Bow varieties, characteristic classes, 3d mirror symmetry

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U. Bonn Oberseminar Representation Theory

Joint work with Yiyan Shou

Work in progress with Tommaso Botta

Related work with A. Varchenko

A. Smirnov

Z. Zhou

L. Rozansky

TOPIC-1

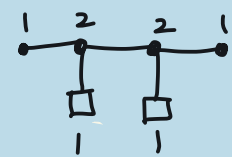
- characteristic classes of singularities
- coincidence

e.g.

elliptic char. classes of singularities on $T^*Gr_2\mathbb{C}^5$

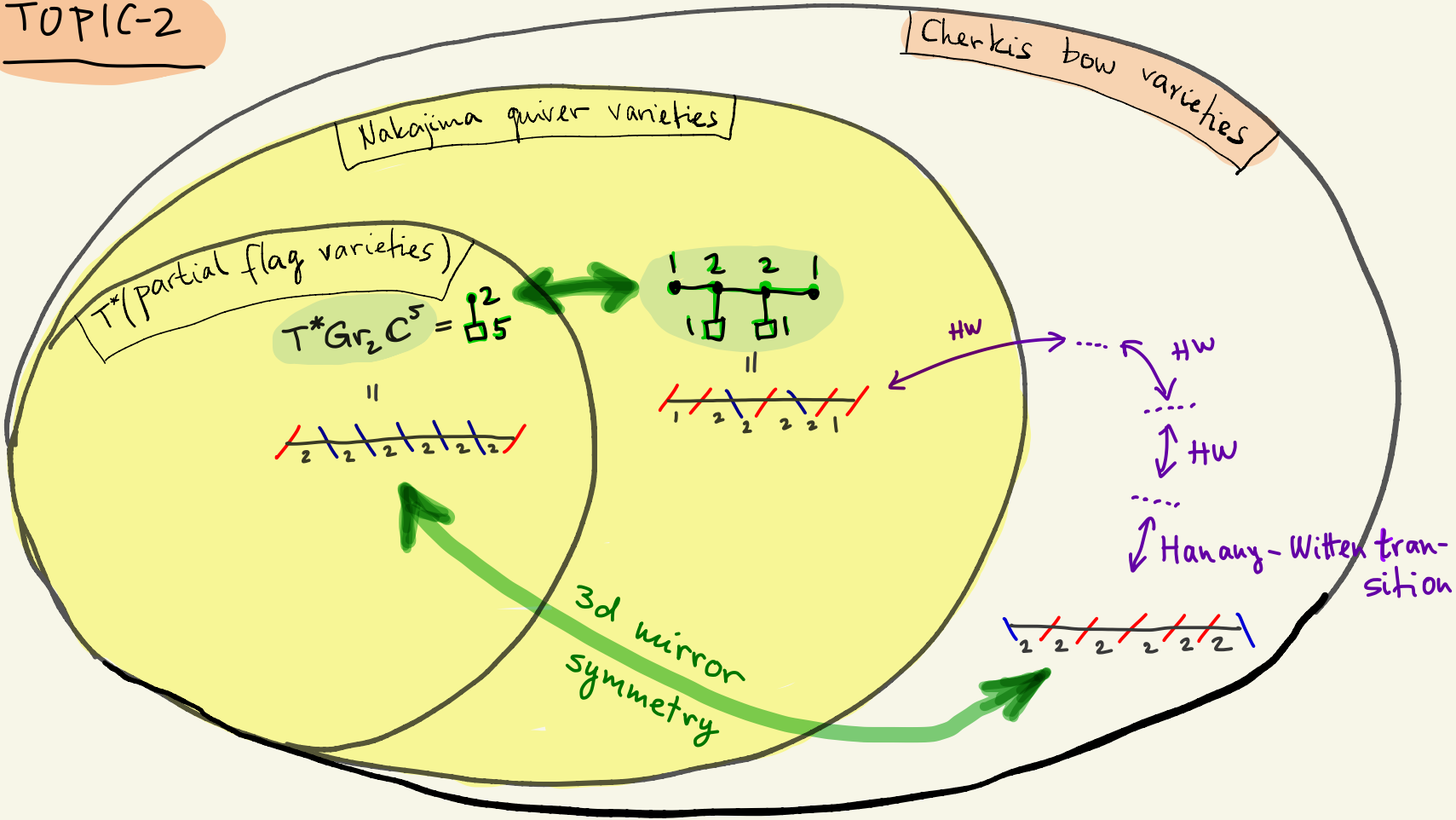
$\longleftrightarrow$

"coincide"

elliptic char. classes of singularities on $N\Big($  $\Big)$

"3D mirror symmetry for characteristic classes"

TOPIC-2

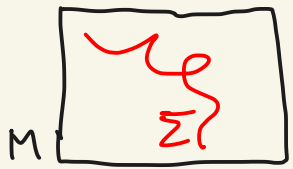

Cherkis bow varieties
Nakajima quiver varieties
T*(partial flag varieties)
$T^*Gr_2 C^5 =$
HW
HW
HW
Hanany-Witten transition
3d mirror symmetry

Class of a subvariety vs geometric enumeration

Key concept

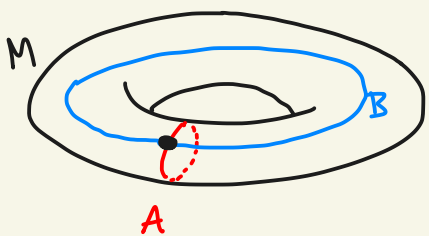
$$\Sigma \subset M \longrightarrow [\Sigma] \in H^*(M)$$

smooth



fundamental class of $\Sigma \subset M$

Example



$$[A], [B] \in H^1(M)$$

$$\int_M [A] \cdot [B] = |A \cap B|$$

counted the right way

Example

$$\underset{\Sigma}{\underset{\parallel}{\{\text{rk} \le 2 \text{ matrices}\}}} \subseteq \underset{M}{\underset{\parallel}{\mathbb{C}^{4\times 4}}} \circlearrowleft GL_4 \times GL_4$$

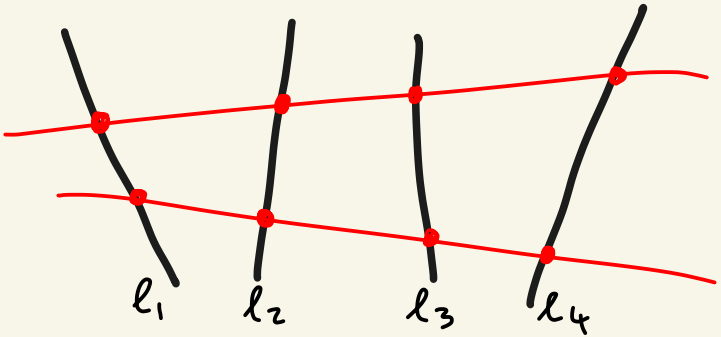
$$[\Sigma] = \ldots + 2\beta_1\beta_2\beta_3\beta_4 + \ldots \in H^*_{GL_4\times GL_4}(M)$$

$$H^*_{GL_4\times GL_4}(M) = \mathbb{Z}[\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\,\beta_1\,\beta_2\,\beta_3\,\beta_4]^{S_4\times S_4}$$

Solution to:

$\ell_1, \ell_2, \ell_3, \ell_4$ lines $\subset \mathbb{P}^3$

How many lines intersect all four?



- $\{\text{rk} \leq 2 \text{ matrices}\} \subseteq \mathbb{C}^{4\times 4}$

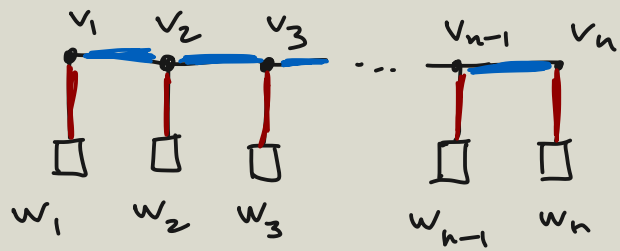
$$\begin{array}{ccc} \{\text{rk} \leq 2 \text{ matrices}\} & \subseteq & \mathbb{C}^{4\times 4} \\ \| & & \| \\ \Sigma & & M \end{array}$$

$\Sigma \rightsquigarrow$ "attracting mfd's"

$M \rightsquigarrow$ quiver varieties

- $[\Sigma] \rightsquigarrow$ Stable envelope class
- $H^* \rightsquigarrow K \rightsquigarrow \text{Ell}$

Nakajima quiver varieties



quiver Q

$\rightsquigarrow$ $\mathcal{N}(Q)$

quiver variety

Ex

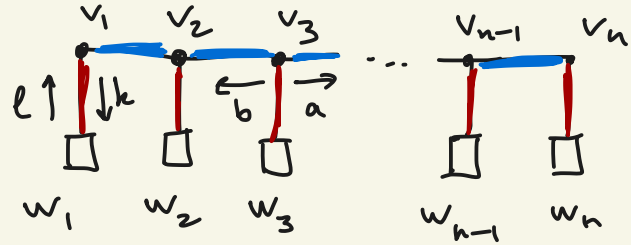
$\mathcal{N}\left(\begin{smallmatrix} k \\ n \end{smallmatrix}\right) = T^*Gr_k\mathbb{C}^n$

$\mathcal{N}\left(k_1 \le k_2 \le k_3 \;;\; n\right) = T^*\mathcal{F}_{k_1,k_2,k_3,n}$

$\mathcal{N}\left(\begin{smallmatrix} 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right) = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

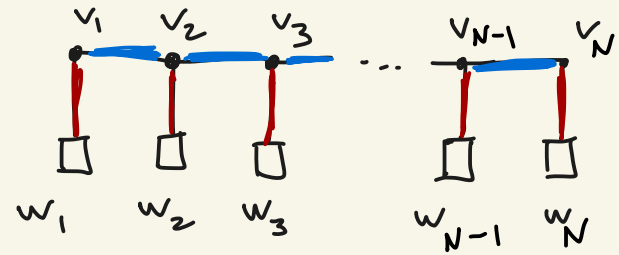
def

$$\mathcal{N}\left(\quad\right) = \cdots$$



- $R := \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$
- $\mu : R \oplus R^* \longrightarrow \bigoplus_i \mathrm{End}(\mathbb{C}^{v_i})$ $\qquad \mu = [a, b] - \ell k$
- $\mathcal{N}(Q) := \mu^{-1}(0)^{ss} \big/ \times_i GL_{v_i}$

$\mathcal{N}(Q)$ (type A)



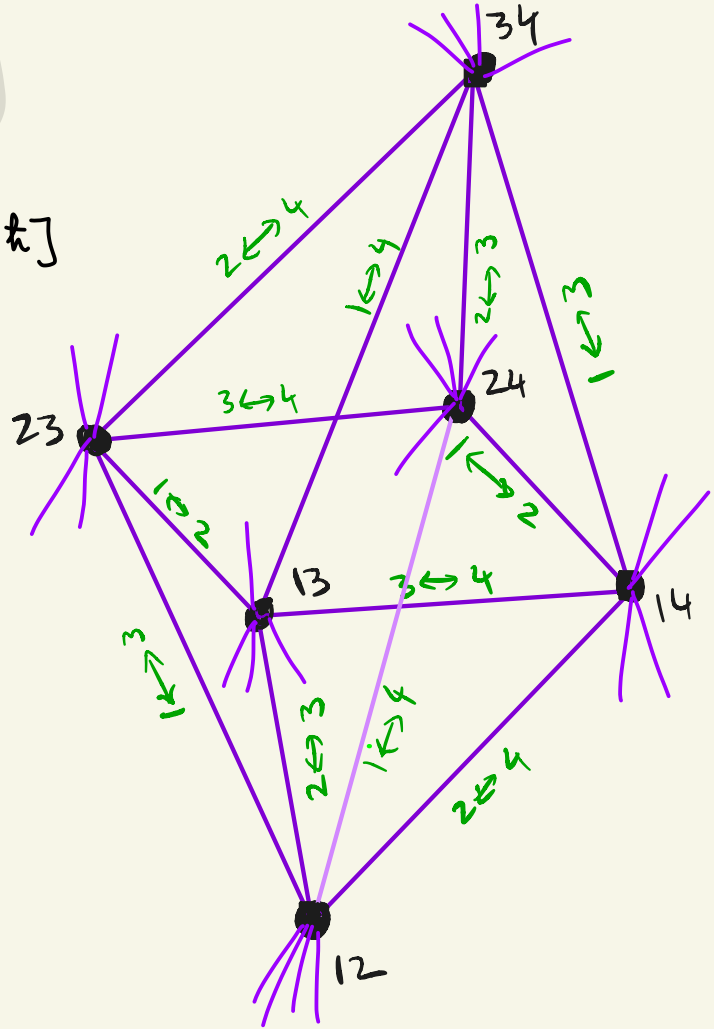
- smooth
- holomorphic symplectic
- $T = \left(T^{w_1} \times T^{w_2} \times \dots \times T^{w_N}\right) \times \mathbb{C}^*_\hbar$ action
- finitely many fixed pts
- "tautological" v_1-, v_2-, ..., v_N-bundles

$$H_T^*(N(Q)) \xrightarrow{\text{Loc}} \bigoplus_{p \text{ T-fixed points}} \underbrace{H_T^*(p)}_{\mathbb{C}[z_1,\dots,z_n,\hbar]}$$

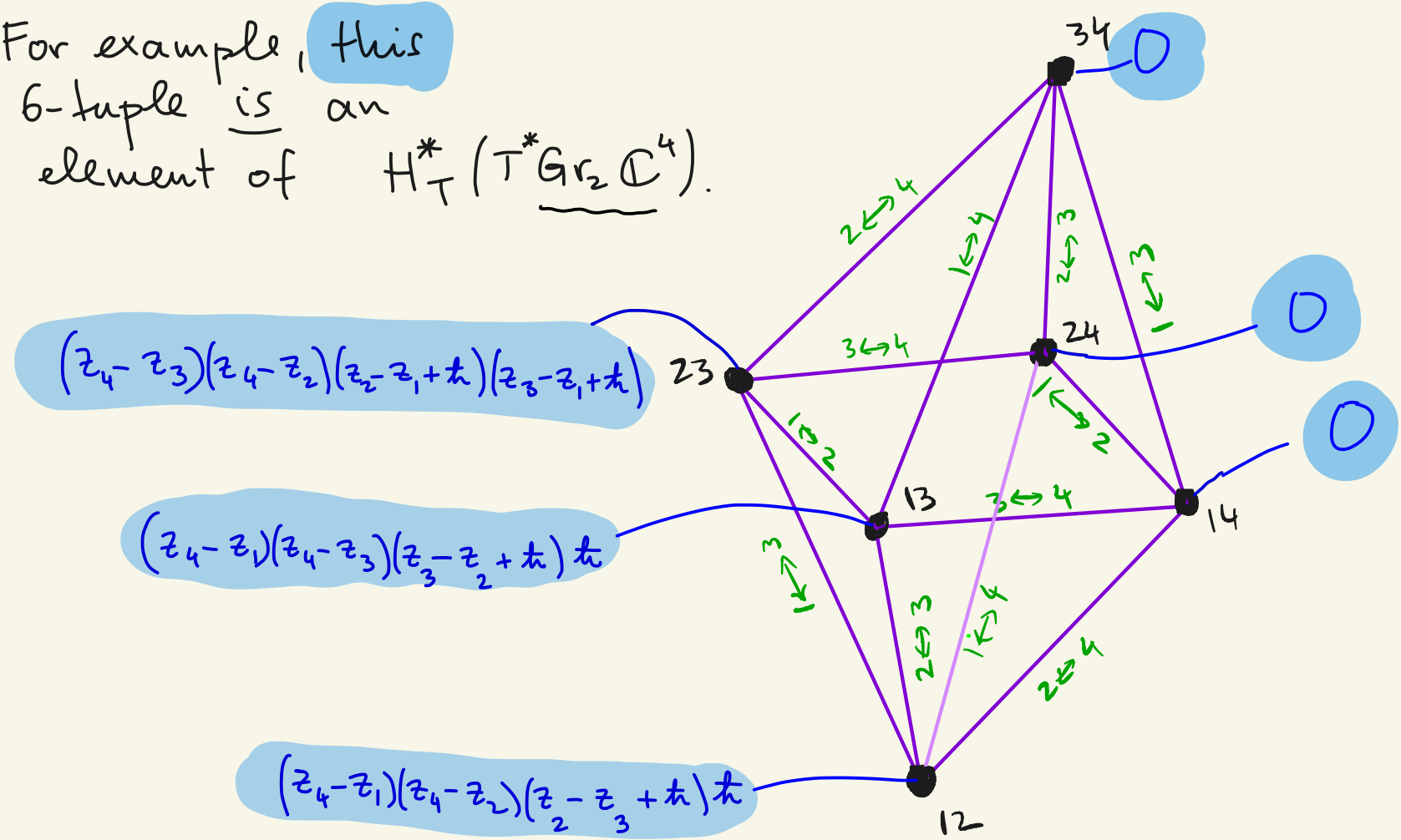
Localization map

$\mathrm{im}(\mathrm{Loc}) = ?$

constraints among the components

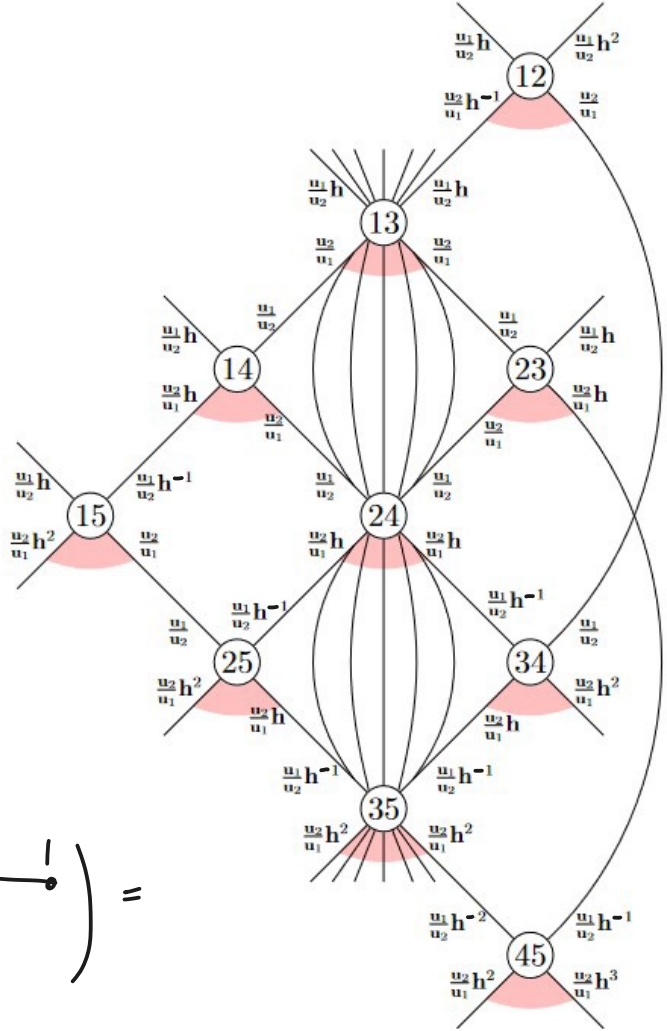


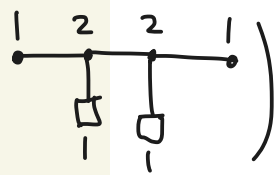
For example, this 6-tuple is an element of $H^*_T(T^*Gr_2\mathbb{C}^4)$.



Warning

- $T^*Gr_2\mathbb{C}^4$ was special ("GKM")
- In general the constraints among components are more restrictive



$\mathcal{N}\Big($  $\Big) =$

Towards $\mathrm{Stab}_p \in H_T^*(\mathcal{N}(Q))$

(torus fixed point)

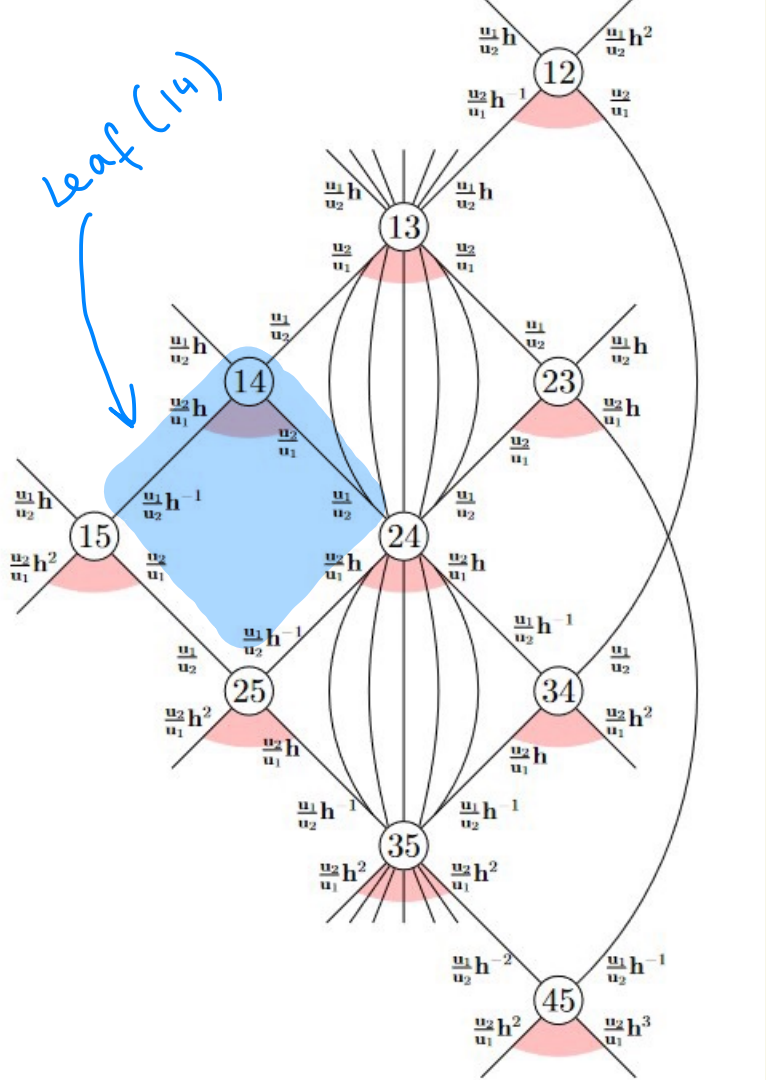
- fix $\mathbb{C}^* \xrightarrow{\sigma} T$

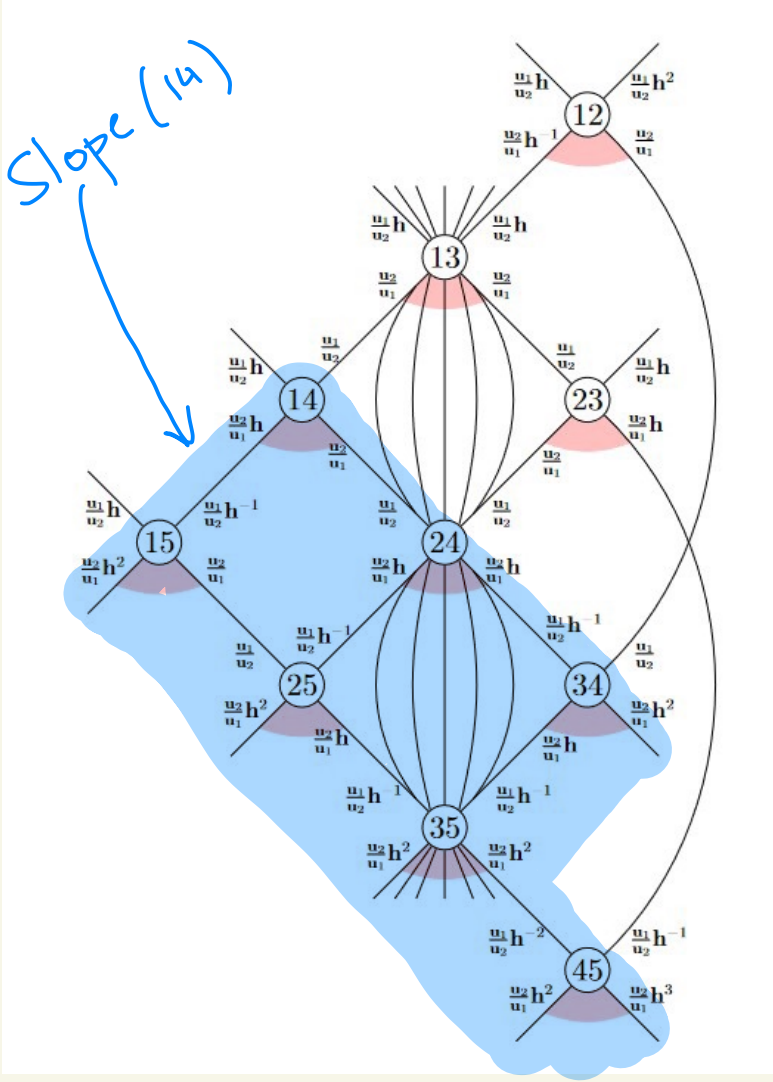
  $z \longmapsto (z, z^2, z^3, \dots, z^n, 1)$

- $p \in \mathcal{N}(Q)^T$ $\quad \mathrm{Leaf}(p) = \{ x \in \mathcal{N}(Q) : \lim_{z \to 0} \sigma(z) x = p \}$

- $p' \leq p$ if $\overline{\mathrm{Leaf}(p)} \ni p'$

- $\mathrm{Slope}(p) := \bigcup_{p' \leq p} \mathrm{Leaf}(p')$


Leaf (14)
12
13
14
23
15
24
25
34
35
45


Slope (14)
12
13
14
23
15
24
25
34
35
45

def [MO] $\mathrm{Stab}_p \in H^*_T(\mathcal{N}(Q))$ is the unique class

Maulik–Okounkov

- **support axiom:**

  supported on $\mathrm{Slope}(p)$

- **normalization axiom:**

  $$\mathrm{Stab}_p\big|_p = e\left(\nu\left(\mathrm{Slope}_p\right)\right)$$

- **boundary axiom:**

  $\mathrm{Stab}_p\big|_q$ divisible by $\hbar$ for $p \neq q$

Stab_{14}

@ 12, 13, 23 $= 0$

@ 14 $= (z_1 - z_2)(z_1 - z_2 + \hbar)$

@ 15 divisible by $(z_1 - z_2 + \hbar)$
divisible by $\hbar$

@ 24 divisible by $(z_1 - z_2)$
divisible by $\hbar$

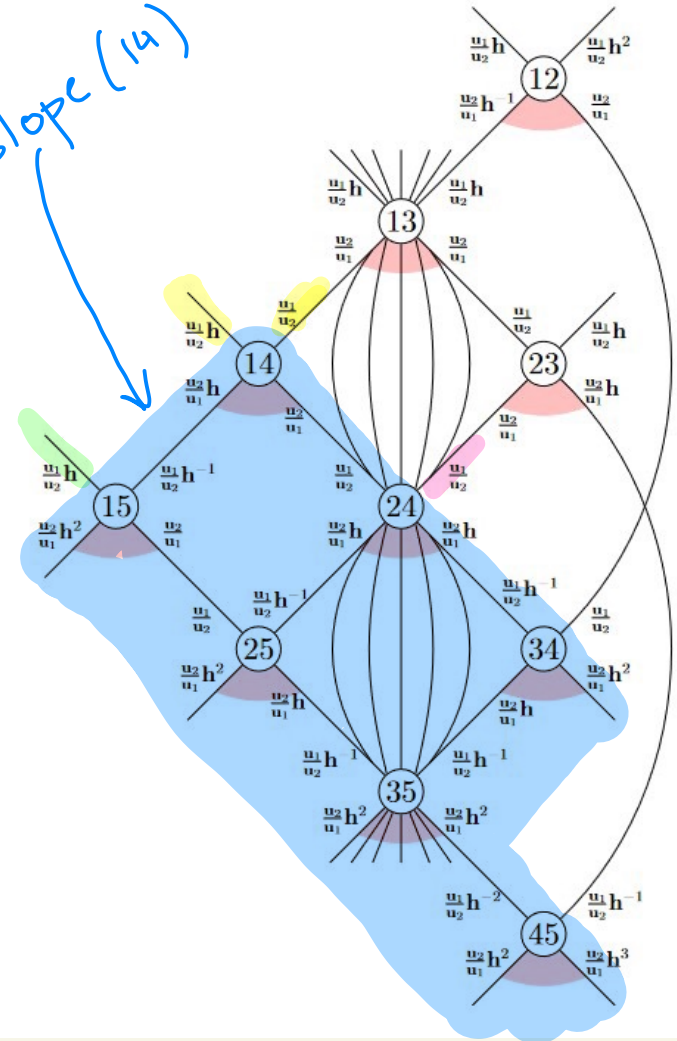
@ 25 divisible by $\hbar$

@ 34 divisible by $(z_1 - z_2)$
divisible by $\hbar$

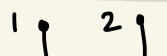
@ 35 divisible by $\hbar$

@ 45 divisible by $(z_1 - z_2 - \hbar)$
divisible by $\hbar$

Slope (14)



Characteristic classes induce quantum group actions

$H_T^*\Big($  $\Big) = \mathbb{C}(z_1, z_2, z_3, z_4)^4$ 4 fixed pts

<u>Stab's</u>: 1_1 $(z_2 - z_1, 0)_{2,3}$ $(z_2 - z_1 + \hbar, \hbar)_{2,3}$ 1_4

<u>Stab's</u>: 1_1 $(z_2 - z_1 + \hbar, \hbar)_{2,3}$ $(0, z_1 - z_2)_{2,3}$ 1_4

<u>Fact</u>: transition matrix:

Yang's R-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Cor</u> $H_T^*\Big($ $\Big)$ is a $Y_\hbar(\mathfrak{gl}_2)$-module geometrically

Summary so far

$$\underbrace{\mathrm{Stab}_p}_{\text{characteristic class of singularity}} \in \underbrace{H_T^*}_{\text{moment graph model}}(\overbrace{N(Q)}^{\text{quiver variety}})$$

<u>rem</u> key ingredients of geometric quantum group actions

Fact some (not all) quiver varieties come in pairs $(X, X^!)$ such that

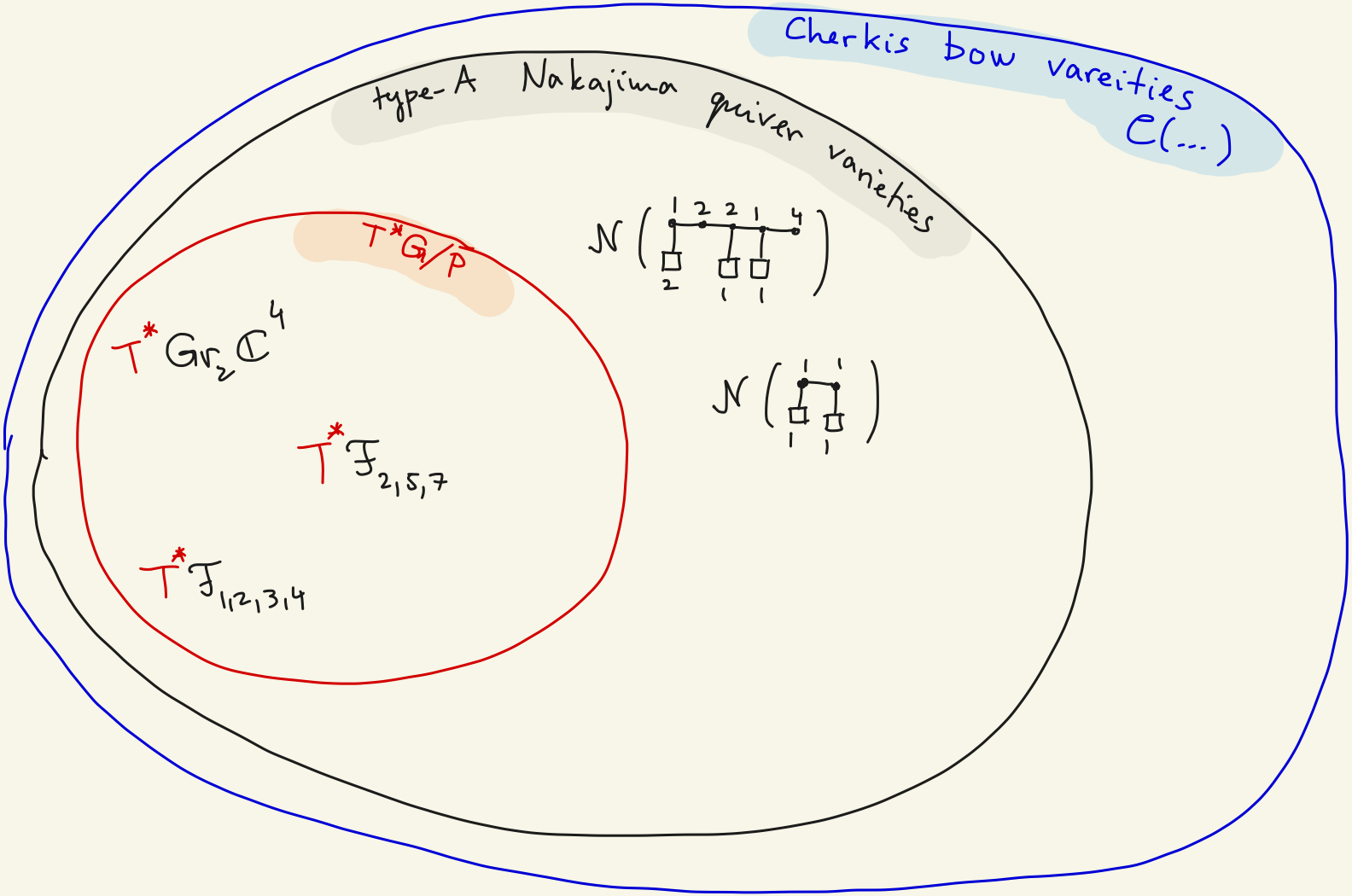
fix-pts: $X \qquad X^!$

$$X^T \xleftrightarrow{\text{bijection}} (X^!)^{(T^!)}$$

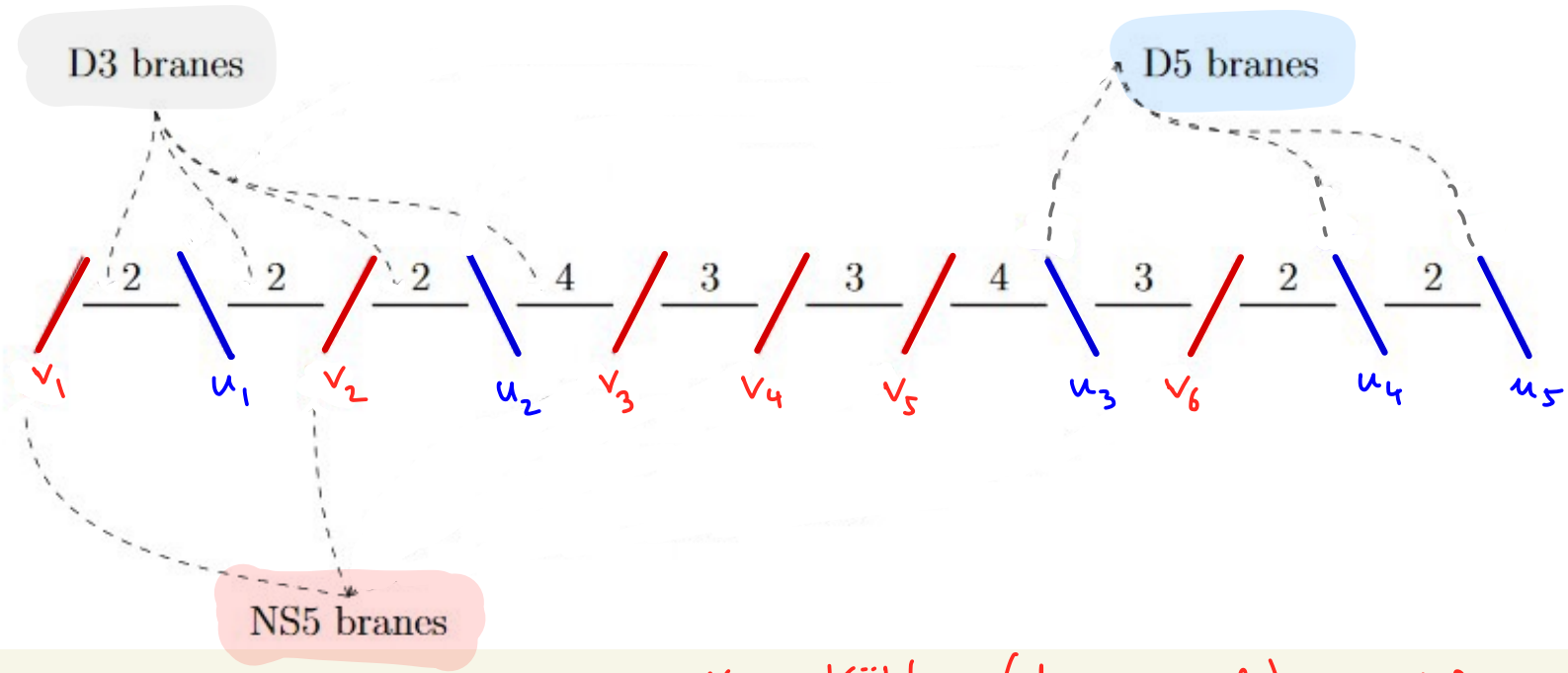
char classes: $\mathrm{Stab}^{\mathrm{Ell}}_P$ "$=$" $\mathrm{Stab}^{\mathrm{Ell}}_{P^!}$

in some sense (later)

dim				
8	$T^*Gr_2\mathbb{C}^4$	$\longleftrightarrow$	$\mathcal{N}\left(\begin{array}{c}1\ \ 2\ \ 1\\ \bullet\!-\!\bullet\!-\!\bullet\\ \square_2\end{array}\right)$	4 [RSVZ]
12	$T^*Gr_2\mathbb{C}^5$	$\longleftrightarrow$	$\mathcal{N}\left(\begin{array}{c}1\ \ 2\ \ 2\ \ 1\\ \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\\ 1\ \ \ \ 1\end{array}\right)$	4
64	$T^*\mathcal{F}_{2,6,10}$	$\longleftrightarrow$	$\mathcal{N}\left(\begin{array}{c}1\ 2\ 3\ 4\ 5\ 6\ 5\ 4\ 2\\ \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\\ \square_2\ \ \square_1\end{array}\right)$	16 [RS]
8	$\mathcal{N}\left(\begin{array}{c}1\ \ 1\ \ 2\ \ 1\\ \bullet\ \ \bullet\!-\!\bullet\!-\!\bullet\\ 2\ \ 2\ \ \ \ 2\end{array}\right)$	$\longleftrightarrow$	$\mathcal{N}\left(\begin{array}{c}1\ \ 1\ \ 2\ \ 1\ \ 1\\ \bullet\ \ \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\\ 1\ \ 1\ \ 2\ \ 1\end{array}\right)$	10
	T^*G/B	$\longleftrightarrow$	T^*G^L/B^L	[RW 2020]
32	$T^*\mathcal{F}_{2,5,7}$	$\longleftrightarrow$	$\mathcal{N}(\nexists)$	


Cherkis bow vareities
$\mathcal{C}(\ldots)$
type-A Nakajima quiver varieties
T^*G/P
$T^*Gr_2\mathbb{C}^4$
$T^*\mathcal{F}_{2,5,7}$
$T^*\mathcal{F}_{1,2,3,4}$
$\mathcal{N}\Big($ 1 2 2 1 4 / 2 1 1 $\Big)$
$\mathcal{N}\Big($ 1 1 / 1 1 $\Big)$

Brane diagrams



v_i : Kähler (dynamical) variables

u_i : equivariant variables

brane diagram $\mathcal{D}$ $\longrightarrow$ $\mathcal{C}(\mathcal{D})$

Cherkis bow variety

Cherkis:

moduli space of unitary instantons on multi-Taub-NUT spaces

(key: Nahm's equation)

Nakajima-Takayama

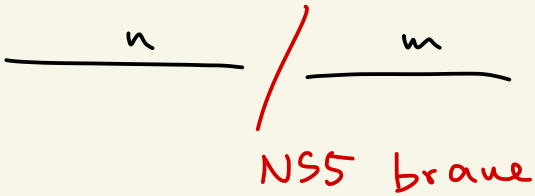
Hamiltonian reduction of representations of certain quivers with relations

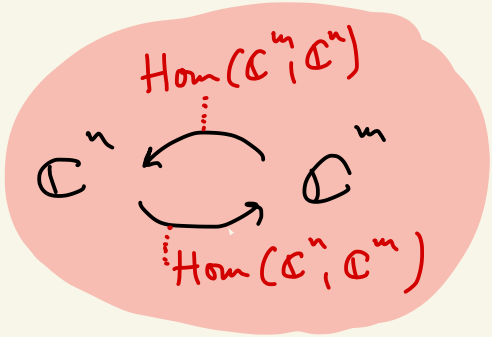
$\sim$

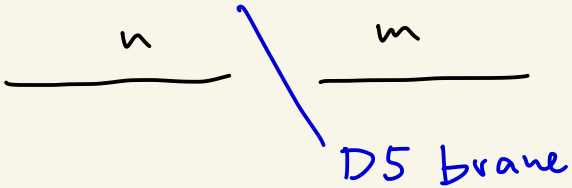
Rozansky-R

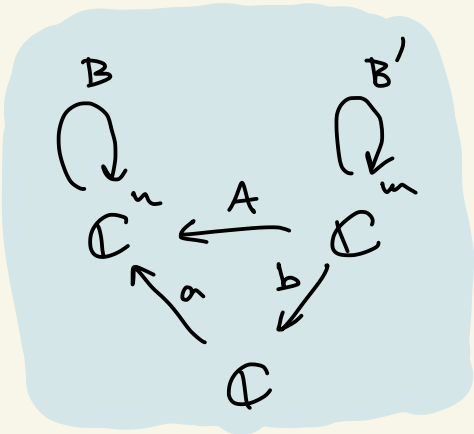
"symplectic intersection" of generalized Lagrange varieties

Nakajima–Takayama description (sketch)









... then ... $\mu^{-1}(0)^{ss} / \prod GL(n)$ ...

mod

$$BA + AB' + ab = 0$$

tautological bundles

(D3 branes)

$\mathcal{D}$ $\longrightarrow$ $\mathcal{C}(\mathcal{D})$ $\circlearrowleft$ $T \times \mathbb{C}^*_\hbar$

brane diagram

bow variety

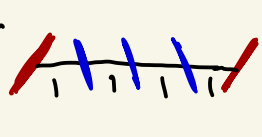
D5 branes

smooth holomorphic symplectic

$$\dim\big(\mathcal{C}(\mathcal{D})\big) = \sum_{u \ D5} \Big[(d_{u_-}+1)\, d_{u_-} + (d_{u_+}+1)\, d_{u^+} \Big]$$

$$+ \sum_{v \ NS5} 2\, d_{v^+} d_{v^-} \qquad - 2 \sum_{x \ D3} d_x^2$$

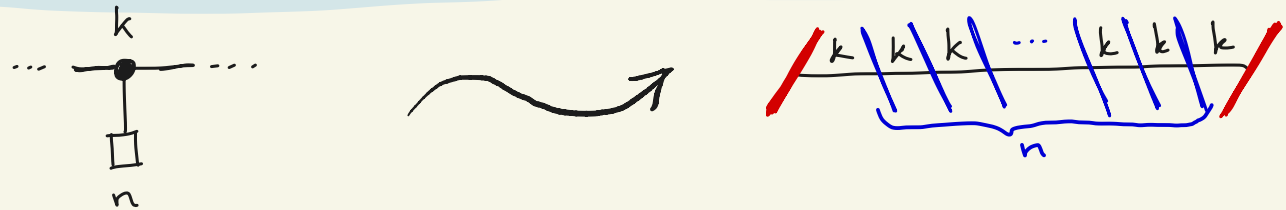
<u>example</u>

$\dim\Big(\mathcal{C}\big($  $\big)\Big) = 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 1$

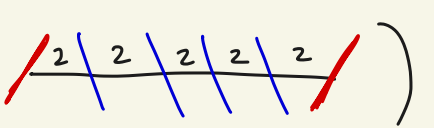
$+ 2\cdot 0\cdot 1 + 2\cdot 1\cdot 0 \quad - 2\left(1^2 + 1^2 + 1^2 + 1^2\right)$

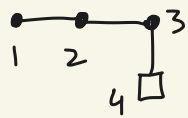
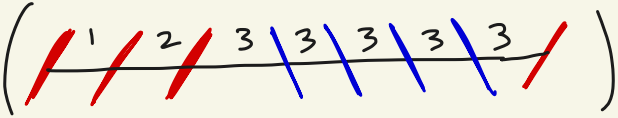
$= 4$

How are $\mathcal{N}$(quiver) special cases?



<u>Examples</u> $T^*\mathbb{P}^1 = \mathcal{N}\Big($ $\Big) = \mathcal{C}\Big($ $\Big)$

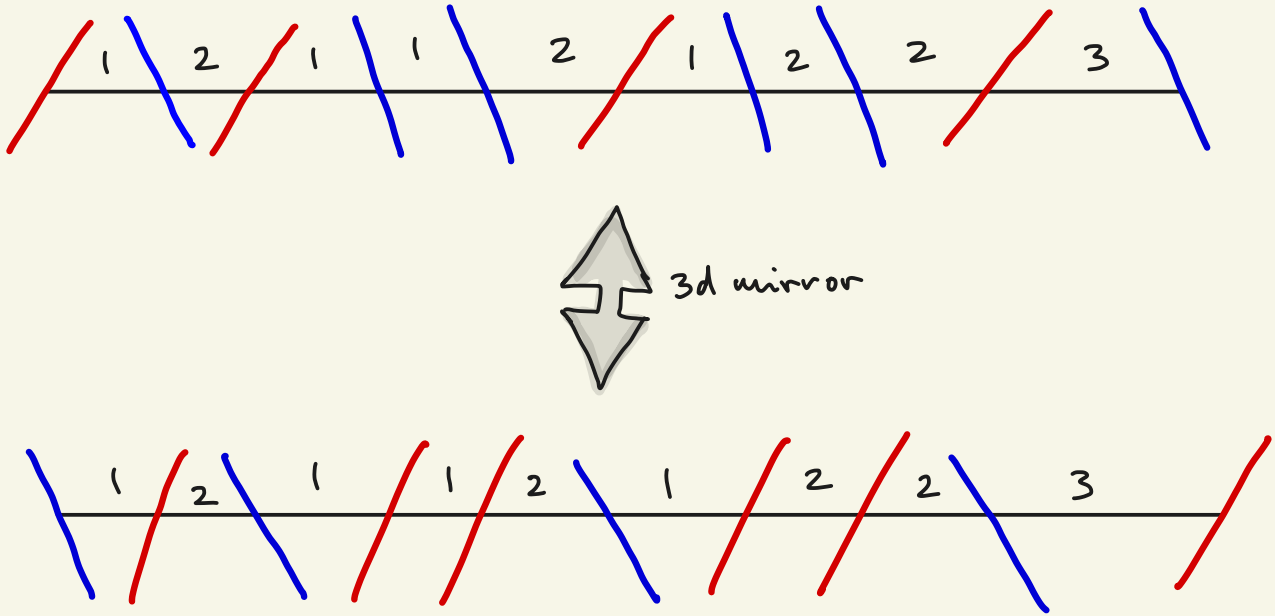
$T^*Gr_2\mathbb{C}^4 = \mathcal{N}\Big($ $\Big) = \mathcal{C}\Big($  $\Big)$

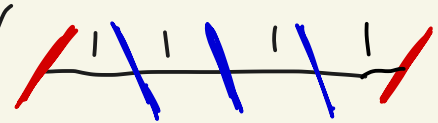
$T^*\mathcal{F}_{1,2,3,4} = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$\mathcal{N}\Big($ $\Big) = \mathcal{C}\Big($ $\Big)$

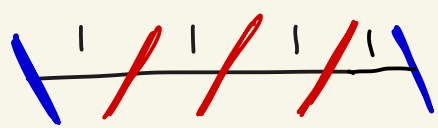
<u>Observe</u> "cobalanced brane diagram"

3D mirror symmetry for bow varieties:



Ex $T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} & 1 \\ 1 & \\ \square_3 & \end{smallmatrix}\right) = \mathcal{C}\Big($  $\Big)$ dim 4

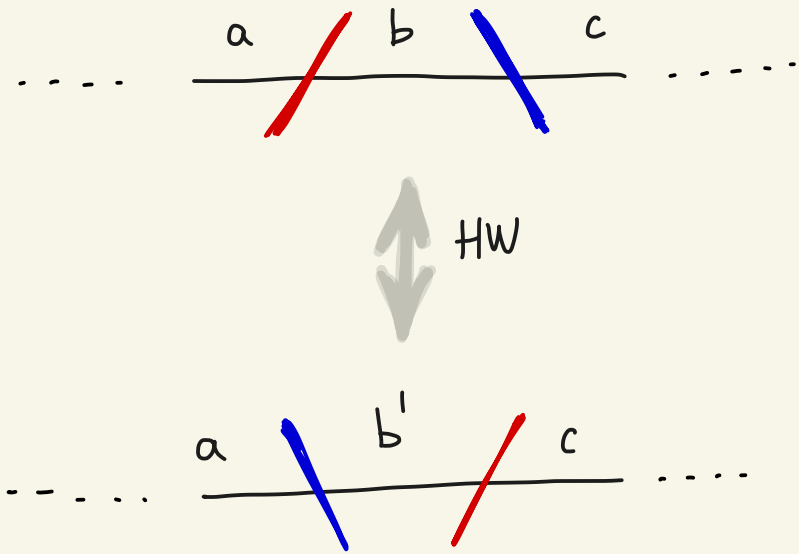
3d mirror

$\mathcal{C}\Big($  $\Big)$ dim 2

not cobalanced, ie not $\mathcal{N}(\ldots)$

... but ... <to be continued>

Hanany-Witten transition on brane diagrams.

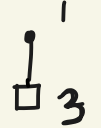
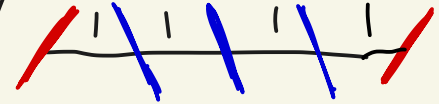


$$b + b' = a + c + 1$$

(why? later: "brane charge")

Thm $\mathcal{C}(\mathcal{D}) \cong \mathcal{C}(\mathrm{HW}(\mathcal{D}))$

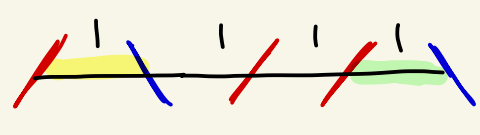
Ex

$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

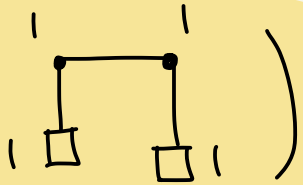
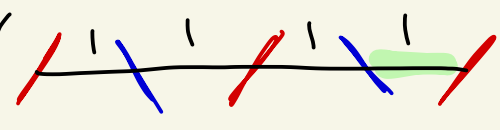
3d mirror

$\mathcal{C}\Big($ $\Big)$

$\|$ Hanany - Witten

$\mathcal{C}\Big($  $\Big)$

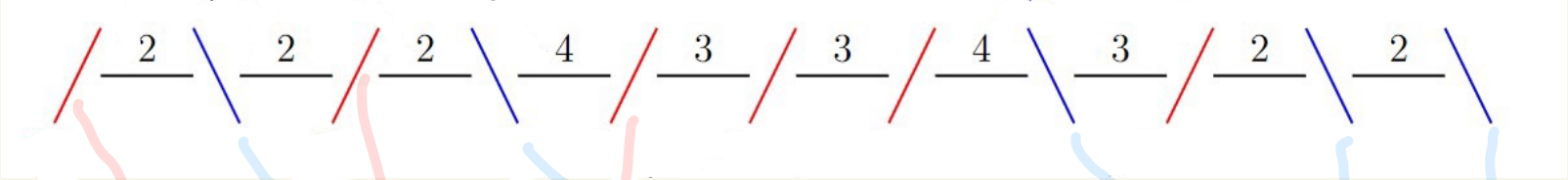
$\|$ Hanany - Witten

$\mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

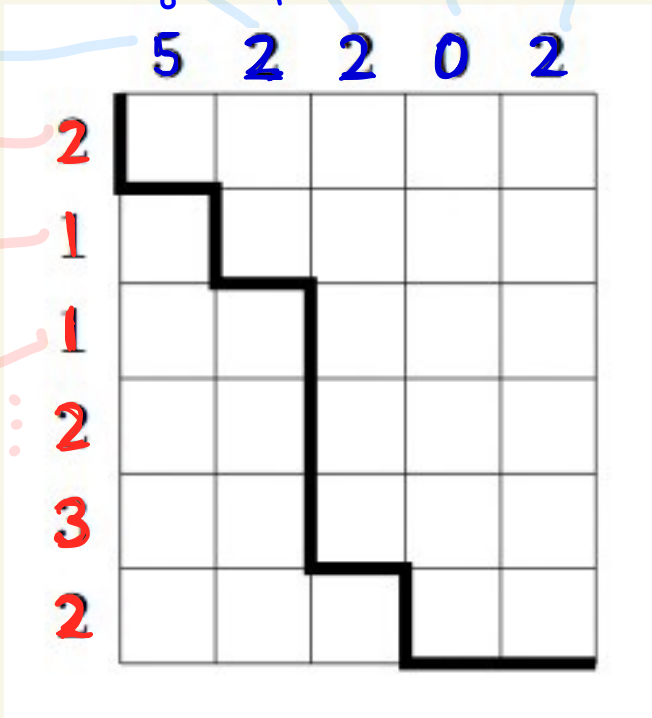
def brane charge

$$\text{charge}\Big(\underbrace{\text{NS5 brane}}_{k \,/\, \ell}\Big) := \ell - k + \#\{\text{D5-branes left of it}\}$$

$$\text{charge}\Big(\underbrace{\text{D5 brane}}_{k \,\backslash\, \ell}\Big) := k - \ell + \#\{\text{NS5-branes right of it}\}$$



charges of D5 branes



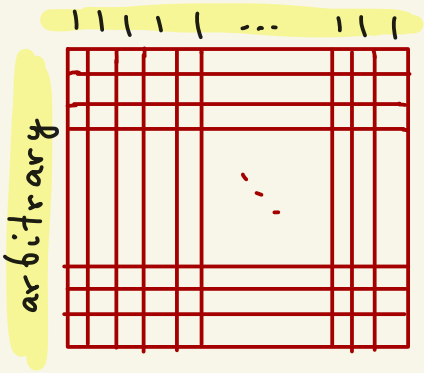
charges of NS5 branes

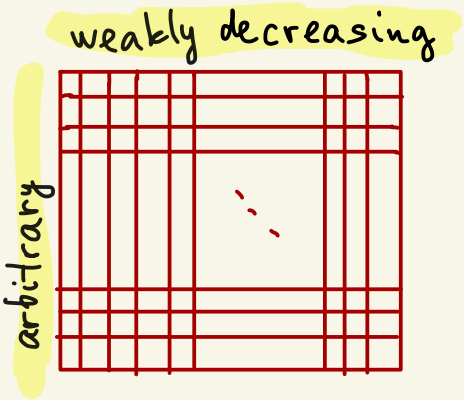
<u>Thm</u> The two charge vectors is a complete invariant of HW class

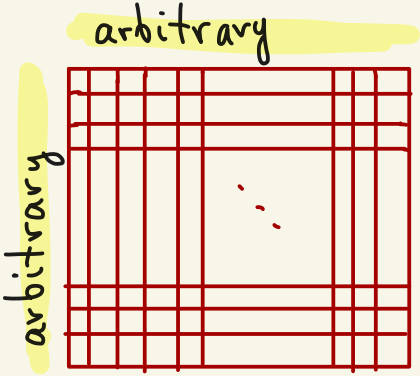
Thm (up to Hanany-Witten transition)

$T^*G/P \subset \mathcal{N}(\text{quiver}) \subset \mathcal{C}(\text{brane diagram})$

$\updownarrow$ $\updownarrow$ $\updownarrow$







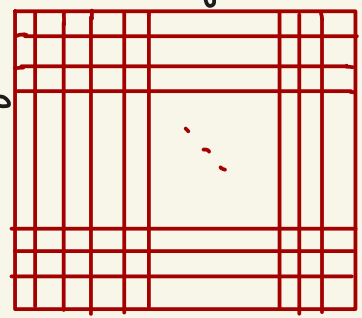
new operation!
transpose

Fact : quantum groups* act on H_T^* (moduli spaces)

② which representation

arbitrary

arbitrary

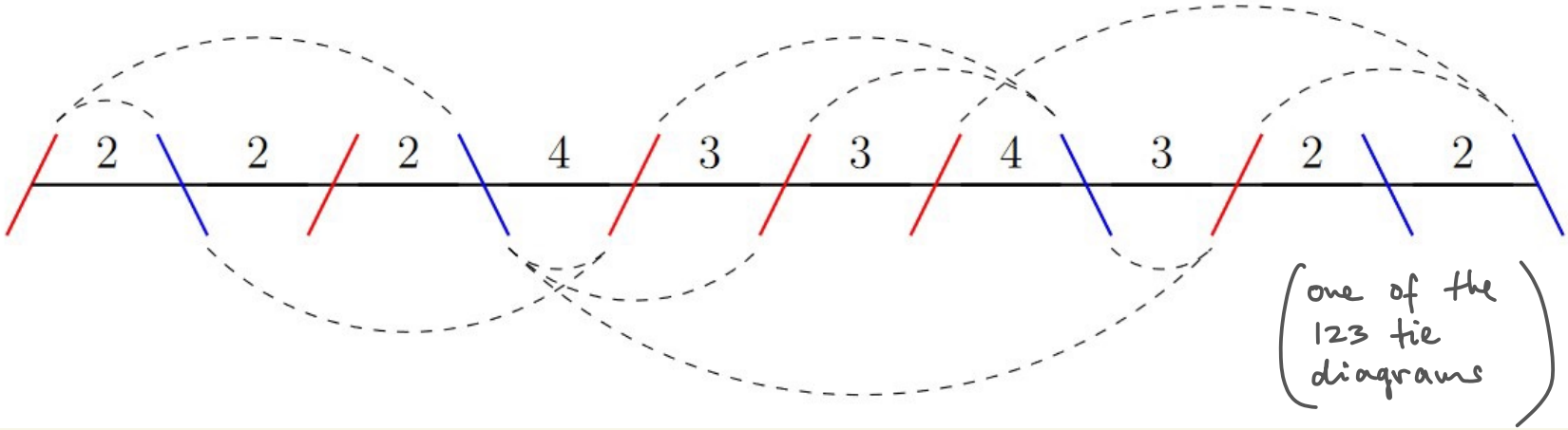


③ which weight space of the representation

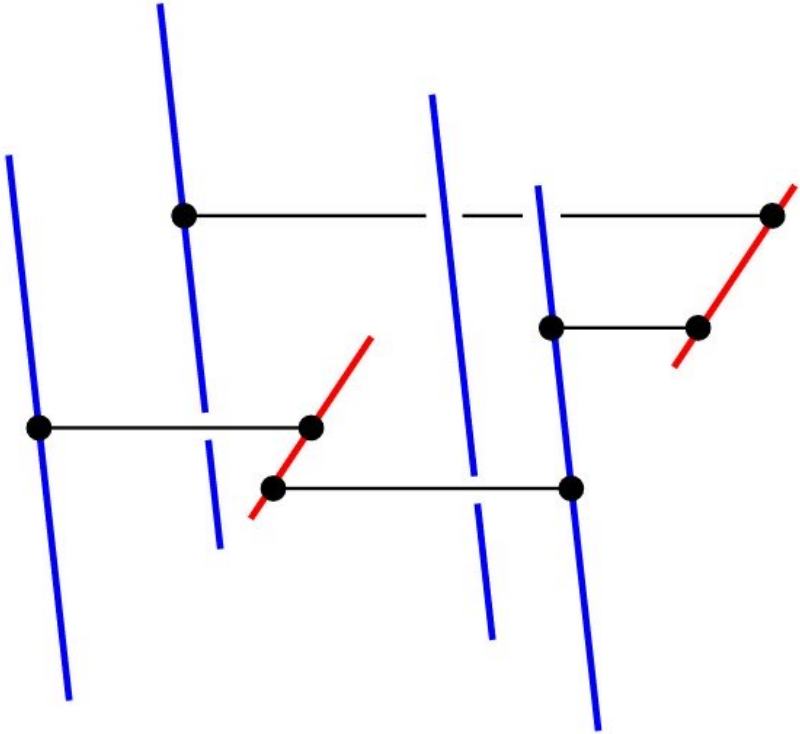
① size : which quantum group

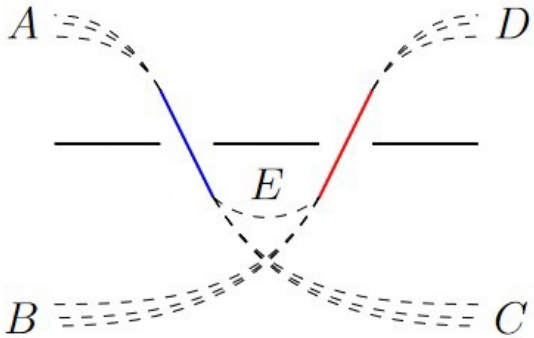
* $\exists$ super-algebra version [R-Rozansky], not in today's talk

fixed points $\overset{1:1}{\longleftrightarrow}$ tie diagrams



- a tie must connect 5-branes of different kinds
- each D3 brane to be covered as many times as its multiplicity

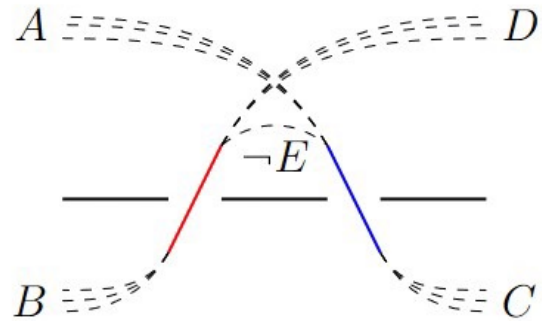


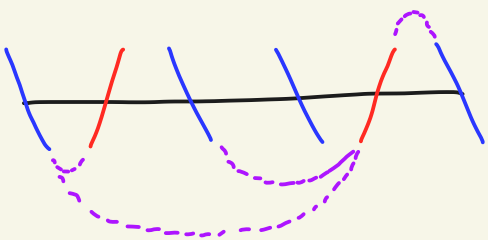


HW transition

on fixpoints

R-III

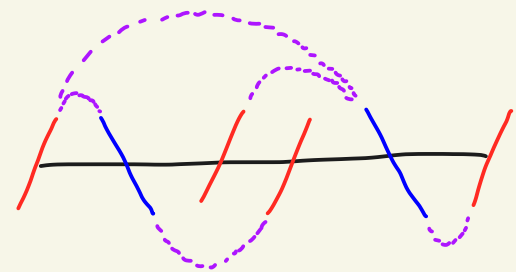


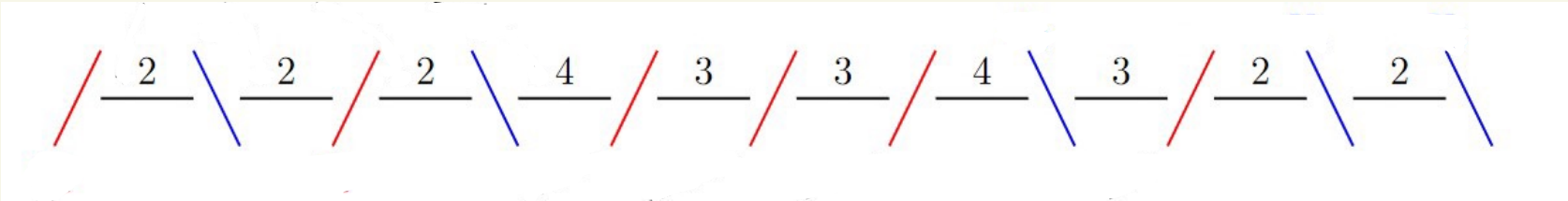


3d mirror

on fixedpoints

horizontal reflection





binary contingency tables

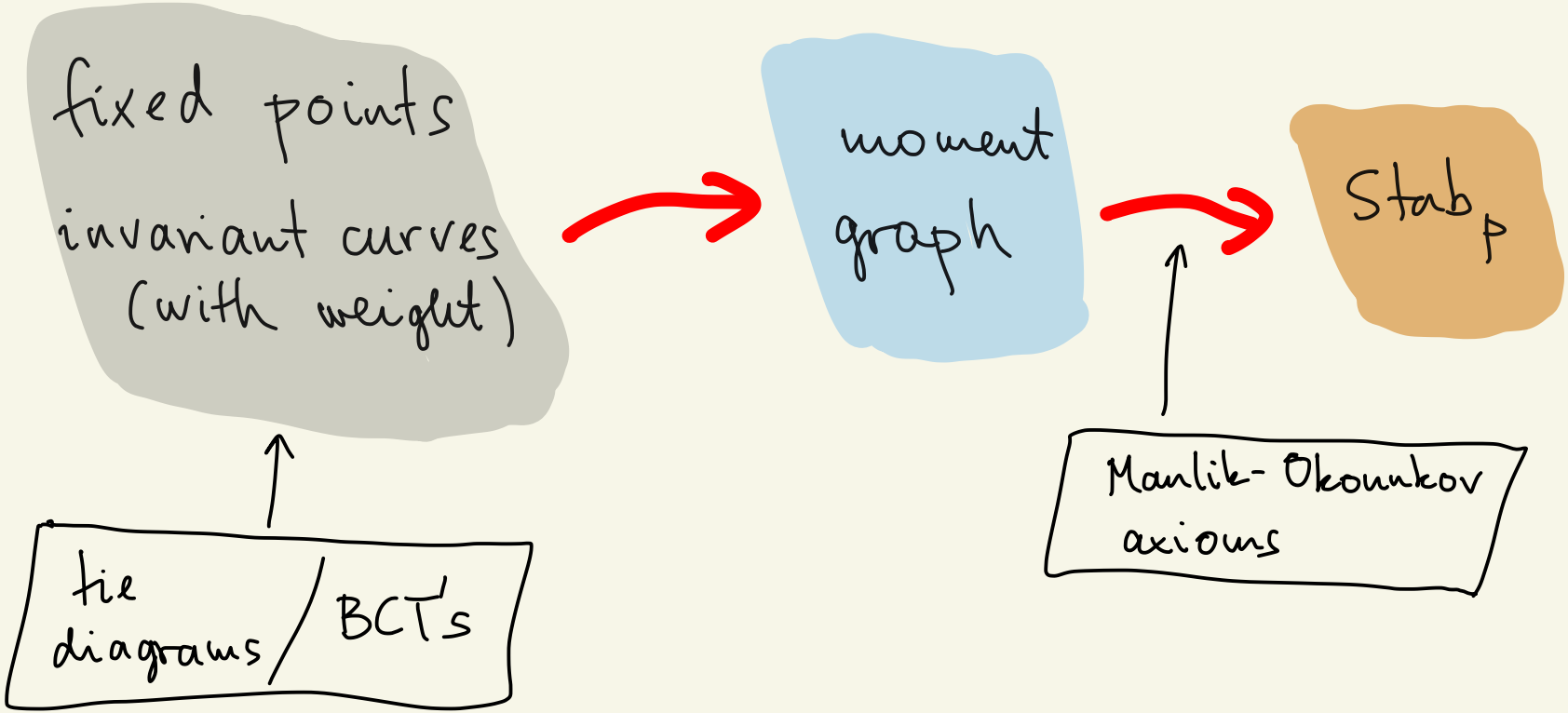
BCT : 0-1-matrix with row & column sums the charge vectors

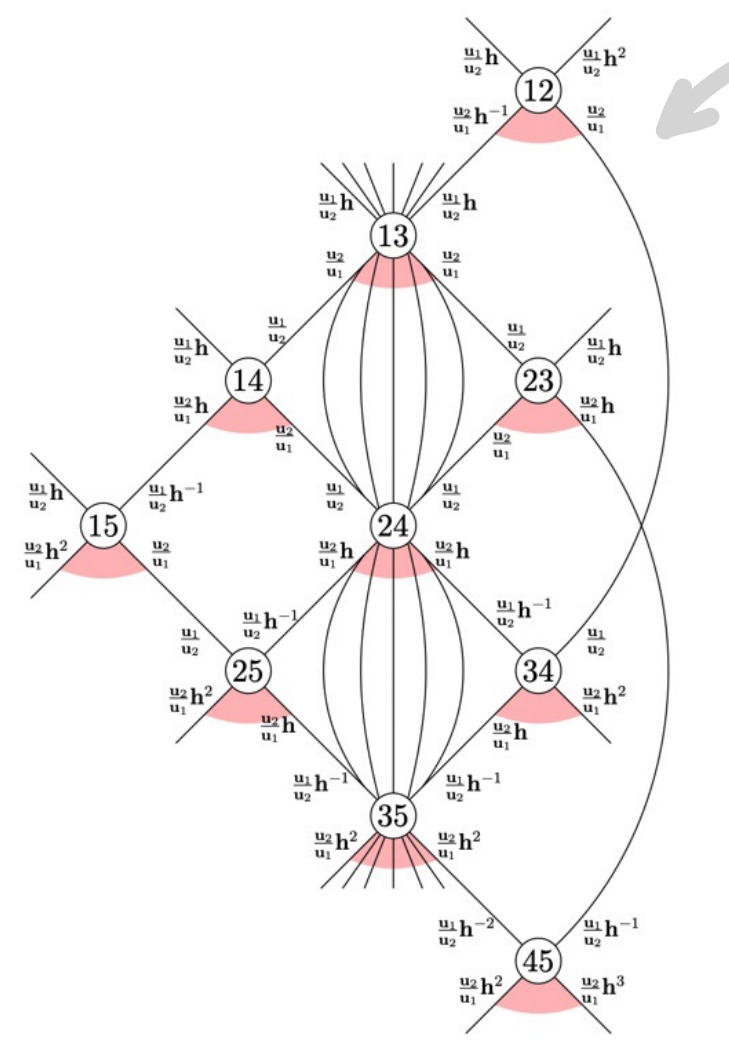
Thm

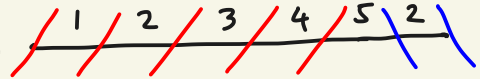
fix pts $\xleftrightarrow{1:1}$ BCT's

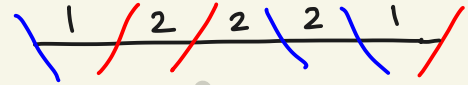
one of the 123 BCTs

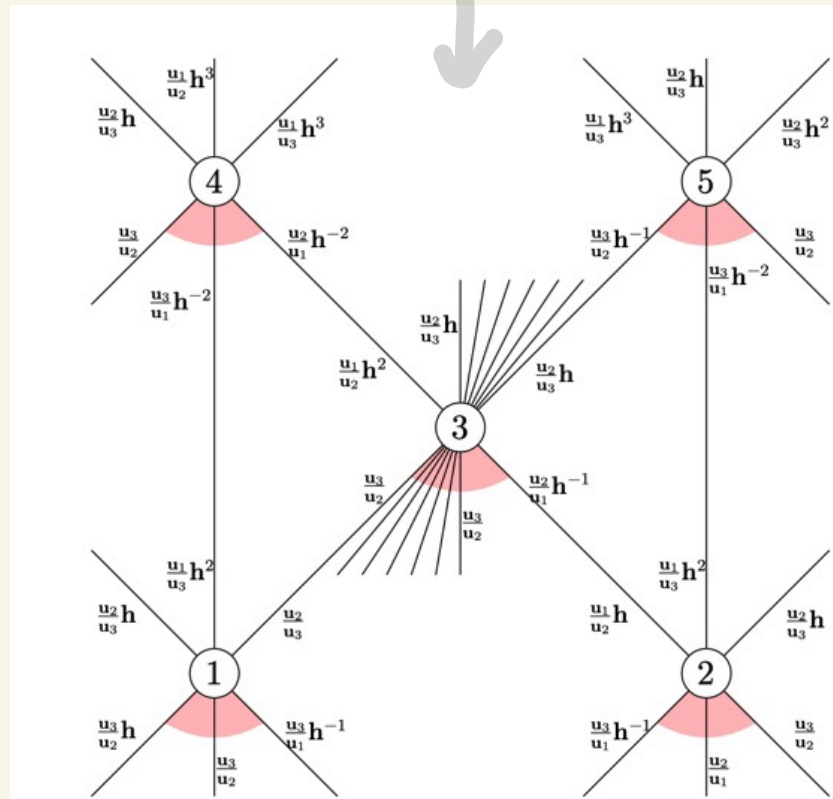
	5	2	2	0	2
2	1	1	0	0	0
1	1	0	0	0	0
1	0	0	1	0	0
2	1	0	1	0	0
3	1	1	0	0	1
2	1	0	0	0	1


fixed points
invariant curves
(with weight)
moment
graph
Stab_p
tie
diagrams / BCTs
Maulik-Okounkov
axioms


12
13
14
23
15
24
25
34
35
45


1 2 3 4 5 2


1 2 2 2 1

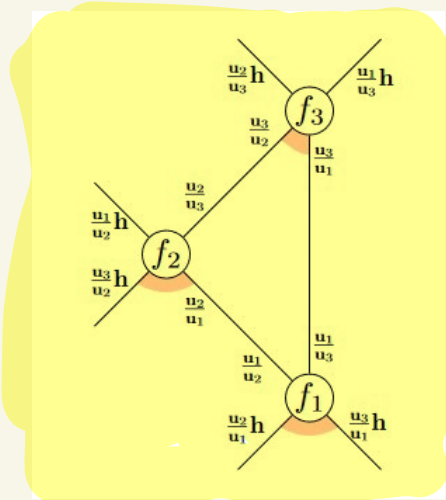

4
5
3
1
2

$= T^*\mathbb{P}^2$

	f_1	f_2	f_3
f_1	$\theta(\frac{u_1}{u_2})\theta(\frac{u_1}{u_3})\theta(\frac{v_2}{v_1}h^4)$	0	0
f_2	$\theta(h)\theta(\frac{u_1}{u_3})\theta(\frac{u_2v_2}{u_1v_1}h^3)$	$\theta(\frac{u_1}{u_2}h)\theta(\frac{u_2}{u_3})\theta(\frac{v_2}{v_1}h^3)$	0
f_3	$\theta(h)\theta(\frac{u_2}{u_1}h)\theta(\frac{u_3v_2}{u_1v_1}h^2)$	$\theta(h)\theta(\frac{u_1}{u_2}h)\theta(\frac{u_3v_2}{u_2v_1}h^2)$	$\theta(\frac{u_2}{u_3}h)\theta(\frac{u_1}{u_3}h)\theta(\frac{v_2}{v_1}h^2)$

combinatorics of tie diagrams BCTs

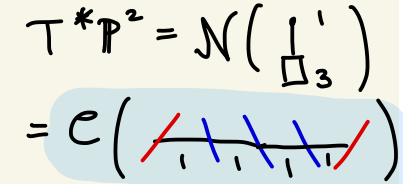
axioms of characteristic classes of singularities (Aganagic–Okounkov)



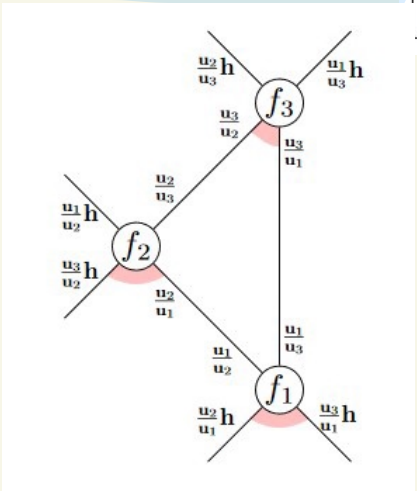
Jacobi theta function

$$\Theta(x) = \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1-q^n x)(1-q^n x^{-1})$$

$(q \in \mathbb{C}^\times \quad |q|<1 \text{ fixed})$

$T^*\mathbb{P}^2 = \mathcal{N}($  $)$

	f_1	f_2	f_3
f_1	$\theta(\frac{u_1}{u_2})\theta(\frac{u_1}{u_3})\theta(\frac{v_2}{v_1}h^4)$	0	0
f_2	$\theta(h)\theta(\frac{u_1}{u_3})\theta(\frac{u_2v_2}{u_1v_1}h^3)$	$\theta(\frac{u_1}{u_2}h)\theta(\frac{u_2}{u_3})\theta(\frac{v_2}{v_1}h^3)$	0
f_3	$\theta(h)\theta(\frac{u_2}{u_1}h)\theta(\frac{u_3v_2}{u_1v_1}h^2)$	$\theta(h)\theta(\frac{u_1}{u_2}h)\theta(\frac{u_3v_2}{u_2v_1}h^2)$	$\theta(\frac{u_2}{u_3}h)\theta(\frac{u_1}{u_3}h)\theta(\frac{v_2}{v_1}h^2)$

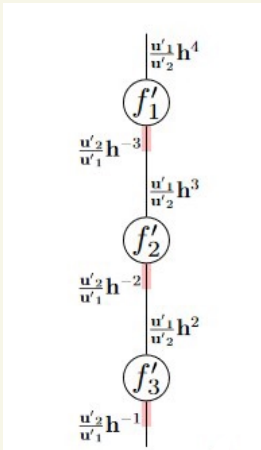


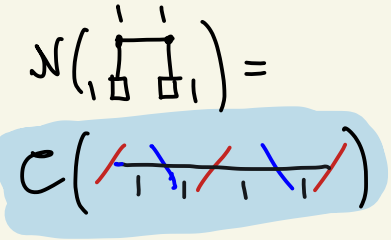
Fact same after

$\pm(\ )^T$

$u. \longleftrightarrow v.$

$h \longleftrightarrow h^{-1}$



$\mathcal{N}($  $)$

	f'_1	f'_2	f'_3
f'_1	$\theta(\frac{u'_1}{u'_2}h^4)\theta(\frac{v'_2}{v'_1})\theta(\frac{v'_3}{v'_1})$	$\theta(h)\theta(\frac{v'_3}{v'_1})\theta(\frac{v'_2u'_2}{v'_1u'_1}h^{-3})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h^{-1})\theta(\frac{v'_3u'_2}{v'_1u'_1}h^{-2})$
f'_2	0	$\theta(\frac{u'_1}{u'_2}h^3)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3}{v'_2})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3u'_2}{v'_2u'_1}h^{-2})$
f'_3	0	0	$\theta(\frac{u'_1}{u'_2}h^2)\theta(\frac{v'_3}{v'_2}h)\theta(\frac{v'_3}{v'_1}h)$

3d mirror symmetry for characteristic classes:

$$\mathcal{C}(\mathcal{D}) \xleftrightarrow{\text{NS5-D5 switch}} \mathcal{C}(\mathcal{D}^!)$$

$$\mathcal{C}(\mathcal{D})^T \xleftrightarrow[p \longleftrightarrow p^!]{} \mathcal{C}(\mathcal{D}^!)^{T^!}$$

$$\mathrm{Stab}\, p\big|_q \;=\; \pm\, \mathrm{Stab}\, q^!\big|_{p^!}$$

switch: equivariant $\leftrightarrow$ dynamical

$\hbar \longleftrightarrow \hbar^{-1}$

3d mirror symmetry for stable envelopes is proved in special cases

- $T^*Gr_k\mathbb{C}^n \longleftrightarrow$ its dual [R-Smirnov-Varchenko-Zhou]
- $T^*G/B \longleftrightarrow T^*G^L/B^L$ < type A [RSVZ], general [R-Weber]
- hypertoric $\longleftrightarrow$ dual hypertoric [Smirnov-Zhou]
- finitely many other cases [R-Shou]

Why do we think 3d mirror symmetry for char. classes hold?

$X \rightsquigarrow H_T^*(X)$-valued functions of $(\underline{z}, \underline{q}, \hbar)$

- 2 sets of equations

$f(z_i+1) = K_i(\underline{z}, \underline{q}, \hbar)\, f$

essentially R-matrix

qKZ equations in equivariant parameters

$\left[q_i \frac{\partial}{\partial q_i} - (c_1 *)\right] f = 0$

quantum multiplication

"dynamical connection" in $\underline{q}$ (Kähler parameters)

- <u>Fact</u>: compatible

- <u>Fact</u>: solution set $= \langle$ vertex functions $\rangle$

curve counting p.s. on X

Higgs branch vs Coulomb branch story ... $\Rightarrow$

$$X \longleftrightarrow X^!$$

$$\begin{array}{ccc} qKZ(X) & & qKZ(X^!) \\ \text{dynamical}(X^!) & \times & \text{dynamical}(X^!) \end{array}$$

curve counting vertex functions (X) $\longleftrightarrow$ curve counting vertex functions $(X^!)$

transition matrix

$$\left[(\mathrm{Stab}_p)_q \right]_{p,q \in X^T}$$

Summary

- $X \circlearrowleft T$ $\qquad \mathrm{Stab}_p$

  enumerative geometry

  representations of quantum groups

  differential equations
  q-difference equations

- $X \xleftrightarrow{\text{3d mirror}} X^!$ $\qquad \mathrm{Stab} \sim \mathrm{Stab}^!$

- bow varieties — closed for 3d mirror symmetry

  rich combinatorics
  (more "complete" than Schubert Cal. / quiver varieties)

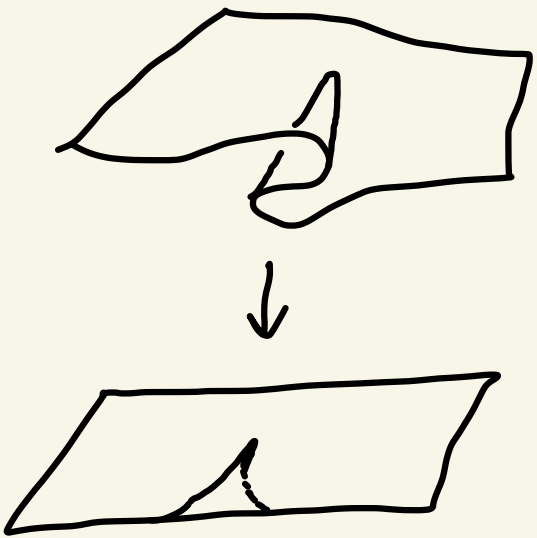
Thank you !

backup slides ↓

$$GL_2 \times GL_2 \circlearrowright \left(\Sigma \subset M \right)$$

$\Sigma = \{ \text{those} \sim (x,y) \mapsto (x^3 + xy, y) \}$

$M = \{ (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \text{ holomorphic germs} \}$



$$[\bar{\Sigma}] \in H^*_{GL_2 \times GL_2}(pt)$$

$$\| $$

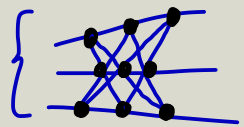
$$c_1^2 + c_2$$

<u>Cor</u> generic map

$$\mathbb{RP}^2 \longrightarrow \mathbb{R}^2$$

has an <u>odd</u> number of cusps

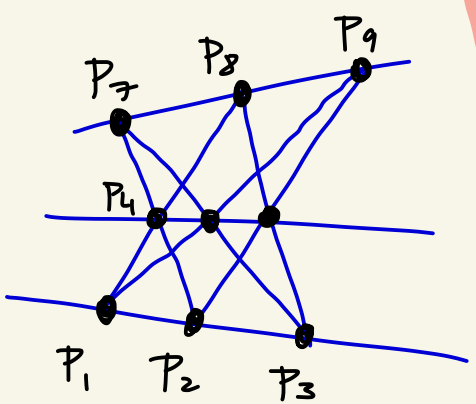
$$GL_3 \times D^9 \circlearrowright \left(\Sigma \subset M \right)$$

Σ = 

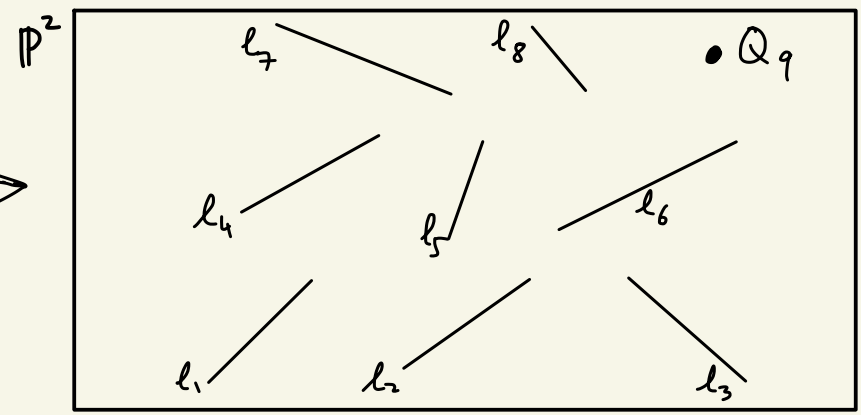
$M = \mathrm{Hom}(\mathbb{C}^9, \mathbb{C}^3)$

$$[\Sigma] \in H^*_{GL_3 \times D^9}(pt) = \mathbb{Z}[c_1\, c_2\, c_3\, d_1\, d_2 \cdots d_9]$$

$$= \cdots + 5\, d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9 + \cdots$$



# of maps



X $\longrightarrow$ difference qKZ-equations

$\longrightarrow$ differential dynamical equations

(consistent!)

on vector-valued functions in $(\underbrace{z}_{\text{equivar}}, \underbrace{q}_{\text{Kähler}})$

$X = Gr_0 \mathbb{C}^2 = pt$

qKZ

$$f(z_1 + \kappa, z_2, q_1, q_2) = q_1 f(z_1, z_2, q_1, q_2)$$
$$f(z_1, z_2 + \kappa, q_1, q_2) = q_1 f(z_1, z_2, q_1, q_2)$$

dynamical

$$\kappa q_1 \frac{\partial f}{\partial q_1} = (z_1 + z_2) f$$

$$\kappa q_2 \frac{\partial f}{\partial q_2} = f$$

$$f = q_1^{\frac{z_1 + z_2}{\kappa}} q_2^{\frac{1}{\kappa}}$$

Kähler parameters keeping track of deg curve

- curve counting in X : $V \in K_T(X)[\underline{q}]$ <u>vertex function</u>
- $p \in X^T$ $\quad V_p \in K(\text{point})[\underline{q}]$

equivariant parameters $\underline{u}$ (mero-)

Kähler parameters $\underline{v}$ (holo-)

- V_p satisfy difference equations $\begin{cases} \text{in } \underline{z} \text{ variables} \\ \text{in } \underline{q} \text{ variables} \end{cases}$

diff eq'ns <u>switched</u> for $X^!$

- $V^!_q := \sum \mathrm{Stab}_q|_p \cdot V_p$ vertex function on $X^!$

fix $q \in \mathbb{C}^*$ $\|q\| < 1$

$$\delta(a,b) := \frac{\vartheta(ab)}{\vartheta(a)\,\vartheta(b)}$$

Jacobi theta function:

$$\vartheta(x) = \underbrace{\left(x^{1/2} - x^{-1/2}\right)}_{\substack{\sim \sin X \\ \text{K theory}}} \underbrace{\prod_{n \geq 1} (1 - q^n x)(1 - q^n x^{-1})}_{q\text{-decoration}}$$

Why do elliptic characteristic classes <u>necessarily</u> depend on a new set of variables? Intuitive answer

Fay's trisecant identity : Ell $\quad x_1x_2x_3 = y_1y_2y_3 = 1 \Rightarrow$

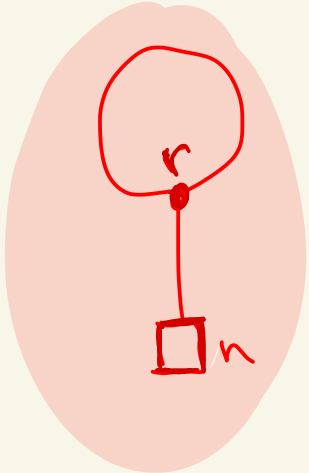
$$\delta(x_1, y_2)\,\delta\left(x_2, \frac{1}{y_1}\right) + \delta(x_2, y_3)\,\delta\left(x_3, \frac{1}{y_2}\right) + \delta(x_3, y_1)\,\delta\left(x_1, \frac{1}{y_3}\right) = 0$$

Trigonometric (K-theory) limit $\quad x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0 \Rightarrow$

$$\underbrace{\cot(x_1)\cot(x_2) + \cot(x_2)\cot(x_3) + \cot(x_3)\cot(x_1)}_{1} = \underbrace{\cot(y_1)\cot(y_2) + \cot(y_2)\cot(y_3) + \cot(y_3)\cot(y_1)}_{1}$$

Rational (H^*) limit $\quad x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0 \Rightarrow$

$$\underbrace{\frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \frac{1}{x_2x_3}}_{0} = \underbrace{\frac{1}{y_1y_2} + \frac{1}{y_1y_3} + \frac{1}{y_2y_3}}_{0}$$



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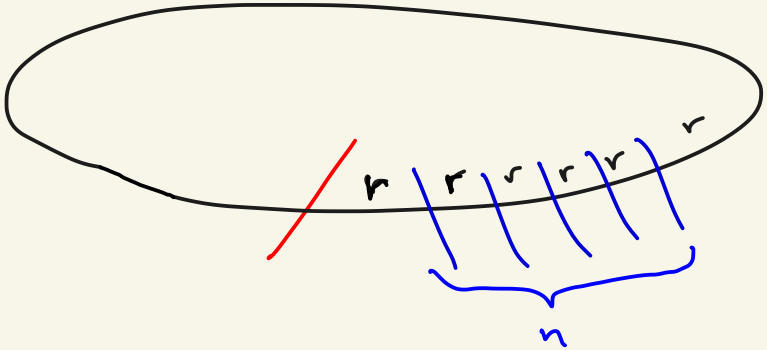
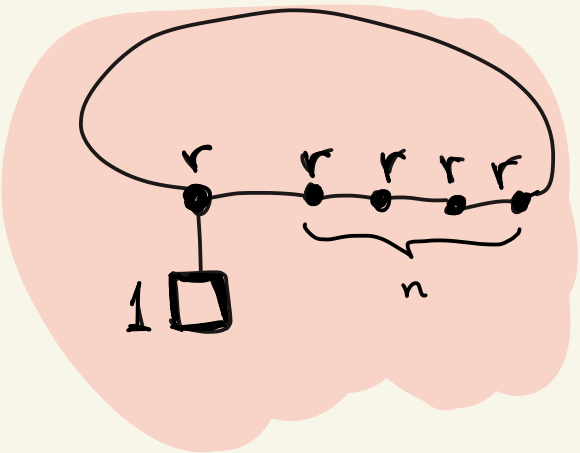


3d mirror

r=1
$\mathrm{Hilb}^{[n]}$
self dual



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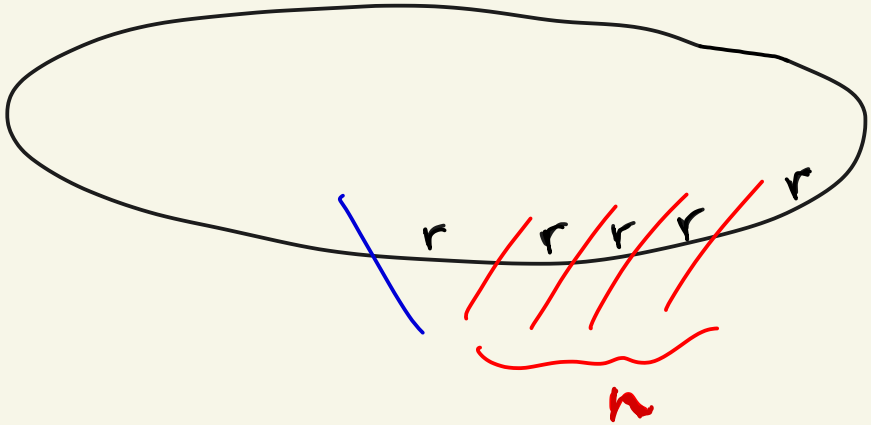

recall

$$[\bar{\Omega}_I] = \operatorname*{Sym}_{t_1 \dots t_k} \left(\prod_{a=1}^{k} \left(\prod_{b=i_a+1}^{n} (z_b - t_a) \right) \prod_{1 \le a \le b \le k} \frac{1}{(t_b - t_a)} \right)$$

essentially the Vandermonde $\frac{\det(\dots)}{\det(\dots)}$ formula for Schur functions

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fact

$$c^{sym}(\Omega_I) = \operatorname*{Sym}_{t_1 \dots t_k} \left(\prod_{a=1}^{k} \left(\prod_{b=1}^{i_a-1} (z_b - t_a + \hbar) \prod_{b=i_a+1}^{n} (z_b - t_a) \right) \prod_{1 \le a \le b \le k} \frac{1}{(t_b - t_a)(t_a - t_b + \hbar)} \right)$$

"weight functions"