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Wien - Budapest AG

# Enumerative geometry on Cherkis bow varieties

Richárd Rimányi



THE UNIVERSITY  
*of* NORTH CAROLINA  
*at* CHAPEL HILL

Joint work with

Yiyan Shou

Work in progress with

Tommaso Botta

Related work with

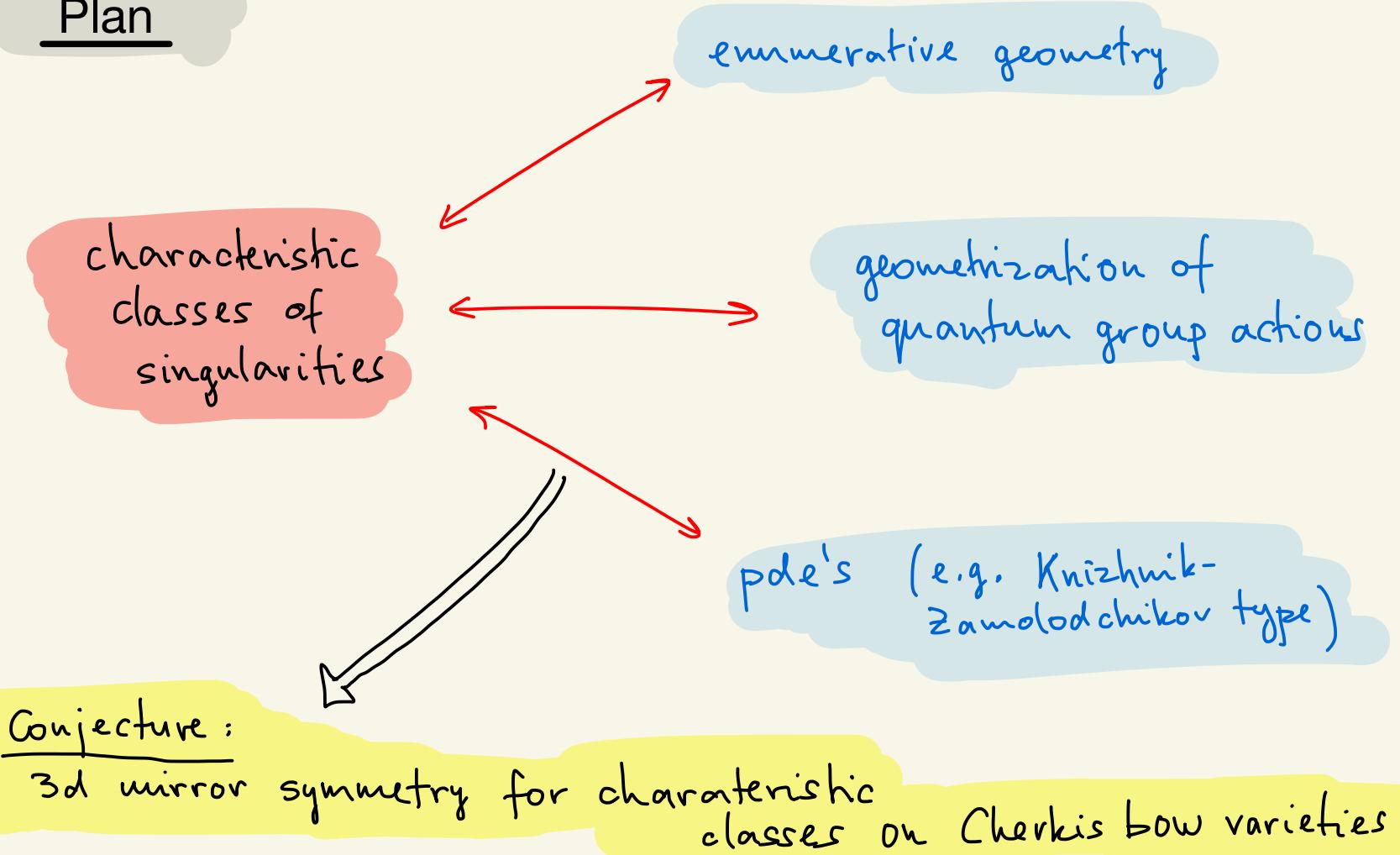
A. Varchenko

A. Smirnov

Z. Zhou

L. Rozansky

# Plan

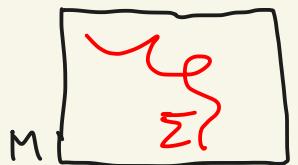


# Class of a subvariety vs geometric enumeration

## Key concept

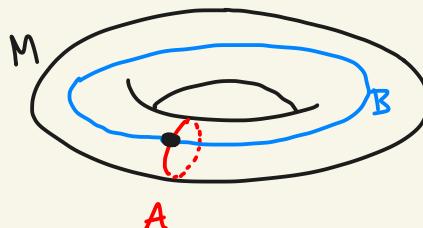
$$\Sigma \subset M \quad \xrightarrow{\text{smooth}} \quad [\Sigma] \in H^*(M)$$

*smooth*



↑  
fundamental class  
of  $\Sigma \subset M$

## Example



$$[A], [B] \in H^*(M)$$
$$\int_M [A] \cdot [B] = |A \cap B|$$

counted the right way

## Example

$$\{ \text{rk} \leq 2 \text{ matrices} \} \subseteq \mathbb{C}^{4 \times 4} \xrightarrow{\parallel M} GL_4 \times GL_4$$

$\Sigma$

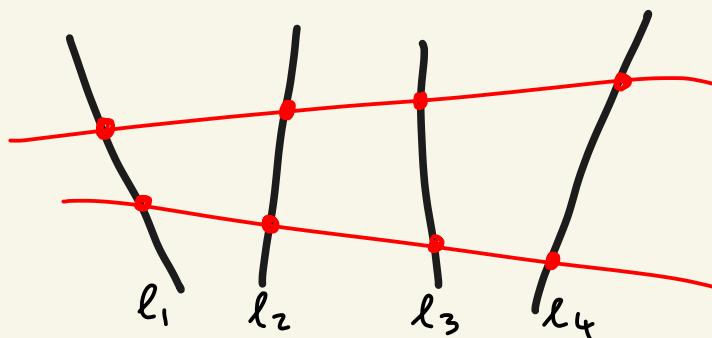
$$[\Sigma] = \dots + 2\beta_1\beta_2\beta_3\beta_4 + \dots \in H^*_{GL_4 \times GL_4}(M)$$

$\cong [s_1 s_2 s_3 s_4 \beta_1 \beta_2 \beta_3 \beta_4]^{S_4 \times S_4}$

Solution to :

$l_1, l_2, l_3, l_4$  lines  $\subset \mathbb{P}^3$

How many lines  
intersect all four?



## New phenomenon: 3D mirror symmetry

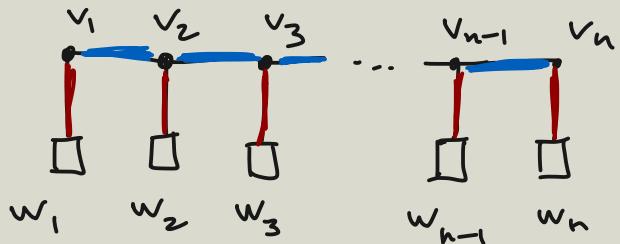
characteristic classes  
of singularities in  $T^* \mathrm{Gr}_2 \mathbb{C}^5$

// equal (in a sophisticated sense)

characteristic classes  
of singularities in

$$N \left( \begin{array}{cccc} ! & 2 & 2 & ! \\ \bullet & - & - & \bullet \\ | & \square & \square & | \end{array} \right)$$

# Nakajima quiver varieties



quiver  $Q$

$\mathcal{N}(Q)$

quiver variety

Ex

$$\mathcal{N}\left(\begin{smallmatrix} k & \bullet \\ n & \square \end{smallmatrix}\right) = T^* \mathrm{Gr}_k \mathbb{C}^n$$

$$\mathcal{N}\left(\begin{smallmatrix} k_1 \leq k_2 \leq k_3 \\ \bullet \quad \bullet \quad \bullet \\ , \quad \square \quad n \end{smallmatrix}\right) = T^* \mathcal{F}_{k_1, k_2, k_3, n}$$

$$\mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ , \quad \square \\ , \quad \square \end{smallmatrix}\right) \underset{\sim}{=} \widetilde{\mathbb{C}^2 / \mathbb{Z}_3}$$

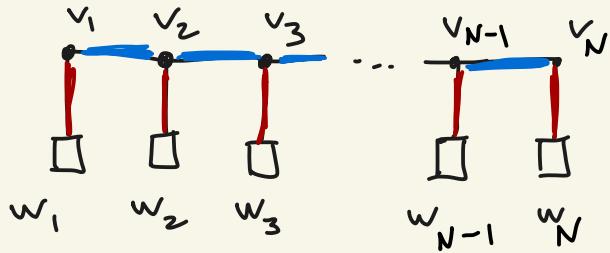
def

$$\mathcal{N} \left( \begin{array}{ccccccc} v_1 & v_2 & v_3 & & v_{n-1} & v_n \\ \downarrow k & \downarrow b & \downarrow a & \cdots & & \\ w_1 & w_2 & w_3 & & w_{n-1} & w_n \end{array} \right) = \dots$$

- $R := \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$
- $\mu: R \oplus R^* \rightarrow \bigoplus_i \text{End}(\mathbb{C}^{v_i})$        $\mu = [a, b] - lk$
- $N(Q) := \tilde{\mu}(0)^{ss} / \bigtimes_i GL_{v_i}$

$N(Q)$

(type A)



- smooth
- holomorphic symplectic
- $T = (T^{w_1} \times T^{w_2} \times \dots \times T^{w_N}) \times \mathbb{C}_{\hbar}^*$  action
- finitely many fixed pts
- "tautological"  $v_1, v_2, \dots, v_N$ -bundles

$$H_T^*(N(Q))$$

Loc

$$\bigoplus_{\text{T-fixed points}} H_T^*(P)$$

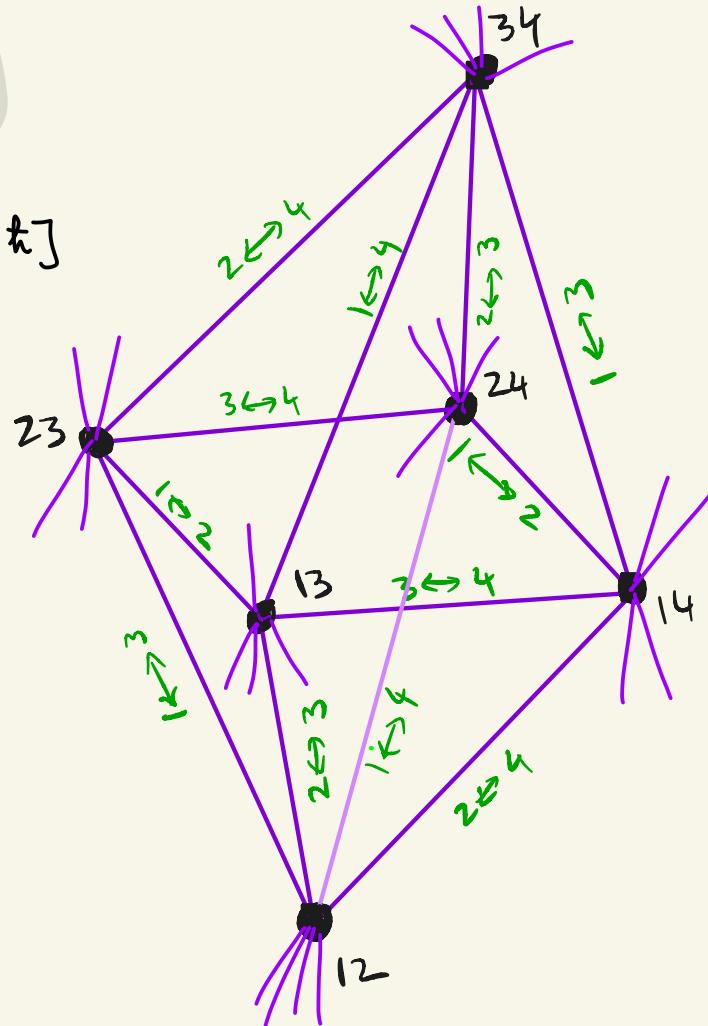
$\bigoplus_{\text{T-fixed points}}$

$$\mathbb{C}[z_1, \dots, z_n, t]$$

Localization map

$$\text{im}(Loc) = ?$$

constraints among the components

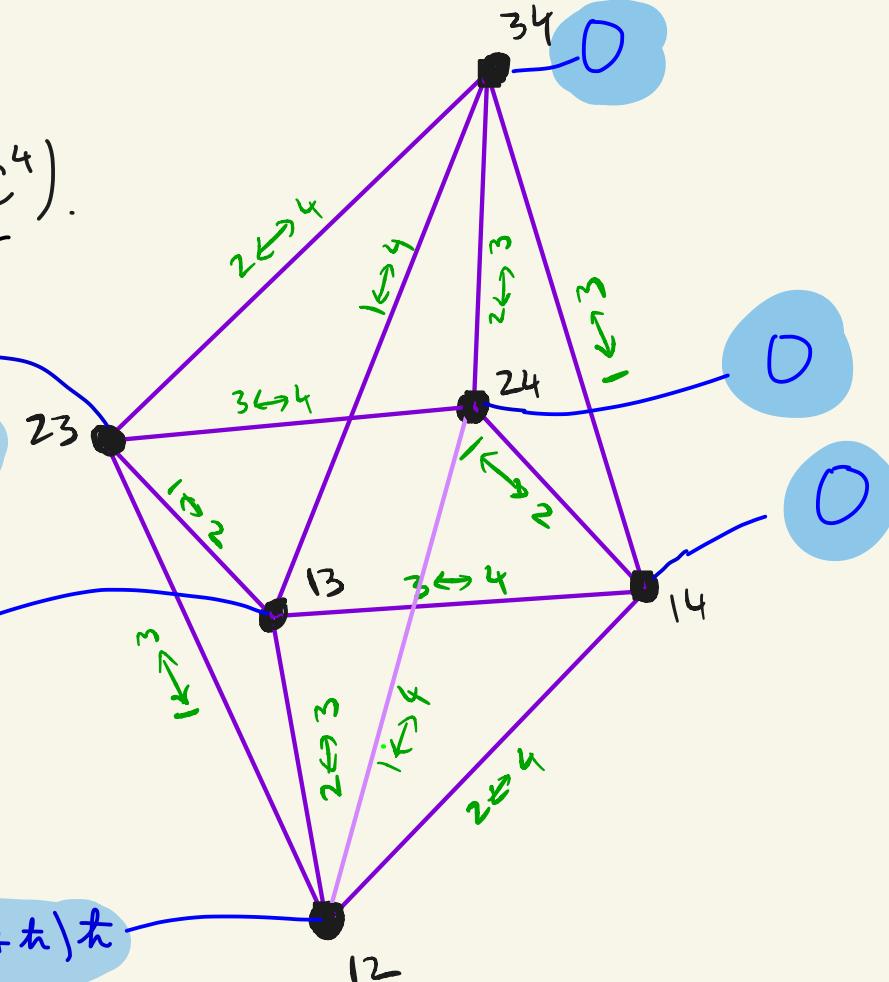


For example, this 6-tuple is an element of  $H_T^*(T^* \underline{\text{Gr}}_2 \mathbb{C}^4)$ .

$$(z_4 - z_3)(z_4 - z_2)(z_2 - z_1 + \hbar)(z_3 - z_1 + \hbar)$$

$$(z_4 - z_1)(z_4 - z_3)(z_3 - z_2 + \hbar) \hbar$$

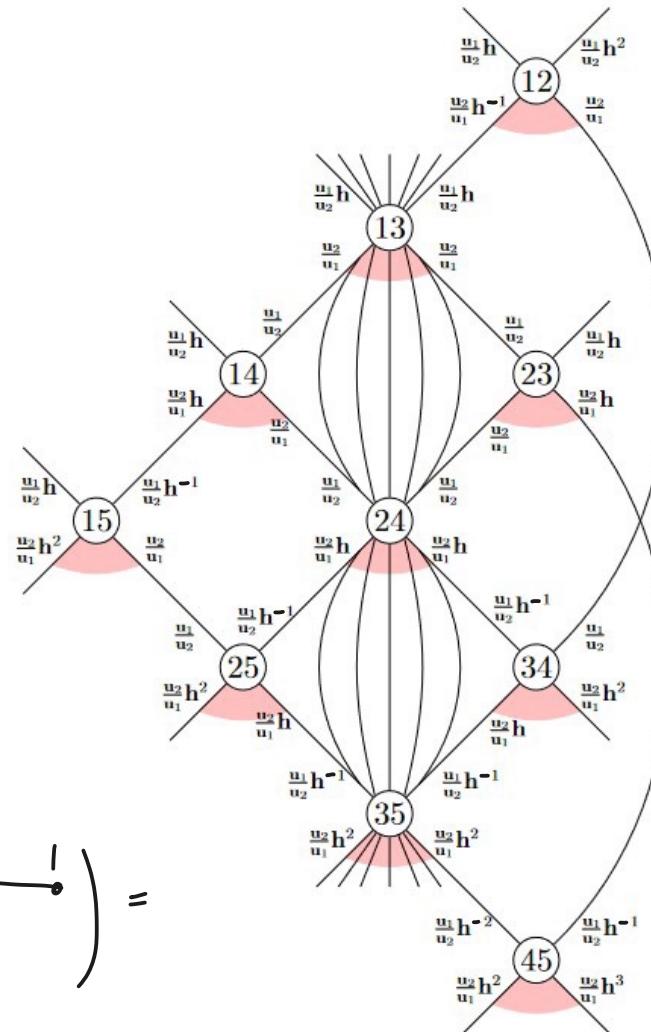
$$(z_4 - z_1)(z_4 - z_2)(z_2 - z_3 + \hbar) \hbar$$



## Warning

- $T^* \text{Gr}_2 \mathbb{C}^4$  was special ("GKM")
- In general the constraints among components are more restrictive

$$\mathcal{N}\left(\begin{array}{c|ccccc} & & & & \\ \bullet & 2 & 2 & 1 & & \\ & | & | & | & & \\ & \square & \square & \square & & \\ & | & | & | & & \\ & 1 & 1 & 1 & & \end{array}\right) =$$

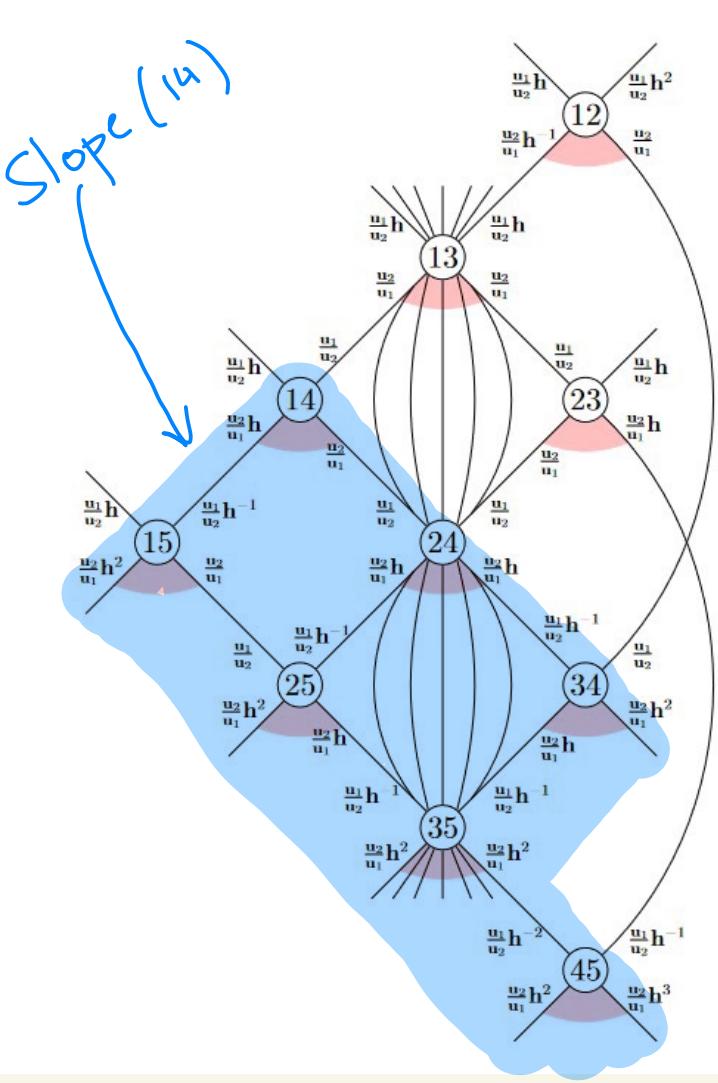
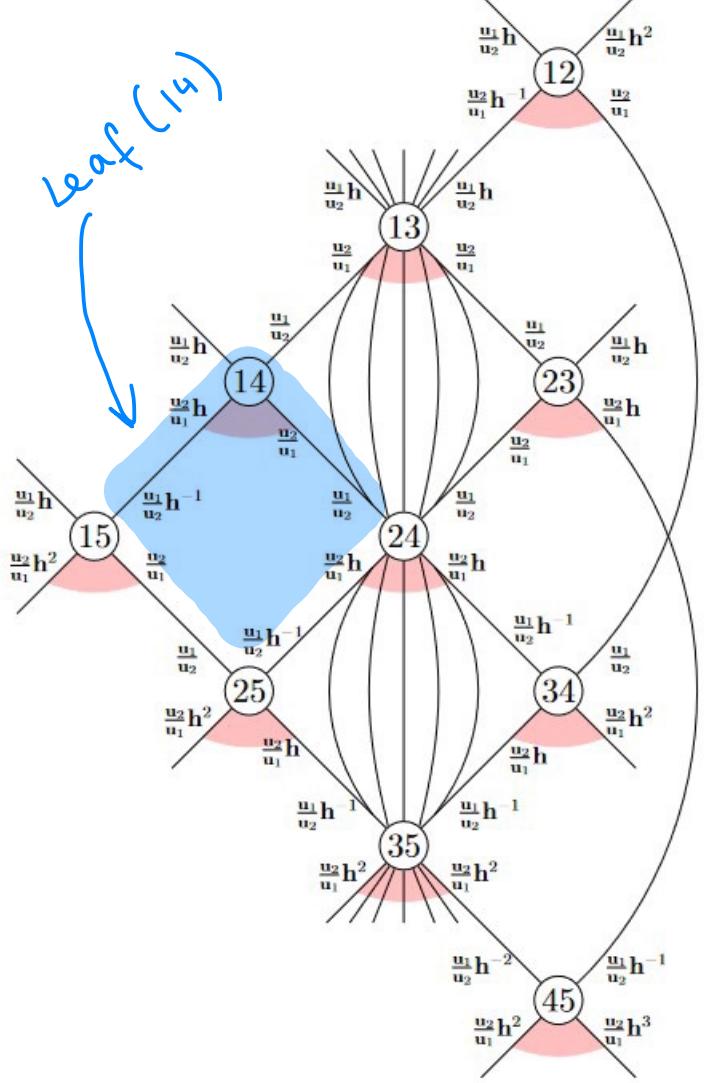


Towards

$$\text{Stab}_p \in H_T^*(N(Q))$$

↑  
(torus fixed point)

- fix  $\mathbb{C}^* \xrightarrow{\delta} T$   
 $z \mapsto (z, z^2, z^3, \dots, z^n, 1)$
- $p \in N(Q)^T$   $\text{Leaf}(p) = \{x \in N(Q) : \lim_{z \rightarrow 0} \delta(z)x = p\}$
- $p' \leq p$  if  $\overline{\text{Leaf}(p)} \ni p'$
- $\text{slope}(p) := \bigcup_{p' \leq p} \text{Leaf}(p')$



# Malik-Okounkov

def  
[MO]  $\text{Stab}_p \in H_+^*(N(Q))$  is the unique class

- support axiom:

supported on  $\text{Slope}(p)$

- normalization axiom:

$$\text{Stab}_p|_p = e(\nu(\text{Slope}_p))$$

- boundary axiom:

$\text{Stab}_p|_q$  divisible by  $t$  for  $p \neq q$

Stab<sub>14</sub>

@ 12, 13, 23 = 0

@ 14 =  $(z_1 - z_2)(z_1 - z_2 + h)$

@ 15 divisible by  $(z_1 - z_2 + h)$

divisible by  $h$

@ 24 divisible by  $(z_1 - z_2)$

divisible by  $h$

@ 25 divisible by  $h$

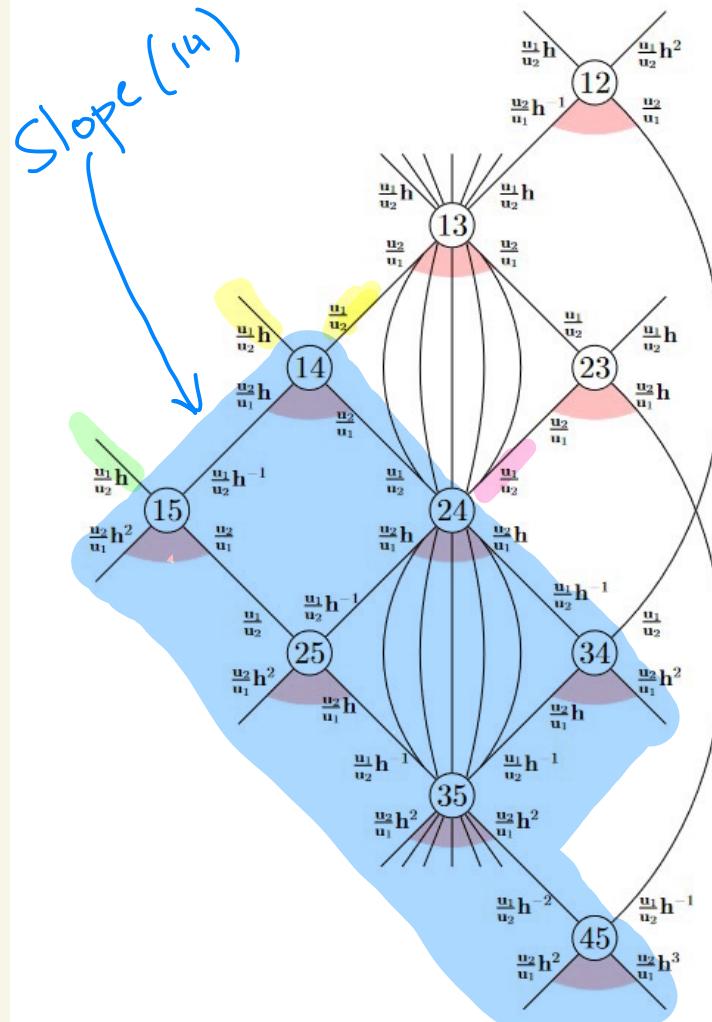
@ 34 divisible by  $(z_1 - z_2)$

divisible by  $h$

@ 35 divisible by  $h$

@ 45 divisible by  $(z_1 - z_2 - h)$

divisible by  $h$



Characteristic classes induce quantum group actions

$$H_T^* \left( \begin{smallmatrix} 0 & 1 & 2 \\ 2 & 2 & 2 \end{smallmatrix} \right) = \mathbb{C} (z_1, z_2, z_3, z_4)^4 \quad 4 \text{ fixed pts}$$

Stab's:    1.     $(z_2 - z_1, 0)$      $(z_2 - z_1 + \hbar, \hbar)$      $|_4$

Stab's:    1.     $(z_2 - z_1 + \hbar, \hbar)$      $(0, z_2 - z_1)$      $|_4$

Fact : transition matrix:

Yang's R-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Cor  $H_T^* \left( \begin{smallmatrix} 0 & 1 & 2 \\ 2 & 2 & 2 \end{smallmatrix} \right)$  is a  $\mathbb{Y}_{\hbar}(gl_2)$ -module geometrically

## Summary so far

$$\text{Stab}_P \in H_T^*(N(Q))$$

characteristic class of singularity

quiver variety

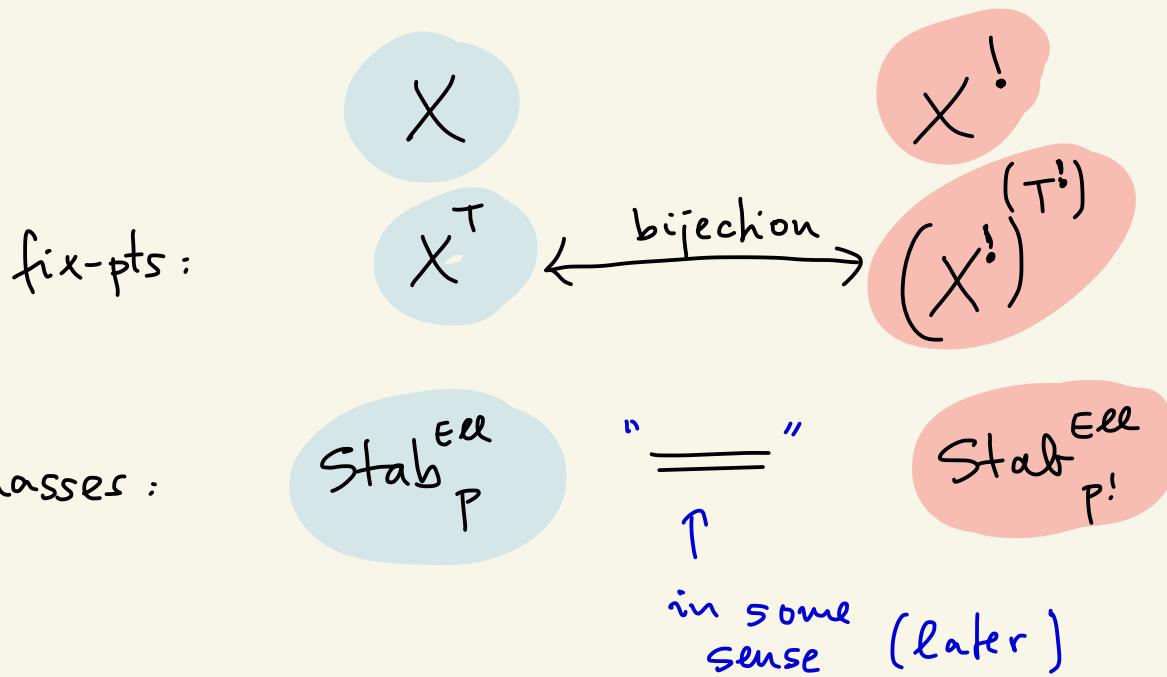
moment graph model

The diagram consists of four components arranged in a square-like shape. In the top-left corner is the expression  $\text{Stab}_P \in H_T^*(N(Q))$ . In the top-right corner is the word "quiver variety" with a red arrow pointing from the top edge of the square to it. In the bottom-left corner is the phrase "characteristic class of singularity" in green. In the bottom-right corner is the phrase "moment graph model" in blue. A green curved arrow points from "Stab\_P" to "characteristic class of singularity". A blue curved arrow points from  $H_T^*(N(Q))$  to "moment graph model".

rem key ingredients of  
geometric quantum group actions

## Fact

Some (not all) quiver varieties come in pairs  $(X, X')$  such that



8

$$T^* \text{Gr}_2 \mathbb{C}^4$$



$$\mathcal{N}\left(\begin{array}{c} 1 \\ 2 \\ 1 \end{array} \middle| \begin{array}{c} \square_2 \end{array}\right)$$

4

[RSV2]

12

$$T^* \text{Gr}_2 \mathbb{C}^5$$



$$\mathcal{N}\left(\begin{array}{c} 1 \\ 2 \\ 2 \\ 1 \end{array} \middle| \begin{array}{c} \square_1 \\ \square_1 \end{array}\right)$$

4

64

$$T^* \mathcal{F}_{2,6,10}$$



$$\mathcal{N}\left(\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 5 \\ 4 \\ 2 \end{array} \middle| \begin{array}{c} \square_2 \\ \square_1 \end{array}\right)$$

16

8

$$\mathcal{N}\left(\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array} \middle| \begin{array}{c} \square_2 \\ 2 \\ \square_2 \\ 2 \end{array}\right)$$



$$\mathcal{N}\left(\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{array} \middle| \begin{array}{c} \square_2 \\ 1 \\ \square_2 \\ 2 \\ 1 \end{array}\right)$$

10

$$T^* G/B$$



$$T^* G^L/B^L$$

[RW 2020]

32

$$T^* \mathcal{F}_{2,5,7}$$



$$\mathcal{N}\left(\begin{array}{c} 3 \\ 1 \end{array}\right)$$

dim

Cherkis bow varieties  
 $C(\dots)$

type-A Nakajima quiver varieties

$$N \left( \begin{array}{c} | & 2 & 2 & 1 & 4 \\ \bullet & - & - & - & - \\ \square & \square & \square & & \\ \downarrow & \downarrow & \downarrow & & \end{array} \right)$$

$$N \left( \begin{array}{c} | & | \\ \bullet & - \\ \square & \square \\ \downarrow & \downarrow \end{array} \right)$$

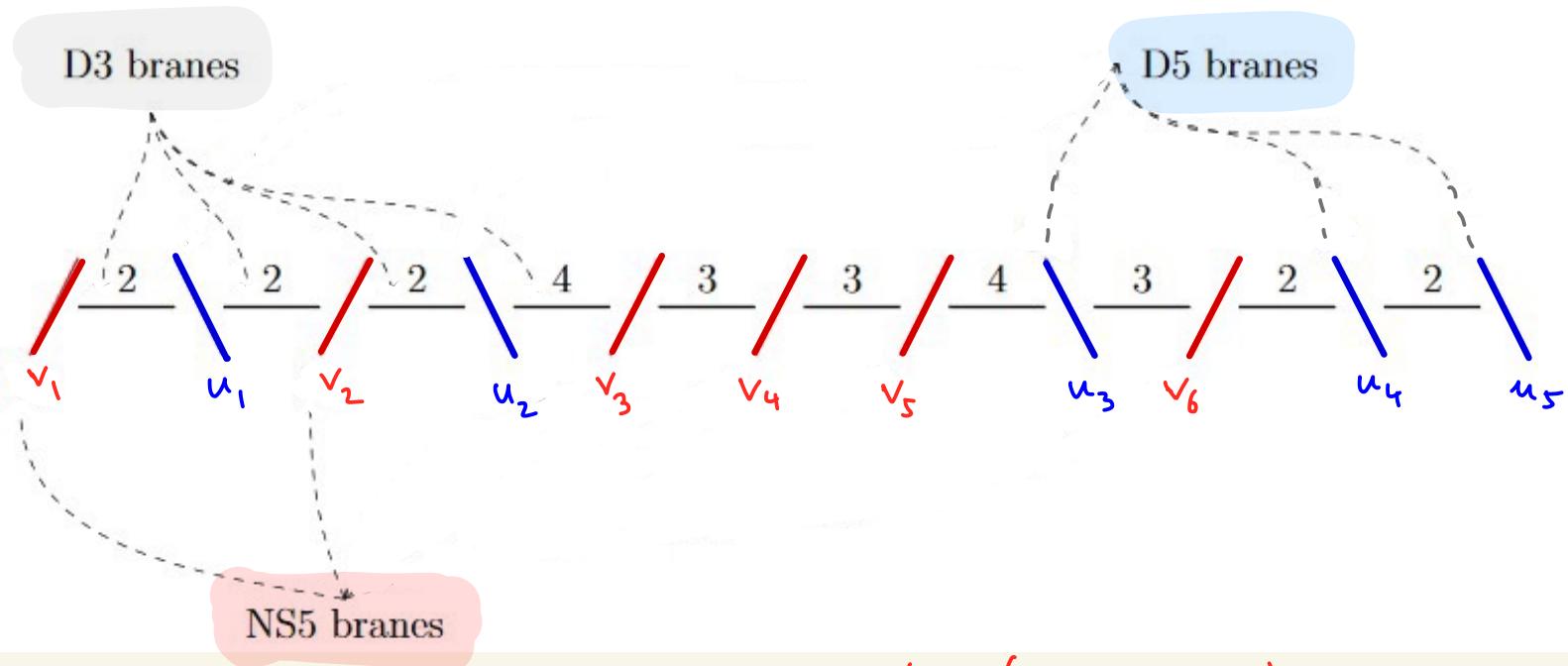
$$T^* \text{Gr}_2 \mathbb{C}^4$$

$$T^* \mathcal{F}_{2,5,7}$$

$$T^* \mathcal{F}_{1,2,3,4}$$

$$T^* G/P$$

## Brane diagrams



$v_i$ : Kähler (dynamical) variables  
 $u_i$ : equivariant variables

brane  
diagram  
 $\mathcal{D}$



$C(\mathcal{D})$

Cherkis bow  
variety



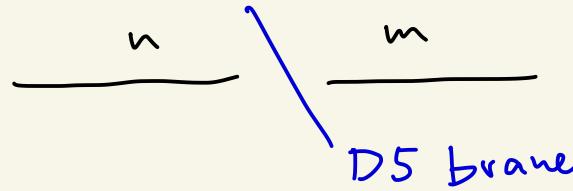
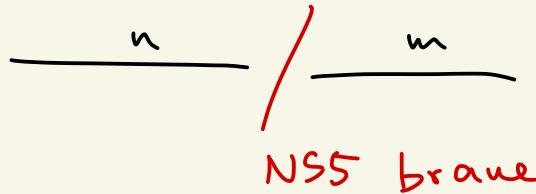
Cherkis:  
moduli space of  
unitary instantons  
on multi-Taub-NUT  
spaces  
(key: Nahm's  
equation)

Nakajima-Takayama  
Hamiltonian reduction  
of representations  
of certain quivers  
with relations

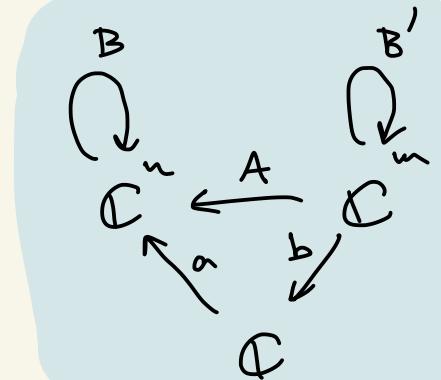
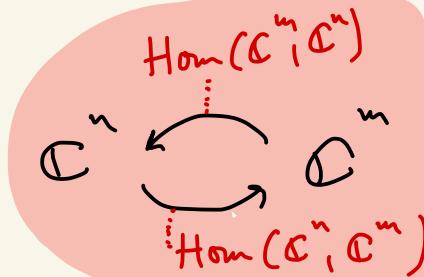
~

Rozansky = R  
"symplectic  
intersection"  
of generalized  
Lagrange  
varieties

# Nakajima - Takayama description (sketch)



... then ...  $\frac{\mu^{-1}(0)^{ss}}{\mathrm{PGL}(n)} \cdots$



mod  
 $BA + AB' + ab = 0$

D  
brane  
diagram



$C(D)$

bow variety

tautological bundles  
(D3 branes)

$T \times C_k^*$   
D5  
branes

smooth  
holomorphic symplectic

$$\dim(C(D)) = \sum_{U \in D5} \left[ (d_{u_-} + 1)d_{u_-} + (d_{u_+} + 1)d_{u_+} \right]$$

$$+ \sum_{V \in NS5} 2 d_{v^+} d_{v^-} - 2 \sum_{X \in D3} d_X^2$$

example

$$\begin{aligned} \dim(C(\text{Diagram})) &= 2 \cdot 1 + 2 \cdot 1 \\ &\quad + 2 \cdot 0 \cdot 1 + 2 \cdot 1 \cdot 0 - 2(1^2 + 1^2 + 1^2 + 1^2) \\ &= 4 \end{aligned}$$

How are  $\mathcal{N}$ (quiver) special cases?



Examples  $T^* \mathbb{P}^1 = \mathcal{N}\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = C\left(\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}\right)$

$$T^* \text{Gr}_2 \mathbb{C}^4 = \mathcal{N}\left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right) = C\left(\begin{smallmatrix} 2 & 2 & 2 & 2 \end{smallmatrix}\right)$$

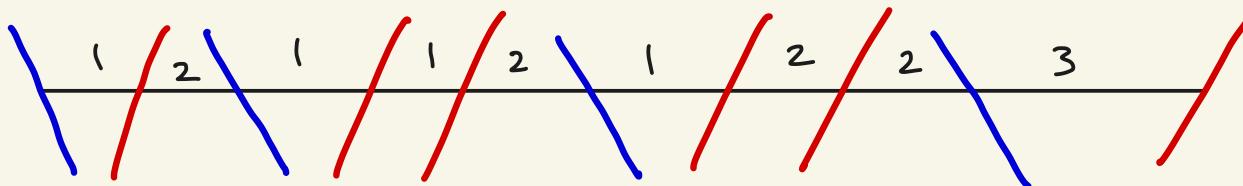
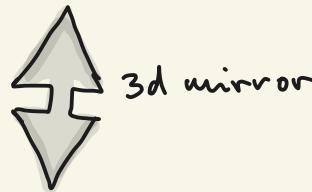
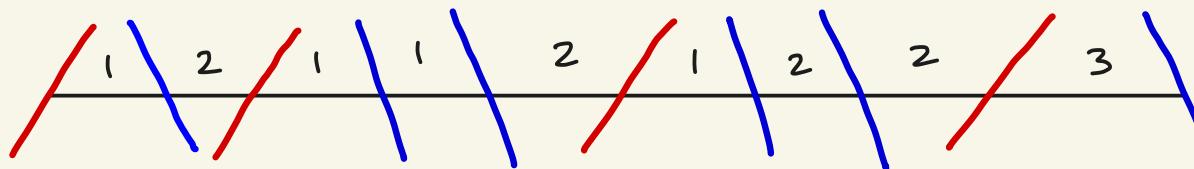
$$T^* \mathcal{F}_{1,2,3,4} = \mathcal{N}\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{smallmatrix}\right) = C\left(\begin{smallmatrix} 1 & 2 & 3 & 3 & 3 & 3 & 3 \end{smallmatrix}\right)$$

$$\mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) = C\left(\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right)$$

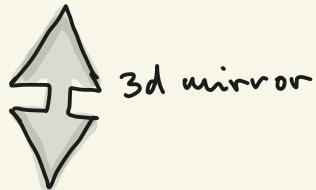
Observe  $\frac{k}{k}$

"cobalanced brane diagram"

3D mirror symmetry for bow varieties:



$$\underline{\text{Ex}} \quad T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} & 1 \\ 1 & \\ \square & 3 \end{smallmatrix}\right) = C\left(\cancel{1} / \cancel{1} / \cancel{1} / \cancel{1} / \cancel{1}\right) \quad \text{dim } 4$$

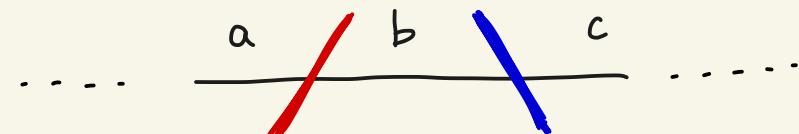


$$C\left(\cancel{1} / \cancel{1} / \cancel{1} / \cancel{1} / \cancel{1}\right) \quad \text{dim } 2$$

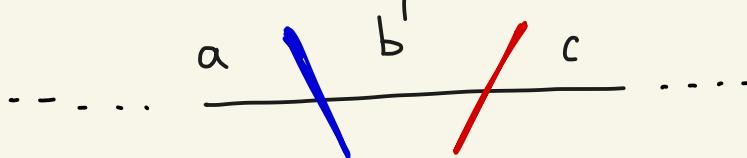
not cobalanced, ie not  $\mathcal{N}(\dots)$

... but ... <to be continued>

Hanany - Witten transition on brane diagrams.



↔  
HW

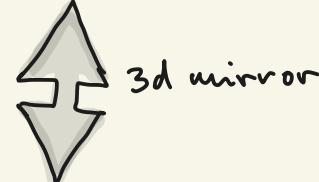


$$b + b' = a + c + 1$$

(why? later:  
"brane charge")

Thm  $C(\mathcal{D}) \approx C(HW(\mathcal{D}))$

$$\underline{\text{Ex}} \quad T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ \square & 3 \end{smallmatrix}\right) = C\left(\begin{array}{c|c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}\right)$$



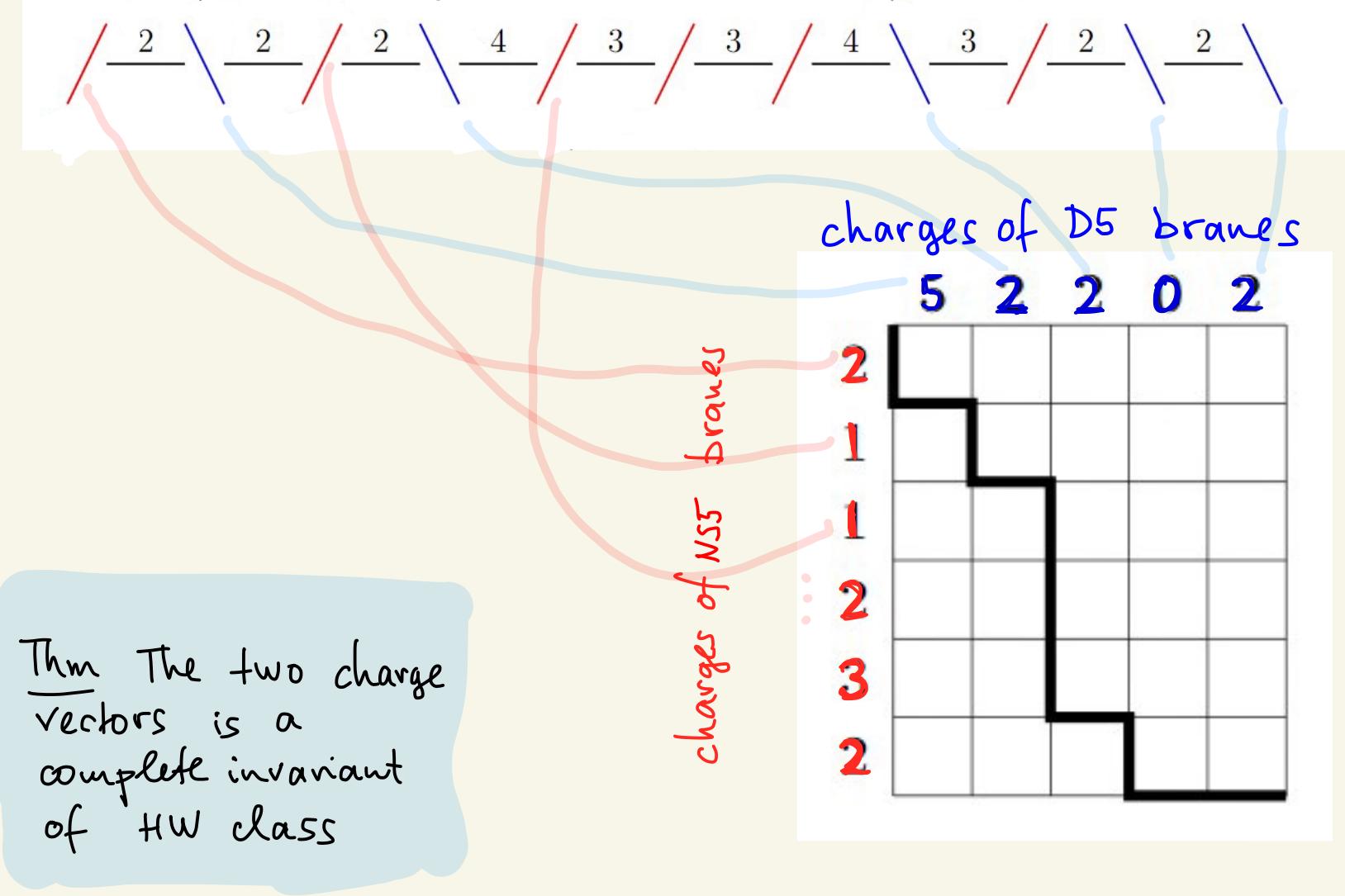
$$\begin{aligned} & \xrightarrow{\text{HW}} C\left(\begin{array}{c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \end{array}\right) \\ \curvearrowleft & C\left(\begin{array}{c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \end{array}\right) \stackrel{\text{HW}}{=} C\left(\begin{array}{c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \end{array}\right) \\ & \qquad \qquad \qquad \stackrel{''}{=} \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ \square & \square \\ 1 & 1 \end{smallmatrix}\right) \end{aligned}$$

$$\Rightarrow T^*\mathbb{P}^2 \quad \xleftarrow{\text{3d mirror}} \quad \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ \square & \square \\ 1 & 1 \end{smallmatrix}\right)$$

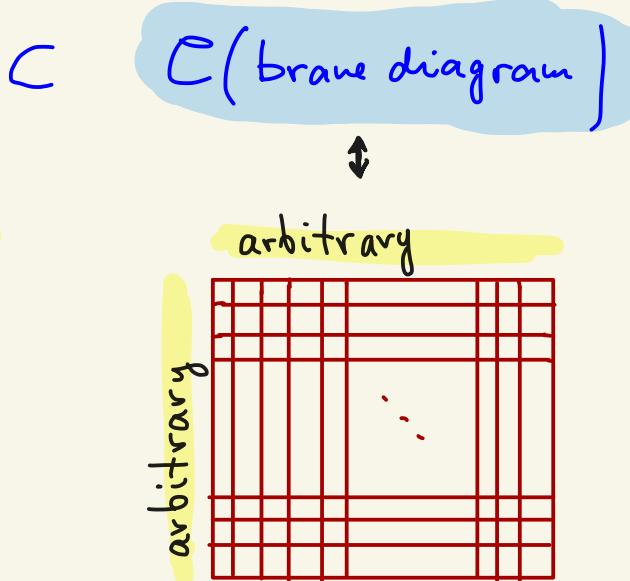
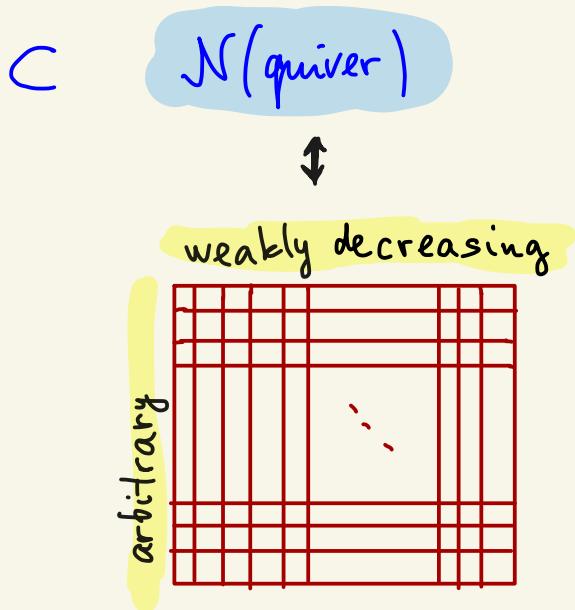
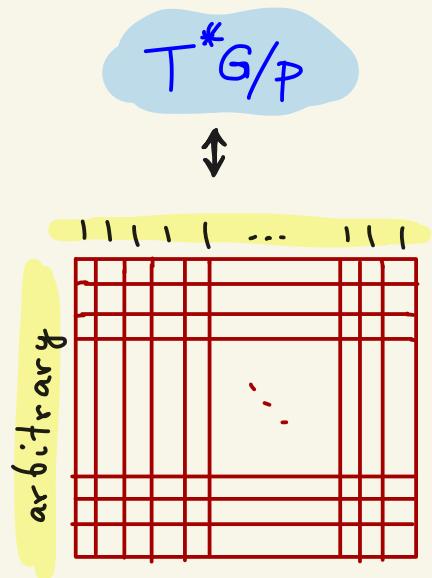
def brane charge

$$\text{charge} \left( \begin{array}{c} \text{NS5 brane} \\ \hline k \cancel{/} l \end{array} \right) := l - k + \#\{\text{D5-branes left of it}\}$$

$$\text{charge} \left( \begin{array}{c} \text{D5 brane} \\ \hline k \cancel{/} l \end{array} \right) := k - l + \#\{\text{NS5-branes right of it}\}$$



Thm (up to Hanany-Witten transition)



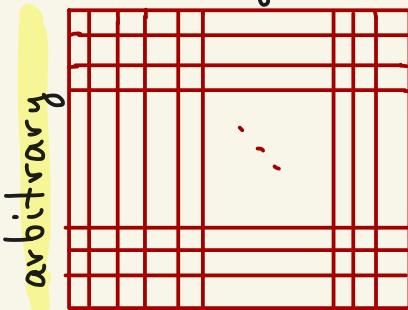
new operation:  
transpose !

Fact : quantum groups\* act on  $H_T^*$  (moduli spaces)

which representation ②

arbitrary

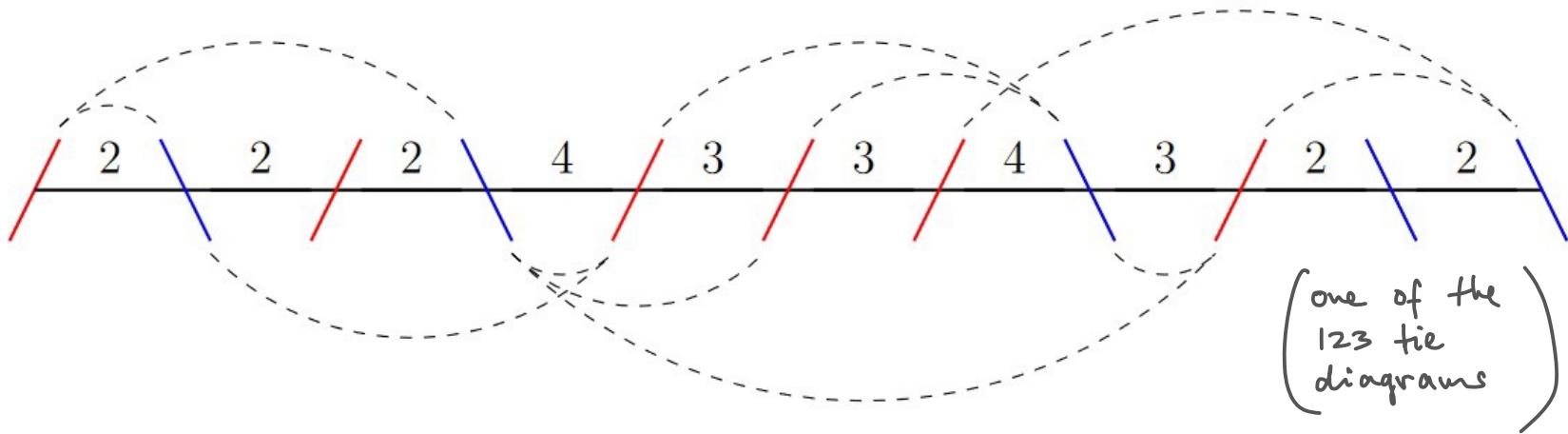
which weight space of the representation ③



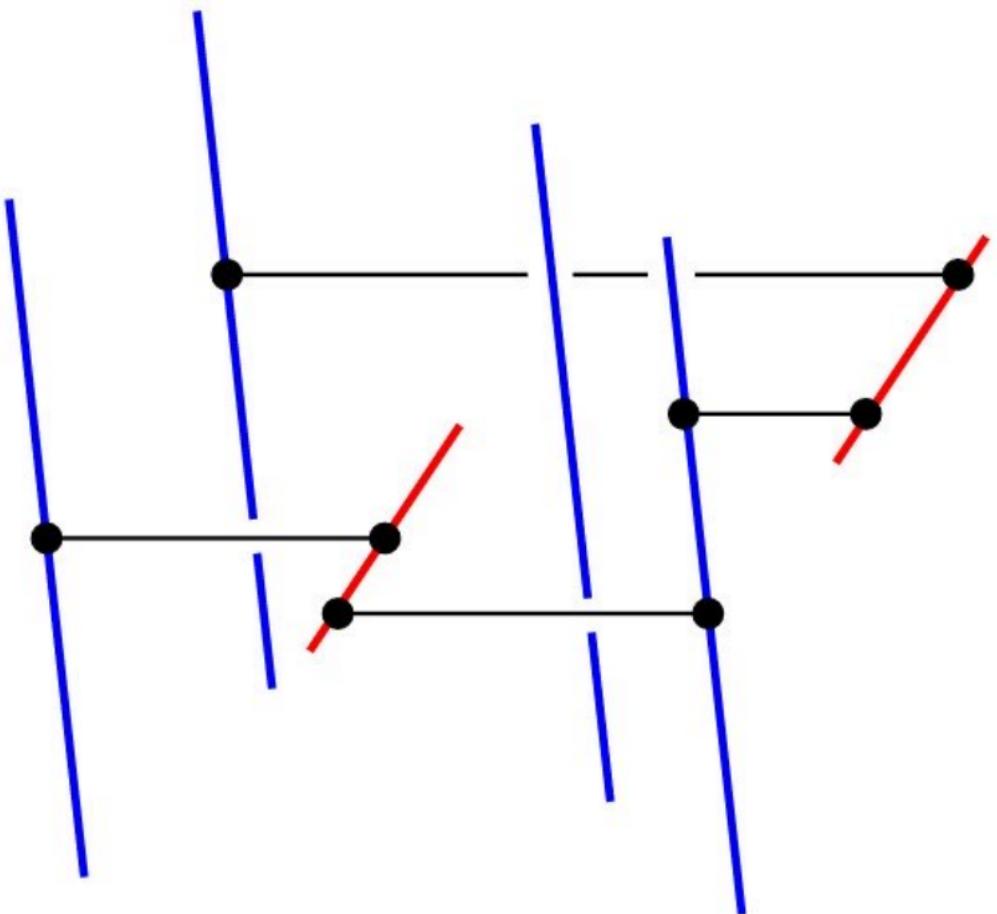
size : which quantum group ①

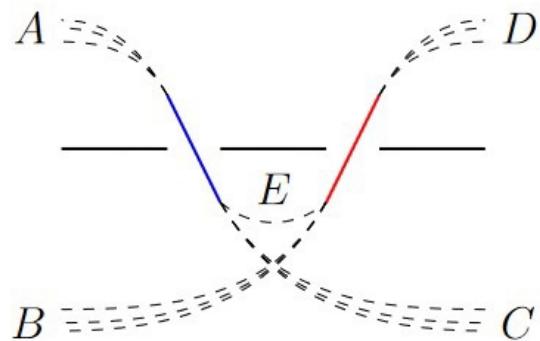
\*  $\exists$  super-algebra version [R-Rozanty], not in today's talk

fixed points  $\overset{1:1}{\leftrightarrow}$  tie diagrams

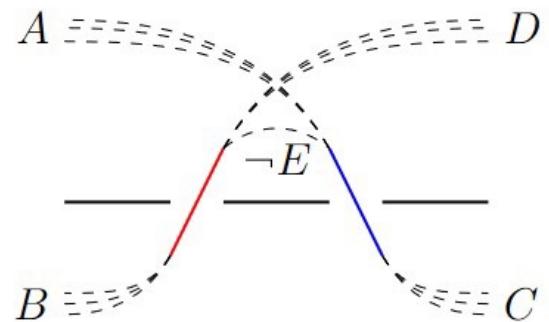


- a tie must connect 5-branes of different kinds
- each D3 brane to be covered as many times as its multiplicity

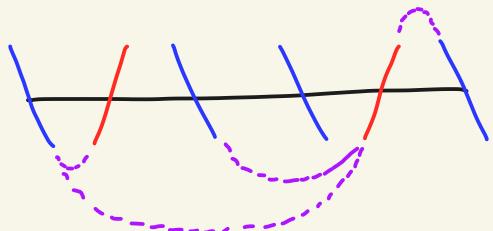




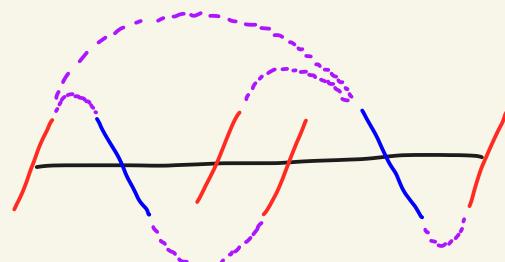
HW transition  
on fixpoints



R-III



3d mirror  
on fixedpoints  
horizontal  
reflection



$$\begin{array}{c} / \quad 2 \quad \backslash \quad 2 \quad / \quad 2 \quad \backslash \quad 4 \quad / \quad 3 \quad / \quad 3 \quad / \quad 4 \quad \backslash \quad 3 \quad / \quad 2 \quad \backslash \quad 2 \quad \backslash \\ \text{Diagram showing a sequence of red and blue segments forming a bracket-like structure.} \end{array}$$

## binary contingency tables

BCT : 0-1-matrix  
with row &  
column sums  
the charge vectors

Thm

fix pts  $\longleftrightarrow$  BCT's

one of the 123 BCTs

	5	2	2	0	2
2	1	1	0	0	0
1	1	0	0	0	0
1	0	0	1	0	0
2	1	0	1	0	0
3	1	1	0	0	1
2	1	0	0	0	1

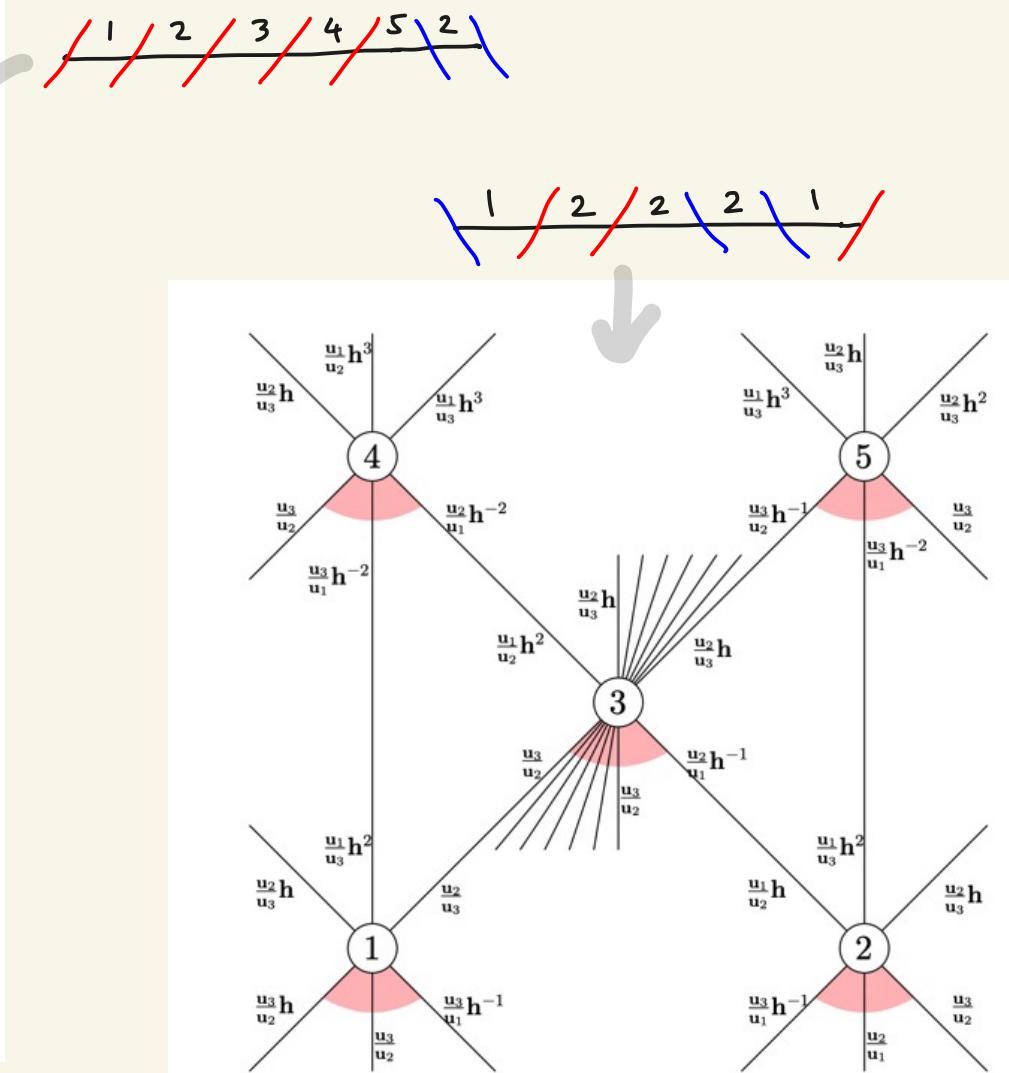
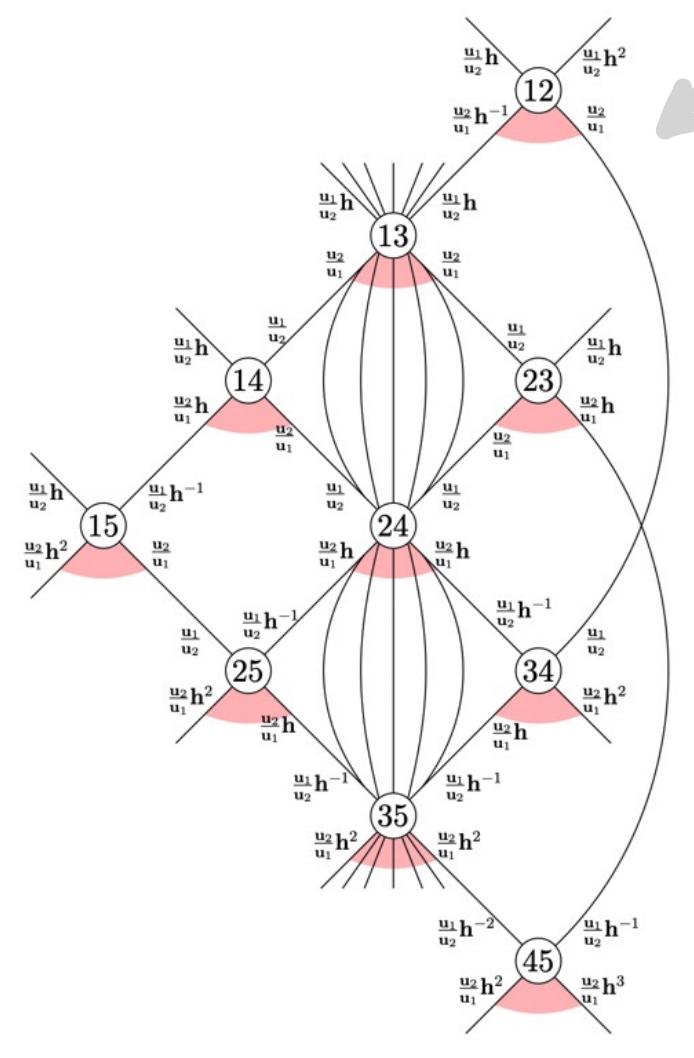
fixed points  
invariant curves  
(with weight)



moment  
graph



$\text{Stab}_P$

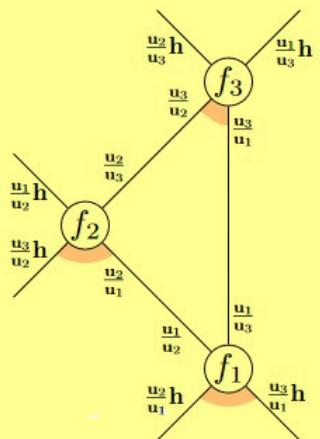


$$\begin{array}{c} / \backslash \backslash \backslash \backslash / \\ | \quad | \quad | \quad | \\ = T^* \mathbb{P}^2 \end{array}$$

	$f_1$	$f_2$	$f_3$
$f_1$	$\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^4\right)$	0	0
$f_2$	$\theta(h)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2 v_2}{u_1 v_1}h^3\right)$	$\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^3\right)$	0
$f_3$	$\theta(h)\theta\left(\frac{u_2}{u_1}h\right)\theta\left(\frac{u_3 v_2}{u_1 v_1}h^2\right)$	$\theta(h)\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_3 v_2}{u_2 v_1}h^2\right)$	$\theta\left(\frac{u_2}{u_3}h\right)\theta\left(\frac{u_1}{u_3}h\right)\theta\left(\frac{v_2}{v_1}h^2\right)$

combinatorics  
of  
tie diagrams  
BCTs

actions of  
characteristic  
of singularities  
(Aganagic-Olson-Kontsevich)

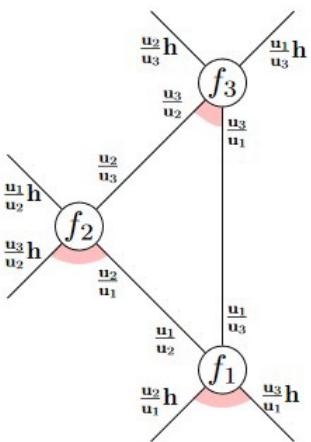


Jacobi theta function

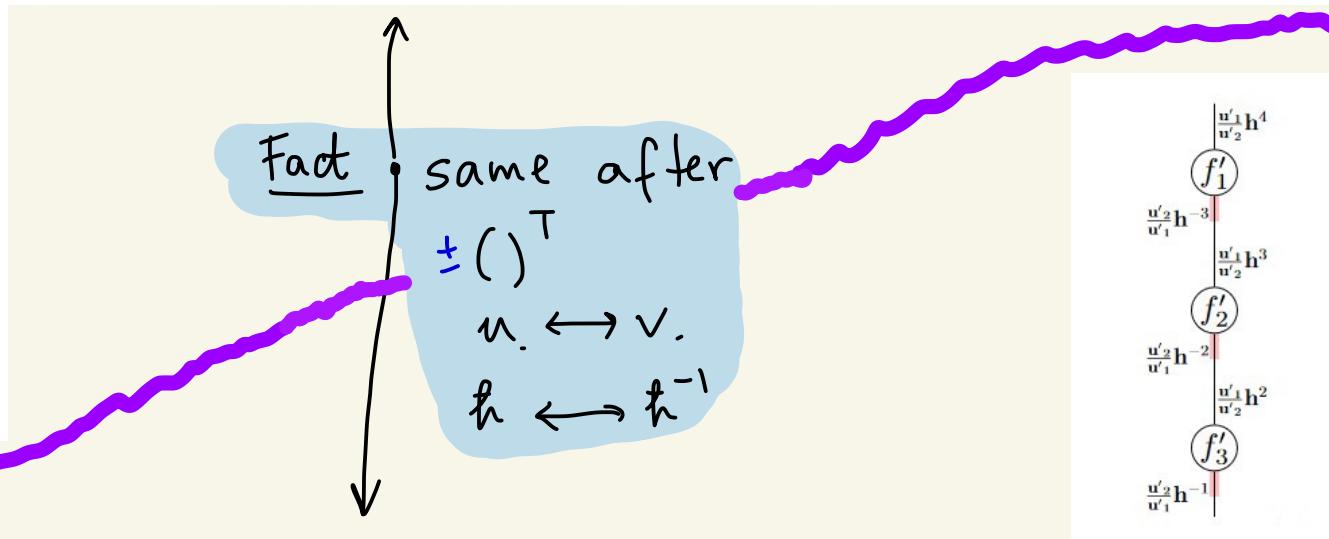
$$\Theta(x) = \left( x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) \prod_{n=1}^{\infty} (1 - q^n x)(1 - q^n x^{-1})$$

( $q \in \mathbb{C}^x \mid |q| < 1$  fixed)

$$\begin{aligned} T^*\mathbb{P}^2 &= \mathcal{N} \left( \begin{array}{c|cc} & 1 & \\ \hline & \square_3 & \end{array} \right) \\ &= \mathcal{C} \left( \begin{array}{ccccccc} \diagdown & \diagup & \diagup & \diagup & \diagup & \diagup & \diagup \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right) \end{aligned}$$



	$f_1$	$f_2$	$f_3$
$f_1$	$\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^4\right)$	0	0
$f_2$	$\theta(h)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2 v_2}{u_1 v_1}h^3\right)$	$\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^3\right)$	0
$f_3$	$\theta(h)\theta\left(\frac{u_2}{u_1}h\right)\theta\left(\frac{u_3 v_2}{u_1 v_1}h^2\right)$	$\theta(h)\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_3 v_2}{u_2 v_1}h^2\right)$	$\theta\left(\frac{u_2}{u_3}h\right)\theta\left(\frac{u_1}{u_3}h\right)\theta\left(\frac{v_2}{v_1}h^2\right)$



$$\begin{array}{c} \frac{u'_1}{u'_2}h^4 \\ \frac{u'_2}{u'_1}h^{-3} \\ \frac{u'_1}{u'_2}h^3 \\ \frac{u'_2}{u'_1}h^{-2} \\ \frac{u'_1}{u'_2}h^2 \\ \frac{u'_2}{u'_1}h^{-1} \end{array}$$

	$f'_1$	$f'_2$	$f'_3$
$f'_1$	$\theta\left(\frac{u'_1}{u'_2}h^4\right)\theta\left(\frac{v'_2}{v'_1}\right)\theta\left(\frac{v'_3}{v'_1}\right)$	$\theta(h)\theta\left(\frac{v'_3}{v'_1}\right)\theta\left(\frac{v'_2 u'_2}{v'_1 u'_1}h^{-3}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h^{-1}\right)\theta\left(\frac{v'_3 u'_2}{v'_1 u'_1}h^{-2}\right)$
$f'_2$	0	$\theta\left(\frac{u'_1}{u'_2}h^3\right)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3}{v'_2}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3 u'_2}{v'_2 u'_1}h^{-2}\right)$
$f'_3$	0	0	$\theta\left(\frac{u'_1}{u'_2}h^2\right)\theta\left(\frac{v'_3}{v'_2}h\right)\theta\left(\frac{v'_3}{v'_1}h\right)$

$$\begin{aligned} \mathcal{N} \left( \begin{array}{c|cc} & 1 & \\ \hline & \square_1 & \square_1 \end{array} \right) &= \\ \mathcal{C} \left( \begin{array}{ccccccc} \diagdown & \diagup & \diagup & \diagup & \diagup & \diagup & \diagup \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right) & \end{aligned}$$

3d mirror symmetry for stable envelopes  
is proved in special cases

- $T^* \text{Gr}_k \mathbb{C}^n$   $\leftrightarrow$  its dual [R-Smirnov-Varchenko-Zhou]
- $T^* G/B$   $\leftrightarrow$   $T^* G^L/B^L$   $\begin{cases} \text{type A [RSVZ]} \\ \text{general [R-Weber]} \end{cases}$
- hypertoric  $\leftrightarrow$  dual hypertoric [Smirnov-Zhou]
- finitely many other cases [R-Shou]

# Why do we think 3d mirror symmetry for char. classes hold?

$X \rightsquigarrow H_T^*(X)$ -valued functions of  $(\underline{z}, \underline{q}, \underline{t})$

- 2 sets of equations

$$f(z_i+1) = K_i(\underline{z}, \underline{q}, \underline{t}) f$$

↑ essentially R-matrix

$q^K Z$  equations in equivariant parameters

$$\left[ q_i \frac{\partial}{\partial q_i} - (c_i *) \right] f = 0$$

↑ quantum multiplication

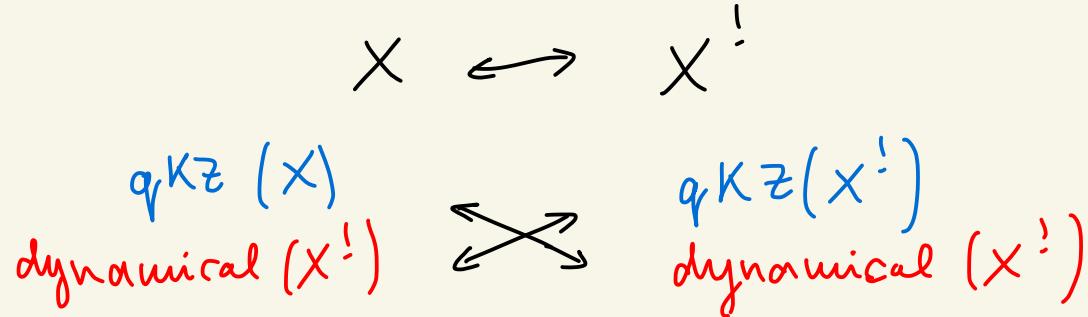
"dynamical connection"  
in  $\underline{q}$  (Kähler parameters)

- Fact: compatible

- Fact: solution set = < vertex functions >

↑ curve counting p.s. on  $X$

String theory ... Higgs branch vs Coulomb branch ...



curve counting  
vertex functions ( $X$ )



curve counting  
vertex functions ( $X'$ )

transition  
matrix

$$\left[ (\text{Stab}_p)_{q_j} \right]_{p, q \in X^T}$$

# Summary

- $X \hookrightarrow T$

$\text{Stab}_p$

enumerative geometry

representations of  
quantum groups

differential equations  
 $q$ -difference equations

- $X \xleftrightarrow[3d\text{ mirror}]{}$

$X^!$

$\text{Stab} \sim \text{Stab}^!$

- bow varieties

closed for 3d mirror symmetry

rich combinatorics

(more "complete" than

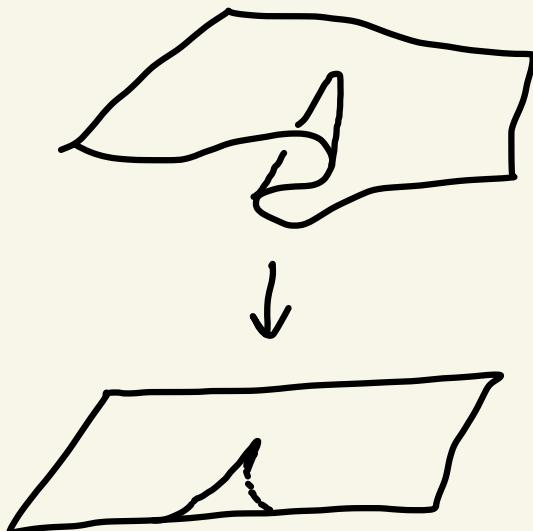
Schubert Cal.  
quiver varieties)



Thank you !

backup slides ↴

$$GL_2 \times GL_2 \subset (\Sigma \parallel M) \quad \{ \text{those } \sim (x,y) \mapsto (x^3 + xy, y) \} \quad \{ (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \text{ holomorphic germs} \}$$



$$[\bar{\Sigma}] \in H^*_{GL_2 \times GL_2} (pt)$$

$$\parallel \\ C_1^2 + C_2$$

Cor generic map  
 $\mathbb{R}\mathbb{P}^2 \rightarrow \mathbb{R}^2$   
has an odd number of cusps

$$GL_3 \times D^9 \subset \left( \sum_{\parallel} \subset M \right)$$

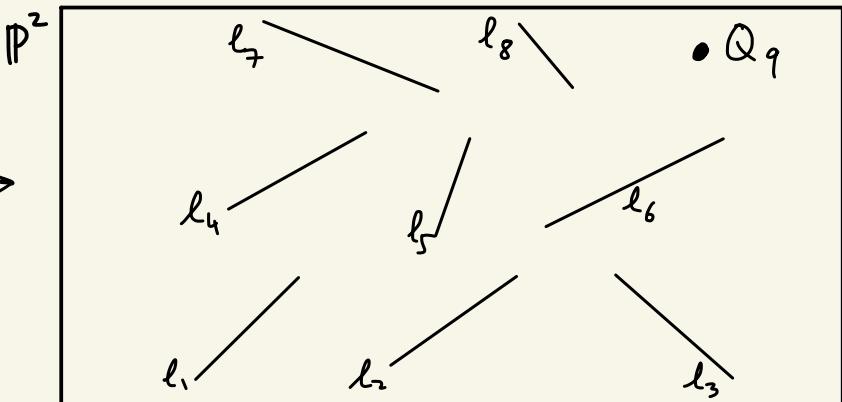
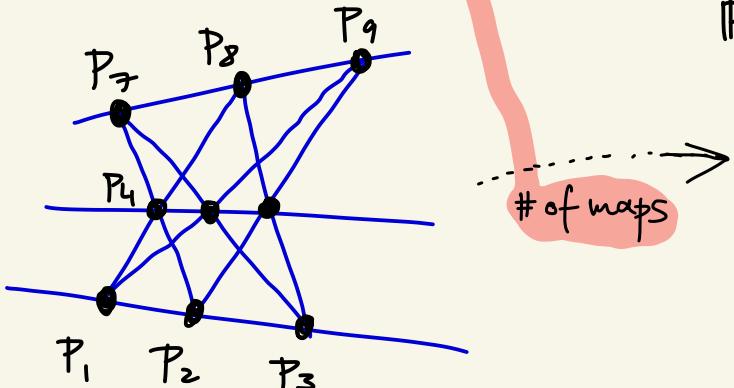
$\left\{ \begin{array}{c} \text{Diagram showing 9 points in a 3x3 grid with many blue lines connecting them.} \\ \end{array} \right\}$

$\text{Hom}(\mathbb{C}^9, \mathbb{C}^3)$

$$[\Sigma] \in H_{GL_3 \times D^9}^*(pt) = \mathbb{Z}[c_1, c_2, c_3, d_1, d_2, \dots, d_9]$$

↓

$$= \dots + 5d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9 + \dots$$



X

difference  $q_r K\mathbb{Z}$ -equations

(consistent!)

differential dynamical equations

on vector-valued functions in  $(\underbrace{z}, \underbrace{q_r})$

equivar

Kähler

$$X = \text{Gr}_0 \mathbb{C}^2 = \text{pt}$$

$q_r K\mathbb{Z}$

$$f(z_1 + K, z_2, q_{r1}, q_{r2}) = q_{r1} f(z_1, z_2, q_{r1}, q_{r2})$$

$$f(z, z_2 + K, q_{r1}, q_{r2}) = q_{r1} f(z_1, z_2, q_{r1}, q_{r2})$$

$$K q_{r1} \frac{\partial f}{\partial q_{r1}} = (z_1 + z_2) f$$

dynamical

$$K q_{r2} \frac{\partial f}{\partial q_{r2}} = f$$

$$f = q_{r1}^{\frac{z_1 + z_2}{K}} q_{r2}^{\frac{1}{K}}$$

Kähler parameters  
keeping track of  
deg curve

- curve counting in  $X$  :  $V \in K_T(X)[q]$  vertex function

- $p \in X^T$   $V_p \in K(\text{point})[q]$

↑  
equivariant  
parameters  $u$   
(mero-)

↑  
Kähler parameters  $v$   
(holo-)

- $V_p$  satisfy difference equations

in  $\mathbb{Z}$  variables

in  $q$  variables

diff eq's  
switched  
for  $X'$

- $V_q^! := \sum \text{Stab}_{q/p} \cdot V_p$  vertex function on  $X^!$

fix  $q \in \mathbb{C}^* \quad \|q\| < 1$

$$\mathfrak{f}(a, b) := \frac{\mathfrak{f}(ab)}{\mathfrak{f}(a) \mathfrak{f}(b)}$$

Jacobi theta function:

$$\mathfrak{f}(x) = \left( x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) \prod_{n \geq 1} (1 - q^n x)(1 - q^n x^{-1})$$

  $\sim \sin X$

K theory

  $q_f$ -decoration

Why do elliptic characteristic classes necessarily depend on a new set of variables? **Intuitive answer**



Fay's trisecant identity : Ell  $x_1x_2x_3 = y_1y_2y_3 = 1 \Rightarrow$

$$\delta(x_1, y_2) \delta(x_2, \frac{1}{y_1}) + \delta(x_2, y_3) \delta(x_3, \frac{1}{y_1}) + \delta(x_3, y_1) \delta(x_1, \frac{1}{y_3}) = 0$$

Trigonometric (K-theory) limit

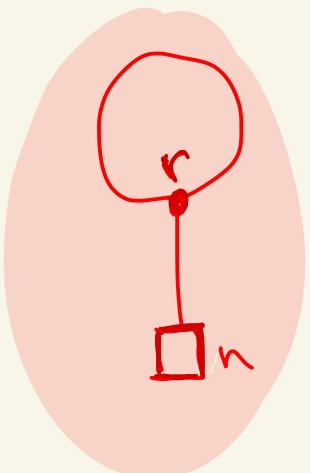
$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0 \Rightarrow$$

$$\cot(x_1)\cot(x_2) + \cot(x_2)\cot(x_3) + \cot(x_3)\cot(x_1) = \cot(y_1)\cot(y_2) + \cot(y_2)\cot(y_3) + \cot(y_3)\cot(y_1)$$

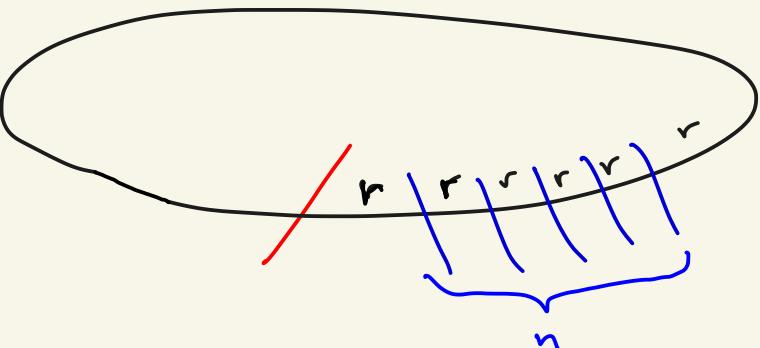
Rational ( $H^*$ ) limit

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0 \Rightarrow$$

$$\frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \frac{1}{x_2x_3} = \frac{1}{y_1y_2} + \frac{1}{y_1y_3} + \frac{1}{y_2y_3}$$

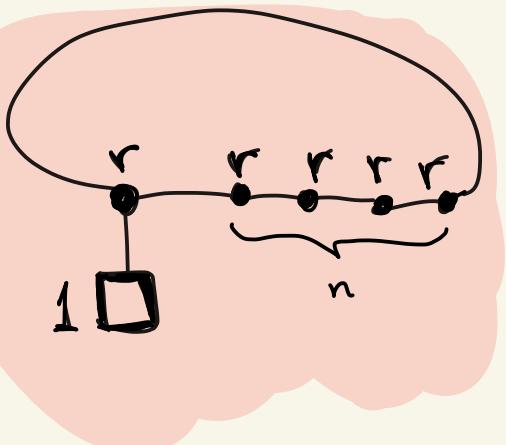


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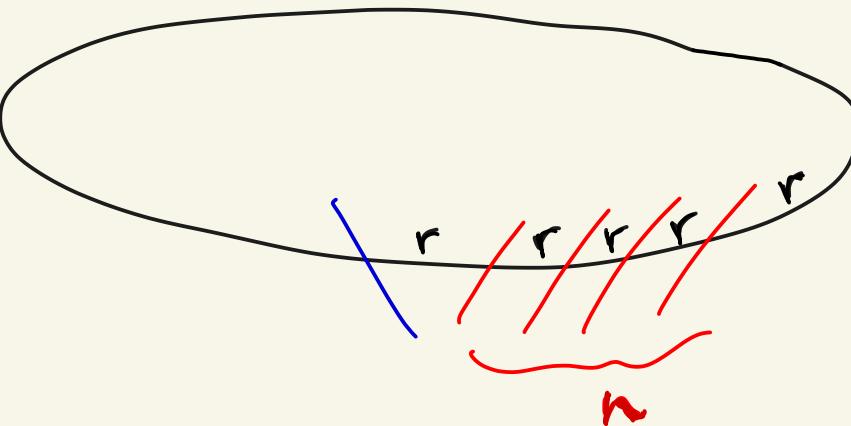


3d mirror

$r=1$   
 $\text{Hilb}^{[n]}$   
self dual



=



recall

$$[\bar{\Omega}_I] = \text{Sym}_{t_1, \dots, t_k} \left( \prod_{a=1}^k \left( \prod_{b=i_a+1}^n (z_b - t_a) \right) \prod_{1 \leq a \leq b \leq k} \frac{1}{(t_b - t_a)} \right)$$

essentially the  
Vandermonde  
 $\frac{\det(\dots)}{\det(\dots)}$  formula  
for Schur  
functions

---

fact

$$c^{\text{Sym}}(\bar{\Omega}_I) = \text{Sym}_{t_1, \dots, t_k} \left( \prod_{a=1}^k \left( \prod_{b=1}^{i_a-1} (z_b - t_a + h) \right) \prod_{b=i_a+1}^n (z_b - t_a) \prod_{1 \leq a \leq b \leq k} \frac{1}{(t_b - t_a)(t_a - t_b + h)} \right)$$

"weight  
functions"