $P$ be the midpoint of $C'D'$. Let $Q$ be the center of the $BCC'B'$ square. Express $\overrightarrow{AP}$, $\overrightarrow{AQ}$, $\overrightarrow{AD'}$, $\overrightarrow{BD}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

**2.6.10.** Let $ABCD$ be a parallelogram. Let the points $A, X_1, X_2, B$ divide $AB$ into three equal parts. Let $C, Y_1, Y_2, D$ divide $CD$ into three equal parts. Let $X_2, U, V, Y_2$ divide $X_2Y_2$ into three equal parts. Express $\overrightarrow{AV}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

**2.6.11.** Let $P_1, \ldots, P_n$ be points, and $\mu_1, \ldots, \mu_n$ be real numbers with $\sum_{i=1}^{n} \mu_i = 1$. For a point $O$ consider $\mathbf{v} = \sum_{i=1}^{n} \mu_i \overrightarrow{OP_i}$. Let $S$ be the end point of the vector $\mathbf{v}$ if it is measured from $O$. Prove that the point $S$ does not depend on the choice of $O$. (Hint: choose two different $O_1$ and $O_2$ and calculate the vector between the obtained "two" $S$ points. You should get $\mathbf{0}$.)

**2.6.12.** Let the reflection of the point $A$ over the point $B$ be $C$. Express $C$ in terms of $A$ and $B$.

**2.6.13.** (cont.) Let $\triangle ABC$ be a triangle. The reflection of $A$ over $B$ is $A'$. The reflection of $B$ over $C$ is $B'$. The reflection of $C$ over $A$ is $C'$. Prove that the centroid of $\triangle ABC$ and the centroid of $\triangle A'B'C'$ coincide.

**2.6.14.** Let $P$ be different from the vertices of the triangle $\triangle ABC$. Let $PBLC$, $PCMA$, $PANB$ be parallelograms. Prove that the segments $AL$, $BM$, $CN$ bisect each other.

**2.6.15.** Points $P, Q, R$ lie on the sides of the $\triangle ABC$ and are such that
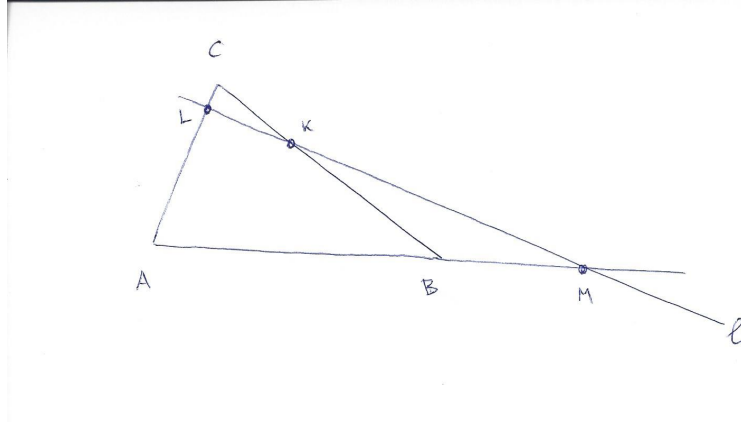
$$(\overrightarrow{BP} : \overrightarrow{PC}) = (\overrightarrow{CQ} : \overrightarrow{QA}) = (\overrightarrow{AR} : \overrightarrow{RB}).$$

Prove that the centroids of the triangles $PQR$ and $ABC$ coincide.

## 2.7   Menelaus' theorem

**Theorem 2.7.1** (Menelaus' theorem). *Let $ABC$ be a triangle and let a transversal line $\ell$ intersect the lines of the sides $AB$, $BC$, $CA$ in $M$, $K$, $L$, respectively. We assume that none of $K, L, M$ coincide with $A, B$, or $C$. Then*

$$(\overrightarrow{AM} : \overrightarrow{MB}) \cdot (\overrightarrow{BK} : \overrightarrow{KC}) \cdot (\overrightarrow{CL} : \overrightarrow{LA}) = -1.$$



*Proof.* We have $K = xB + x'C$, $L = yC + y'A$, $M = zA + z'B$ with $x + x' = y + y' = z + z' = 1$. Since $K, L, M$ are collinear, according to Proposition 2.4.2 we know that there are real numbers $p, q, r$ not all 0 such that $p + q + r = 0$ and $pK + qL + rM = 0$. We have

$$p(xB + x'C) + q(yC + y'A) + r(zA + z'B) = 0,$$

or rearranged, we have

$$(rz + qy')A + (px + rz')B + (qy + px')C = 0. \tag{2.2}$$

The sum of the coefficients in this last expression is

$$(rz + qy') + (px + rz') + (qy + px') = p(x + x') + q(y + y') + r(z + z') = p + q + r = 0. \tag{2.3}$$

According to Corollary 2.4.3 (2.2) and (2.3) can only hold if all three

$$rz + qy' = px + rz' = qy + px' = 0.$$

Therefore we have

$$y'/z = -r/q, \qquad z'/x = -p/r, \qquad x'/y = -q/p$$

(verify that from the hypotheses we know that none of the $x, y, z, p, q, r$ are 0). Therefore

$$(\overrightarrow{AM} : \overrightarrow{MB}) \cdot (\overrightarrow{BK} : \overrightarrow{KC}) \cdot (\overrightarrow{CL} : \overrightarrow{LA}) = \frac{z'}{z} \cdot \frac{x'}{x} \cdot \frac{y'}{y} = \frac{z'}{x} \cdot \frac{x'}{y} \cdot \frac{y'}{z} = \frac{-p}{r} \cdot \frac{-q}{p} \cdot \frac{-r}{q} = -1$$

what we wanted to prove.                                                                    □

**Theorem 2.7.2** (reverse Menelaus' theorem). *Let $ABC$ be a triangle and let $M, K, L$ be points on the lines $AB, BC, CA$ (but different from $A, B, C$) such that*

$$(\overrightarrow{AM} : \overrightarrow{MB}) \cdot (\overrightarrow{BK} : \overrightarrow{KC}) \cdot (\overrightarrow{CL} : \overrightarrow{LA}) = -1. \qquad (2.4)$$

*Then $K, L, M$ are collinear.*

*Proof.* Let $\ell$ be the line connecting $L$ and $K$. We claim that $\ell$ is not parallel with the $AB$ line—this will be proved in Problem 2.7.1 below.
Let the intersection point be $M'$. According to Menelaus' theorem we have

$$(\overrightarrow{AM'} : \overrightarrow{M'B}) \cdot (\overrightarrow{BK} : \overrightarrow{KC}) \cdot (\overrightarrow{CL} : \overrightarrow{LA}) = -1.$$

Comparing this with the assumption (2.4) on $K, L, M$ we conclude that

$$(\overrightarrow{AM'} : \overrightarrow{M'B}) = (\overrightarrow{AM} : \overrightarrow{MB}).$$

Proposition 2.5.3 then implies that $M = M'$, hence the fact that $K, L, M$ are collinear.    □

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**Problems**

**2.7.1.** Let $ABC$ be a triangle and let $M, K, L$ be points on the lines $AB, BC, CA$ (but different from $A, B, C$) such that $(\overrightarrow{AM} : \overrightarrow{MB}) \cdot (\overrightarrow{BK} : \overrightarrow{KC}) \cdot (\overrightarrow{CL} : \overrightarrow{LA}) = -1$. Prove that the line $\ell$ connecting $L$ and $K$ is not parallel with $AB$. [Hint: Problem 2.6.2 may be of use.]

**2.7.2.** Let $D$ be a point on the line of $BC$, and let $E$ be a point on the line of $AC$ of the $\triangle ABC$ (but let $D, E$ be distinct from the vertices). Assume the lines $AD$ and $BE$ intersect in a point $P$. Prove that if $(\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = -1$ then $CP$ is parallel with $AB$.

## 2.8 Barycentric coordinates

**Theorem 2.8.1.** *Let $A$, $B$ and $C$ be non-collinear points in the plane. For any point $P$ we may write*

$$P = xA + yB + zC$$

*where the real coefficients $x, y, z$ satisfy*

$$x + y + z = 1.$$

*Moreover, $x, y, z$ are uniquely determined by the point $P$.*

We call $x, y, z$ the barycentric coordinates of $P$ with respect to the triangle $ABC$.
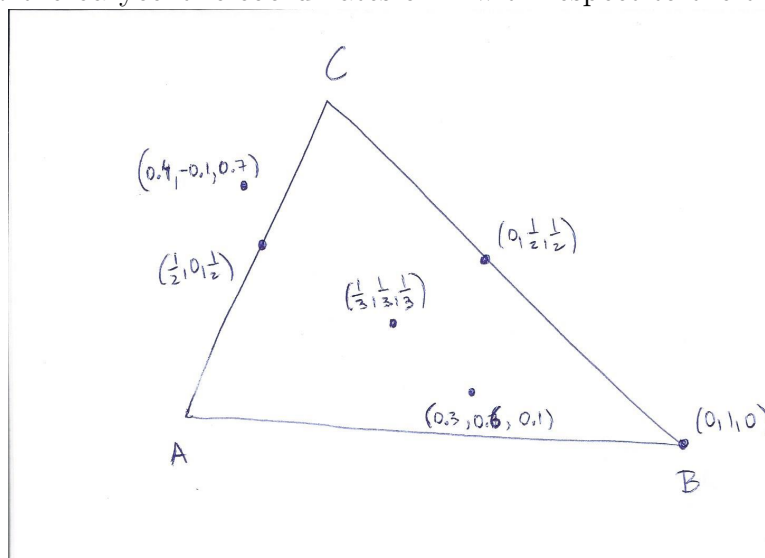


*Proof.* The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel. Hence any vector can be written as a linear combination of them, for example $\overrightarrow{AP} = p\overrightarrow{AB} + q\overrightarrow{AC}$ (cf. Proposition 2.2.2). Using that $\overrightarrow{AP} = P - A$, $\overrightarrow{AB} = B - A$, $\overrightarrow{AC} = C - A$ we can rearrange it to

$$P = (1 - p - q)A + pB + qC,$$

and hence $x = 1 - p - q, y = p, z = q$ satisfy the requirements.

To prove the uniqueness of barycentric coordinates assume that $x, y, z$ and $x', y', z'$ are such that

$$P = xA + yB + zC, \qquad x + y + z = 1,$$
$$P = x'A + y'B + z'C, \qquad x' + y' + z' = 1.$$
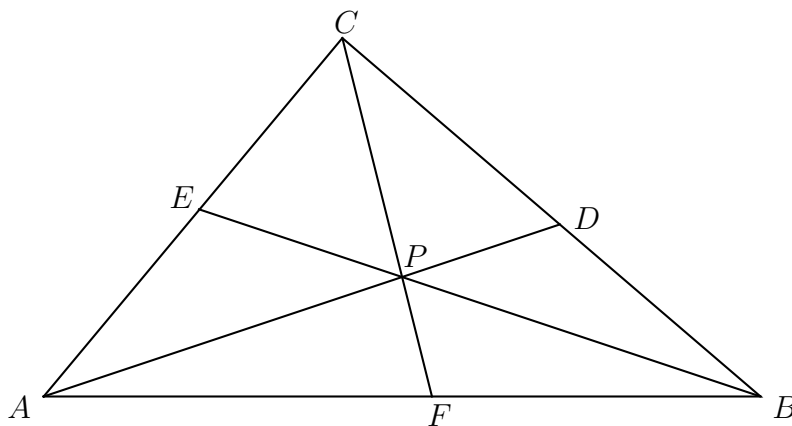
Figure 2.7: The Ceva configuration

Then we have

$$\mathbf{0} = (x - x')A + (y - y')B + (z - z')C, \qquad (x - x') + (y - y') + (z - z') = 0.$$

Corollary 2.4.3 implies that $x - x' = y - y' = z - z' = 0$ which proves uniqueness. $\qquad\square$

**PROJECT 2.** *Observe the similarity between Proposition 2.4.1 and Theorem 2.8.1. They are the 1D and 2D cases of a general n-dimensional theorem. If you learned linear algebra (specifically the notions of linear independence, generating set, basis) then find and prove this general n-dimensional theorem.*

## 2.9  Ceva's theorem

**Theorem 2.9.1** (Ceva's Theorem[1])**.** *Let ABC be a triangle and let P be a point in the plane which does not lie on any of the sides of $\triangle ABC$. Suppose the lines AP, BP and CP meet the opposite sides of ABC at D, E and F, respectively. Then*

$$(\overrightarrow{AF} : \overrightarrow{FB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = 1.$$

Note that $P$ does not need to lie inside the triangle.

*Proof.* Using barycentric coordinates, we write $P$ as

$$P = xA + yB + zC,$$

---

[1]Geometer Giovanni Ceva (1647–1734) is credited with this theorem

where $x + y + z = 1$. From our assumptions it follows that none of $x, y, z, x + y, x + z, y + z$ are 0 (verify!). Let us consider the point

$$V = \frac{x}{x+y}A + \frac{y}{x+y}B. \tag{2.5}$$

This expression implies that $V$ is on the line $AB$. Calculation shows that

$$V = \frac{1}{x+y}P + \frac{-z}{x+y}C,$$

and the sum of the coefficients $1/(x+y)+(-z)/(x+y) = (1-z)/(x+y) = (x+y)/(x+y) = 1$. Hence $V$ is also on the line $CP$. We conclude that the point $V$ is the intersection of $AB$ and $CP$, hence $V = F$. Moreover, from (2.5) we obtain that

$$(\overrightarrow{AF} : \overrightarrow{FB}) = \frac{\frac{y}{x+y}}{\frac{x}{x+y}} = \frac{y}{x}.$$

Similarly, we find that

$$(\overrightarrow{BD} : \overrightarrow{DC}) = \frac{z}{y}, \tag{2.6}$$

$$(\overrightarrow{CE} : \overrightarrow{EA}) = \frac{x}{z}. \tag{2.7}$$

Hence

$$(\overrightarrow{AF} : \overrightarrow{FB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = \frac{y}{x}\frac{z}{y}\frac{x}{z} = 1.$$

$\square$

**Theorem 2.9.2** (Reverse Ceva's theorem). *Suppose $ABC$ is a triangle, $D, E, F$ are points on the lines of the sides (but none of them coincide with a vertex) such that*

$$(\overrightarrow{AF} : \overrightarrow{FB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = 1.$$

*Then $AD$, $BE$, $CF$ are either concurrent, or they are pairwise parallel.*

*Proof.* If $AD$, $BE$, $CF$ are pairwise parallel, then the theorem is proved. Assume that two of these three lines intersect. Without loss of generality we assume that it is $AD$ and $BE$. Let $P = AD \cap BE$, and assume that $CP$ intersects $AB$ in the point $F'$ (verify that our assumpitons imply that $AB$ is not parallel with $CP$!). By Ceva's Theorem 2.9.1 we have

$$(\overrightarrow{AF'} : \overrightarrow{F'B}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = 1.$$

Comparing this with the condition in the theorem we obtain that

$$(\overrightarrow{AF'} : \overrightarrow{F'B}) = (\overrightarrow{AF} : \overrightarrow{FB}).$$

Proposition 2.5.3 then implies that $F = F'$, hence the fact that $AD, BE, CF$ are concurrent.
$\square$

For fun, let us include here another "high-school style" proof of Ceva's Theorem 2.9.1. This proof does not use vectors at all. Instead it uses the notion of area, and the obvious fact that the area of a triangle is half the product of base and height.

*Proof.* For simplicity let $D, E, F$ be on the sides (and not outside) of the triangle $ABC$, and let $P = AB \cap CF = BC \cap AD = CA \cap BE$. The triangles $AFC$ and $FBC$ have "bases" $AF$ and $FB$ and they share the same height $m_C$. Hence the ratio of their areas equals the ratio of their bases:

$$\frac{\text{Area}(AFC)}{\text{Area}(FBC)} = \frac{AF}{FB}.$$

Similar argument for the triangles $AFP$ and $FBP$ gives

$$\frac{\text{Area}(AFP)}{\text{Area}(FBP)} = \frac{AF}{FB}.$$

From the two equations above simple algebra implies

$$\frac{\text{Area}(AFC) - \text{Area}(AFP)}{\text{Area}(FBC) - \text{Area}(FBP)} = \frac{AF}{FB}.$$

The difference of the triangles $AFC$ and $AFP$ is the triangle $CAP$. The difference of the triangles $FBC$ and $FBP$ is the triangle $BCP$. Hence we obtained

$$\frac{\text{Area}(CAP)}{\text{Area}(BCP)} = \frac{AF}{FB}. \tag{2.8}$$

We obtained (2.8) by considering the $AB$ side of the triangle the "base". Repeating the same argument but now considering the $BC$ side or the $CA$ side to be the "base" we obtain the equations.

$$\frac{\text{Area}(ABP)}{\text{Area}(CAP)} = \frac{BD}{DC},$$

$$\frac{\text{Area}(BCP)}{\text{Area}(ABP)} = \frac{CE}{EA}.$$

From the last three equations we obtain

$$(\overrightarrow{AF} : \overrightarrow{FB}) \cdot (\overrightarrow{BD} : \overrightarrow{DC}) \cdot (\overrightarrow{CE} : \overrightarrow{EA}) = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} =$$

$$= \frac{\text{Area}(CAP)}{\text{Area}(BCP)} \cdot \frac{\text{Area}(ABP)}{\text{Area}(CAP)} \cdot \frac{\text{Area}(BCP)}{\text{Area}(ABP)} = 1,$$

which proves Ceva's theorem in the case when $P$ is inside the triangle. Similar arguments work when $P$ is outside. □

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**Problems**

**2.9.1.** Consider the $\triangle ABC$ and non-zero real numbers $k_1, k_2, k_3$. Let $P_{AB}$ and $P'_{AB}$ be on the $AB$ line, the first in the $AB$ segment, the second outside of the $AB$ segment, such that

$$\frac{|AP_{AB}|}{|P_{AB}B|} = \frac{|AP'_{AB}|}{|P'_{AB}B|} = \frac{k_1}{k_2}.$$

Define $P_{BC}$ and $P'_{BC}$ on the $BC$ line similarly with the ratio $k_2/k_3$; and define $P_{CA}$ and $P'_{CA}$ on the $CA$ line similarly with the ratio $k_3/k_1$. Show that the lines $AP_{BC}$, $BP_{CA}$, $CP_{AB}$ are concurrent.

**2.9.2.** (cont.) Prove that the line $P_{CA}P_{AB}$ contains the point $P'_{BC}$.

**2.9.3.** (cont.) Prove that $P'_{AB}$, $P'_{BC}$, and $P'_{CA}$ are collinear.

**2.9.4.** Let $P$ be an interior point of $\triangle ABC$. The lines connecting $P$ with the vertices cut $\triangle ABC$ into six smaller triangles. We color every second of these six triangles with red, the rest with blue. Prove that the product of the areas of the red triangles is the same as the product of the areas of the blue triangles.

## 2.10 Desargues' theorem—a few affine versions

Desargues' theorem is a remarkable theorem on incidences of certain lines and points involving two triangles. The key notions are as follows.

- For $\triangle ABC$ and $\triangle A'B'C'$ we may consider the three lines connecting the corresponding vertexes: $AA'$, $BB'$, and $CC'$. We will consider the condition that these three lines are concurrent (or are pairwise parallel). If concurrent, we call the intersection point the center of perspectivity.

- For $\triangle ABC$ and $\triangle A'B'C'$ we may consider the intersections of the corresponding sides $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$. We will consider the condition that these three points (exist and) are collinear—or none of the three exist. If collinear, we call the obtained line the axis of perspectivity.

**Theorem 2.10.1.** *Let $ABC$ and $A'B'C'$ be triangles such that $AB||A'B'$, $AC||A'C'$, $BC||B'C'$. Then the three lines $AA'$, $BB'$, $CC'$ are either concurrent or pairwise parallel.*

*Proof.* Because of the conditions on parallel lines we can write

$$B - C = k_1(B' - C'), \qquad C - A = k_2(C' - A'), \qquad A - B = k_3(A' - B'), \qquad (2.9)$$

for some real numbers $k_1, k_2, k_3$. Adding together these three equalities (and rearranging the right hand side) we obtain

$$\mathbf{0} = (k_2 - k_3)A' + (k_3 - k_1)B' + (k_1 - k_2)C'. \qquad (2.10)$$

The coefficients of (2.10) add up to 0, hence Corollary 2.4.3 implies that

$$k_2 - k_3 = k_3 - k_1 = k_1 - k_2 = 0, \qquad \text{and hence} \qquad k_1 = k_2 = k_3.$$

Let $k$ be the common value of $k_1, k_2$ and $k_3$. Then from (2.9) we can deduce

$$A - kA' = B - kB' = C - kC'. \qquad (2.11)$$

We can consider two cases. If $k = 1$ then (2.11) implies that the lines $AA'$, $BB'$, $CC'$ are pairwise parallel. If $k \neq 1$ then (2.11) can be rearranged to

$$P = \frac{1}{1-k}A + \frac{-k}{1-k}A' = \frac{1}{1-k}B + \frac{-k}{1-k}B' = \frac{1}{1-k}C + \frac{-k}{1-k}C'$$

showing that the point $P$ is on all three lines $AA'$, $BB'$, $CC'$—proving that $AA'$, $BB'$, $CC'$ are concurrent. $\qquad\square$

**Theorem 2.10.2.** *Let $ABC$ and $A'B'C'$ be triangles such that $AA'$, $BB'$, $CC'$ are concurrent. Assume that the point $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist. Then the points $K, L, M$ are collinear.*

*Proof.* The intersection point on $AA'$, $BB'$, $CC'$ can be written as

$$k_1A + (1 - k_1)A' = k_2B + (1 - k_2)B' = k_3C + (1 - k_3)C'. \qquad (2.12)$$

Rearranging, say, the first equality we obtain

$$k_1 A - k_2 B = (1 - k_2)B' - (1 - k_1)A'.$$

If $k_1 = k_2$ then this formula says $k_1(A - B) = (1 - k_1)(B' - A')$, meaning that $AB||A'B'$ which is not the case. So we know that $k_1 \neq k_2$. Hence, we may divide by $k_1 - k_2$ and write

$$\frac{k_1}{k_1 - k_2}A + \frac{-k_2}{k_1 - k_2}B = \frac{1 - k_2}{k_1 - k_2}B' + \frac{-(1 - k_1)}{k_1 - k_2}A'.$$

The sum of the coefficients on the left hand side is 1, and the sum of the coefficients on the right hand side is also 1 (check it!). Therefore the left hand side expression is a point on the $AB$ line, and the right hand side expression is a point on the $A'B'$ line. Hence the common value must be the intersection $AB \cap A'B'$. We obtained that

$$K = \frac{k_1}{k_1 - k_2}A + \frac{-k_2}{k_1 - k_2}B,$$

equivalently

$$(k_1 - k_2)K = k_1 A - k_2 B. \tag{2.13}$$

We deduced (2.13) from the fact that the first expression and the second expression in (2.12) are equal. Similarly, the fact that the second and third, as well as the first and third expressions in (2.12) are equal we obtain

$$(k_2 - k_3)L = k_2 B - k_3 C, \qquad (k_3 - k_1)M = k_3 C - k_1 A. \tag{2.14}$$

Adding together all three equalities in (2.13) and (2.14) we get

$$\mathbf{0} = (k_1 - k_2)K + (k_2 - k_3)L + (k_3 - k_1)M.$$

Observe that none of the three coefficients are 0, and they add up to 0. According to Proposition 2.4.2 this means that $K, L, M$ are collinear, what we wanted to prove. $\square$

**Theorem 2.10.3.** *Let $ABC$ and $A'B'C'$ be triangles such that $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist and are collinear. Then $AA'$, $BB'$, $CC'$ are either concurrent or are pairwise parallel.*

It is possible to prove this theorem with the techniques we used in the last two proofs—and it may be a good practice for students to write down such a proof. However, for a change we are going to prove it by reduction to Theorem 2.10.2—showing that in some sense Desargues' theorem and its reverse are the same, in other words, Desargues' theorem is "self-dual".

*Proof.* If $AA'$, $BB'$, $CC'$ are pairwise parallel, then we are done. If not, then two of them intersect, say, $AA'$ intersects $BB'$.
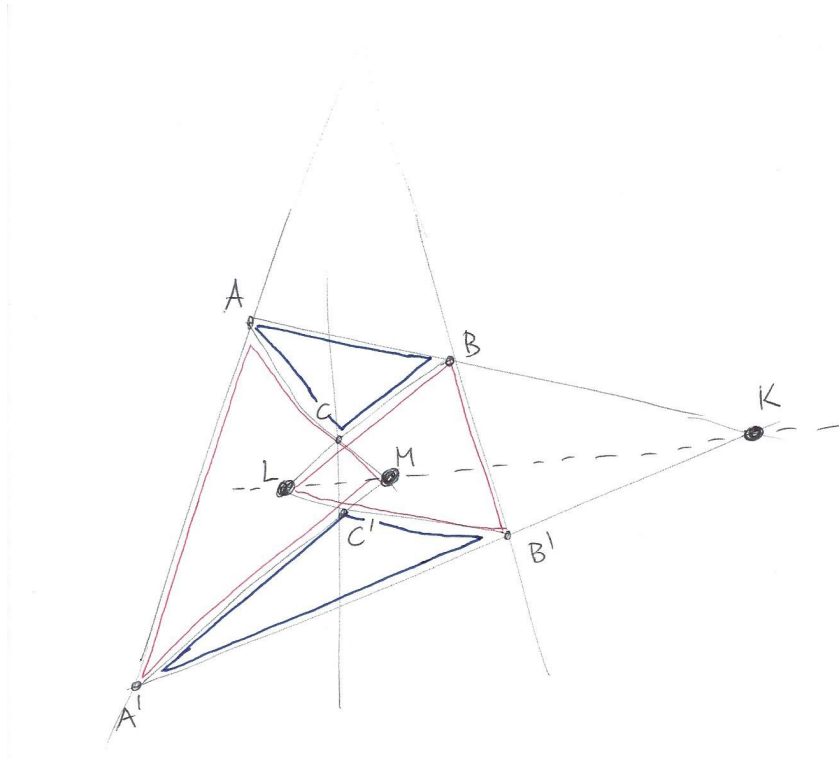


Consider the triangles $AA'M$ and $BB'L$. By looking at the picture one can see that the lines $AB$, $A'B'$, $ML$ connecting the corresponding vertexes are concurrent. Theorem 2.10.2 can be applied to the triangles $AA'M$ and $BB'L$, and we obtain that $C = AM \cap BL$, $C' = MA' \cap LB'$, and $AA' \cap BB'$ are collinear. That is, $CC'$ passes through the intersection of $AA' \cap BB'$, and hence $AA', BB'$, and $CC'$ are concurrent.                    □

In a later chapter we will see a simple and elegant way of phrasing Desargues' theorem—in projective geometry. All of the three theorems above (and more) are some special cases of that projective Desargues' theorem.

**PROJECT 3.** *We can connect two points $A$ and $B$ of the plane if we have a straightedge. Now suppose that $B$ is "hidden", it is only given by portions of two intersecting lines, but we cannot go close to the intersection point $B$; for example it is outside of the margin of our paper. How can we connect $A$ and $B$ with a straightedge? How to connect two hidden points?*

**Problems**

**2.10.1.** Assume that for $\triangle ABC$ and $\triangle A'B'C'$ we have $AA'||BB'||CC'$, and that the points $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist. Prove that $K, L, M$ are collinear.

**2.10.2.** The triangles $\triangle A_1 B_1 C_1$, $\triangle A_2 B_2 C_2$, and $\triangle A_3 B_3 C_3$ have their corresponding sides parallel. Hence each pair of triangles has a center of perspectivity (assume that these centers of perspectivities exist). Prove that the three centers of perspectivities are collinear.

**2.10.3.** Use a search engine to learn about "Monge's theorem". Try to relate it to Problem 2.10.2.

**2.10.4.** A line drawn through the vertex $A$ of a parallelogram $ABCD$ cuts $CB$ in $P$ and $CD$ in $Q$. A line through $C$ cuts $AB$ in $R$ and $AD$ in $S$. Prove that $PR$ and $QS$ are parallel.

## 2.11   Desargues triangles in intersecting planes

In this section we go out of our way again and show an interesting "high-school style" argument in relation with Desargues' theorem.

Consider two planes $P_1$ and $P_2$ in 3 dimensions, intersecting in the line $\ell$. Let $\triangle ABC$ be in $P_1$ and let $\triangle A'B'C'$ be in $P_2$. We will analyse the conditions and the claim of Theorem 2.10.3 for these two triangles.
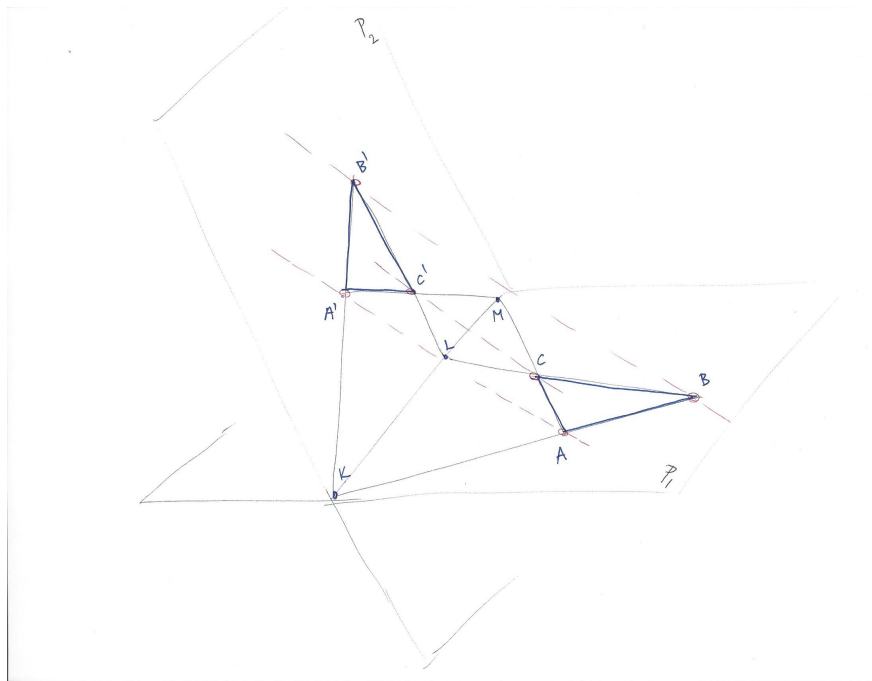
The condition is about the three points $K = AB \cap A'B$, $L = BC \cap B'C'$, $M = CA \cap C'A'$. Observe that $AB \subset P_1$, $A'B' \subset P_2$, hence $K \in P_1 \cap P_2 = \ell$. Similarly, $L$ and $M$ must also lie on $\ell$. So the assumption of Theorem 2.10.3 that $K, L, M$ are collinear does not even have to be assumed! It automatically holds.

The statement of Theorem 2.10.3 is about the three lines $AA'$, $BB'$, $CC'$. It will be useful to consider three more planes. Let $S_K$ be the plane containing the intersecting lines $AB$, $A'B'$. Let $S_L$ be the plane containing the intersecting lines $BC$, $B'C'$. Let $S_M$ be the plane containing the intersecting lines $AC$, $A'C'$.

Observe that both $A$ and $A'$ are contained in $S_K$ and in $S_M$. If two point are contained in two planes, then their connecting line must be the intersection of the two planes. We have $AA' = S_K \cap S_M$. Similarly $BB' = S_K \cap S_L$, and $CC' = S_L \cap S_M$.

We obtained that the three lines $AA'$, $BB'$, $CC'$ are the pairwise intersections of three planes in space. Let's see how can three (pairwise intersecting) planes look like in three space. There are two possibilities: (i) either the third one is parallel with the intersection line of the first two, or (ii) the third one intersects the intersection line of the first two. Theses two configurations are illustrated in the picture.
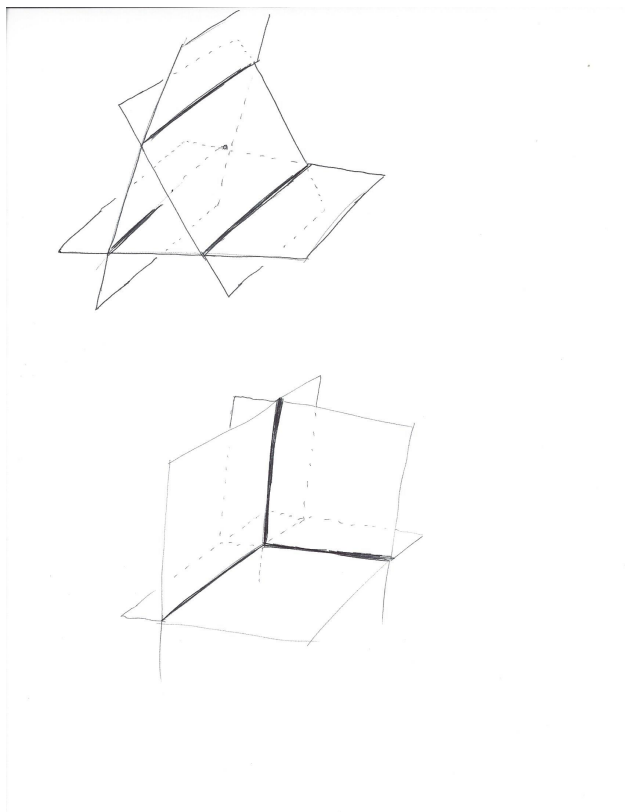
In the first case the three intersection lines are pairwise parallel: $AA'$, $BB'$, $CC'$ are pairwise parallel. In the second case the three intersection lines are concurrent: $AA'$, $BB'$, $CC'$ are concurrent.

What we found is that the 3D version of Desargues theorem 2.10.3 is a tautology.

**PROJECT 4.** *Analyses 3D versions of the other two versions of Desargues' theorem above.*

**PROJECT 5.** *Find a no-calculation proof of e.g. Theorem 2.10.3, by first moving one of the triangles out of plane into 3D.*

## 2.12 Dot product: algebra and geometry

Let us recall the notion of dot product from Calculus. The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a number denoted by $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$.

Geometrically $\mathbf{ab} = |\mathbf{a}||\mathbf{b}| \cos \phi$, where $|\mathbf{x}|$ denotes the length of a vector $\mathbf{x}$ and $\phi$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Especially, $\mathbf{ab} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Algebraically $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$.

*Problem*: recall from calculus why the above geometric and algebraic definition agree. The following properties are easily verified from the algebraic definition.

- $\mathbf{ab} = \mathbf{ba}$

- $(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}$

- $(\lambda\mathbf{a})\mathbf{b} = \lambda(\mathbf{ab})$

- $\mathbf{aa} \geq 0$, and $\mathbf{aa} = 0$ if and only if $\mathbf{a} = \mathbf{0}$

- $\mathbf{ab} = 0$ for all $\mathbf{b}$ implies that $\mathbf{a} = 0$.

The power of dot product that we will repeatedly use in geometry is the duality: its clear geometric meaning and its simple algebraic properties. (What we will not use any further is the $a_1 b_1 + a_2 b_2$ expression.)

**Definition 2.12.1.** The length of a vector $\mathbf{a}$ is defined to be $|\mathbf{a}| = \sqrt{\mathbf{aa}}$. (The square root makes sense because of the the non-negativity property above. Also this definition is consistent with the geometric interpretation of dot product above.) The length of a segment $AB$ is defined to be $d(AB) = |\overrightarrow{AB}|$. The distance of two sets $P, Q \subset \mathbb{R}^2$ is defined to be $\inf\{d(A, B) : A \in P, B \in Q\}$.

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## Problems

**2.12.1.** Verify the algebraic properties of the dot product above.

**2.12.2.** Prove that for a parallelogram the sum of squares of the sides is equal to the sum of the squares of the diagonals.

**2.12.3.** Let $O$ be the center of the $ABCDEF$ regular hexagon whose side length is 1. Find

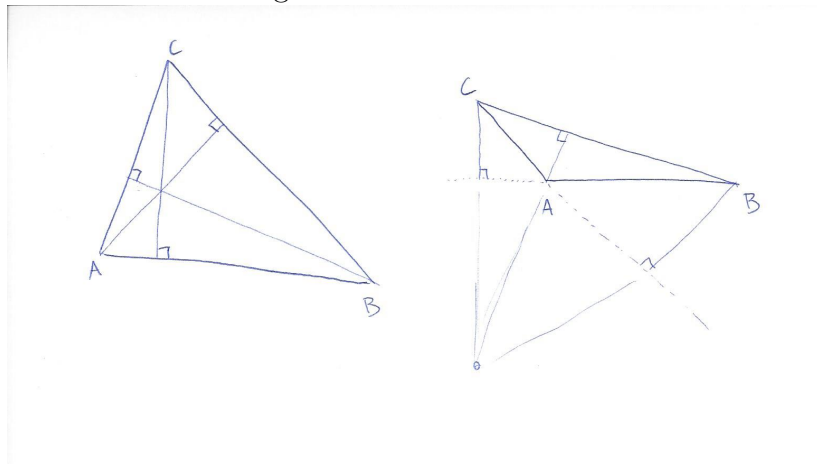$$\overrightarrow{AB} \cdot \overrightarrow{AO}, \overrightarrow{AB} \cdot \overrightarrow{AC}, \overrightarrow{BC} \cdot \overrightarrow{EF}, \overrightarrow{FC} \cdot \overrightarrow{BD}, \overrightarrow{FC} \cdot \overrightarrow{EF}.$$

**2.12.4.** Let $A, B, C, D$ be points in the plane, let $A'$ be the midpoint of $BC$, let $B'$ be the midpoint of $CA$, and let $C'$ be the midpoint of $AB$. Prove that

$$(D - A')(C - B) + (D - B')(A - C) + (D - C')(B - A) = 0.$$

## 2.13   Altitudes of a triangle are concurrent

Let $ABC$ be a triangle. A line passing through the vertex and perpendicular to the opposite side is called an altitude. A triangle has three altitudes.



**Theorem 2.13.1.** *The three altitudes of a triangle are concurrent.*

*Proof.* Let $D$ be the intersection of the altitudes containing the vertexes $A$ and $B$. Then $AD \perp BC$ and $BD \perp AC$. Hence we have

$$(D - A)(B - C) = 0, \qquad (D - B)(C - A) = 0.$$

Adding these two equations together, and using the algebraic properties of dot product we obtain

$$0 = (D - A)(B - C) + (D - B)(C - A) = \ldots = -(D - C)(A - B).$$

Therefore $D - C \perp A - B$, that is, the line $DC$ is the altitude containing $C$. All three altitudes pass through $D$.  $\square$

The intersection of the three altitudes is called the orthocenter of the triangle.
Consider the vertexes and the orthocenter. It is remarkable that each of theses four points is the orthocenter of the triangle formed by the other three points. Such a set of four points will be called an orthocentric tetrad.

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### Problems

**2.13.1.** Suppose that the segment connecting the midpoints of $AB$ with $CD$, and the segment connecting the midpoints of $BC$ with $DA$ are of the same length. Prove that $AC$ is perpendicular to $BD$.

## 2.14    The Feuerbach circle

**Lemma 2.14.1.** *If $A, B, C, D$ is an orthocentric tetrad then*

$$(A + B - C - D)^2 = (A - B + C - D)^2 = (A - B - C + D)^2 =$$

$$(-A + B + C - D)^2 = (-A + B - C + D)^2 = (-A - B + C + D)^2.$$

*Proof.* The six numbers above are in fact three pairs, e.g. $(A + B - C - D)^2$ and $(-A - B + C + D)^2$ are clearly equal, because they are length squares of a vector and its opposite vector. What we need to prove is that two numbers not in the same pair are also equal. Without loss of generality let us choose the first two. Calculation shows that

$$(A + B - C - D)^2 - (A - B + C - D)^2 = 4(A - D)(B - C).$$

Since $A, B, C, D$ is an orthocentric tetrad $A - D$ is orthogonal to $B - C$, and hence $(A - D)(B - C) = 0$, showing that $(A + B - C - D)^2 = (A - B + C - D)^2$. $\square$

**Theorem 2.14.2** (Feuerbach circle)**.** *Let $D$ be the orthocenter of the ABC triangle. Consider the following nine points (a) the midpoints of the sides, (b) the midpoints of the segments connecting vertexes to the orthocenter, (c) the feet of the altitudes. These nine points are on one circle.*
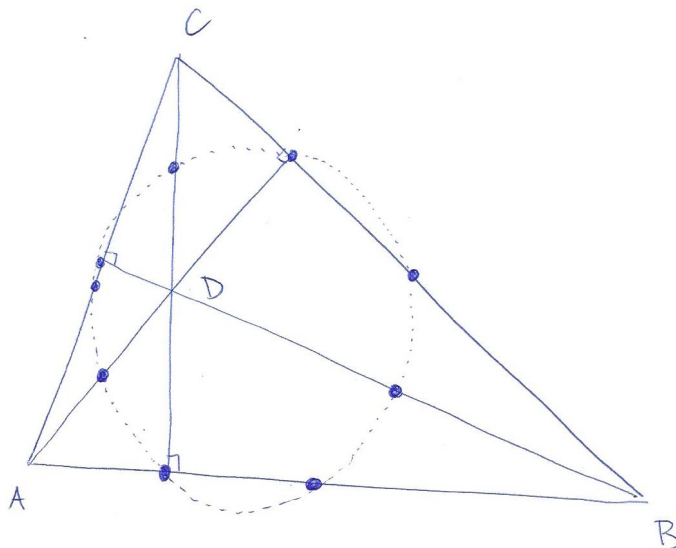
*Proof.* We will only prove that points (a) and (b) are on one circle. The fact that points (c) are also on the same circle is left as an exercise.

Observe that the six points in (a) and (b) are the midpoints of the six segments connecting two of $A, B, C, D$ where $A, B, C, D$ form an orthocentric tetrad. Hence these points are

$$(A + B)/2, (A + C)/2, (A + D)/2, (B + C)/2, (B + D)/2, (C + D)/2.$$

Let $N = (A + B + C + D)/4$. The vectors connecting $N$ to the six points are

$$(-A - B + C + D)/4, (-A + B - C + D)/4, (-A + B + C - D)/4,$$

$$(A - B - C + D)/4, (A - B + C - D)/4, (A + B - C - D)/4.$$

These six vectors have the same length because of Lemma 2.14.1. Therefore all six points are of the same distance from $N$: they are on one circle. $\square$

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### Problems

**2.14.1.** (Thales' theorem) Let $O$ be the midpoint of $AC$. Prove that $\angle(ABC) = \pi/2$ if and only if $d(AO) = d(BO)$. (That is $\angle(ABC) = \pi/2$ if and only if $B$ is on the circle with center $O$ and radius $d(AO)$.)

**2.14.2.** Finish the proof of Feuerbach's Theorem 2.14.2, i.e. prove that points (c) are also on the same circle as points (a) and (b). [Hint: Thales' theorem is useful.]

## 2.15 Angle sum of a triangle

**Lemma 2.15.1.** *Let the line $m$ intersect a pair of parallel lines $\ell$, $\ell'$; and let $\alpha$ and $\alpha'$ be the angles obtained as in Figure 2.8 (a). Then $\alpha = \alpha'$.*

*Proof.* Take unit vectors $\mathbf{x}$, $-\mathbf{x}$ and $\mathbf{u}$ in the lines $\ell$, $\ell'$, and $m$ as in the picture. Then

$$\alpha = \arccos(x \cdot u), \qquad \alpha' = \arccos((-x) \cdot (-u)),$$

so they obviously agree. $\square$

**Theorem 2.15.2.** *The sum of the angles of a triangle is $\pi$.*

*Proof.* Let the line $\ell'$ be parallel to $AB$ and pass through the point $C$, see Figure 2.8 (b). According to Lemma 2.15.1 $\alpha = \alpha'$ and $\beta = \beta'$ hence we have

$$\alpha + \beta + \gamma = \alpha' + \beta' + \gamma = \pi.$$
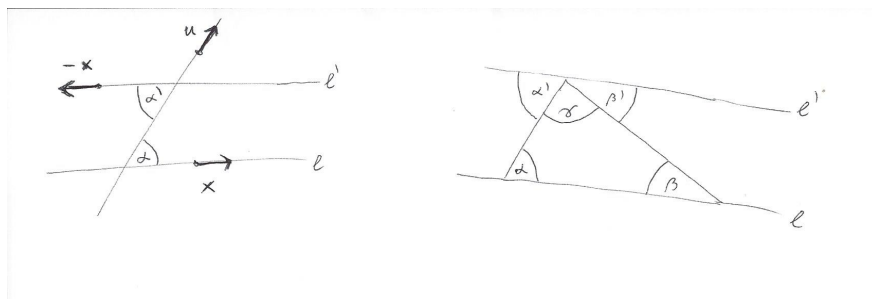
$\square$

Figure 2.8: The sum of the angles of a triangle

## 2.16   Law of cosines, law of sines

For a triangle $\triangle ABC$ the sides opposite to $A, B, C$ will be denoted by $a, b, c$ respectively, and the angles at $A, B, C$ will be called $\alpha, \beta, \gamma$ respectively.

**Theorem 2.16.1** (Law of Cosines). *We have*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

*Proof.* We have $c^2 = \mathbf{c}^2 = (\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{ab} = a^2 + b^2 - 2ab\cos\gamma$. ☐

**Corollary 2.16.2** (Pythagorean theorem). *In a right triangle with hypothenuse $c$ we have* $c^2 = a^2 + b^2$.

*Proof.* This is the Law of Cosines for $\gamma = \pi/2$. ☐

**Corollary 2.16.3** (Triangle inequality). *For three points $A, B, C$ in the plane we have*

$$d(AB) \leq d(BC) + d(CA).$$

The proof follows from the Law of Cosines, details are left as an exercise.

**Theorem 2.16.4** (Law of Sines). *We have*

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

We will give two proves: the first one shows that the Law of Sines is a formal consequence of the Law of Cosines. The second proof is geometric.

*Proof.* Proof1. From the Law of Cosines we get $\cos\gamma = (a^2 + b^2 - c^2)/(2ab)$. Using $\sin\gamma = \sqrt{1 - \cos^2\gamma}$ we have

$$\sin\gamma = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} =$$

$$\sqrt{\frac{4a^2b^2 - (a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2)}{4a^2b^2}} = \frac{\sqrt{-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2}}{2ab}.$$

Dividing both sides by $c$ we obtain

$$\frac{\sin\gamma}{c} = \text{an expression symmetric in } a, b, c.$$

Therefore we will get the *same* expression on the right hand side, if we start with $\alpha$ or $\beta$, not $\gamma$. This proves that $\sin\gamma/c = \sin\alpha/a = \sin\beta/b$. □

*Proof.* Proof2. The altitude $m$ passing through vertex $C$ is a side of two right triangles (See picture) yielding the two expressions: $m = b\sin\alpha$, $m = a\sin\beta$. Putting the right hand sides equal and rearranging gives $\sin\alpha/a = \sin\beta/b$. □

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**Problems**

**2.16.1.** Prove Corollary 2.16.3, the triangle inequality.

## 2.17 Angle bisectors, perpendicular bisectors

Consider an angle $\alpha$ less than $\pi$. The ray inside the angle that cuts $\alpha$ into two angles of measure $\alpha/2$ is called the angle bisector.
The distance $d(P, \ell)$ of a point $P$ to a line $\ell$ is the the infimum of the distances between $P$ and $A$ where $A \in \ell$.

**Lemma 2.17.1.** *The distance of a point to a line is obtained on the perpendicular segment dropped from the point to the line.*

*Proof.* The Pythagorean theorem proves that $x < x'$ in the figure on the left. □

**Lemma 2.17.2.** *The points in an angle that are of the same distance from the two rays of the angle are exactly the points of the angle bisector.*

(a)                                    (b)
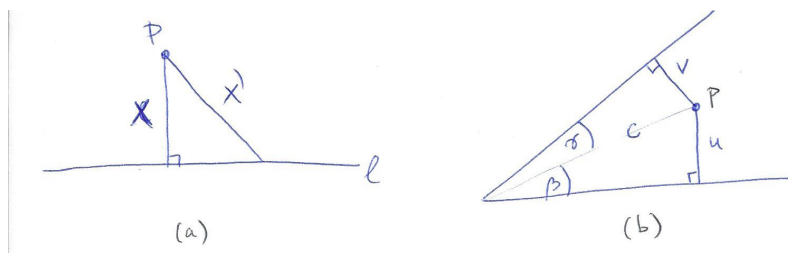
*Proof.* For a point $P$ as in the figure on the right, its distance to the two sides is $u$ and $v$ according to Lemma 2.17.1. We have $u = c\sin(\beta)$, $v = c\sin(\gamma)$. Hence $u = v$ holds if and only if $\sin(\beta) = \sin(\gamma)$. Well known properties of the sin function imply that this holds if and only if $\beta = \gamma$.                                                                                 $\square$
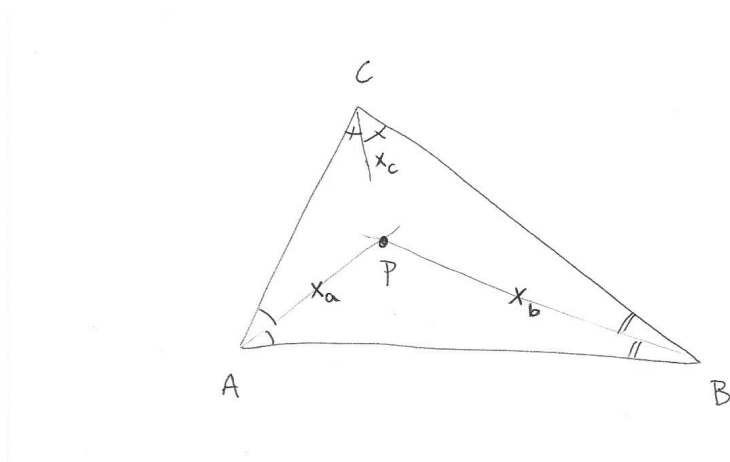
**Theorem 2.17.3.** *The three angle bisectors of a triangle are concurrent.*



*Proof.* Let $x_a, x_b, x_c$ be the angle bisectors through the vertexes $A, B, C$. Let $P$ be the intersection of the $x_a$ and $x_b$ (verify that they cannot be parallel!). Then

$$\left. \begin{array}{lll} P \in x_a & \Rightarrow & d(P,b) = d(P,c) \\ P \in x_b & \Rightarrow & d(P,a) = d(P,c) \end{array} \right\} \Rightarrow d(P,a) = d(P,b) \Rightarrow P \in x_c,$$

where three of the four $\Rightarrow$ implications above use Lemma 2.17.2. Since the intersection of $x_a$ and $x_b$ is on $x_c$, we have that $x_a, x_b, x_c$ are concurrent.                              $\square$

A byproduct of the theorem is that the intersection of the angle bisectors has the same distance to the sides. In other words there is a circle with this center that touches the sides of the triangle: the so-called circle inscribed in the triangle.
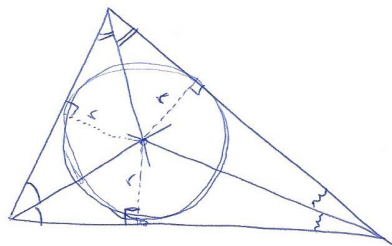
For a segment $AB$, the line passing through the midpoint of $AB$ and perpendicular to $AB$ is called the perpendicular bisector.

**Theorem 2.17.4.** *The three perpendicular bisectors of the sides of a triangle are concurrent.*

The proof is left as an exercise.

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**Problems**

**2.17.1.** Find and prove a lemma analogous to Lemma 2.17.2 but it it about the perpendicular bisector of a segment.

**2.17.2.** Using Problem 2.17.1 find a proof of Theorem 2.17.4 (logically similar to the proof of Theorem 2.17.3).

**2.17.3.** Find a byproduct of your proof in Problem 2.17.2, analogous to the byproduct of the proof of Theorem 2.17.3.

**2.17.4.** Prove that a triangle is equilateral if two of its circumcenter, centroid, and orthocenter coincide.

**2.17.5.** Let $\mathcal{C}$ be a circle in the plane, and let $\lambda$ be a number. Prove that $\lambda\mathcal{C} = \{\lambda x \in \mathbb{R}^2 : x \in \mathcal{C}\}$ is also a circle.

**2.17.6.** (cont.) Reflect the orthocenter of $\triangle ABC$ over the midpoints of the sides. Prove that the obtained three points are on the circumscribed circle. (Hint: use Problem 2.17.5 for a well chosen origin and $\lambda$, as well as the Feuerback circle.)

**2.17.7.** Let $O$ be the center of circumscribed circle of $\triangle ABC$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vectors pointing from $O$ to the vertexes. Let $M$ be the endpoint of $\mathbf{a} + \mathbf{b} + \mathbf{c}$ measured from $O$. Prove that $M$ is the orthocenter of $\triangle ABC$.

**2.17.8.** (cont.) Let $O$ be the center of the circumscribed circle of $\triangle ABC$. Let $M$ be the orthocenter, and let $S$ be the centroid of $\triangle ABC$. Prove that $O, M, S$ are collinear. (The obtained line is called the Euler-line of $\triangle ABC$.) Find $\overrightarrow{OS}/\overrightarrow{SM}$.

**2.17.9.** Let $D$ be a point on the side $BC$ of $\triangle ABC$. Prove that

$$\frac{BD}{DC} = \frac{|AB|\sin(DAB\angle)}{|AC|\sin(DAC\angle)}.$$

**2.17.10.** (cont.) Prove the Angle Bisector Theorem: If the angle bisector from $A$ intersects $BC$ in $D$, then

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}.$$

**2.17.11.** (cont.) Reprove that angle bisectors are concurrent by using the Angle Bisector Theorem and Ceva's theorem.

**2.17.12.** Let $AB = 6$, $BC = 10$ in the $\triangle ABC$. The angle bisector at $B$ intersects $AC$ in $D$. Connect $D$ with the midpoint of $AB$, let the intersection point with $BC$ be $E$. What is the length $|BE|$?

**2.17.13.** Let the lengths of the sides of $\triangle ABC$ be $a, b, c$ ($a$ is opposite of $A$, etc). If $O$ is the center of the inscribed circle then prove that

$$O = \frac{aA + bB + cC}{a + b + c}.$$
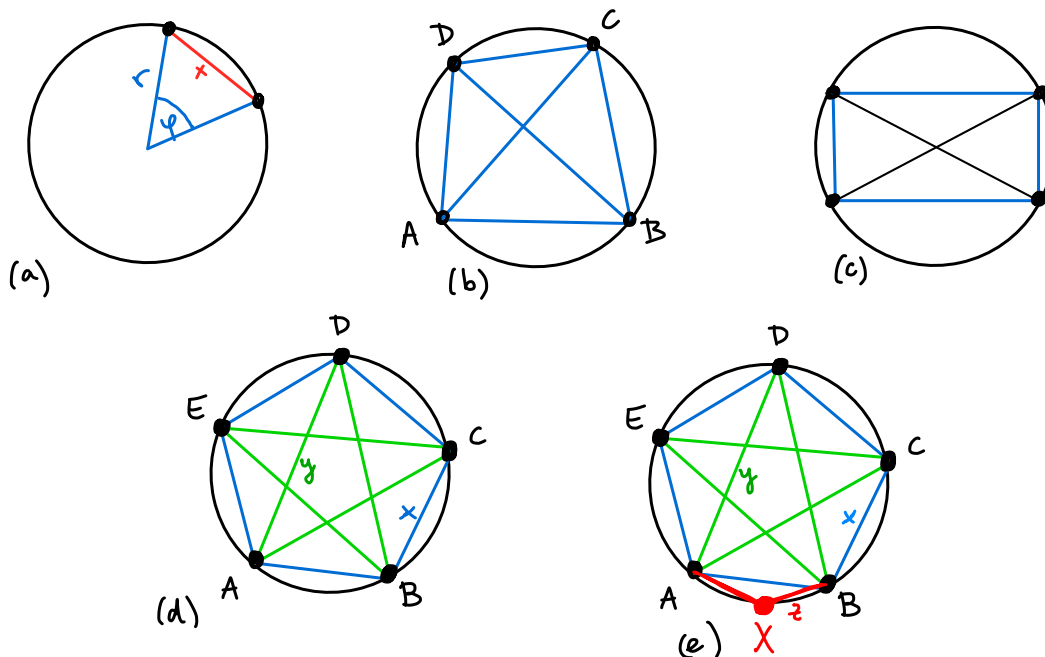
**2.17.14.** In Figure (a) above find the length $x$ in terms of $r$ and $\varphi$.

**2.17.15.** [Ptolemy's theorem] Let $A$, $B$, $C$, $D$ be points on a circle in this order. Use Problem 2.17.14 and trigonometric identities to prove

$$d(A, B)d(C, D) + d(A, D)d(B, C) = d(A, C)d(B, D),$$

see Figure (b).

**2.17.16.** Which familiar theorem does Ptolemy's theorem reduce to if $A, B, C, D$ is a rectangle? (See Figure (c).)

**2.17.17.** Apply Ptolemy's theorem to four vertices of a regular pentagon (see Figure (d)), and find the ratio $y/x$. (Is this number familiar from somewhere else?)

**2.17.18.** Find the ratio $d/z$ where $d$ is the diameter of the circle, and $z$ is the side of a regular 10-gon (see Fidure (e)). [Hint: use Ptolemy's theorem for the $AXBD$ quadrilateral.

## 2.18    Rotation, applications

For $\alpha$ an angle and $\mathbf{v} = \overrightarrow{OA}$ a plane vector let $R_\alpha(\mathbf{v})$ denote the vector obtained from rotating $\mathbf{v} = \overrightarrow{OA}$ around the origin in the counterclockwise direction, see Picture 2.9 (a). Thus $R_\alpha$ is a map from vectors to vectors. Calculation shows that algebraically

$$R_\alpha : \langle x, y \rangle \mapsto \langle x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha \rangle.$$

If a vector is given by an arrow $\mathbf{v} = \overrightarrow{PQ}$ where $P$ is not the origin, then to get $R_\alpha(\mathbf{v})$ we formally need to translate $\overrightarrow{PQ}$ to $\overrightarrow{OA}$, then rotate this $\overrightarrow{OA}$ by $\alpha$. Figure 2.9 (b) shows that this procedure is not necessary: $R_\alpha(\mathbf{v})$ is also obtained by rotating $\overrightarrow{PQ}$ around $P$ by $\alpha$.



Figure 2.9: Rotations

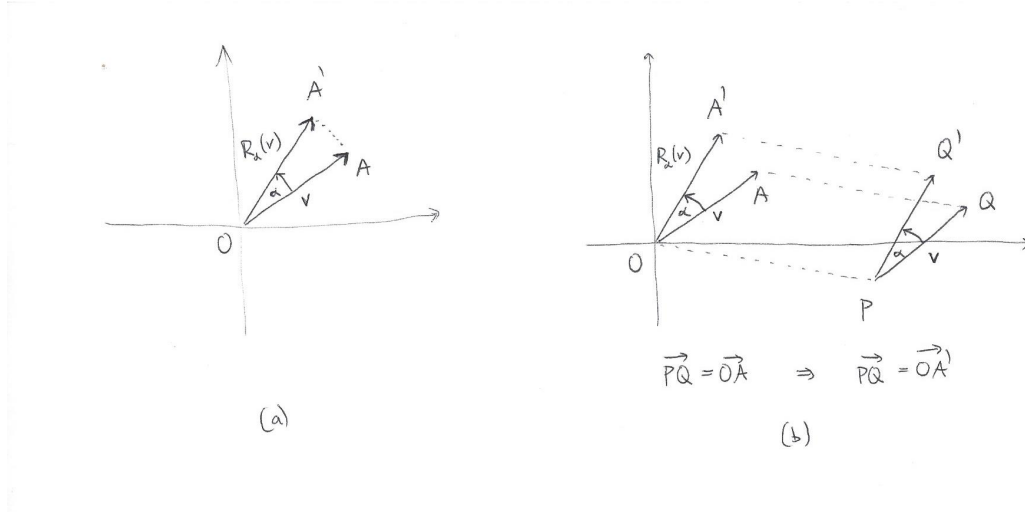**Proposition 2.18.1.** *The rotation operator on vectors is consistent with vector operations as follows,*

- $R_\alpha(\mathbf{a} + \mathbf{b}) = R_\alpha(\mathbf{a}) + R_\alpha(\mathbf{b})$,

- $R_\alpha(\mathbf{a} - \mathbf{b}) = R_\alpha(\mathbf{a}) - R_\alpha(\mathbf{b})$,

- $R_\alpha(\lambda\mathbf{a}) = \lambda R_\alpha(\mathbf{a})$,

- $R_\alpha(\mathbf{a})R_\alpha(\mathbf{b}) = \mathbf{a}\mathbf{b}$,

- $R_\beta(R_\alpha(\mathbf{a})) = R_{\alpha+\beta}(\mathbf{a})$.

*Proof.* All can be calculated from the algebraic description of the operations. □

**Theorem 2.18.2.** *Let $\triangle ABC$ be a triangle, and let $T1$ and $T2$ squares on the sides $AC$ and $BC$ outside the triangle. Let $K$ and $L$ be the centers of $T1$ and $T2$, and $X$ is the midpoint of $AB$. Then $XK$ and $XL$ have the same length.*
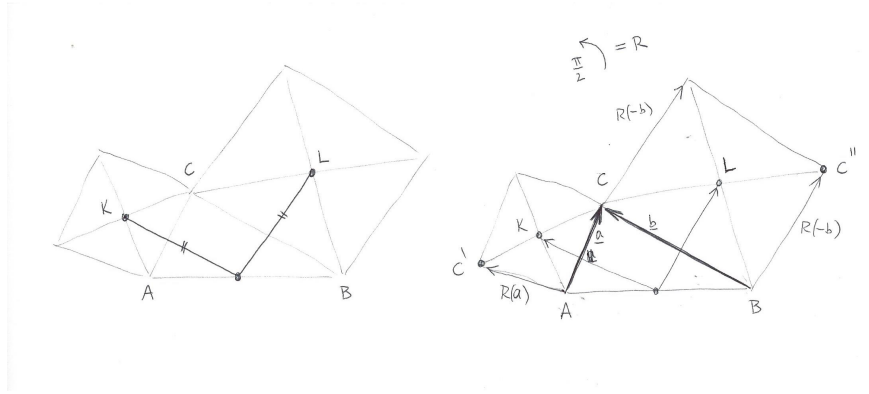


*Proof.* Let $\mathbf{a} = \overrightarrow{AC}$, $\mathbf{b} = \overrightarrow{BC}$, and let $R = R_{\pi/2}$ be the rotation by $\pi/2$ operator. Observe that we can express all relevant vectors in our picture using $\mathbf{a}, \mathbf{b}, R$: for example $\overrightarrow{AC'} = R(\mathbf{a})$, $\overrightarrow{BC''} = -R(\mathbf{b})$.
We have

$$\overrightarrow{XK} = \overrightarrow{XA} + \overrightarrow{AK} = \frac{\overrightarrow{BA}}{2} + \frac{\overrightarrow{AC} + \overrightarrow{AC'}}{2} = \frac{-\mathbf{a} + \mathbf{b}}{2} + \frac{\mathbf{a} + R(\mathbf{a})}{2} = \frac{R(\mathbf{a}) + \mathbf{b}}{2},$$

$$\overrightarrow{XL} = \overrightarrow{XB} + \overrightarrow{BL} = \frac{\overrightarrow{AB}}{2} + \frac{\overrightarrow{BC} + \overrightarrow{BC''}}{2} = \frac{-\mathbf{b} + \mathbf{a}}{2} + \frac{\mathbf{b} - R(\mathbf{b})}{2} = \frac{\mathbf{a} - R(\mathbf{b})}{2}.$$

The idea of the proof is that we suspect that not only $XK$ and $XL$ are of the same length but one is the $\pi/2$ rotation of the other. Hence we calculate

$$R(\overrightarrow{XL}) = R\left(\frac{\mathbf{a} - R(\mathbf{b})}{2}\right) = \frac{R(\mathbf{a}) - R(R(\mathbf{b}))}{2}.$$

Observe that applying $R$ twice on a vector is the same as multiplication by $-1$. Indeed, $R_{\pi/2}R_{\pi/2} = R_\pi = $multiplication by $(-1)$. Hence

$$R(\overrightarrow{XL}) = \frac{R(\mathbf{a}) + \mathbf{b}}{2} = \overrightarrow{XK},$$

what we wanted to prove. □

The proof above is not the shortest or most elegant proof, but illustrates the main point: naming sufficient vectors and operations (but not more) that determine the picture we can express any other vectors in terms of the named ones. Then we can make comparisons among any two. A more "elegant" version of the same proof will be given in Problem 2.18.3.

**Theorem 2.18.3** (Napoleon Bonaparte[2]). *Let $\triangle ABC$ be an arbitrary triangle and let $T_a$, $T_b$, $T_c$ by regular (a.k.a. equilateral) triangles on the sides of $a, b, c$, outside of $ABC$. Let $A', B', C'$ be the centers of $T_a$, $T_b$, $T_c$. Then $\triangle A'B'C'$ is a regular triangle.*
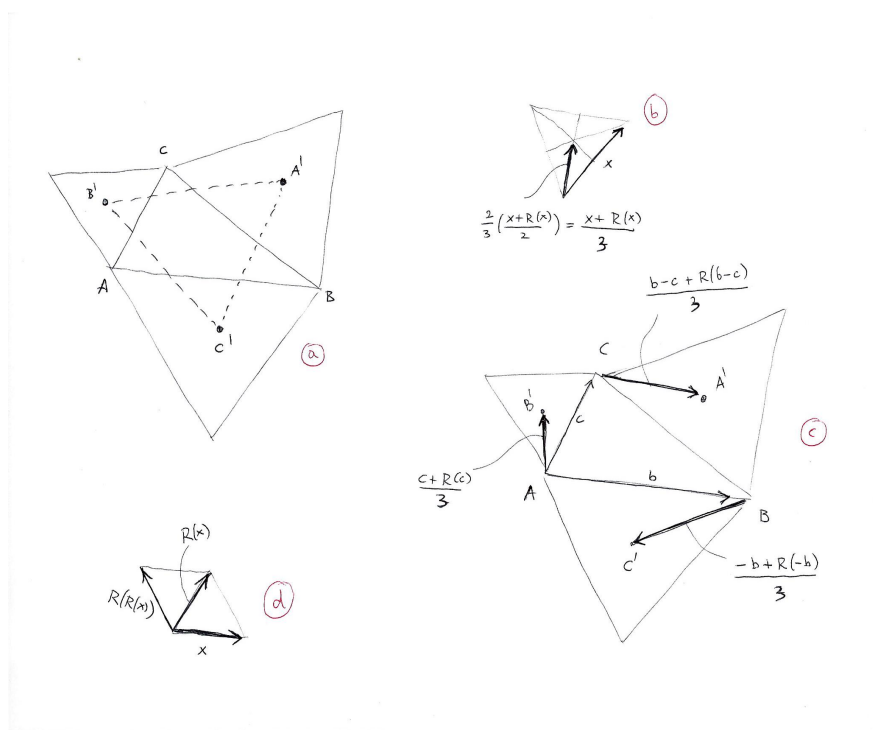


Figure 2.10: Napoleon's theorem

*Proof.* Let $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$, and let $R = R_{\pi/3}$ be the rotation by $\pi/3$ operator. Our goal is to express relevant vectors, namely $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ in terms of these.

---

[2]yes, him

First observe that if $\mathbf{x}$ is a side vector of a regular triangle then the vector pointing from a vertex to its center as drawn in Picture 2.10 (b) is $(\mathbf{x} + R(\mathbf{x}))/3$. Therefore we have

$$\overrightarrow{AB'} = \frac{\mathbf{c} + R(\mathbf{c})}{3}, \qquad \overrightarrow{CA'} = \frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3}, \qquad \overrightarrow{BC'} = \frac{-\mathbf{b} + R(-\mathbf{b})}{3},$$

see Picture 2.10 (c). Now we can express

$$\overrightarrow{A'B'} = -\frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3} - \mathbf{c} + \frac{\mathbf{c} + R(\mathbf{c})}{3} = \frac{-1}{3}\mathbf{b} - \frac{1}{3}\mathbf{c} - \frac{1}{3}R(\mathbf{b}) + \frac{2}{3}R(\mathbf{c}),$$

$$\overrightarrow{A'C'} = -\frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3} - \mathbf{c} + \mathbf{b} + \frac{-\mathbf{b} + R(-\mathbf{b})}{3} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c} - \frac{2}{3}R(\mathbf{b}) + \frac{1}{3}R(\mathbf{c}).$$

What we want to prove is $R(\overrightarrow{A'B'}) = \overrightarrow{A'C'}$ so let us calculate

$$R(\overrightarrow{A'B'}) = R\left(\frac{-1}{3}\mathbf{b} - \frac{1}{3}\mathbf{c} - \frac{1}{3}R(\mathbf{b}) + \frac{2}{3}R(\mathbf{c})\right) = \frac{-1}{3}R(\mathbf{b}) - \frac{1}{3}R(\mathbf{c}) - \frac{1}{3}R(R(\mathbf{b})) + \frac{2}{3}R(R(\mathbf{c})).$$

Looking at Picture 2.10 (d) we see that $R(\mathbf{x}) = \mathbf{x} + R(R(\mathbf{x}))$, and hence $R(R(\mathbf{x})) = R(\mathbf{x}) - \mathbf{x}$. We further have

$$R(\overrightarrow{A'B'}) = \frac{-1}{3}R(\mathbf{b}) - \frac{1}{3}R(\mathbf{c}) - \frac{1}{3}(R(\mathbf{b}) - \mathbf{b}) + \frac{2}{3}(R(\mathbf{c}) - \mathbf{c}) = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c} - \frac{2}{3}R(\mathbf{b}) + \frac{1}{3}R(\mathbf{c}).$$

This last expression is the same as the expression for $\overrightarrow{A'C'}$ above, hence $R(\overrightarrow{A'B'}) = \overrightarrow{A'C'}$ what we wanted to prove. $\qquad\square$

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### Problems

**2.18.1.** Let $R$ denote the rotation by $30^o$ $(= \pi/6)$ counterclockwise. Simplify the expressions $R^{12}(\mathbf{x})$, $R^{13}(\mathbf{x})$, $R^{100}(\mathbf{x})$, $R^6(\mathbf{x})$. Consider the polynomial $p(x) = x^6 + x^0$. What is $p(R)(\mathbf{x})$ for any $\mathbf{x}$? Consider the polynomial $p(x) = x^4 - x^2 + x^0$. What is $p(R)(\mathbf{x})$ for any $\mathbf{x}$?

**2.18.2.** Let $R$ denote the rotation by $\pi/9$. Write down a (non-zero) polynomial $p(R)$ in $R$, such that for all $\mathbf{x}$ we have $p(R)(\mathbf{x}) = \mathbf{0}$. Let $R$ denote the rotation by $\pi/10$. Write down a (non-zero) polynomial $p(R)$ in $R$, such that for all $\mathbf{x}$ we have $p(R)(\mathbf{x}) = \mathbf{0}$. Let $R$ denote the rotation by $\pi/12$. Write down a (non-zero) polynomial $p(R)$ in $R$, such that for all $\mathbf{x}$ we have $p(R)(\mathbf{x}) = \mathbf{0}$.

**PROJECT 6.** *Try to find the smallest degree polynomials for each of the three $R$'s in the exercise above. Read about "cyclotomic polynomials."*

**2.18.3.** Consider the situation of Theorem 2.18.2. Express the vector $\overrightarrow{XL}$ in terms of the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC''}$ (e.g. using Problem 2.6.7). Express the vector $\overrightarrow{XK}$ in terms of the vectors $\overrightarrow{AC'}$ and $\overrightarrow{BC}$. Look at the expression you got for $\overrightarrow{XL}$, and term by term apply the rotation $R$. You should get your expression for $\overrightarrow{XK}$. Write down the obtained proof for Theorem 2.18.2.

**2.18.4.** Let $P, Q, R, S$ be the centers of the squares that are described externally on the sides of a quadrilateral (in this order). Prove that $PR$ and $QS$ are of the same length, and are perpendicular to each other.

## 2.19   Wedge product of two plane vectors

In Calculus we learn the geometry and the algebra of the notion of cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ in 3-space. Geometrically $\mathbf{a} \times \mathbf{b}$ has length $|\mathbf{a}||\mathbf{b}| \sin \theta$ (where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$), it lies in the line orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ and its direction satisfies the right-hand rule. Algebraically

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Plane vectors $\langle a_1, a_2 \rangle$ can be considered space vectors by $\langle a_1, a_2, 0 \rangle$. If we take the cross product of two such plane-space vectors then we obtain a vector of the form $\langle 0, 0, * \rangle$. We do not want to keep carrying the $(0, 0)$-part, hence we give a new definition capturing only the third coordinate.

**Definition 2.19.1.** The wedge product $\mathbf{a} \wedge \mathbf{b}$ of two plane vectors $\mathbf{a} = \langle a_1, a_2 \rangle, \mathbf{b} = \langle b_1, b_2 \rangle$ is the third coordinate of the cross product $\langle a_1, a_2, 0 \rangle \times \langle b_1, b_2, 0 \rangle$.

From the arguments above we obtain that

- (algebra) $\langle a_1, a_2 \rangle \wedge \langle b_1, b_1 \rangle = a_1 b_2 - a_2 b_1$,

- (geometry) $\mathbf{a} \wedge \mathbf{b} = \pm |\mathbf{a}||\mathbf{b}| \sin \theta$. Since $|\mathbf{a}||\mathbf{b}| \sin \theta$ is the area of a the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ (see figure below), we have

$$\mathbf{a} \wedge \mathbf{b} = \pm \text{Area(parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}).$$

  Analysing the right-hand rule mentioned above we can determine wether + or - stand in the formula above: if the direction of $\mathbf{b}$ is obtained from the direction of $\mathbf{a}$ by a counterclockwise rotation by not more than $\pi$ then the sign is positive, otherwise negative.
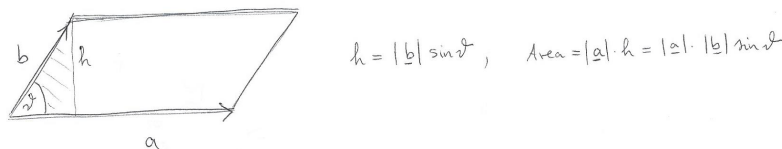
The algebraic interpretation easily proves the following properties.

**Proposition 2.19.2.** *We have*

(i) *(anti-symmetry)* $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, $\mathbf{a} \wedge \mathbf{a} = 0$,

(ii) *(bilinearity)*

$$(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}, \qquad \mathbf{c} \wedge (\mathbf{a} + \mathbf{b}) = \mathbf{c} \wedge \mathbf{a} + \mathbf{c} \wedge \mathbf{b},$$

$$(\lambda \cdot \mathbf{a}) \wedge \mathbf{b} = \lambda \cdot \mathbf{a} \wedge \mathbf{b}, \qquad \mathbf{a} \wedge (\lambda \cdot \mathbf{b}) = \lambda \cdot \mathbf{a} \wedge \mathbf{b}.$$

(iii) *We have* $\mathbf{a} \wedge \mathbf{b} = 0$ *if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*

Again, the power of this operation is the duality between the properties just listed (proved by algebra) and the geometric interpretation. Here is an application of the the wedge product.

**Proposition 2.19.3.** *The points* $A, B, C$ *are collinear if and only if* $A \wedge B + B \wedge C + C \wedge A = 0$.

*Proof.* The points $A, B, C$ are collinear if and only if the vectors $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are in one line. They are in one line if and only if their spanned parallelogram degenerates to a segment, i.e. has area 0. Hence $A, B, C$ are in one line if and only if

$$0 = (A - B) \wedge (B - C) = A \wedge B - A \wedge C - B \wedge B + B \wedge C = A \wedge B + B \wedge C + C \wedge A.$$

$\square$

Before Proposition 2.19.3 our algebraic interpretations of collinearity were Propositions 2.4.1 and 2.4.2. All the theorems proved using those two propositions have alternative proofs using our new algebraic interpretation Proposition 2.19.3. We will not reprove earlier theorems though (students may find it a good exercise), but rather give some new incidence theorem, and as a change we will prove them using Proposition 2.19.3.

**Theorem 2.19.4** (Newton-Gauss line)**.** *Let* $a, b, c, d$ *be four pairwise intersecting lines (this configuration is called a complete quadrilateral). The pairwise intersections are six points. Three pairs of these six points are not connected by the lines* $a, b, c, d$, *these are called diagonals (PS, RQ, UV in the picture). The midpoints of the three diagonals are collinear.*
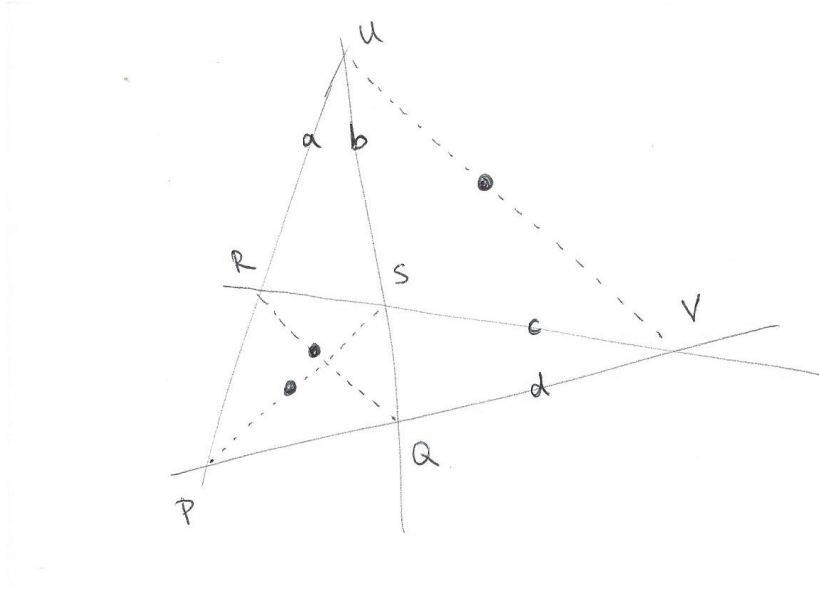
*Proof.* Using the notation of the picture consider the sum of the following twelve terms

$$
\begin{array}{cccc}
P \wedge R & P \wedge Q & S \wedge R & S \wedge Q \\
R \wedge U & Q \wedge V & R \wedge V & Q \wedge U \\
U \wedge P & V \wedge P & V \wedge S & U \wedge S
\end{array}
$$

We will view this sum in two different ways.

First: The sum of the terms in each column is zero, because the triples of points $(P, R, U)$, $(P, Q, V)$, $(S, R, V)$, $(S, Q, U)$ are collinear, see Proposition 2.19.3. Hence the total sum is zero.

Second: The first row is 4 times $(P + S)/2 \wedge (R + Q)/2$, the second row is 4 times $(R + Q)/2 \wedge (U + V)/2$. The third row is 4 times $(U + V)/2 \wedge (P + S)/2$.

We conclude that

$$
\frac{P + S}{2} \wedge \frac{R + Q}{2} + \frac{R + Q}{2} \wedge \frac{U + V}{2} + \frac{U + V}{2} \wedge \frac{P + S}{2} = 0,
$$

and hence according to Proposition 2.19.3 $(P+S)/2$, $(R+Q)/2$, $(U+V)/2$ are collinear.   $\square$

**Theorem 2.19.5** ("parallel case of Pappus' theorem"). *Let $A, B, C$ be collinear points and let $A', B', C'$ be collinear. Then two of*

$$
AB' \| A'B, \qquad BC' \| B'C, \qquad AC' \| A'C
$$

*imply the third.*
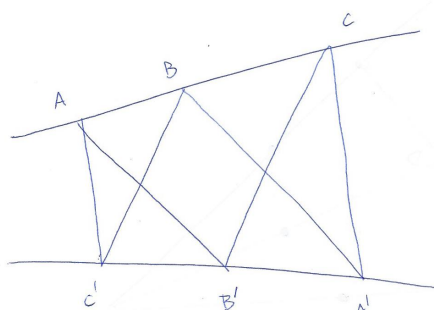
*Proof.* Consider the following three numbers

$$(A - B') \wedge (A' - B), \qquad (B - C') \wedge (B' - C), \qquad (C - A') \wedge (C' - A).$$

The vanishing of these three numbers is equivalent to the three parallelity conditions of the theorem, according to Proposition 2.19.2(iii).

However, one can distribute the *sum* of these three numbers, use the anti-symmetry property of $\wedge$ and conclude that the total sum is 0. Hence the vanishing of two of them indeed implies the vanishing of the third one. $\qquad\qquad\square$

We will learn more on Pappus' theorem in Section 3.

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**Problems**

**2.19.1.** In the $\triangle ABC$ let the side lengths be $a = 5, b = 6, c = 7$. Let $K$ be a point on the $AB$ side with distance 2 from $A$. Let $L$ be a point on the $AC$ side with distance 3 from $A$. What is the ratio of the areas of $\triangle ABC$ and $\triangle AKL$?

**2.19.2.** Let $D, E, F$ be points on the sides $AB$, $BC$, $CA$ of a triangle, dividing the sides in the ratios $k_1 : 1$, $k_2 : 1$, $k_3 : 1$. Show that

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABC)} = \frac{1 + k_1 k_2 k_3}{(1 + k_1)(1 + k_2)(1 + k_3)}.$$

## 2.20 A 3D view of plane geometry, triple product

In Calculus we learned the geometry and algebra of the triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ of three *space* vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. It will be convenient to use the following notation $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, and we may call it the triple product, or triple wedge product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. In particular the following hold

- $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$;

- $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is plus or minus the volume of the parallelepiped spanned by the space vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$;

- $$(a_1, a_2, a_3) \wedge (b_1, b_2, b_3) \wedge (c_1, c_2, c_3) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix};$$

- (3-linearity) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is linear in each of the variables. Linearity in the first variable means

$$(\mathbf{a} \pm \mathbf{a}') \wedge \mathbf{b} \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \pm \mathbf{a}' \wedge \mathbf{b} \wedge \mathbf{c},$$

$$(\lambda \mathbf{a}) \wedge \mathbf{b} \wedge \mathbf{c} = \lambda \cdot \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c},$$

and linearity in the second and third variables are similar;

- (antisymmetry) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{a} = \mathbf{c} \wedge \mathbf{a} \wedge \mathbf{b} = -\mathbf{a} \wedge \mathbf{c} \wedge \mathbf{b} = -\mathbf{c} \wedge \mathbf{b} \wedge \mathbf{a} = -\mathbf{b} \wedge \mathbf{a} \wedge \mathbf{c}$.

In view of the second property above it makes no sense of considering the triple product of vectors lying in the $(x, y, 0)$ plane. A useful tool, however is considering our plane as the $z = 1$ plane in 3-space. That is, if a point was $(a_1, a_2)$ earlier, now we consider it as $(a_1, a_2, 1)$.

What we gained is a new operation: we can form the triple product $A \wedge B \wedge C$ for three points. What we lost is that we partially lost our earlier operations: for example $A + B$ does not make sense any more since $(a_1, a_2, 1) + (b_1, b_2, 1) = (a_1 + b_1, a_2 + b_2, 2)$ is not in the $z = 1$ plane any more. However, for example $xA + yB$ makes sense if $x + y = 1$.

**Proposition 2.20.1.** *For a triangle $\triangle ABC$ in the $z = 1$ plane we have*

$$\text{Area}(\triangle ABC) = \pm \frac{1}{2} A \wedge B \wedge C,$$

*and the sign is positive if and only if going around the triangle in the order $A$, $B$, $C$ is counterclockwise.*

The proof is left as an exercise.

**Corollary 2.20.2.** *The three points $A, B, C$ of the $z = 1$ plane are collinear if and only if $A \wedge B \wedge C = 0$.*

*Proof.* Both conditions are equivalent to the condition that the volume of the parallelepiped spanned by the space vectors $A, B, C$ is zero. $\qquad \square$

Corollary 2.20.2 is now our 4th algebraic interpretation of collinearity of three points in the plane—however this in new settings. Again, all theorems that were proved using any of the earlier three interpretations (Propositions 2.4.1, 2.4.2, 2.19.3) can be proved with this new one too—just we need to be careful using operation that make sense in the $z = 1$ plane. Let us illustrate this with the following new proof Menelaus' theorem 2.7.1.

*Proof.* Consider our triangle in the $z = 1$ plane. We have $K = xB + x'C$, $L = yC + y'A$, $M = zA + z'B$ with $x + x' = y + y' = z + z' = 1$. According to Corollary 2.4.3 we have

$$0 = K \wedge L \wedge M = (xB + x'C) \wedge (yC + y'A) \wedge (zA + z'B).$$

Using 3-linearity and antisymmetry of the triple product we can distribute the above expression and obtain

$$0 = (xyz + x'y'z')A \wedge B \wedge C.$$

Since $ABC$ is a triangle (with non-zero area) we have $A \wedge B \wedge C \neq 0$. Therefore $xyz + x'y'z' = 0$ which is a rearrangement of Menelaus' theorem. $\square$

---

## Problems

**2.20.1.** If $A', B', C'$ are the midpoints of $BC$, $CA$, $AB$ respectively, then show that

$$4A' \wedge B' \wedge C' = A \wedge B \wedge C.$$

Deduce that $\text{Area}(\triangle A'B'C') = \frac{1}{4}\,\text{Area}(\triangle ABC)$.

**2.20.2.** Reprove Problem 2.19.2 using the triple wedge operation.

**2.20.3.** Give a proof of Proposition 2.20.1. [Hint: Let $T$ be the tetrahedron with vertexes $(0, 0, 0), A, B, C$. Use the geometric interpretation of the triple product to conclude that the volume of $T$ is plus or minus one sixth of the triple product $ABC$. Finish the proof by observing that the volume of $T$ is one third of the area of the $\triangle ABC$.]

**2.20.4.** Reprove Ceva's theorem, using the triple wedge operation. Hint: Try to rephrase the "high-school style" proof from the end of Section 2.9.

**2.20.5** (P)**.** Let the side lengths of the $\triangle ABC$ be $a, b, c$ ($a$ is opposite with $A$ etc), and let the angles be $\alpha, \beta, \gamma$. Let the foot of the altitude from $A$ be $D$. Prove that

$$aD = (b \cos \gamma)B + (c \cos \beta)C.$$

Deduce that the area of the triangle formed by the feet of the altitudes is

$$2 \cos \alpha \cos \beta \cos \gamma \cdot \text{Area}(\triangle ABC).$$

# Chapter 3

# Projective Geometry

## 3.1 Projective plane as extended affine plane

In relation with some exercises in Chapter 1 we developed the intuitive idea that we would like to send a point in a configuration to "infinity". This idea is made precise by defining a new object, the projective plane $\mathbb{P}^2$. Just like the affine plane has points and lines and the incidence relation between them, the projective plane will also have points and lines and incidence relation between them. However, the projective plane will have "more" points and lines. Moreover the incidence relation will be somewhat more "complete" and more "symmetric" than in the affine case.

Let $a$ be a line on the Euclidean plane. The collection of all lines parallel with $a$ will be called a "parallelism class" of lines on $\mathbb{R}^2$, and let's call this object $I_a$. So, if $a||b$ then $I_a = I_b$. One way of thinking of the $I_*$'s as "symbols" associated with lines in such a way that parallel lines have the same symbol.

**Definition 3.1.1.** Let points of $\mathbb{P}^2$ be

- points of $\mathbb{R}^2$ (these will be called affine points),

- all the $I_a$'s (these will be called ideal points).

Let the lines of $\mathbb{P}^2$ be

- extended lines of $\mathbb{R}^2$: extend each line in $\mathbb{R}^2$ with its own parallelism class, that is extend $a$ with $I_a$,

- the collection of all ideal points is declared to be on one new line, the ideal line.

Hence we added to the points of the plane $\mathbb{R}^2$ infinitely many new points, one for each parallelism class of lines on $\mathbb{R}^2$. We also added one new line. Observe that the lines coming from lines in $\mathbb{R}^2$ are exactly one point "longer" on $\mathbb{P}^2$.

**PROJECT 7.** One way of imagining $I_a$ is imagining it at the "end" of $a$. Since $a$ has two "ends", we must visualize it at both ends of $a$ at the same time. In this visualization, if a point "goes to infinity" in one direction of the projective line $a \cup \{I_a\}$, then after it disappears in one direction, it comes back at infinity at the other "end" of $a$. Let $A, B, C$ be the vertices of a triangle in a counterclockwise order. Imagine that we send this triangle to "infinity" in one direction, and as described above, it reappears from the opposite direction. Will the reappearing triangle be named $ABC$ clockwise or counterclockwise?

## 3.2   Incidence properties of $\mathbb{P}^2$

The following observation can be established by checking all possible cases.

**Proposition 3.2.1.** *For any two points there is a unique line passing through them. For any two lines there is a unique point contained in both of them.*

## 3.3   Remarkable incidence theorems
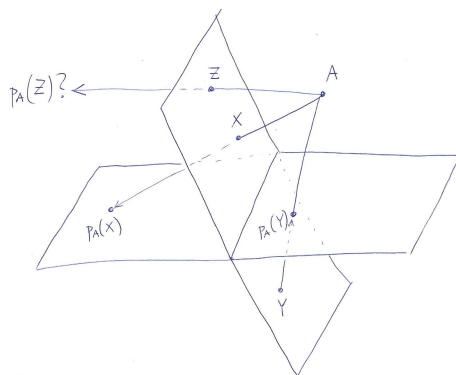
Desargues' theorem holds in $\mathbb{P}^2$.

**Theorem 3.3.1** (Desargues' theorem—projective)**.** *Two triangles are perspective with respect to a point if and only if they are perspective with respect to an axis.*

**PROJECT 8.** Learn the "vector calculus" of projective geometry—by searching for "projective coordinates". Then prove the projective Desargues' theorem. The calculation should be similar to the calculation in the proof of the affine Desargues' theorems in Chapter 2.

**PROJECT 9.** Let $S_1$ and $S_2$ be planes in $\mathbb{R}^3$ intersecting in a line. Let $A$ be a point in $\mathbb{R}^3$, not on these planes. There is a natural notion of projecting points of $S_1$ to points of $S_2$ from $A$ (see Figure ?). We obtain a map $p_A$ from a subset of $S_1$ to $S_2$. Observe that this map is not defined for *all* points of $S_1$ and that it is not surjective. Find a natural extension

$$\tilde{p}_A : \mathbb{P}^2(S_1) \to \mathbb{P}^2(S_2)$$

where $\mathbb{P}^2(S_i)$ is the projective plane obtained by extending $S_i$, and conclude that $\tilde{p}_A$ is bijective. Interpret the outcome as a visualization of the ideal line of $S_1$ in $S_2$. Use this idea to prove the projective Desargues' theorem, based on the fact that we already proved the affine Desargues' theorem in Chapter 2.

Pappus' theorem holds in $\mathbb{P}^2$.

**Theorem 3.3.2** (Pappus' theorem—projective version)**.** *If $A, B, C$ are collinear and $A', B', C'$ are collinear, then $K = AB' \cap A'B$, $L = BC' \cap B'C$, and $M = CA' \cap C'A$ are also collinear.*

**PROJECT 10.** Prove the projective Pappus' theorem using projective coordinates, c.f. Project 8.

**PROJECT 11.** Prove the projective Pappus' theorem using the idea of Project 9.

## 3.4 Problems

**3.4.1.** Prove Proposition 3.2.1.

The next three problems refer to the following part of the projective Desargues theorem: *If $\triangle ABC$ and $\triangle A'B'C'$ are perspective w.r.t. a point $P$, then $K = AB \cap A'B'$, $L = BC \cap B'C'$, and $M = CA \cap C'A'$ are collinear.*

**3.4.2.** Phrase the affine geometry theorem that we obtain from this theorem if $K$ is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

**3.4.3.** Phrase the affine geometry theorem that we obtain from this if theorem $A$ is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

**3.4.4.** Phrase the affine geometry theorem that we obtain from this if theorem $P, A, A'$ is an ideal point (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

The next two problems refer to the projective Pappus' Theorem 3.3.2.

**3.4.5.** Phrase the affine geometry theorem that we obtain from this theorem if $K$ and $L$ are ideal points (and none of $A, B, C, A', B', C'$ are ideal). You do not need to prove this theorem, just phrase it.

**3.4.6.** Phrase the affine geometry theorem that we obtain from this theorem if $A, B, C$ are ideal points (and none of the other points are ideal). You do not need to prove this theorem, just phrase it.

**3.4.7.** Let $ABCD$ be a quadrilateral in $\mathbb{P}^2$. Let $A_1, B_1, C_1, D_1$ be on the $AB, BC, CD, DA$ sides, respectively, so that $A_1 D_1$, $BD$ and $B_1 C_1$ are concurrent. Prove that $A_1 B_1$, $AC$ and $C_1 D_1$ are also concurrent.

# Chapter 4

# Spherical Geometry

Now we will study the geometry on the sphere of radius $R$. Let us fix the positive number $R$ for the whole chapter.

## 4.1 Points, lines, triangles, polarity

In spherical geometry the role of the "plane" will be played by the sphere

$$S^2 = \{x \in \mathbb{R}^3 : ||x|| = R\}.$$

When we say point, we mean a point on $S^2$. For points $P$ and $-P$ in $S^2$ we will say they are *antipodal*.

The role of straight lines will be played by the "great circles" in $S^2$. A great circle is defined to be the intersection $S^2 \cap P$ where $P$ is a plane in $\mathbb{R}^3$ containing the origin.

Now we have a geometry of "points" and "lines". The following point-line incidence properties are obvious.

**Proposition 4.1.1.** *In spherical geometry*

- *given two non-antipodal points there is a unique line passing through them;*

- *the intersection of any two lines is exactly two points.*

**Definition 4.1.2.**

- For two (not antipodal) points of $S^2$ there is exactly one great circle connecting them. The two points divide this great circle to two arcs. The shorter of the two will be called a *spherical segment* connecting the two points.

- For three spherical points $A, B, C$ (that are not contained in a spherical line) the union of the three spherical segments $AB$, $BC$, $CA$ is called a *spherical triangle*.

- For a spherical line $a = S^2 \cap P$ there are two points $X, Y$ on $S^2$ with $X \perp P$, $Y \perp P$.
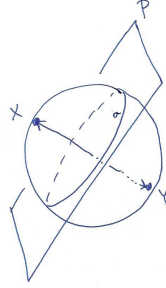


    We will call these two points the poles of $a$.

- Let $\triangle ABC$ be spherical triangle. Let $C'$ be a pole of $AB$ contained in the same hemisphere as $C$. Let $B'$ be a pole of $CA$ contained in the same hemisphere as $B$. Let $A'$ be a pole of $BC$ contained in the same hemisphere as $A$. The spherical triangle $\triangle A'B'C'$ is called the *polar triangle* of $\triangle ABC$.

**Theorem 4.1.3** (Bipolar Theroem). *The polar triangle of the polar triangle of $\triangle ABC$ is itself.*

*Proof.* For the purpose of this proof, if $X$ and $Y$ are on $S^2$ and $X \perp Y$ then we will call the spherical segment $XY$ a quadrant.
We have that $AB'$ is a quadrant because $B'$ is a pole of $AC$. The segment $AC'$ is a quadrant, because $C'$ is a pole of $AB$.
Since $AB'$ and $AC'$ are quadrant, $A$ must be a pole of $B'C'$. What remains to be proved is that $A$ is on the same side of $B'C'$ as $A'$.
The points $A$ and $A'$ are on the same side of $BC$ by definition of $A'$. Moreover $A'$ is a pole of $BC$. Hence the segment $AA'$ is less than a quadrant.
Since $AA'$ is less than a quadrant, and $A$ is a pole of $B'C'$ (proved above), we must have that $A$ and $A'$ is on the same side of $B'C'$.                                              $\square$

## 4.2   Length, tangent vectors, angle

**Definition 4.2.1.** The distance $d(X, Y)$ between points $X, Y \in S^2$ is defined to be

$$d(X, Y) = R \arccos\left(\frac{X}{||X||} \cdot \frac{Y}{||Y||}\right) = R \arccos\left(\frac{X \cdot Y}{R^2}\right).$$

Because of the geometric interpretation of dot product, $d(X, Y)$ is in fact $R$ times the angle between the vectors $X$ and $Y$. This number is then the same as the length (in the Calculus sense) of the spherical segment $XY$, see the figure below.
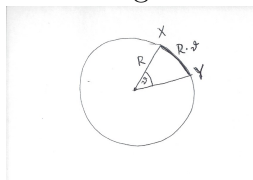


Thus, the odd looking definition of distance above is natural: it is the length measured *on the sphere*.

**Definition 4.2.2.** A vector $\mathbf{v} \in \mathbb{R}^2$ is called tangent to $S^2$ at $X \in S^2$ if $\mathbf{v} \perp X$. The collection of tangent vectors to $S^2$ at $X$ is called the tangent plane $T_X S^2$.

It is natural to imagine arrows representing tangent vectors in such a way that the arrows start at $X$. Thus $T_X S^2$ coincides with the usual notion of a plane tangent to $S^2$ at $X$. Given a spherical segment connecting $X$ and $Y$ there is a unique unit vector $\mathbf{v} \in T_X S^2$ which is determined by $Y = \alpha X + \beta \mathbf{v}$ with $\beta > 0$. The vector $\mathbf{v}$ will be called the unit tangent vector at $X$ to the segment from $X$ to $Y$.
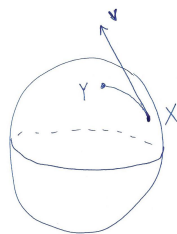


Let $\mathbf{v} \in T_X S^2$ be a unit vector and consider the parameterized curve

$$s(t) = \cos\left(\frac{t}{R}\right) X + R \sin\left(\frac{t}{R}\right) \mathbf{v}. \tag{4.1}$$

First observe that this curve lies in $S^2$. Indeed, for any $t$ we have

$$
\begin{aligned}
s(t)s(t) &= \left(\cos\left(\frac{t}{R}\right) X + R \sin\left(\frac{t}{R}\right)\mathbf{v}\right)\left(\cos\left(\frac{t}{R}\right) X + R \sin\left(\frac{t}{R}\right)\mathbf{v}\right)\\
&= \underbrace{XX}_{R^2}\cos^2\left(\frac{t}{R}\right) + \underbrace{X\mathbf{v}}_{0}\,2R\cos\left(\frac{t}{R}\right)\sin\left(\frac{t}{R}\right) + \underbrace{\mathbf{vv}}_{1}\,R^2\sin^2\left(\frac{t}{R}\right)\\
&= R^2\left(\cos^2\left(\frac{t}{R}\right) + \sin^2\left(\frac{t}{R}\right)\right) = R^2.
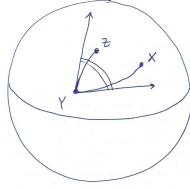\end{aligned}
$$

The curve is also in the plane spanned by $X$ and $\mathbf{v}$. Hence the curve is in the intersection of a plane with the sphere, that is, in a spherical line.

Now we are going to study the parametrization (4.1) with respect to the distance notion defined above. Let $a$ be in $(0, R\pi]$. Then $s([0, a])$ is a spherical segment and $\mathbf{v}$ is the unit vector tangent to the spherical segment $Xs(a)$ at $X$. We claim that the length of $s([0, a])$ is $a$. Indeed,

$$d(X, s(a)) = R\arccos\left(\frac{X \cdot \left(\cos\left(\frac{a}{R}\right)X + R\sin\left(\frac{a}{R}\right)\mathbf{v}\right)}{R^2}\right) = R\arccos\left(\cos\left(\frac{a}{R}\right)\right) = a.$$

Moreover, for any $t_1, t_2$ with $|t_1 - t_2| < R\pi$ we have that the length of the $s(t_1)s(t_2)$ segment is $|t_1 - t_2|$ (Ex. 4.6.1 below.) This remarkable property of the parametrization (4.1) is called "arc-length parametrization", or "natural parametrization".

**Definition 4.2.3.** The angle $XYZ\angle$ is defined to be the angle of the following two vectors: the unit tangent vector at $Y$ to the segment $YX$, and the unit tangent vector at $Y$ to the segment $YZ$.



## 4.3   Sides and angles of polar triangles

**Theorem 4.3.1.** *Let $a, b, c$ be the sides and $\alpha, \beta, \gamma$ be the angles of a triangle $\triangle ABC$. Let $a', b', c'$ be the sides and $\alpha', \beta', \gamma'$ be the angles of the polar triangle $\triangle A'B'C'$. Then*

$$\frac{a'}{R} = \pi - \alpha, \qquad\qquad \frac{b'}{R} = \pi - \beta, \qquad\qquad \frac{c'}{R} = \pi - \gamma,$$

$$\frac{a}{R} = \pi - \alpha', \qquad\qquad \frac{b}{R} = \pi - \beta', \qquad\qquad \frac{c}{R} = \pi - \gamma'.$$

*Proof.* Because of symmetry and the Bipolar Theorem 4.1.3 it is enough to prove one of the six statements. Let $K = AB \cap B'C'$, $L = AC \cap B'C'$. Since $A$ is a pole of $B'C'$ we have $KL = R\alpha$.
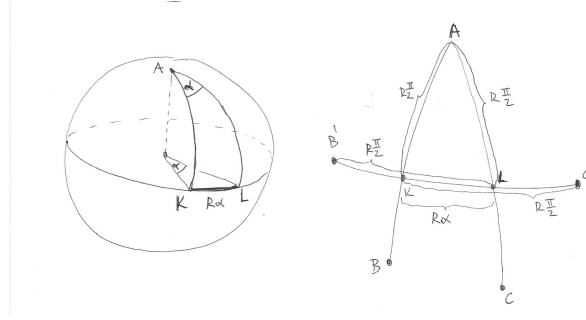
Then $B'C' = B'L + KC' - KL = R\pi/2 + R\pi/2 - R\alpha$, proving $a' = R(\pi - \alpha)$.                      □

# 4.4   Laws of Cosines and Sines

**Theorem 4.4.1** (Spherical Law of Cosines). *In $\triangle ABC$ on the sphere (with usual notations) we have*

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right) + \sin\left(\frac{a}{R}\right)\sin\left(\frac{b}{R}\right)\cos\gamma.$$

*Proof.* Let $\mathbf{u}$ be the unit tangent vector of $CA$ at $C$. Let $\mathbf{v}$ be the unit tangent vector of $CB$ at $C$. Then

$$A = \cos\left(\frac{b}{R}\right)\cdot C + R\sin\left(\frac{b}{R}\right)\cdot\mathbf{u}, \qquad B = \cos\left(\frac{a}{R}\right)\cdot C + R\sin\left(\frac{a}{R}\right)\cdot\mathbf{v}.$$

Therefore

$$c = d(A,B) = R\arccos\left(\frac{A\cdot B}{R^2}\right)$$

$$= R\arccos\left(\frac{\overbrace{C\cdot C}^{R^2}\cos\left(\frac{b}{R}\right)\cos\left(\frac{a}{R}\right) + R^2\sin\left(\frac{b}{R}\right)\sin\left(\frac{a}{R}\right)\overbrace{\mathbf{u}\cdot\mathbf{v}}^{\cos\gamma}}{R^2}\right),$$

and rearrangement gives the Theorem.                      □

**Corollary 4.4.2** (Spherical Pythagorean Theorem). *In $S^2$, a triangle is a right triangle (with hypotenuse c) if and only if $\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right)$.*

*Proof.* This is the Spherical Law of Cosines for $\gamma = \pi/2$.                      □

**Project.** Write $\kappa = 1/R$ in the Spherical Law Cosines and in the Spherical Pythagorean Theorem. Then expand in Taylor series with respect to $\kappa$. Study the coefficients of $1, \kappa, \kappa^2$ on the two sides of the statements.

**Theorem 4.4.3** (Spherical Triangle Inequality)**.** *In a spherical triangle we have* $c < a + b$.

*Proof.* We have

$$\cos\left(\tfrac{c}{R}\right) = \cos\left(\tfrac{a}{R}\right)\cos\left(\tfrac{b}{R}\right) + \sin\left(\tfrac{a}{R}\right)\sin\left(\tfrac{b}{R}\right)\cos\gamma;$$
$$\cos\left(\left(\tfrac{a}{R}\right) + \left(\tfrac{b}{R}\right)\right) = \cos\left(\tfrac{a}{R}\right)\cos\left(\tfrac{b}{R}\right) - \sin\left(\tfrac{a}{R}\right)\sin\left(\tfrac{b}{R}\right).$$

The first line is the Law of Cosines, the second line is a trig identity. From trig we have $\sin(a/R)\sin(b/R) \in (0,1)$, $\cos\gamma \in (-1,1)$. Therefore the right hand side of the first line is larger than the right hand side of the second line. Hence the same holds for the left hand sides, i.e. $\cos(c/R) > \cos((a+b)/R)$. Since cos is strictly monotone decreasing on $(0,\pi)$ this implies $c/R < (a+b)/R$ and hence $c < a + b$. $\qquad\square$

**Theorem 4.4.4** (Dual Law of Cosines)**.** *In* $\triangle ABC$ *on the sphere (with usual notations)*

$$-\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\left(\frac{c}{R}\right).$$

*Proof.* Apply the Law of Cosines to the polar triangle. $\qquad\square$

*Remark* 4.4.5. The dual Law of Cosines calculates $\gamma$ if $\alpha, \beta$ and $c$ are given. In Euclidean geometry one can calculate $\gamma$ as long as $\alpha$ and $\beta$ are given, with no need for $c$! Also, this theorem shows that the sum of the angles of a triangle is not constant, keeping $\alpha$ and $\beta$ the same, but changing $c$ *does* change $\gamma$. We will learn more about the sum of the angles $\alpha + \beta + \gamma$ in the next section.

**Theorem 4.4.6** (Spherical Law of Sines)**.** *In* $\triangle ABC$ *on the sphere (with usual notations)*

$$\frac{\sin\alpha}{\sin\left(\tfrac{a}{R}\right)} = \frac{\sin\beta}{\sin\left(\tfrac{b}{R}\right)} = \frac{\sin\gamma}{\sin\left(\tfrac{c}{R}\right)}.$$

*Proof.* From the Law of Cosines we have

$$\cos\gamma = \frac{\cos\left(\tfrac{c}{R}\right) - \cos\left(\tfrac{a}{R}\right)\cos\left(\tfrac{b}{R}\right)}{\sin\left(\tfrac{a}{R}\right)\sin\left(\tfrac{b}{R}\right)}.$$

From this, using $\sin^2 + \cos^2 = 1$ we have

$$\sin\gamma = \frac{\sqrt{1 - \cos^2\left(\tfrac{a}{R}\right) - \cos^2\left(\tfrac{b}{R}\right) - \cos^2\left(\tfrac{c}{R}\right) + 2\cos\left(\tfrac{a}{R}\right)\cos\left(\tfrac{b}{R}\right)\cos\left(\tfrac{c}{R}\right)}}{\sin\left(\tfrac{a}{R}\right)\sin\left(\tfrac{b}{R}\right)}.$$

Dividing both sides by $\sin\left(\frac{c}{R}\right)$ we see that the right hand side becomes symmetric in $a, b, c$. This proves the theorem. □

**Theorem 4.4.7.** *Let $R = 1$. The common value of $\sin\alpha/\sin a = \sin\beta/\sin b = \sin\gamma/\sin c$ is equal*

$$\frac{6\,\text{Vol}(OABC\text{-}tetrahedron)}{\sin a \sin b \sin c},$$

*in other words the $\sqrt{\ }$-expression in the proof above is equal 6 times the volume of the parallelepiped.*

*Proof.* We can move the $\triangle ABC$ into the special position $A = (1, 0, 0), B = (*, \text{positive}, 0), C = (*, *, \text{positive})$. Then a good picture and calculation shows that

$$A = (1, 0, 0), B = (\cos c, \sin c, 0), C = (\cos b, \sin b \cos\alpha, \sin b \sin\alpha).$$

Using the determinant formula for the volume of the $OABC$ tetrahedron we obtain the theorem. □

## 4.5 Anglesum, area, Girard's formula

If two great circles intersect in angle $\theta$, then the domain between the two (on both sides) will be called a $\theta$-lune.
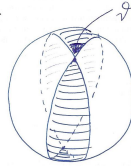


**Lemma 4.5.1.** *The area of a $\theta$-lune is $4\theta R^2$.*

*Proof.* The area of a $\theta$-lune is a linear function of $\theta$. For $\theta = \pi$ we get the surface area of $S^2$, which is $4\pi R^2$. □

**Theorem 4.5.2** (Girard's formula)**.** *The sum of the angles of a spherical triangle is $\pi$ plus the area of the triangle, ie.*

$$\alpha + \beta + \gamma = \pi + \frac{\text{Area}(\triangle ABC)}{R^2}.$$

*Proof.* Consider a spherical triangle $T$ with angles $\alpha, \beta, \gamma$. The $\alpha$-lune, $\beta$-lune, and $\gamma$-lune together cover the sphere 1-sheeted, except they cover $T$ 3 times, as well as a congruent copy $T'$ of $T$ "on the other side" also 3 times.
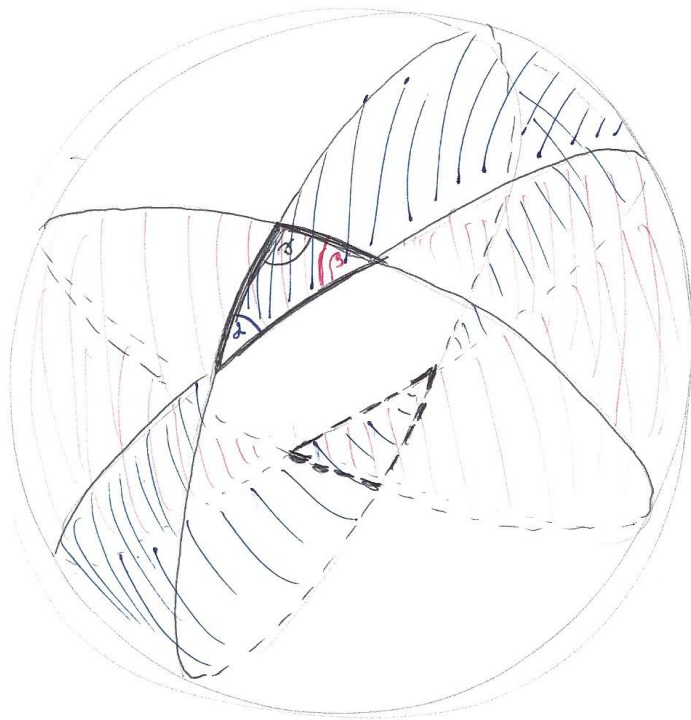


Hence

$$\text{Area}(\alpha\text{-lune}) + \text{Area}(\beta\text{-lune}) + \text{Area}(\gamma\text{-lune}) = \text{Area}(S^2) + 2\,\text{Area}(T) + 2\,\text{Area}(T').$$

And therefore, using Lemma 4.5.1 we have

$$4\alpha R^2 + 4\beta R^2 + 4\gamma R^2 = 4R^2\pi + 4\,\text{Area}(T),$$

which proves the theorem.                                                                 □

**Corollary 4.5.3.** *The sum of the angles of a spherical triangle is always strictly larger than $\pi$.*

*Remark* 4.5.4. In Euclidean geometry the sum of the angles of a triangle is $\pi$ $(= 180^o)$, so Girard's formula says that in spherical geometry the sum of angles is the Euclidean value

*plus a correction term.* The correction term equals the area divided by $R^2$. For small (area) triangles this correction term is small and the sum of the angles is close to the expected Euclidean value $\pi$.

*Remark* 4.5.5. It is tempting to regard the *flat* Euclidean plane as the surface of the sphere with $R = \infty$ (imagine spheres with larger and larger radii, and how they will get less and less curved). Girard's formula is consistent with this point of view. In the $R \to \infty$ limit Girard's formula recovers the Euclidean $\alpha + \beta + \gamma = \pi$ theorem.

*Remark* 4.5.6. In Calculus language the "correction term" Area $/R^2$ can be written as

$$\iint_{\triangle ABC} K dA,$$

where $K = 1/R^2$, what we call the *Gaussian curvature* of the sphere. Phrasing Girard's formula as

$$\alpha + \beta + \gamma = \pi + \iint_{\triangle ABC} K dA$$

has a far-reaching generalization in differential geometry, called *Gauss-Bonnet theorem.*

A spherical triangle is always contained in a hemisphere. Thus, the area of a spherical triangle is at most half the area of $S^2$, that is $2\pi R^2$. Hence, from Girard's formula we obtain an upper bound for the angle sum as well, namely

$$\pi < \alpha + \beta + \gamma < 3\pi.$$

**Proposition 4.5.7.** *For the perimeter $p$ of a triangle on the sphere we have $0 < p < 2\pi R$.*

*Proof.* Applying the $\pi < \alpha + \beta + \gamma < 3\pi$ estimate for the polar triangle we obtain the statement. □

**Theorem 4.5.8.** *The sum of the angles of a spherical quadrilateral is $2\pi$ plus the area divided by $R^2$.*

*Proof.* Cut the quadrilateral by a diagonal to two triangles $T_1$ and $T_2$. For the two triangles we have Girard's formula

$$\alpha + \beta_1 + \delta_1 = \pi + \text{Area}(T_1)/R^2,$$

$$\gamma + \beta_2 + \delta_2 = \pi + \text{Area}(T_2)/R^2.$$
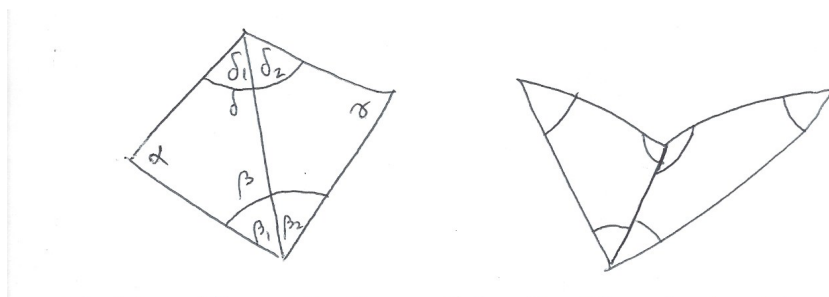
Adding together we obtain

$$\alpha + \beta + \gamma + \delta = 2\pi + \text{Area}(T_1 \cup T_2)/R^2.$$

$\square$

**Theorem 4.5.9.** *The sum of the angles of a spherical n-gon is the Euclidean value $(n-2)\pi$ plus the area of the n-gon divided by $R^2$.*

*Proof.* The idea of the preceding proof can be applied iteratively. $\square$

---

# 4.6 Problems

**4.6.1.** Consider the parametrization (4.1). Prove that for $|t_1 - t_2| < R\pi$ we have that the length of the $s(t_1)s(t_2)$ segment is $|t_1 - t_2|$.

**4.6.2.** The coordinates of Chapel Hill NC are 35.93°N, 79.03°W. Look up what these coordinates mean. Look up the coordinates of Paris, France. Find the distance between Chapel Hill and Paris (call the radius of the Earth $R$). Hint: the Law of Cosines will be useful.

**4.6.3.** Let $L$ and $W$ be points on the Earth (radius $R$), with coordinates 38.8°N, 9.15°W and 38.8°N, 77°W (why L and W?). What is their spherical distance?

**4.6.4.** The points $L$ and $W$ of the exercise above are on the same latitude (38.8°N). What is their distance measured on that latitude?

**4.6.5.** Prove Theorem 4.5.9 in detail.

**4.6.6.** We saw above that $\pi$ < angle sum of a spherical triangle < $3\pi$. Prove that this estimate is sharp.

**4.6.7.** Let $P$ be a plane not containing the origin. Prove that if $P \cap S^2$ is not empty, then it is a circle in spherical geometry. That is, prove that there exist a point $C \in S^2$ and a nonnegative number $r$ such that $P \cap S^2 = \{A \in S^2 : d(A, C) = r\}$.

**4.6.8.** Let $R = 1$.

(a) Let the sides of a triangle in spherical geometry be .3, .4, and .5. Find its angles.

(b) Let the sides of a triangle in spherical geometry be .03, .04, and .05. Find its angles.

(c) Let the sides of a triangle in spherical geometry be .003, .004, and .005. Find its angles.

**4.6.9.** Consider Chapel Hill, Chicago, and Dallas. Find the distances between any two of these cities online. Using these distances find the angle between the Chapel Hill-Chicago and the Chapel Hill-Dallas segments.

**4.6.10.** Consider Chapel Hill, Chicago, and Dallas. Find the distances between any two of these cities online. Calculate the area of the triangle with these vertices on the surface of the Earth.

**4.6.11.** Let $a, b, c$ and $\alpha, \beta, \gamma$ be the side lengths and angles of a spherical triangle. Assume $\gamma = \pi/2$ (that is, we have a right triangle). Express $\sin \alpha$ in terms of the sides. Verify that for small sides (compared to $R$) your result is close to the Euclidean value of $a/c$.

**4.6.12.** Give a proof of the spherical Law of Sines based on the result of Exercise 4.6.11 along the lines of Proof 2 of the Euclidean Law of Sines (Theorem 2.16.4).

**4.6.13.** Let $a, b, c$ and $\alpha, \beta, \gamma$ be the side lengths and angles of a spherical triangle. Assume $\gamma = \pi/2$ (that is, we have a right triangle). Express $\cos \alpha$ in terms of the sides. Verify that for small sides (compared to $R$) your result is close to the Euclidean value of $b/c$.

**4.6.14.** Let $a, b, c$ and $\alpha, \beta, \gamma$ be the side lengths and angles of a spherical triangle. Assume $\gamma = \pi/2$ (that is, we have a right triangle). Express $\tan \alpha$ in terms of the sides. Verify that for small sides (compared to $R$) your result is close to the Euclidean value of $a/b$.

**4.6.15.** Given two distinct points $X, Y \in S^2$, find the set of points of $S^2$ which are at equal distance from $X$ and $Y$.

**4.6.16.** Define the notion of perpendicular bisector of a spherical segment. Prove that the perpendicular bisectors of the sides of a triangle are concurrent, and that their intersection is of the same distance from the vertices. Deduce the existence of circumscribed circle of $\triangle ABC$.

**4.6.17.** Let $P$ be a point and $l$ be a line on $S^2$. Assume $P$ is not a pole of $l$. Prove that there is a unique line $k$ passing through $P$ and intersecting $l$ in a right angle. Show that the shortest spherical distance connecting $P$ with $l$ is the segment$\subset k$.

**4.6.18.** Given two lines $l_1, l_2 \subset S^2$, find the set of points of $S^2$ which are at equal distance from $l_1$ and $l_2$.

**4.6.19.** Define the notion of angle bisectors of a triangle. Prove that the angle bisectors of the a triangle are concurrent, and that their intersection is of the same distance from the sides. Deduce the existence of inscribed circle of $\triangle ABC$.

**4.6.20.** Let $l_1$ and $l_2$ be different lines on $S^2$. Prove that there is a unique line $k$ perpendicular to both.

**4.6.21.** Consider the $ABC$ spherical triangle, and its medians. Consider also the "flat" triangle in $\mathbb{R}^3$ whose vertices are $ABC$. Using Euclidean properties of the flat $\triangle ABC$ prove that the medians of the spherical $\triangle ABC$ are concurrent.

**4.6.22.** Let a spherical triangle have two angles $\pi/2$. Prove that two of its sides are $R\pi/2$.

**4.6.23.** Let the equilateral spherical triangle have area $R^2\pi/2$. What is its side length?

# Chapter 5

# Hyperbolic Geometry

## 5.1   Minkowski space, Lorentz transformations

Let us fix a positive number $\mathfrak{c}$, we will call it the "speed of light".

**Definition 5.1.1** (Minkowski[1] Inner Product)**.** For any vectors $\mathbf{x}$ and $\mathbf{y}$ in 3-space the Minkowski inner product is

$$\Phi((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1 y_1 + x_2 y_2 - \mathfrak{c}^2 x_3 y_3.$$

The space $\mathbb{R}^3$ equipped with the Minkowski inner product is called the Minkowski space, the Minkowski space-time, or the space of special relativity. We will call a vector $\mathbf{x}$ in Minkowski space

- time-like if $\Phi(\mathbf{x}, \mathbf{x}) < 0$,

- light-like if $\Phi(\mathbf{x}, \mathbf{x}) = 0$, and

- space-like if $\Phi(\mathbf{x}, \mathbf{x}) > 0$.

Observe that light-like vectors $\mathbf{x} = (x_1, x_2, x_3)$ satisfy

$$(\mathfrak{c}\, x_3)^2 = x_1^2 + x_2^2$$

and thus form a double cone, called the "light-cone." Vectors $\vec{OA}$ with $A$ inside the double cone are time-like, and with $A$ outside the light-cone are space-like, see Figure 5.1.
Let us study "natural" linear transformations of the Minkowski space.

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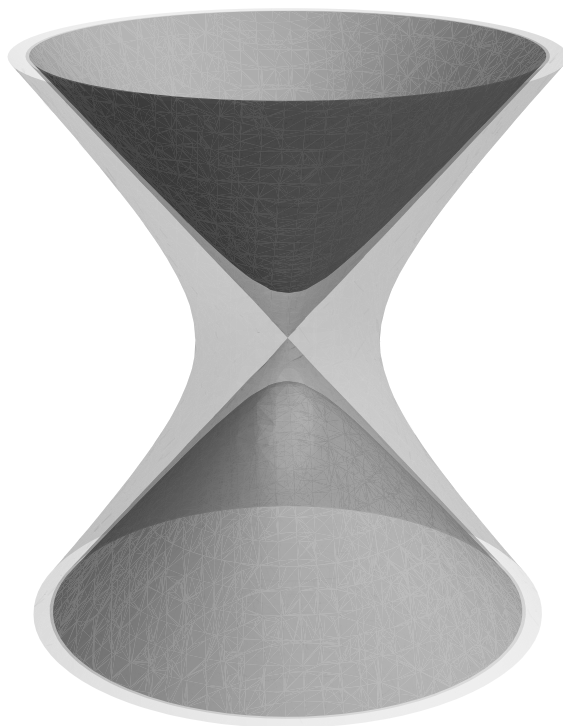[1]Hermann Minkowski, 1864-1909

Figure 5.1: The light-cone formed by all light-like vectors. On the outside is the set of all space-like vectors $\mathbf{x}$ satisfying $\Phi(\mathbf{x}, \mathbf{x}) = 5$. On the inside is the set of all time-like vectors $\mathbf{x}$ satisfying $\Phi(\mathbf{x}, \mathbf{x}) = -1$. The top component of this 2-sheeted hyperboloid is the hyperbolic plane $\mathbb{H}^2$.

**Definition 5.1.2.** A $3 \times 3$ matrix $A$ is called a Lorentz matrix, if

$$\Phi(Ax, Ay) = \Phi(x, y)$$

for any two vectors $x, y \in \mathbb{R}^3$. The map $\mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$ is called a Lorentz transformation.

Lorentz matrices form a group. That is, the identity matrix is a Lorentz matrix, the product of Lorentz matrices is a Lorentz matrix, Lorentz matrices are invertible and their inverses are also Lorentz matrixes. These claims are left as Exercise 5.10.5. The group of Lorentz matrices is called the Lorentz group, $O(2, 1)$.

The matrices

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} \frac{1}{\sqrt{1-(v/\mathfrak{c})^2}} & 0 & \frac{-v}{\sqrt{1-(v/\mathfrak{c})^2}} \\ 0 & 1 & 0 \\ \frac{-v/\mathfrak{c}^2}{\sqrt{1-(v/\mathfrak{c})^2}} & 0 & \frac{1}{\sqrt{1-(v/\mathfrak{c})^2}} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-(v/\mathfrak{c})^2}} & \frac{-v}{\sqrt{1-(v/\mathfrak{c})^2}} \\ 0 & \frac{-v/\mathfrak{c}^2}{\sqrt{1-(v/\mathfrak{c})^2}} & \frac{1}{\sqrt{1-(v/\mathfrak{c})^2}} \end{pmatrix} \tag{5.2}$$

are Lorentz matrices for any $\alpha, v \in \mathbb{R}$ with $|v| < \mathfrak{c}$. The last two are called the "Lorentz boosts".

**PROJECT 12.** *Read an introduction to Special Relativity, and see how in that theory the Lorentz transformations (in particular the Lorentz boosts) are derived as laws of physics, from Einstein's "fundamental postulates."*

**Convention.** For the rest of the book we will assume that $\mathfrak{c} = 1$. That is, we simply rescale the third coordinate of the Minkowski space by $\mathfrak{c}$, and hence achieve that $\Phi$ has the form

$$\Phi(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

## 5.2 The hyperbolic plane

**Definition 5.2.1** (Hyperbolic plane, hyperboloid model)**.** We define $\mathbb{H}^2$ to be the top component of the 2-sheeted hyperboloid $x_1^2 + x_2^2 - x_3^2 = -1$:

$$\mathbb{H}^2 = \{x \in \mathbb{R}^3 : \Phi(x, x) = -1, x_3 > 0\}.$$

**Definition 5.2.2.** Lines of $\mathbb{H}^2$ are defined to be the non-empty intersection of $\mathbb{H}^2$ with any plane $P$ containing the origin.

Now we have another geometry of points and lines, so we are interested in the incidence properties, analogous to Proposition 4.1.1. To find those incidence properties a new "model" of $\mathbb{H}^2$ will be useful.

**Klein model.** Let $\mathbb{K}^2 = \{(x_1, x_2, 1) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$. Consider the following map $f$ from $\mathbb{H}^2$ to $\mathbb{K}^2$: for $A \in \mathbb{H}^2$ let $f(A) = OA \cap \mathbb{K}^2$.

**Proposition 5.2.3.** *The map $f$ is a one-to-one correspondence (a.k.a. bijection) between $\mathbb{H}^2$ and $\mathbb{K}^2$.*

*Proof.* Observe that for $x = (x_1, x_2, x_3) \in \mathbb{H}^2$ we have $f(x) = (x_1/x_3, x_2/x_3, 1)$. The injectivity and the surjectivity of this map follow from straightforward calculations.      $\square$

Observe that the $f$-image of a line in $\mathbb{H}^2$ is an open cord of $\mathbb{K}^2$ (that is, an open segment connecting two points on the boundary circle of $\mathbb{K}^2$). Looking at point-line incidence on the Klein model, the following statements are obvious.

**Proposition 5.2.4** (Incidence properties of $\mathbb{H}^2$). *In $\mathbb{H}^2$*

   *(1) for any two points there is exactly one line connecting them.*

   *(2) The intersection of any two lines is either empty or 1 point.*

   *(3) For a line $l$ and point $P \notin l$ there are infinitely many lines $m$ passing through $P$ parallel to $l$ (i.e. not intersecting $l$).*

Recall that the first two of these incidence properties hold in Euclidean geometry too. The Euclidean counterpart of the third one is

   *(3') In the Euclidean plane, for a line $l$ and point $P \notin l$ there is exactly one line $m$ passing through $P$ parallel to $l$ (i.e. not intersecting $l$).*

Intellectuals of the 19th century argued whether we can decide empirically/theoretically whether (3) or (3') holds in reality. You can read about that debate, its history, and its philosophical significance, if you search for the *problem of parallels* or the names of some of the main constibutors: J. Bolyai, N. Lobachevsky, C.-F. Gauss.

## 5.3 The hyperbolic plane is homogeneous and isotropic

When we studied spherical geometry we thought of it as a "homogeneous space": there were no distinguished points. This holds because there are dot product preserving transformations of $\mathbb{R}^3$ mapping any point of the sphere to any other point. The space $S^2$ is also "isotropic". Namely, there are no distinguished directions at a given point: all directions (say unit tangent vectors) at the given point play the same role. This holds because for any two tangent unit vectors $\mathbf{u}, \mathbf{v}$ at $X$ there is a dot product preserving transformation keeping $X$ fixed but mapping $\mathbf{u}$ to $\mathbf{v}$.

In our $\mathbb{H}^2$ model some points look different than others. Say, the "vertex" of the hyperboloid, $(0, 0, 1)$ looks "special". However, this is only an illusion. The illusion is caused by looking at $\mathbb{H}^2$ with the Euclidean notion of length and angle in our mind. The Euclidean notions of length and angle are derived from the dot product. The dot product is not relevant in the theory of $\mathbb{H}^2$. We should look at natural transformations of $\mathbb{R}^3$ preserving not the dot product, but the Minkowski inner product. Such natural transformations of the Minkowski space $(\mathbb{R}^3, \Phi)$ are the Lorentz transformations from Section 5.1.

For the Lorentz transformations, the hyperbolic plane $\mathbb{H}^2$ turns out to be homogeneous and isotropic:

- For any two points $X, Y \in \mathbb{H}^2$ there is a Lorentz matrix $A$ with $AX = Y$ (Exercise 5.10.8).

- We can additionally require that $A$ maps given directions at $X$ and $Y$ to each other (Exercise 5.10.9).

## 5.4 Digression: Hyperbolic trig functions

**Definition 5.4.1.** Define $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x / \cosh x$.

To get familiar with these hyperbolic trigonometric functions do the following exercises.
- Draw their graphs. Prove that sinh is odd, cosh is even.
- Prove that $\cosh^2 x - \sinh^2 x = 1$ for all $x$.
- Prove that $\cosh' x = \sinh x$, $\sinh' x = \cosh x$.
- Define arccosh as the inverse function of $\cosh|_{[0,\infty)}$. Draw its graph. Determine its domain and range.
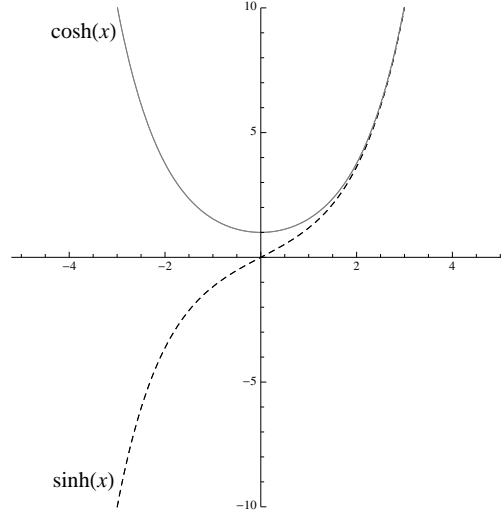
## 5.5   Length, angle

**Definition 5.5.1.** The *distance* between two points $X, Y \in \mathbb{H}^2$ is defined by

$$d(X, Y) = \operatorname{arccosh}(-\Phi(X, Y)).$$

We need to verify that this definition makes sense, that is, $-\Phi(X, Y)$ is in the domain of arccosh.

**Theorem 5.5.2.** *For any $X, Y \in \mathbb{H}^2$, the Minkowski inner product satisfies $\Phi(X, Y) \leq -1$.*

*Proof.* By Cauchy-Schwartz, we observe

$$\Phi(X, Y) = x_1 y_1 + x_2 y_2 - x_3 y_3 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} - x_3 y_3 = \sqrt{x_3^2 - 1}\sqrt{y_3^2 - 1} - x_3 y_3.$$

Now we only need to show that $(x_3^2 - 1)(y_3^2 - 1) \leq (x_3 y_3 - 1)^2$. By rearranging terms, we see that this condition is equivalent to $(x_3 - y_3)^2 \geq 0$. ∎

**Proposition 5.5.3.** *We have $d(X, Y) \geq 0$, and $d(X, Y) = 0$ if and only if $X = Y$.*

*Proof.* The first statement is obvious from the definition of arccosh, and the second one follows from careful examination of the estimates in the proof above. ∎

**Definition 5.5.4.** For $X \in \mathbb{H}^2$ define $X^\perp = \{v \in \mathbb{R}^3 : \Phi(X, v) = 0\}$.

Observe that $X^\perp$ is a plane in $\mathbb{R}^3$.

**Theorem 5.5.5.** *We have $X^\perp = T_X \mathbb{H}^2$, that is the tangent plane to $\mathbb{H}^2$ at $X$ in $\mathbb{R}^3$.*

*Proof.* Tangent vectors to $\mathbb{H}^2$ at $X$ are derivatives of curves in $\mathbb{H}^2$ passing through $X$. Let $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ be such a curve with $\phi(0) = X$. Then $\phi(t) \subset \mathbb{H}^2$ for all $t$ implies $\phi_1(t)^2 + \phi_2(t)^2 - \phi_3(t)^2 = -1$. Differentiating (and dividing by 2) gives $\phi_1(t)\phi_1'(t) + \phi_2(t)\phi_2'(t) - \phi_3(t)\phi_3'(t) = 0$. Substituting $t = 0$ gives $\Phi(X, \phi'(0)) = 0$. So we obtained that $T_X \mathbb{H}^2$ is part of $X^\perp$, but both are planes, so they must be equal. $\qquad\square$

**Theorem 5.5.6.** *All non-zero elements in $T_X \mathbb{H}^2$ are spacelike.*

The proof of this theorem is left as an exercise (Problem 5.10.2).

Let $X \in \mathbb{H}^2$, $v \in T_X \mathbb{H}^2$ with $\Phi(v, v) = 1$. Consider the curve

$$s(t) = \cosh t \cdot X + \sinh t \cdot v.$$

**Proposition 5.5.7.** *The curve $s$ is in $\mathbb{H}^2$. We have $s(0) = x$. The curve $s(t)$ is contained in a line in $\mathbb{H}^2$. We have $d(X, s(t)) = t$.*

*Proof.* The calculation

$$\Phi(s(t), s(t)) = \Phi(\cosh t \cdot X + \sinh t \cdot v, \cosh t \cdot X + \sinh t \cdot v) =$$

$$\cosh^2 t \cdot \Phi(X, X) + 0 + 0 + \sinh^2 t \cdot \Phi(v, v) = \sinh^2 t - \cosh^2 t = -1$$

shows that $s(t) \subset \mathbb{H}^2$. Since $s(t)$ is also in the plane spanned by the vectors $X, v$, we have that $s(t)$ is in a hyperbolic line.
The distance formula is proved by

$$d(X, s(t)) = \operatorname{arccosh}(-\Phi(X, \cosh t \cdot X + \sinh t \cdot v)) =$$

$$\operatorname{arccosh}(-\cosh t \cdot \Phi(X, X) + 0) = \operatorname{arccosh}(\cosh t) = t.$$

$\qquad\square$

*Remark* 5.5.8. In Calculus the length of a curve $\gamma$ is defined as $\int_a^b \|\gamma'(t)\| dt$. When defining distance on $\mathbb{H}^2$ we could have used the obvious analogue $\int_a^b \sqrt{\Phi(\gamma'(t), \gamma'(t))} dt$ of this definition (note that all $\gamma'(t)$ vectors are spacelike, so the square root makes sense!). Let us calculate the length of $s([0, t_0])$ above, using this Calculus motivated definition too: $\int_0^{t_0} \sqrt{\Phi(s'(t), s'(t))} dt$. Since $s'(t) = \sinh t \cdot X + \cosh t \cdot v$, we have

$$\Phi(s'(t), s'(t)) = \sinh^2 t \cdot \Phi(X, X) + \cosh^2 t \cdot \Phi(v, v) = -\sinh^2 t + \cosh^2 t = 1.$$

Hence the Calculus motivated definition gives $\int_0^{t_0} \sqrt{1}\,dt = t_0$. This is the same expression we obtained from Definition 5.5.1. *Our strange-looking $d(X, Y) = \operatorname{arccosh}(-\Phi(X, Y))$ definition is in fact consistent with the Calculus motivated formula.*

**Angle.**
We also want to define the notion of angle. Let $X \in \mathbb{H}^2$, $u, v \in T_X \mathbb{H}^2$ ($u$, $v$ non zero), and consider the rays starting at $X$ and going in the $u$, respectively $v$ direction. It is natural to define the angle of these two rays to be the angle of $u$ and $v$ in $T_X \mathbb{H}^2$, see Figure 5.2. Since the vectors of $T_X \mathbb{H}^2$ are space-like, the angle of $u$ and $v$ is naturally defined using the usual calculus notion.
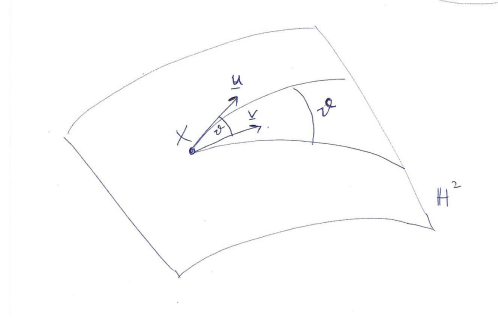


Figure 5.2: Angle in $\mathbb{H}^2$

**Definition 5.5.9.** For non-zero $u, v \in T_X \mathbb{H}^2$, define

$$\angle(u, v) = \arccos\left(\frac{\Phi(u, v)}{\sqrt{\Phi(u, u)\Phi(v, v)}}\right).$$

## 5.6 Laws of Cosines and Sines

**Theorem 5.6.1** (Hyperbolic Law of Cosines)**.** *In $\triangle ABC$ on the $\mathbb{H}^2$ (with usual notations)*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

*Proof.* Let $u$ be the unit tangent vector of $CA$ at $C$. Let $v$ be the unit tangent vector of $CB$ at $C$. Then $A = \cosh b \cdot C + \sinh b \cdot u$, $B = \cosh a \cdot C + \sinh a \cdot v$. Therefore $c = AB = \operatorname{arccosh}(\Phi(A, B)) = \operatorname{arccosh}(\cosh b \cosh a + \sinh b \sinh a \; \Phi(u, v))$, but $\Phi(u, v) = \cos \gamma$. $\square$

**Corollary 5.6.2** (Hyperbolic Pythagorean theorem). *If $\gamma = \pi/2$ then $\cosh c = \cosh a \cosh b$.*

**PROJECT 13.** *In the Hyperbolic Law of Cosines and in the Hyperbolic Pythagorean theorem do the following. Write $a \cdot t$ for $a$ etc for all side measurements. Then expand in Taylor series with respect to $t$. Study the coefficients of $1, t, t^2$.*

**Theorem 5.6.3** (Hyperbolic triangle-inequality). *In a triangle on $\mathbb{H}^2$ we have $c < a + b$.*

*Proof.* We have
$$\begin{aligned}
\cosh c &= \cosh a \cosh b - \sinh a \sinh b \cos \gamma \\
\cosh(a + b) &= \cosh a \cosh b + \sinh a \sinh b
\end{aligned}$$
The first line is the Law of Cosines, the second line is a hyperbolic trig identity (prove it). We have $\sinh a \sinh b > 0$, $\cos \gamma \in (-1, 1)$. Therefore the RHS of the first line smaller than the RHS of the second line. Hence the same holds for the LHSs, i.e. $\cosh c < \cosh(a + b)$. However, $\cosh$ is strictly monotone increasing on $(0, \infty)$. $\qquad \square$

**Theorem 5.6.4** (Hyperbolic Law of Sines). *In $\triangle ABC$ in $\mathbb{H}^2$ (with usual notations)*

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$

*Proof.* From Law of Cosines we have

$$\cos \gamma = \frac{-\cosh c + \cosh a \cosh b}{\sinh a \sinh b}.$$

From this, using $1 + \sinh^2 = \cosh^2$ we have

$$\sin \gamma = \frac{\sqrt{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2 \cosh a \cosh b \cosh c}}{\sinh a \sinh b}.$$

Divide by $\sinh c$ and see that the RHS becomes symmetric in $a, b, c$. $\qquad \square$

**Proposition 5.6.5.** *In $\triangle ABC$ let $\gamma = \pi/2$, and let us use the standard notations. Then we have*

$$\sin \alpha = \frac{\sinh a}{\sinh c}.$$

The proof follows from the Law of Sines.

**Proposition 5.6.6.** *In $\triangle ABC$ let $\gamma = \pi/2$, and let us use the standard notations. We have*

$$\cos \alpha = \frac{\tanh b}{\tanh c}.$$

*Proof.* We have
$$\cos \alpha = \frac{\sqrt{\sinh^2 c - \sinh^2 a}}{\sinh c} = \frac{\sqrt{\cosh^2 c - \cosh^2 a}}{\sinh c}.$$
The first equality follows from the Proposition above, and the second equality is obtained by applying the identity $\sinh^2 +1 = \cosh^2$ in the numerator for both terms. Using the Pythagorean theorem and the $\sinh^2 +1 = \cosh^2$ identity again we obtain

$$\cos \alpha = \frac{\sqrt{\cosh^2 c - (\cosh^2 c/ \cosh^2 b)}}{\sinh c} = \frac{\sqrt{1 - \frac{1}{\cosh^2 b}}}{\frac{\sinh c}{\cosh c}} = \frac{\sqrt{\frac{\sinh^2 b}{\cosh^2 b}}}{\frac{\sinh c}{\cosh c}} = \frac{\tanh b}{\tanh c}.$$

$\square$

**Proposition 5.6.7.** *In $\triangle ABC$ let $\gamma = \pi/2$, and let us use the standard notations. We have*

$$\tan \alpha = \frac{\tanh a}{\sinh b}.$$

The proof is left as exercise, see Problem 5.10.15.

**Theorem 5.6.8** (Hyperbolic Dual Law of Cosines). *In $ABC$ triangle we have*

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c.$$

*Proof.* Drop a perpendicular from $C$ to $AB$ and let the resulting segments be $c_1$ and $c_2$, and the altitude $h$. We have $\cosh a = \cosh h \cosh c_2$, $\cosh b = \cosh h \cosh c_1$ from Pythagoras. Multiply these two equations. Then multiply the equality by $\cosh(c_1+c_2)$ and rewrite $\cosh h^2$ as $\sinh^2 h + 1$. We obtain

$$\cosh a \cosh b(\cosh c_1 \cosh c_2 + \sinh c_1 \sinh c_2) = (1 + \sinh^2 h) \cosh c_1 \cosh c_2 \cosh c.$$

Divide by $\cosh c_1 \cosh c_2$ and rearrange, we obtain

$$\cosh a \cosh b - \cosh c = \sinh^2 h \cosh c - \cosh a \cosh b \tanh c_1 \tanh c_2.$$

Apply the Law of Cosines on the LHS, and sin and cos formulas for right angled triangles above on the RHS and we obtain the formula of the theorem.                                    $\square$

*Remark* 5.6.9. The Hyperbolic Dual Law of Cosines calculates $\gamma$ if $\alpha, \beta$ and $c$ are given. In Euclidean geometry one can calculate $\gamma$ as long as $\alpha$ and $\beta$ are given, no need for $c$! Also, this theorem shows that the sum of the angles of a triangle is not constant, keeping $\alpha$ and $\beta$ the same, but changing $c$ *does* change $\gamma$. More on this in Section 5.8.

## 5.7   Angle of parallelism

Let us call closed half-lines *rays* (by closed we mean that the point where a line is cut into two is an element of the ray). In the Klein model a ray is a closed-open segment whose 'closed end' is inside the disk and its 'open end' is on the boundary of the disk.

**Definition 5.7.1.** Two rays $a$ and $b$ in $\mathbb{H}^2$ are called *limiting parallel* if in the Klein model $\mathbb{K}^2$ their open ends coincide (necessarily on the boundary of the disk), see Picture 5.3. Two lines will be called limiting parallel if rays can be chosen on them that are limiting parallel.
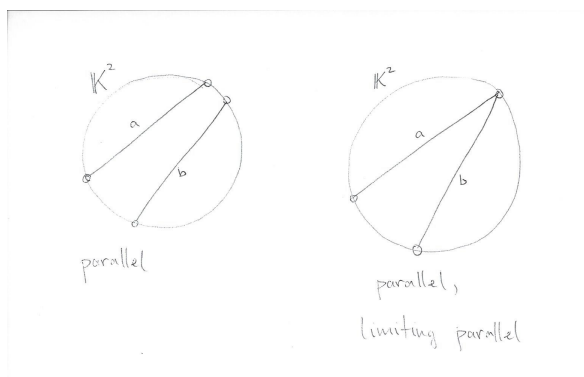


Figure 5.3: Limiting parallel lines in $\mathbb{K}^2$

Recall that the points of the boundary circle are *not* part of the Klein model (points of $\mathbb{K}^2$ are in the *interior* of the unit circle), so limiting parallel rays are disjoint, that is, their lines are parallel.

Consider a point $P$ not on a line $\ell$. If we draw rays connecting $P$ with points $C_1, C_2, \ldots$ of the line $\ell$, and we let $C_i$ go to infinity in one direction of the line $\ell$, then the limit position of the rays will be limiting parallel to (one side of) $\ell$, see Figure 5.4.

For a given $\ell$ and point $P \notin \ell$, there are two rays limiting parallel with the two directions of $\ell$. We will be interesting of the angle formed by these two rays. The next definition gives a name to (half of) this angle.

**Definition 5.7.2.** Consider a line $\ell$ and $A \in \ell$ (see Figure 5.5). Let $m$ be perpendicular to $\ell$ at $A$. Measure distance $d$ on $m$ from $A$, and let the resulting point be $B$. Consider the limiting parallel $x$ to $\ell$ passing through $B$. Denote the angle between $x$ and $m$ by $\Pi(d)$. Define $\Pi(d)$ the *angle of parallelism* corresponding to the distance $d$.
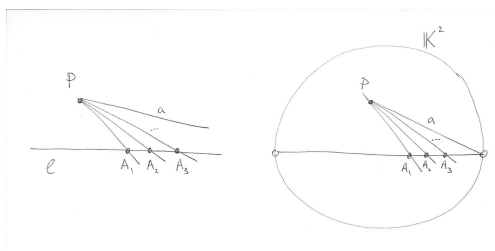
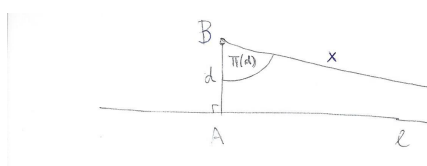Figure 5.4: Limiting parallel line as limit of intersecting lines



Figure 5.5: Angle of parallelism

**Theorem 5.7.3.** *We have* $\Pi(d) = \arcsin(1/\cosh d)$.

*Proof.* Intuitively, use the dual law of cosines to the "degenerate triangle"[2] $ABC$ where $C$ is the "ideal point" where $\ell$ and $x$ meet in angle 0. This is only an intuitive "proof". Here is how we can make it precise.
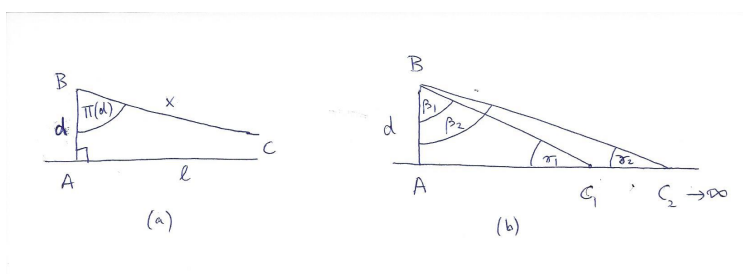


Figure 5.6: Illustration of the proof of Theorem 5.7.3

Consider the triangles in the second picture in Figure 5.6, with $C_1, C_2, \ldots \to \infty$. The dual law of cosines gives

$$\cos \gamma_i = \sin \beta_i \cosh d.$$

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[2]we will call such an object a 1x ideal triangle in Section 5.8

As $i \to \infty$, we have $\gamma_i \to 0$ due to Proposition 5.6.7, and $\beta_i \to \Pi(d)$. Hence in the limit we obtain

$$1 = \sin(\Pi(d)) \cosh d,$$

what proves the Theorem. $\qquad \square$

*Remark* 5.7.4. Using Calculus we can deduce from Theorem 5.7.3 that $\Pi(d)$ is strictly monotone decreasing, its limit in 0 is $\pi/2$ and its limit in $\infty$ is 0. Think over what these properties mean geometrically.

## 5.8 Area and angle-sum of a triangle

Consider a triangle, and let us "move" one of its vertex "to infinity". That is, in the Klein model let us consider a shape, which is a triangle with two vertices *in* the disk, but one vertex is *on the boundary*. This shape in $\mathbb{H}^2$ is called a *1x ideal triangle*. From inside $\mathbb{H}^2$ the description of a 1x ideal triangle is the following: Two points $A$ and $B$, their connecting segment, and a pair of limiting parallel rays starting at $A$ and $B$.

Similarly, if two vertices of a triangle are moved to the boundary circle of $\mathbb{K}^2$ then we call the resulting object a 2x ideal triangle. If all three vertices are moved to the boundary circle of $\mathbb{K}^2$ then we call it a 3x ideal triangle. If we simply say *ideal triangle*, we mean a 3x ideal triangle.



triangle          1x ideal triangle          2x ideal triangle          3x ideal triangle
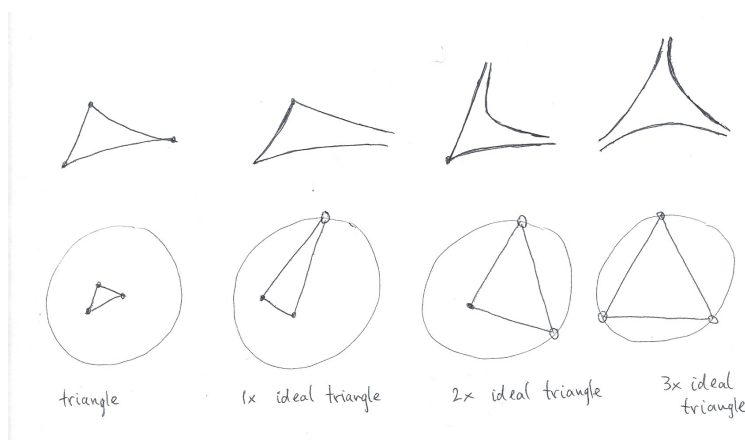
Think over what would be a description of 2x and 3x ideal triangles from inside $\mathbb{H}^2$ (that is, use a description that does not mention the notion "boundary of the Klein model", instead use the notion limiting parallel). In the Euclidean plane 3x ideal triangles do not exist. Do 1x or 2x ideal triangles exist?

**Theorem 5.8.1.** *The area of any ideal triangle in $\mathbb{H}^2$ is $\pi$.*

We will not prove this theorem in this class.

*Remark* 5.8.2. Let us discuss what kind of mathematical reasoning would go into a proof. Recall that the notion of area should be $\iint_T 1 dA$ where $dA$ is an area-form. Hence, first we would need to figure out what an "area form" is on $\mathbb{H}^2$ (based on our knowledge of "length" and "angle"). Then we would need to set up a double integral of $1 dA$ on an ideal triangle. Then we would need to calculate that double integral.[3] The theorem above claims that we would get $\pi$. This result should be surprising for two reasons: why is the integral finite, and why is it independent of which ideal triangle we choose? Well, surprisingly or not, the Theorem holds.

Consider a 2x ideal triangle. It is determined by the angle of the two sides that intersect. Let us call it *the angle* of the 2x ideal triangle.

**Theorem 5.8.3.** *The area of a 2x ideal triangle with angle $\theta$ is $\pi - \theta$.*

*Proof.* Let $f(\theta)$ denote the area of a 2x ideal triangle with angle $\theta$. Thus $f$ is a function defined on $[0, \pi]$, and we want to show that it is actually the $\pi - \theta$ function.
Our first claim is that

$$f(\theta_1 + \theta_2) = f(\theta_1) + f(\theta_2) - \pi \tag{5.3}$$

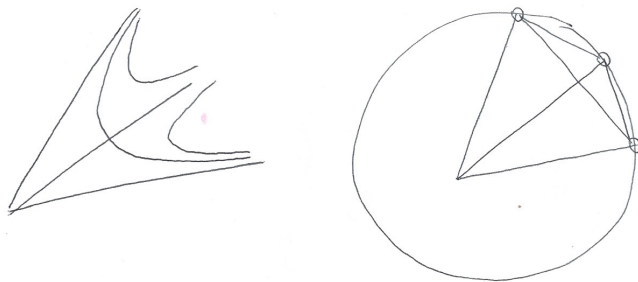holds. This claim follows from Figure 5.7, and Theorem 5.8.1 (think this over).



Figure 5.7: $f(\theta_1 + \theta_2) = f(\theta_1) + f(\theta_2) - \pi$

We will need two more claims:

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[3]A shortcut of the argument above would be if we first show that any two ideal triangles can be mapped into each other by a Lorentz transformation—which is a length preserving, and hence area preserving transformation of $\mathbb{H}^2$. Then we just need to calculate the area of *one* ideal triangle.

- $f(\pi) = 0$ (in fact $\lim_{\theta \to \pi} f(\theta) = 0$);

- $f$ is continuous.

We will not give precise argument for either of these, but both should be intuitively clear: for $\theta = \pi$ the 2x ideal triangle degenerates to a line, whose area is 0; and continuity is the intuitive statement that for small change of $\theta$ the 2x ideal triangle goes through a "small" change, and hence its area goes through a "small" change too.

Now we claim that if a continuous $f$ satisfies (5.3) and $f(\pi) = 0$, then it must be $f(\theta) = \pi - \theta$. This claim is proved by the trick of defining another function $g(\theta) = \pi - f(\theta)$. The properties of $f$ imply that $g$ is continuous, $g(\pi) = \pi$, and

$$g(\theta_1 + \theta_2) = g(\theta_1) + g(\theta_2) \tag{5.4}$$

(think over how we got this functional equation from (5.3)).
From $g(\pi) = \pi$, and iterated application of (5.4) we find that

$$g\left(\frac{1}{2}\pi\right) = \frac{1}{2}\pi, \quad g\left(\frac{1}{4}\pi\right) = \frac{1}{4}\pi, \quad g\left(\frac{1}{8}\pi\right) = \frac{1}{8}\pi, \ldots,$$

eventually $g(1/2^m \cdot \pi) = 1/2^m \cdot \pi$ for any $m$. From this, again by iterated application of (5.4) we obtain that

$$g\left(\frac{k}{2^m}\pi\right) = \frac{k}{2^m}\pi$$

for any $k$ and $m$. However, the numbers of the form $k/2^m$ are dense, that is, for any $\theta$ there is a sequence $k_n/2^{m_n}$ whose limit is $\theta$. We have

$$g(\theta) = g\left(\lim_{n \to \infty} \frac{k_n}{2^{m_n}}\right) = \lim_{n \to \infty} g\left(\frac{k_n}{2^{m_n}}\right) = \lim_{n \to \infty}\left(\frac{k_n}{2^{m_n}}\right) = \theta.$$

In the second equality we used the fact that $g$ is continuous.

Now that we verified that $g(\theta) = \theta$, we also have $f(\theta) = \pi - g(\theta) = \pi - \theta$, what we wanted to prove. $\square$

**Theorem 5.8.4.** *Let ABC be a triangle in $\mathbb{H}^2$ with usual notations. We have*

$$\mathrm{Area}(\triangle ABC) = \pi - (\alpha + \beta + \gamma).$$

*Proof.* The statement follows from Figure 5.8. (The reader is advised to stop reading and construct the proof based on the Figure.)
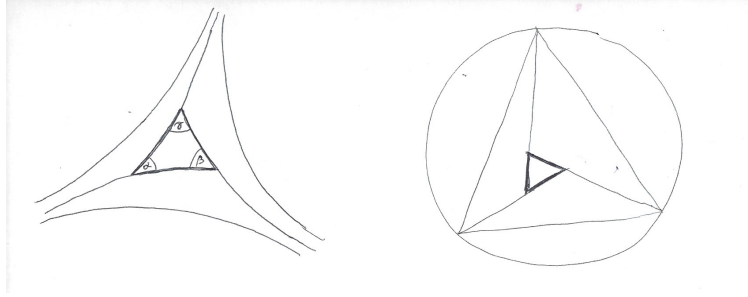
Figure 5.8: Illustration of the proof of Theorem 5.8.4

Starting with the $\triangle ABC$, extend the picture with three 2x ideal triangles (with angles $\pi - \alpha, \pi - \beta, \pi - \gamma$ respectively) as shown in the figure. The union of these four shapes is an ideal triangle. We have

$$\begin{aligned}
\text{Area}(\triangle ABC) &= \pi - (\text{Area(the three 2x ideal triangles of the figure)}) \\
&= \pi - ((\pi - (\pi - \alpha)) + (\pi - (\pi - \beta)) + (\pi - (\pi - \gamma))) \\
&= \pi - (\alpha + \beta + \gamma),
\end{aligned}$$

what we wanted to prove.                                                                      $\square$

**Corollary 5.8.5.** *For a triangle in the hyperbolic plane we have* $\alpha + \beta + \gamma = \pi - \text{Area}$.

**Corollary 5.8.6.** *The sum of the angles of a triangle in* $\mathbb{H}^2$ *is always strictly less than* $\pi$. *For small (area) triangles the sum of angles is close to* $\pi$ *(the "Euclidean value"). For large (area) triangles the sum of angles is close to 0.*

## 5.9   Hyperbolic planes of different curvatures

When studying spherical geometry, we did not only study one of them (the radius 1 sphere), but infinitely many of them at the same time, namely the geometry of the spheres of any (positive) radius. We could have done the same in hyperbolic geometry by starting our studies with

$$\mathbb{H}^2 = \{X \in \mathbb{R}^3 : \Phi(X, X) = -k^2\}$$

for some positive constant $k$. The name of this $k$ is *pseudo-radius*. We restricted ourselves to the $k = 1$ special case in order to keep our calculations and formulas simple. Yet, if we considered the general $k$ case, we would have arrived at similar looking theorems. Here we list them:

- The natural parametrization of a curve is $s(t) = \cosh(t/k)X + k\sinh(t/k)v$ for $X \in \mathbb{H}^2$, $v \in T_X\mathbb{H}^2$ with $\Phi(v, v) = 1$.

- The Law of Cosines is

$$\cosh\left(\frac{c}{k}\right) = \cosh\left(\frac{a}{k}\right)\cosh\left(\frac{b}{k}\right) - \sinh\left(\frac{a}{k}\right)\sinh\left(\frac{b}{k}\right)\cos(\gamma).$$

- Pythagorean theorem: In a triangle $\gamma = \pi/2$ if and only if

$$\cosh\left(\frac{c}{k}\right) = \cosh\left(\frac{a}{k}\right)\cosh\left(\frac{b}{k}\right).$$

- The Law of Sines is

$$\frac{\sin\alpha}{\sinh\left(\frac{a}{k}\right)} = \frac{\sin\beta}{\sinh\left(\frac{b}{k}\right)} = \frac{\sin\gamma}{\sinh\left(\frac{c}{k}\right)}.$$

- The Dual Law of Sines is

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh\left(\frac{c}{k}\right).$$

- The angle of parallelism is

$$\Pi(d) = \arcsin\left(\frac{1}{\cosh\left(\frac{d}{k}\right)}\right).$$

- $\text{Area(ideal triangle)} = k^2\pi$, $\text{Area(2x ideal triangle)} = k^2(\pi - \theta)$.

- We have
$$\text{Area}(\triangle ABC) = k^2\left(\pi - (\alpha + \beta + \gamma)\right),$$
  and hence
$$\alpha + \beta + \gamma = \pi - \frac{\text{Area}(\triangle ABC)}{k^2}.$$

The quantity $-1/k^2$ is called the *Gaussian curvature* of the hyperbolic plane, and is denoted by $K$. In this language the last formula reads

$$\alpha + \beta + \gamma = \pi + K\,\text{Area}(\triangle ABC),$$

or, in calculus language

$$\alpha + \beta + \gamma = \pi + \iint_{\triangle ABC} K dA.$$

Observe that this way of writing the angle sum formula (called Gauss-Bonnet formula) is the *same* as in spherical, or Euclidean geometry (cf. Remark 4.5.6). As this coincidence suggests, indeed there is a way of treating the three geometries together within a more general framework: Riemannian geometry.

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## 5.10   Problems

**5.10.1.** Prove that $\Phi$ is non-degenerate. That is, suppose that $v \in \mathbb{R}^3$ is such that $\Phi(v, u) = 0$ for all $u \in \mathbb{R}^3$. Prove that $v = 0$.

**5.10.2.** Let $X \in \mathbb{H}^2$. Let $X^\perp = \{v \in \mathbb{R}^3 : \Phi(X, v) = 0\}$. Prove that $X^\perp$ is a plane (in the usual Euclidean sense).

**5.10.3.** (cont.) Prove that all vectors in $X^\perp$ are space-like. [Hint-1: find the "steepest" vector in $X^\perp$ and check its angle with the $xy$-plane. Hint-2 (if you learned Linear Algebra): use Sylvester's Law of Inertia.]

**5.10.4.** Consider the curve $\phi(t) = (t, 0, \sqrt{t^2 + 1})$ in $\mathbb{R}^3$.

(a) Prove that this curve is in $\mathbb{H}^2$.

(b) Prove that the image of this curve in $\mathbb{H}^2$ is a line.

(c) What is the speed of this curve at $t = -2, -1, 0, 1, 2$. [Hint: speed="length (using $\Phi$) of the derivative".]

**5.10.5.** Prove that Lorentz matrices form a group.

**5.10.6.** Prove that the matrices (5.1), (5.2) are Lorentz matrices.

**5.10.7.** Complete the proof of Proposition 5.2.3.

**5.10.8.** Prove that for any two points $X, Y \in \mathbb{H}^2$ there is a Lorentz matrix $AX = Y$.

**5.10.9.** Let $X, Y \in \mathbb{H}^2$, $v \in T_X \mathbb{H}^2$, $w \in T_Y \mathbb{H}^2$ with $\Phi(v, v) = 1$, $\Phi(w, w) = 1$. Prove that there is a Lorentzian matrix $A$ such that $AX = Y$, $Av = w$.

**5.10.10.** Find and prove a formula for $\sinh(x + y)$, $\cosh(x + y)$ in terms of $\sinh x$, $\cosh x$, $\sinh y$, $\cosh y$.

**5.10.11.** Let $X = (2, 2, 3)$, $u = (1, 5, 4)$, $v = (-5, -4, -6)$. Verify that $X \in \mathbb{H}^2$, $u, v \in T_X \mathbb{H}^2$. Let $a$ be the ray starting in $X$ in the direction of $u$. Let $b$ be the ray starting at $X$ in the direction of $v$. Give parametrizations of $a$ and $b$ (warning: $u$ and $v$ are not unit vectors w.r.t. $\Phi$). Find the angle between $a$ and $b$.

**5.10.12.** Prove the following version of Desargues' theorem in $\mathbb{H}^2$. The triangles $ABC$ and $A'B'C'$ are perspective w.r.t. to the point $P$. Suppose that $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist. Prove that $K, L, M$ are collinear. [Hint: this problem only considers point-line adjacencies, no metric. Then which model of the the hyperbolic plane is more useful? You will be able to reduce this problem the Desargues version we already know.]

**5.10.13.** Consider a horizontal plane in $\mathbb{R}^3$ whose intersection with $\mathbb{H}^2$ is not empty. (a) Prove that the intersection is a circle in the Euclidean geometry of $\mathbb{R}^3$. (b) Prove that the intersection is a circle in the hyperbolic geometry $\mathbb{H}^2$.

**5.10.14.** Consider a plane in $\mathbb{R}^3$ whose intersection with $\mathbb{H}^2$ is not empty. Prove that the intersection is a circle in the hyperbolic geometry $\mathbb{H}^2$.

**5.10.15.** Consider a right angled triangle in $\mathbb{H}^2$, with right angle $\gamma$. Find a formula for $\tan \alpha$ in terms of $a$ and $b$ (Proposition 5.6.7).

**5.10.16.**

(a) In $\mathbb{H}^2$ find the angles of the triangle with sides $3, 4, 5$.

(b) In $\mathbb{H}^2$ find the angles of the triangle with sides $0.03, 0.04, 0.05$.

(c) Find the angles of the triangle with sides $3, 4, 5$ in Euclidean geometry.

(d) Find the angles of the triangle with sides $0.03, 0.04, 0.05$ in Euclidean geometry.

**5.10.17.** Let $\triangle ABC$ and $\triangle A'B'C'$ have the same corresponding angles in $\mathbb{H}^2$ (that is, they are *similar*). Prove that they have the same corresponding sides as well (that is, they are *congruent*).

**5.10.18.** Let $\triangle ABC$ and $\triangle A'B'C'$ be perspective with respect to a line $\ell$ in $\mathbb{H}^2$, and suppose that $AA'$ and $BB'$ are limiting parallel. Prove that $AA'$ and $CC'$ are also limiting parallel.

**5.10.19.** Prove that the function $\Pi(d)$ is strictly monotone decreasing on $[0, \infty)$, and find its limit at $d = 0$ and $d = \infty$.

**5.10.20.** Find and prove a formula relating the area with the sum of the angles for an $n$-gon in $\mathbb{H}^2$.

**5.10.21.** Suppose we want to *force* that *Hyperbolic Geometry* is in fact *Spherical Geometry* on a sphere with some radius. For this we look at the Pythagorean theorems. We could achieve our goal if we could force $\cosh(x)$ to be equal with $\cos(x/R)$ for some $R$. Use the Taylor expansions of cosh and cos (or, alternatively, exponential formulas for trig and hyperbolic trig functions), to solve

$$\cosh(x) = \cos(x/R) \qquad (\forall x)$$

for $R$. Then complete the sentence: "Hyperbolic Geometry is like Spherical Geometry on a sphere with radius _____".