

Stable envelopes,
3d mirror symmetry,
bow varieties

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partially based on joint works with

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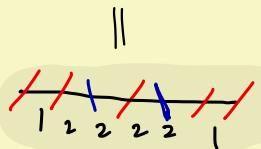
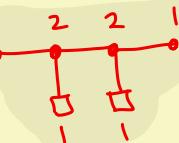
Cherkis bow varieties

Nakajima quiver var's

T^*G/P

$$T^*Gr_2 \mathbb{C}^5 = \begin{array}{c} \bullet \\ \square \\ \bullet \\ \square \\ \bullet \end{array} \begin{matrix} 2 & & & \\ & 5 & & \end{matrix}$$

3d mirror for
char. classes



HW

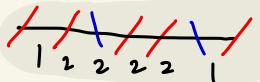
3d mirror



HW

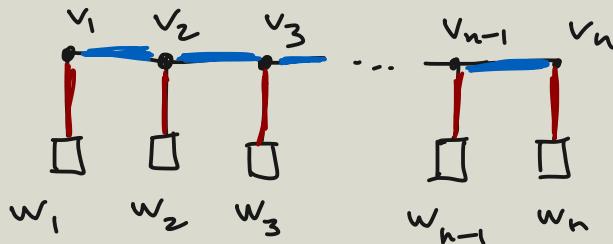


HW



Hanany-Witten transition

Nakajima quiver varieties (type A)



quiver Q

$\mathcal{N}(Q)$

quiver variety

Ex

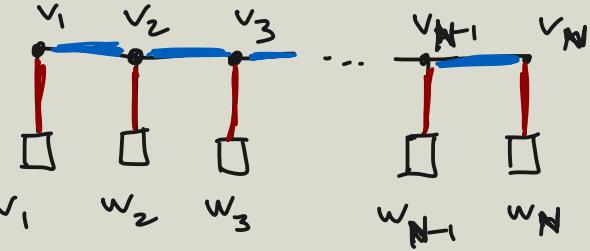
$$\mathcal{N}\left(\begin{smallmatrix} k & \bullet \\ n & \square \end{smallmatrix}\right) = T^* \mathrm{Gr}_k \mathbb{C}^n$$

$$\mathcal{N}\left(\begin{smallmatrix} k_1 \leq k_2 \leq k_3 \\ \bullet \quad \bullet \quad \bullet \\ \square & n \end{smallmatrix}\right) = T^* \mathcal{F}_{k_1, k_2, k_3, n}$$

$$\mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ \square & \square \\ 1 & 1 \end{smallmatrix}\right) = \widetilde{\mathbb{C}^2 / \mathbb{Z}_3}$$

$N(Q)$

- smooth
- holomorphic symplectic
- $T = (T^{w_1} \times T^{w_2} \times \dots \times T^{w_N}) \times \mathbb{C}_{\hbar}^*$ action
- finitely many fixed pts
- "tautological" v_1, v_2, \dots, v_N -bundles



$$H_T^*(N(Q)) = ?$$

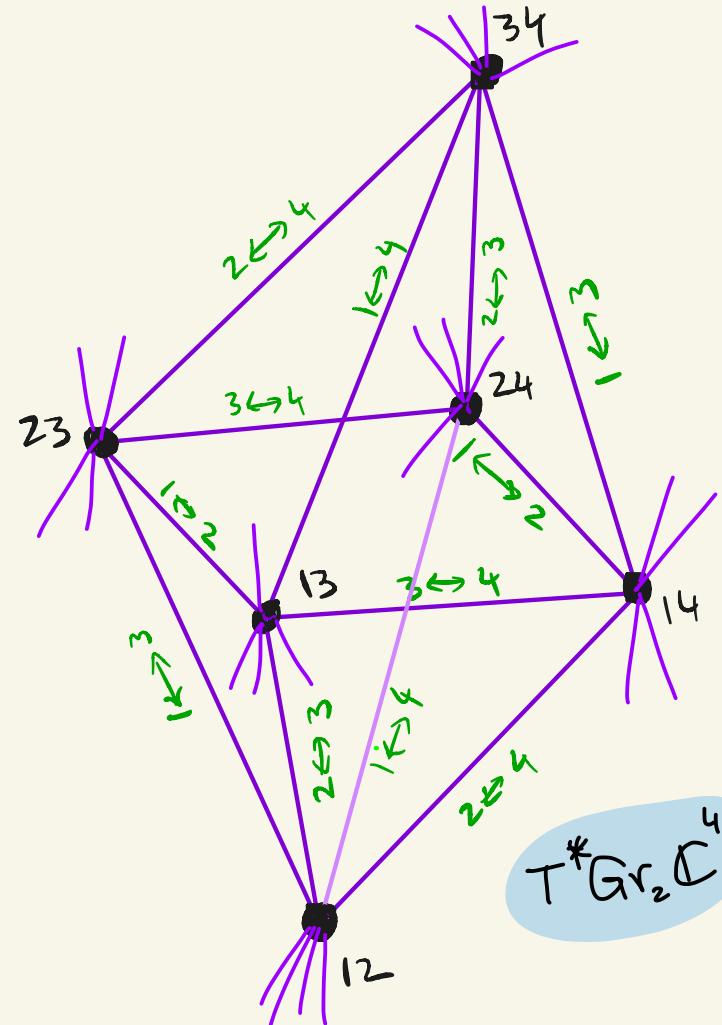
$$H_T^*(N(Q)) \xrightarrow{\text{Loc}} \bigoplus_{\text{T-fix}} H_T^*(\text{pt})$$

equivariant localization map
(restriction to fixpoints)

$\mathbb{C}[u_1, \dots, u_n, t]$

$$\text{im}(\text{Loc}) = ?$$

constraints among the components

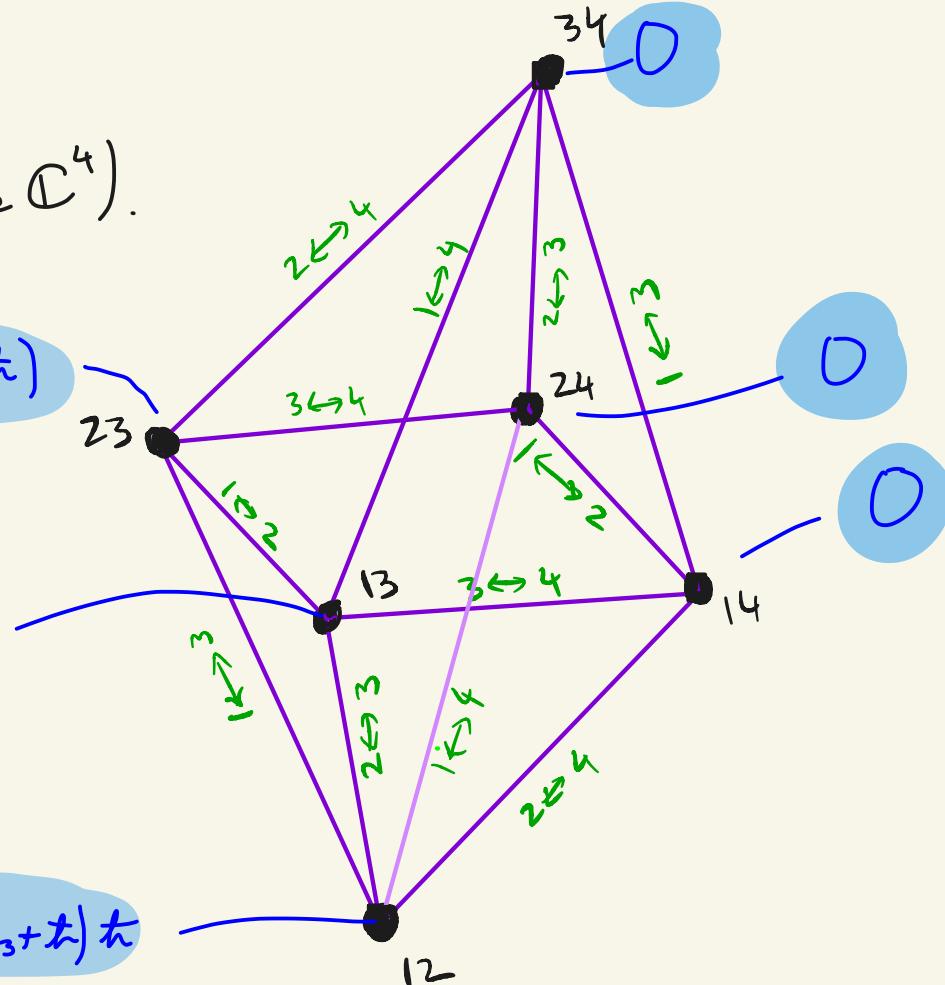


For example, this 6-tuple is an element of $H_T^*(T^*Gr_2 \mathbb{C}^4)$.

$$(u_4 - u_3)(u_4 - u_2)(u_2 - u_1 + t)(u_3 - u_1 + t)$$

$$(u_4 - u_1)(u_4 - u_3)(u_3 - u_2 + t) t$$

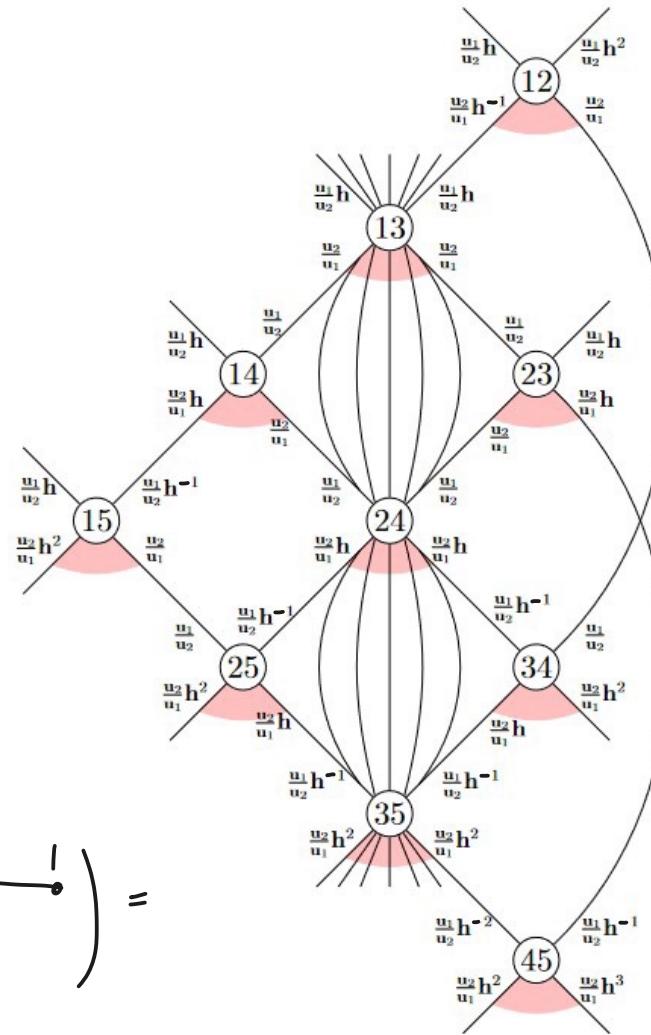
$$(u_4 - u_1)(u_4 - u_2)(u_2 - u_3 + t) t$$



Warning

- $T^* \text{Gr}_2 \mathbb{C}^4$ was special ("GKM")
- In general the constraints among components are more restrictive

$$\mathcal{N}\left(\begin{array}{c|ccccc} & & & & \\ \bullet & 2 & 2 & 1 & & \\ & | & | & | & & \\ & \square & \square & \square & & \\ & | & | & | & & \\ & 1 & 1 & 1 & & \end{array}\right) =$$



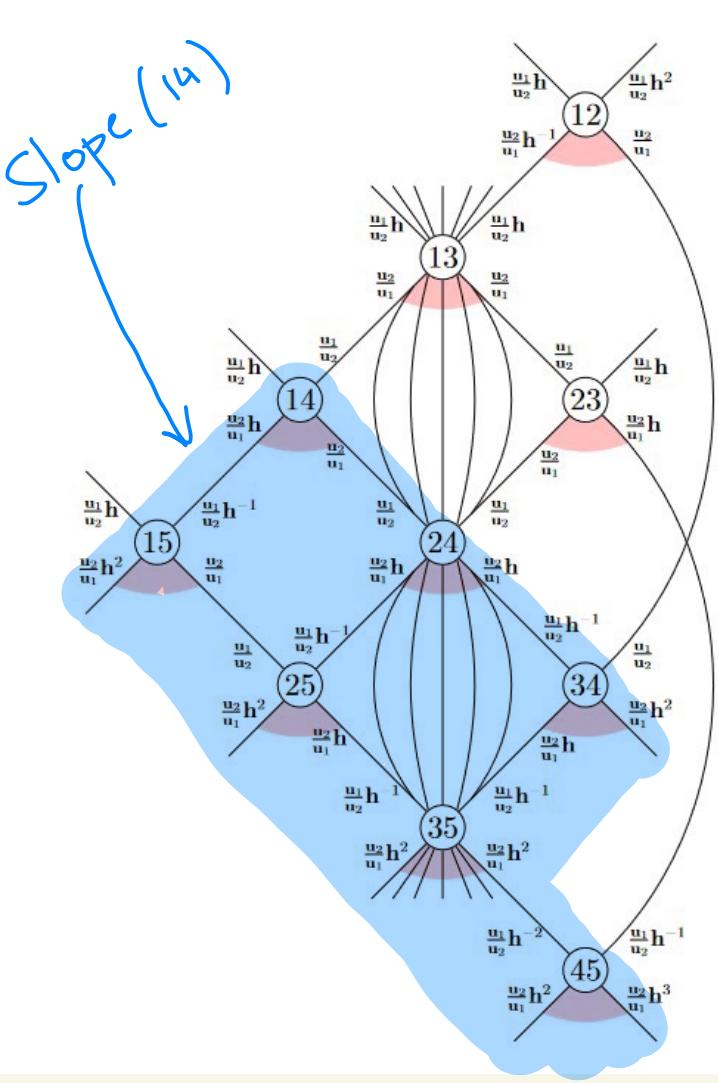
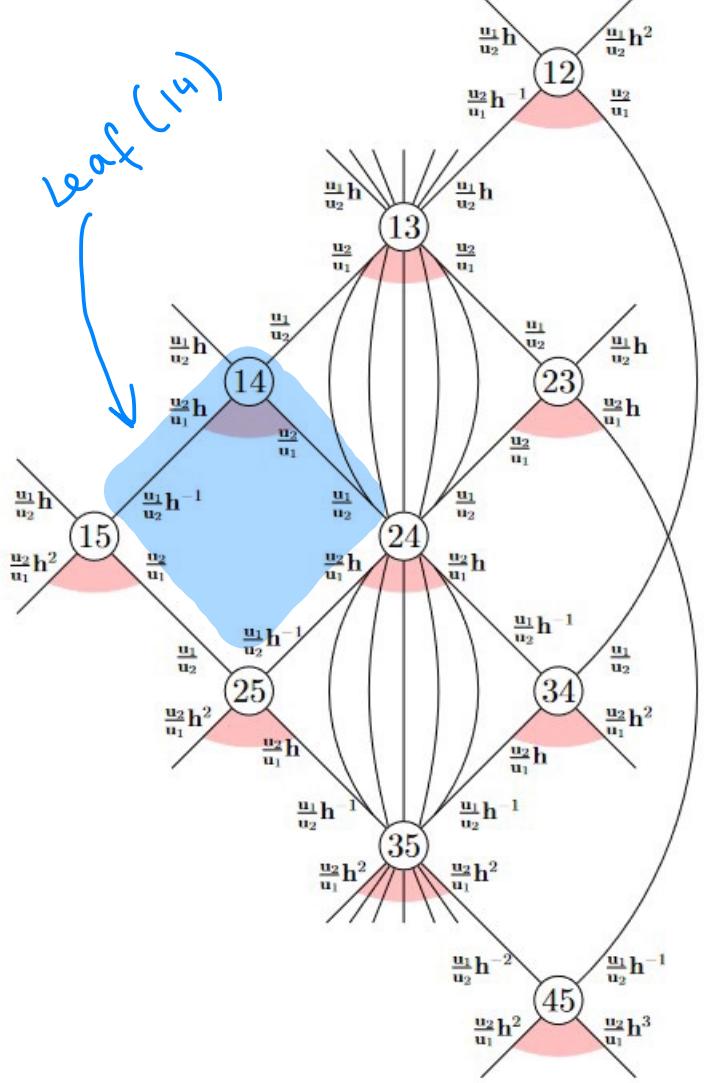
Towards:

$$\text{Stab}_P \in H_T^*(N(Q))$$

"cohomology
stable envelope
class"

$$P \in N(Q)^T$$

- fix $\mathbb{C}^* \xrightarrow{\delta} T$
 $u \mapsto (u^1, u^2, u^3, \dots, u^n; 1)$
- $P \in N(Q)^T$ $\text{Leaf}(P) = \{x \in N(Q) : \lim_{z \rightarrow 0} \delta(z)x = P\}$
- $P' \leq P$ if $\overline{\text{Leaf}(P)} \ni P'$
- $\text{slope}(P) := \bigcup_{P' \leq P} \text{Leaf}(P')$



[Maulik - Okounkov]

def $\text{Stab}_p \in H_+^*(N(Q))$ is the unique class

- support axiom:

supported on $\text{Slope}(p)$

- normalization axiom:

$$\text{Stab}_p|_p = e(\nu(\text{Slope}_p))$$

- boundary axiom:

$\text{Stab}_p|_q$ divisible by t for $p \neq q$

Stab₁₄

@ 12, 13, 23 = 0

@ 14 = $(u_1 - u_2)(u_1 - u_2 + h)$

@ 15
divisible by $u_1 - u_2 + h$
divisible by h

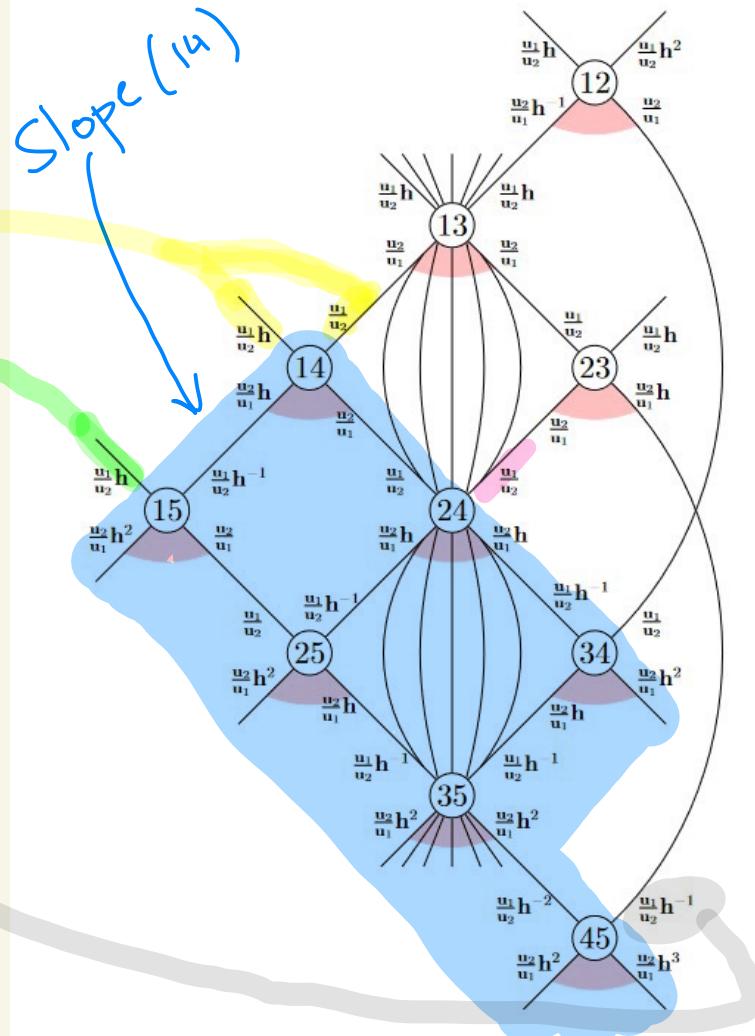
@ 24
divisible by $(u_1 - u_2)$
divisible by h
divisible by h

@ 25
divisible by h

@ 34
divisible by $u_1 - u_2$
divisible by h

@ 35
divisible by h

@ 45
divisible by $u_1 - u_2 - h$
divisible by h



REMARK (2 SLIDES) ON GEOM. REPR. THEORY:

Stab's define geometric R-matrices & quantum group actions.
[MO]

- $\zeta = (z_1, z_2^2, z_3^3, \dots, z_n^n, 1)$ $H_T^*(N(Q)^T) \xrightarrow{\text{Stab}_\zeta} H_T^*(N(Q))$
 $1_p \xrightarrow{\quad} \text{Stab}_{\zeta|p}$

- other 1-parameter subgroups also define Stab's

$$H_T^*(N(Q)^T) \xrightarrow[\text{Stab}_{\zeta'}]{\vdots} H_T^*(N(Q)) \otimes \underline{\mathbb{C}(z, \hbar)}$$

- $\text{Stab}_\zeta^{-1} \circ \text{Stab}_{\zeta'}$ =: "geometric R-matrix"

$$\mathcal{N} := T^* \mathrm{Gr}_0 \mathbb{C}^2 \sqcup T^* \mathrm{Gr}_1 \mathbb{C}^2 \sqcup T^* \mathrm{Gr}_2 \mathbb{C}^2$$

$$H_T^*(\mathcal{N}^\top) \xrightarrow[\text{Stab}_{g'}]{\text{Stab}_g} H_T^*(\mathcal{N})$$

$$\begin{aligned} g &= (z_1, z^2, 1) \\ g' &= (z^2, z_1, 1) \end{aligned}$$

$$T^* \mathrm{Gr}_0 \mathbb{C}^2$$

$$| \mapsto |$$

$$T^* \mathrm{Gr}_1 \mathbb{C}^2$$

$$|_{10} \mapsto |$$

$$|_{01} \mapsto |$$

$$T^* \mathrm{Gr}_2 \mathbb{C}^2$$

$$| \mapsto |$$

$$\begin{array}{c} | \\ (z_2 - z_1, 0) \\ (\hbar, z_1 - z_2 + \hbar) \end{array}$$

$$\begin{array}{c} | \\ (z_2 - z_1 + \hbar, \hbar) \\ (0, z_1 - z_2) \\ | \end{array}$$

$$\text{Stab}_2^{-1} \circ \text{Stab}_{g'} =$$

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

gl_2 Yangian
R matrix



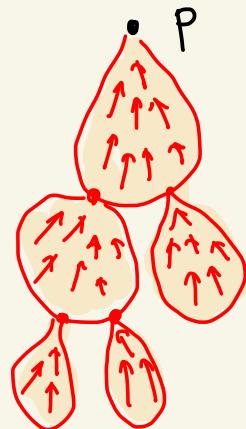
END OF
REMARK

So far :

$$\text{Stab}_P \in H_T^*(\mathcal{N}(Q))$$

T-fixed point of $\mathcal{N}(Q)$

- defined axiomatically, "class of"



- remark : main ingredients of defining quantum group actions on $H_T^*(\mathcal{N}(Q))$.

Fact stable envelope \exists in $H_T^* \hookrightarrow K_T \hookrightarrow \text{Ell}_T$

$\text{Stab}_P^H|_q = \text{polynomial in } u_1, \dots, u_n, t$

$\text{Stab}_P^K|_q = \text{Laurent polynomial in } u_1, \dots, u_n, t$

$\text{Stab}_P^{\text{Ell}}|_q = \text{elliptic function in } u_1, \dots, u_n, t, v_1, \dots, v_m$

$\underbrace{v_1, \dots, v_m}_{\text{dynamical/K\"ahler variables}}$

fix $q \in \mathbb{C}^* \quad \|q\| < 1$

Jacobi theta function:

$$\vartheta(x) = \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) \prod_{n \geq 1} ((1 - q^n x)(1 - q^n x^{-1}))$$

$\sim \sin X$

K theory

q -decoration

$$\vartheta(a, b) := \frac{\vartheta(ab)}{\vartheta(a)\vartheta(b)}$$

Why do elliptic characteristic classes necessarily depend on a new set of variables? Intuitive answer



Fay's trisecant identity :

$$x_1 x_2 x_3 = y_1 y_2 y_3 = 1$$

$$\delta(x_1, y_2) \delta(x_2, \frac{1}{y_1}) + \delta(x_2, y_3) \delta(x_3, \frac{1}{y_2}) + \delta(x_3, y_1) \delta(x_1, \frac{1}{y_3}) = 0$$

Trigonometric (K-theory) limit

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$$

$$\cot(x_1) \cot(x_2) + \cot(x_2) \cot(x_3) + \cot(x_3) \cot(x_1) = \cot(y_1) \cot(y_2) + \cot(y_2) \cot(y_3) + \cot(y_3) \cot(y_1)$$

Rational (H^*) limit

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$$

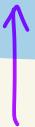
$$\frac{1}{x_1 x_2} + \frac{1}{x_1 x_3} + \frac{1}{x_2 x_3} = \frac{1}{y_1 y_2} + \frac{1}{y_1 y_3} + \frac{1}{y_2 y_3}$$

Fact (3d mirror symmetry for char. classes)

\exists pairs
bijection on

$$\begin{array}{ccc} X & & X' \\ X^T & \xleftrightarrow[p]{p'} & (X')^{T'} \end{array}$$

$$\text{Stab}_{P^q}^{E^H} \Big|_{q'} = \text{Stab}_{q'^{p'}}^{E^H} \Big|_{p'}$$

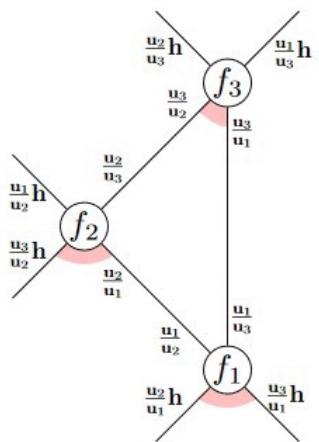


equivariant variables \leftrightarrow Kähler variables

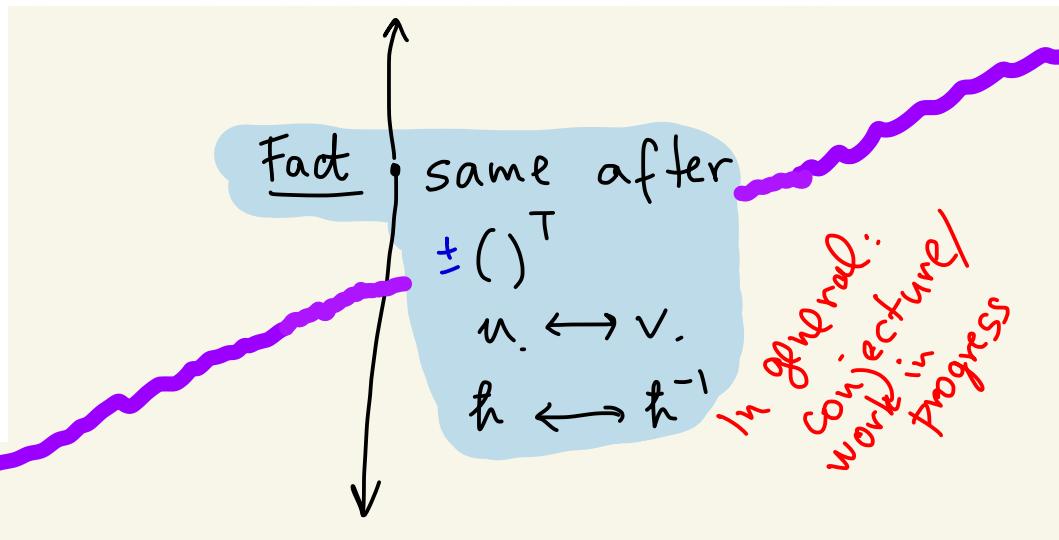
$$t \leftrightarrow t^{-1}$$

Example ↗

$$\tau^* \mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} \sqcup & \sqcap \\ \square_3 \end{smallmatrix}\right)$$



	f_1	f_2	f_3
f_1	$\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^4\right)$	0	0
f_2	$\theta(h)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2 v_2}{u_1 v_1}h^3\right)$	$\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^3\right)$	0
f_3	$\theta(h)\theta\left(\frac{u_2}{u_1}h\right)\theta\left(\frac{u_3 v_2}{u_1 v_1}h^2\right)$	$\theta(h)\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_3 v_2}{u_2 v_1}h^2\right)$	$\theta\left(\frac{u_2}{u_3}h\right)\theta\left(\frac{u_1}{u_3}h\right)\theta\left(\frac{v_2}{v_1}h^2\right)$



$$\mathcal{N}\left(\begin{smallmatrix} \sqcup & \sqcap \\ \square_1 & \square_1 \end{smallmatrix}\right) = \widetilde{\mathfrak{S}}/\mathbb{Z}_3$$

	f'_1	f'_2	f'_3
f'_1	$\theta\left(\frac{u'_1}{u'_2}h^4\right)\theta\left(\frac{v'_2}{v'_1}\right)\theta\left(\frac{v'_3}{v'_1}\right)$	$\theta(h)\theta\left(\frac{v'_3}{v'_1}\right)\theta\left(\frac{v'_2 u'_2}{v'_1 u'_1}h^{-3}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h^{-1}\right)\theta\left(\frac{v'_3 u'_2}{v'_1 u'_1}h^{-2}\right)$
f'_2	0	$\theta\left(\frac{u'_1}{u'_2}h^3\right)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3}{v'_2}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3 u'_2}{v'_2 u'_1}h^{-2}\right)$
f'_3	0	0	$\theta\left(\frac{u'_1}{u'_2}h^2\right)\theta\left(\frac{v'_3}{v'_2}h\right)\theta\left(\frac{v'_3}{v'_1}h\right)$

8

$$T^* \text{Gr}_2 \mathbb{C}^4$$



$$\mathcal{N}\left(\begin{array}{c} 1 \\ 2 \\ 1 \\ \hline \square_2 \end{array}\right)$$

12

$$T^* \text{Gr}_2 \mathbb{C}^5$$



$$\mathcal{N}\left(\begin{array}{cccc} 1 & 2 & 2 & 1 \\ \hline \square_1 & \square_2 & \square_1 \end{array}\right)$$

64

$$T^* \mathcal{F}_{2,6,10}$$



$$\mathcal{N}\left(\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 2 \\ \hline \square_2 & \square_1 \end{array}\right)$$

8

$$\mathcal{N}\left(\begin{array}{cccc} 1 & 1 & 2 & 1 \\ \hline \square_2 & \square_2 & \square_2 & \square_2 \end{array}\right)$$



$$\mathcal{N}\left(\begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \\ \hline \square_1 & \square_1 & \square_2 & \square_1 \end{array}\right)$$

16

$$T^* G/B$$



$$T^* G^L/B^L$$

10

32

$$T^* \mathcal{F}_{2,5,7}$$



$$\mathcal{N}\left(\begin{array}{c} 3 \\ 2 \end{array}\right)$$

Cherkis bow varieties
 $C(\dots)$

type-A Nakajima quiver varieties

$$N \left(\begin{array}{c} | & 2 & 2 & 1 & 4 \\ \bullet & - & - & - & - \\ \square & \square & \square & & \\ \square & \square & \square & & \\ | & | & | & & \end{array} \right)$$

$$N \left(\begin{array}{c} | & | \\ \bullet & - \\ \square & \square \\ | & | \end{array} \right)$$

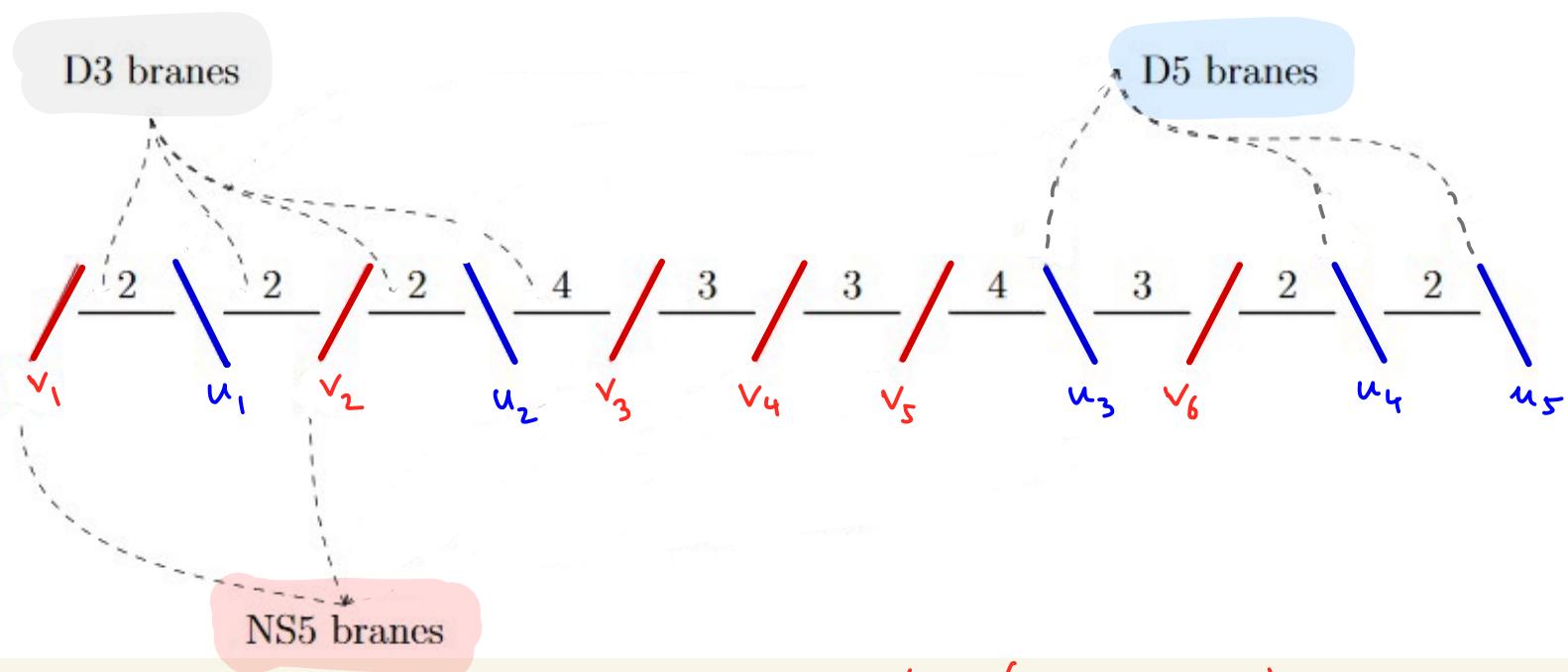
$$T^* \text{Gr}_2 \mathbb{C}^4$$

$$T^* \mathcal{F}_{2,5,7}$$

$$T^* \mathcal{F}_{1,2,3,4}$$

$$T^* G/P$$

Brane diagrams



v_i : Kähler (dynamical) variables
 u_i : equivariant variables

$$\mathbb{C}^0 \xrightarrow[2]{\quad} \mathbb{C}^2 \xrightarrow[2]{\quad} \mathbb{C}^2 \xrightarrow[2]{\quad} \mathbb{C}^4 \xrightarrow[4]{\quad} \mathbb{C}^3 \xrightarrow[3]{\quad} \mathbb{C}^3 \xrightarrow[3]{\quad} \mathbb{C}^4 \xrightarrow[4]{\quad} \mathbb{C}^3 \xrightarrow[3]{\quad} \mathbb{C}^2 \xrightarrow[2]{\quad} \mathbb{C}^2 \xrightarrow[2]{\quad} \mathbb{C}^0$$

$$\frac{\mathbb{C}^n}{\mathbb{C}^m}$$

$$\frac{\mathbb{C}^m}{\mathbb{C}^n}$$

$$T^* \text{Hom}(\mathbb{C}^n, \mathbb{C}^m)$$

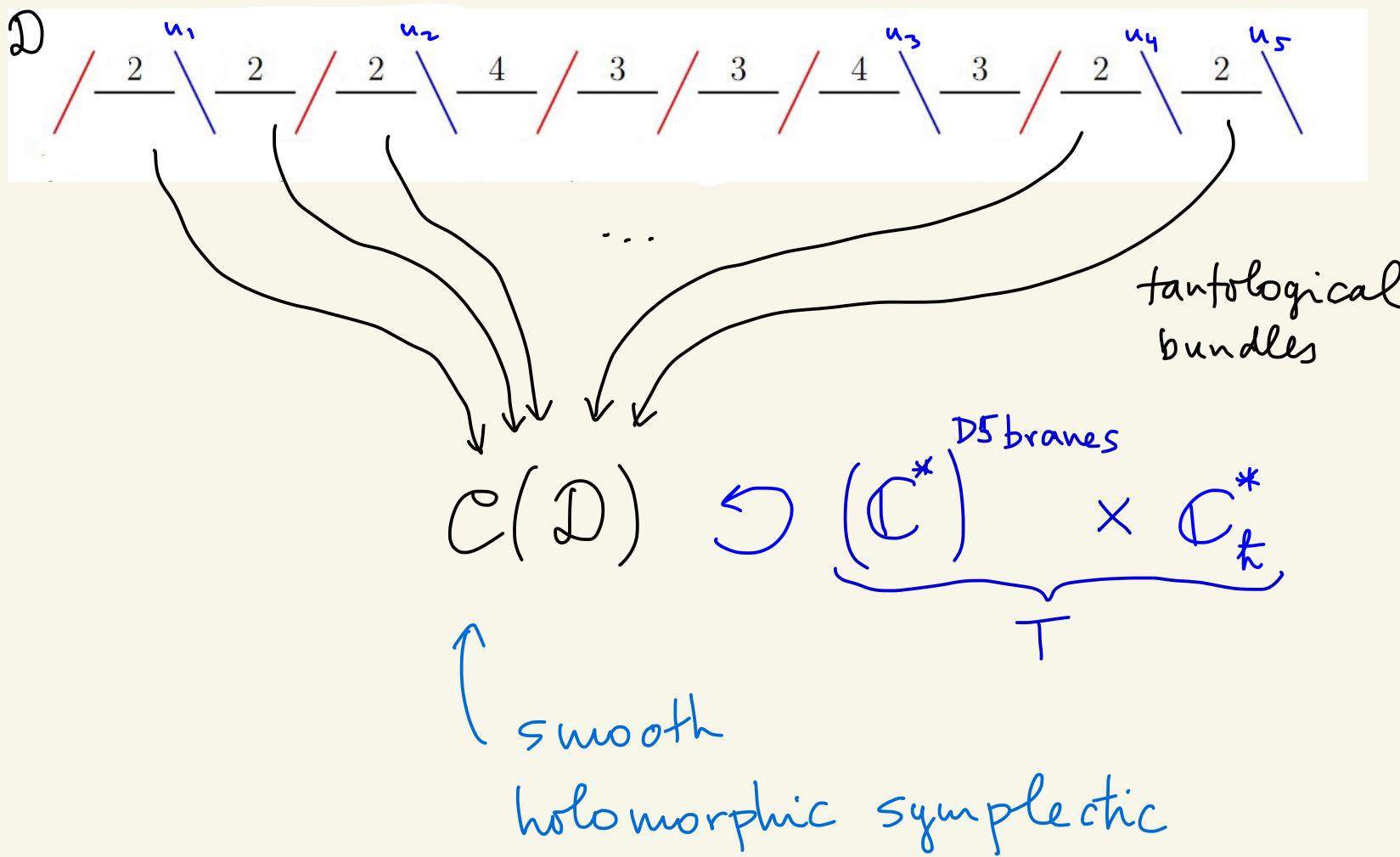
$$\begin{matrix} \uparrow \\ GL_n \times GL_m \end{matrix}$$

$$\mathcal{B}_{n,m} = "T^*(GL_n \times \mathbb{C}^m) // \begin{pmatrix} * & * \\ 0 & I \end{pmatrix}"$$

$$\begin{matrix} \uparrow \\ GL_n \times GL_m \times \mathbb{C}^* \end{matrix}$$

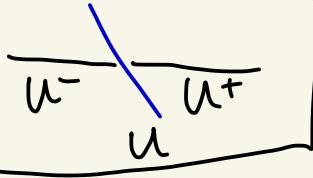
[Nakajima-Takayama]

[R-Rozansky]



$$\dim(C(D)) = \sum_{U \in DS} [(d_{u_-} + 1)d_{u_-} + (d_{u_+} + 1)d_{u_+}]$$

notation



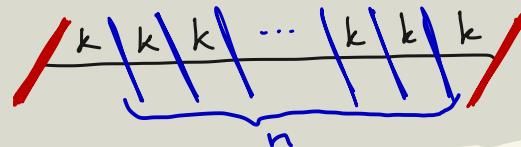
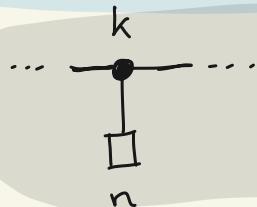
$$+ \sum_{V \in NS5} 2 d_{v^+} d_{v^-} - 2 \sum_{X \in D3} d_X^2$$

example

$$\begin{aligned} \dim(C(\text{red line with blue ticks})) &= 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 \\ &\quad + 2 \cdot 0 \cdot 1 + 2 \cdot 1 \cdot 0 - 2(1^2 + 1^2 + 1^2 + 1^2) \\ &= 4 \end{aligned}$$

$T^* \#^2$

How are \mathcal{N} (quiver) special cases?



Examples $T^* \mathbb{P}^1 = \mathcal{N}\left(\begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix}\right) = \mathcal{C}\left(\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}\right)$

$$T^* \text{Gr}_2 \mathbb{C}^4 = \mathcal{N}\left(\begin{smallmatrix} 2 \\ 2 \\ 4 \end{smallmatrix}\right) = \mathcal{C}\left(\begin{smallmatrix} 2 & 2 & 2 & 2 & 2 \end{smallmatrix}\right)$$

$$T^* \mathcal{F}_{1,2,3,4} = \mathcal{N}\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 4 & \square \end{smallmatrix}\right) = \mathcal{C}\left(\begin{smallmatrix} 1 & 2 & 3 & 3 & 3 & 3 & 3 \end{smallmatrix}\right)$$

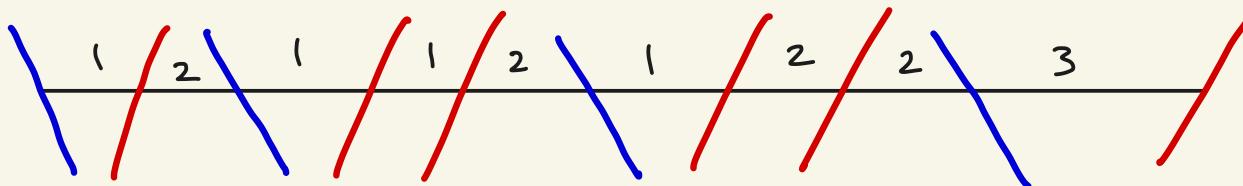
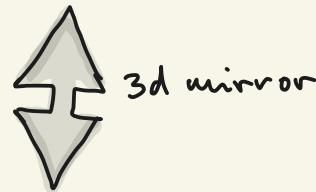
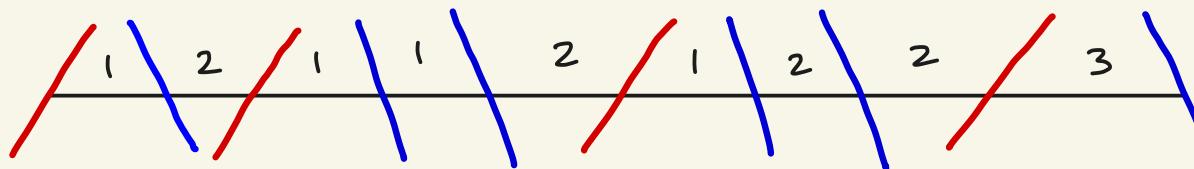
$$\mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ \square & \square \end{smallmatrix}\right) = \mathcal{C}\left(\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right)$$

Observe

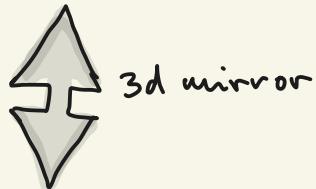
$k \neq k$

"cobalanced brane diagram"

3D mirror symmetry for bow varieties:



$$\underline{\text{Ex}} \quad T^* \mathbb{P}^2 = \mathcal{N} \left(\begin{smallmatrix} & 1 \\ 1 & \\ \square & 3 \end{smallmatrix} \right) = C \left(\begin{array}{c|c|c|c|c|c} \textcolor{red}{1} & 1 & 1 & 1 & 1 & \textcolor{red}{1} \\ \hline & & & & & \end{array} \right) \quad \text{dim 4}$$

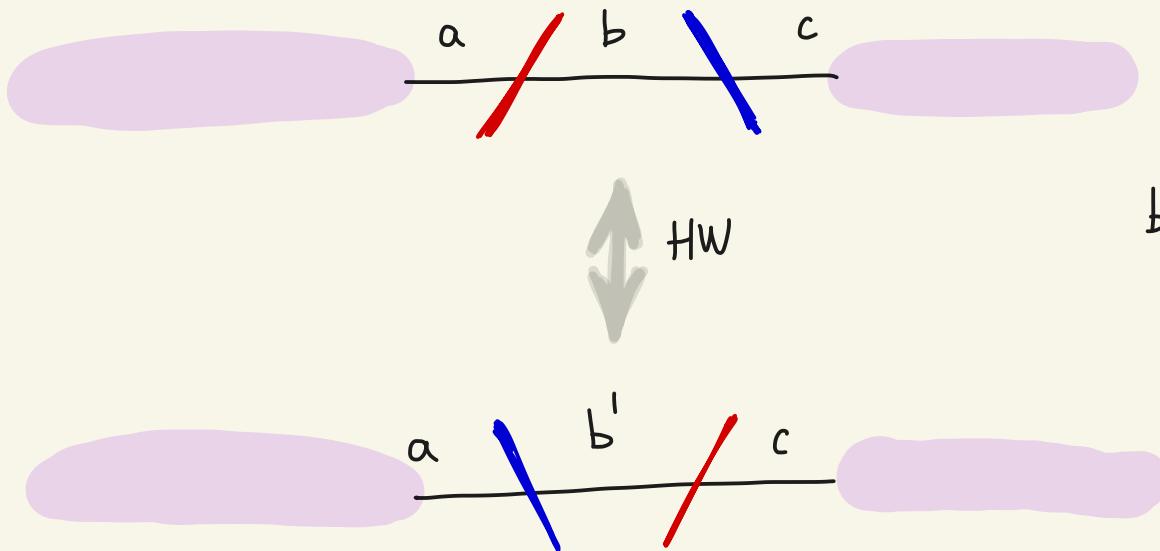


$$C \left(\begin{array}{c|c|c|c|c|c} \textcolor{blue}{1} & \textcolor{red}{1} & \textcolor{red}{1} & \textcolor{red}{1} & \textcolor{red}{1} & \textcolor{blue}{1} \\ \hline & & & & & \end{array} \right) \quad \text{dim 2}$$

not cobalanced, ie not $\mathcal{N}(\dots)$

... but ... <to be continued>

Hanany - Witten transition on brane diagrams.

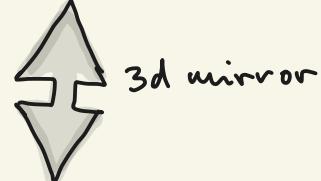


$$b + b' = a + c + 1$$

(why? later:
"brane charge")

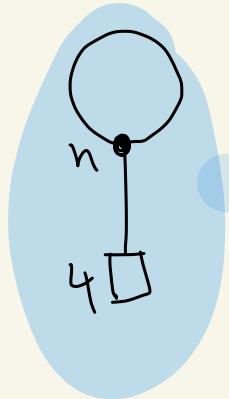
Thm $C(\mathcal{D}) \approx C(HW(\mathcal{D}))$

$$\underline{\text{Ex}} \quad T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right) = C\left(\begin{array}{c|c|c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}\right)$$

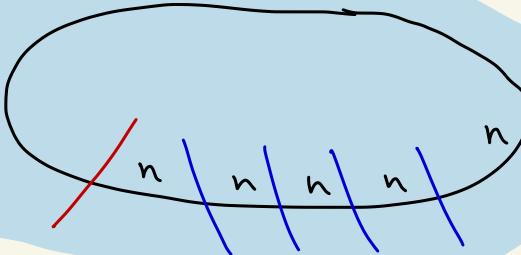


$$\begin{aligned} & \xrightarrow{\text{HW}} C\left(\begin{array}{c|c|c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}\right) \\ & \curvearrowleft C\left(\begin{array}{c|c|c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}\right) \stackrel{\text{HW}}{=} C\left(\begin{array}{c|c|c|c|c|c|c|c} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}\right) \\ & \qquad \qquad \qquad \stackrel{''}{=} \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \end{aligned}$$

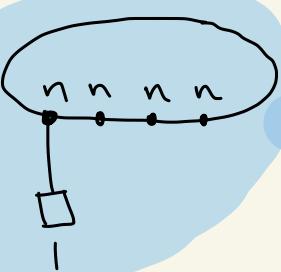
$$\Rightarrow T^*\mathbb{P}^2 \quad \xleftarrow{\text{3d mirror}} \quad \mathcal{N}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$$



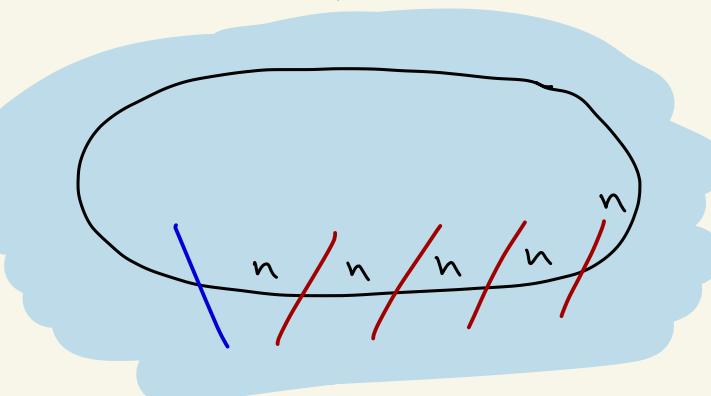
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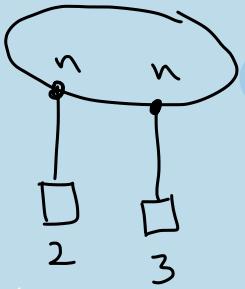


3d mirror

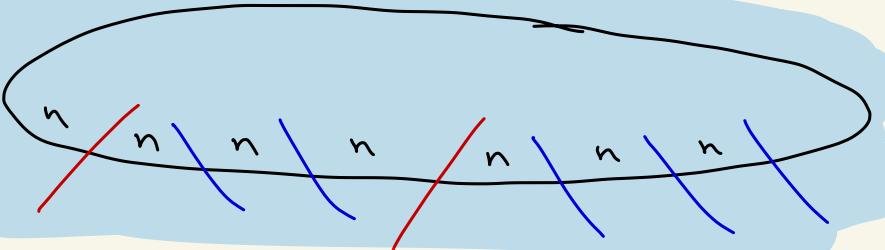


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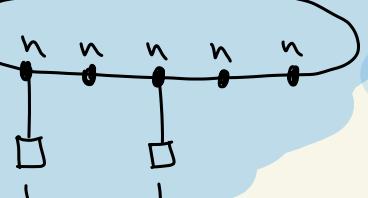




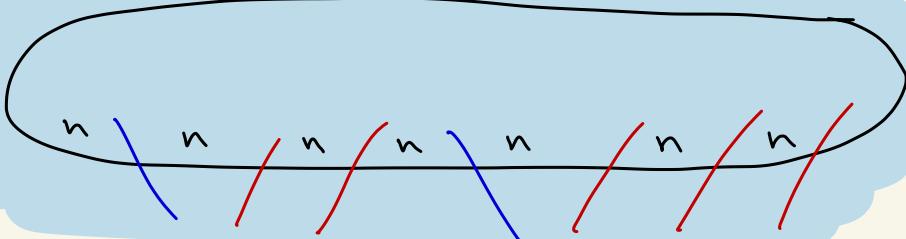
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3d mirror



=



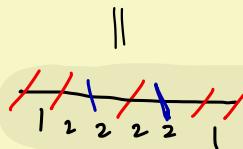
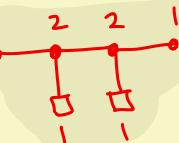
Cherkis bow varieties

Nakajima quiver var's

T^*G/P

$$T^*Gr_2 \mathbb{C}^5 = \begin{array}{c} \bullet \\ \square \\ \bullet \\ \square \\ \bullet \end{array} \begin{matrix} 2 & & & \\ & 5 & & \end{matrix}$$

3d mirror for
char. classes

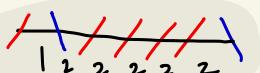


HW

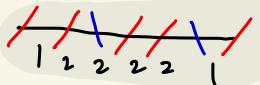
3d mirror



HW



HW

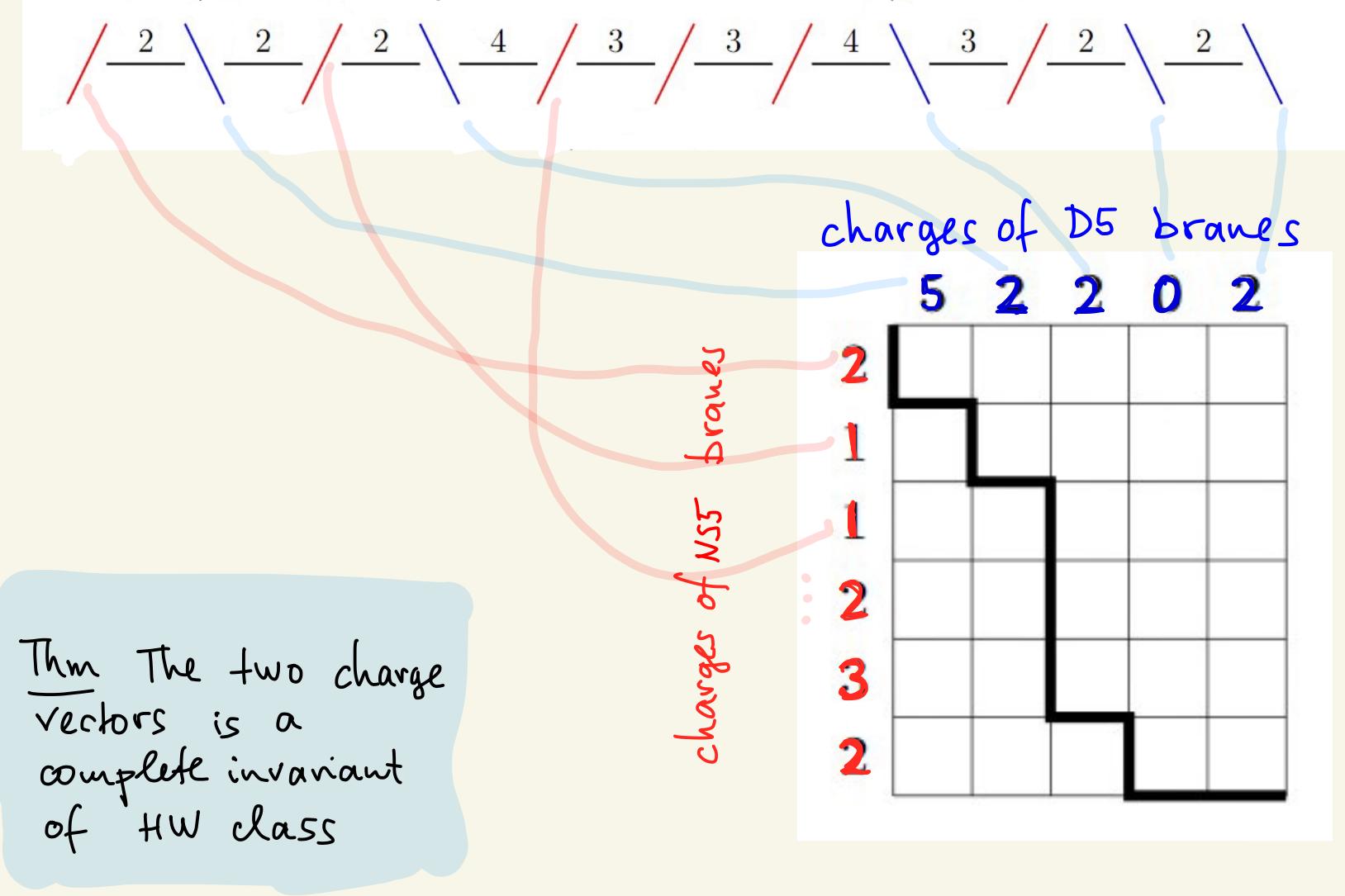


Hanany-Witten transition

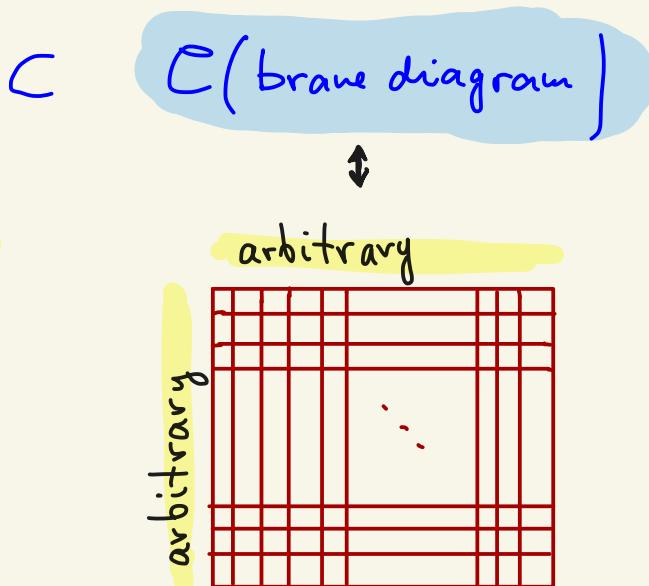
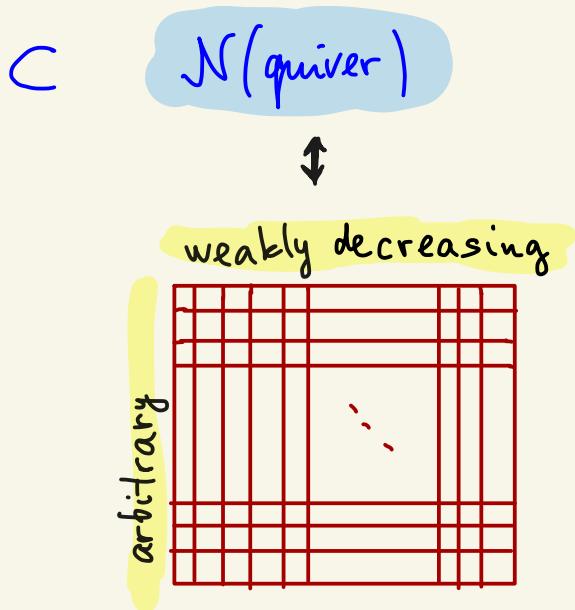
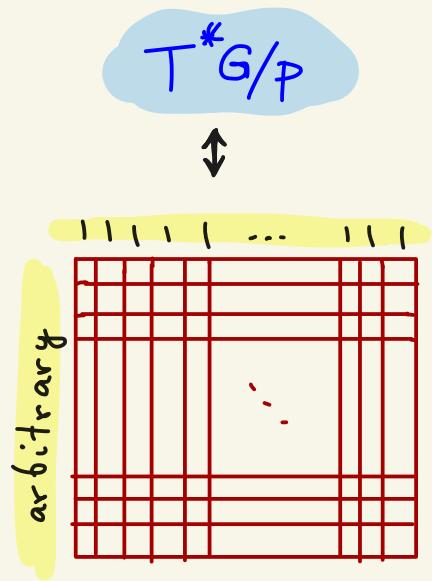
def brane charge

$$\text{charge} \left(\begin{array}{c} \text{NS5 brane} \\ \hline k \cancel{/} l \end{array} \right) := l - k + \#\{\text{D5-branes left of it}\}$$

$$\text{charge} \left(\begin{array}{c} \text{D5 brane} \\ \hline k \cancel{/} l \end{array} \right) := k - l + \#\{\text{NS5-branes right of it}\}$$



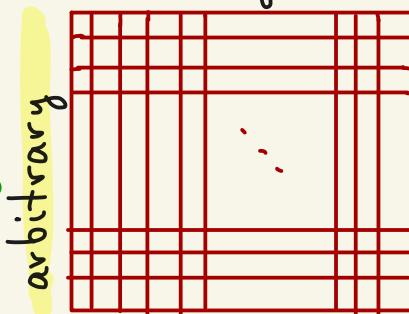
Thm (up to HW transitions)



closed for transpose!

REMARK ON GEOM. REPR. THEORY

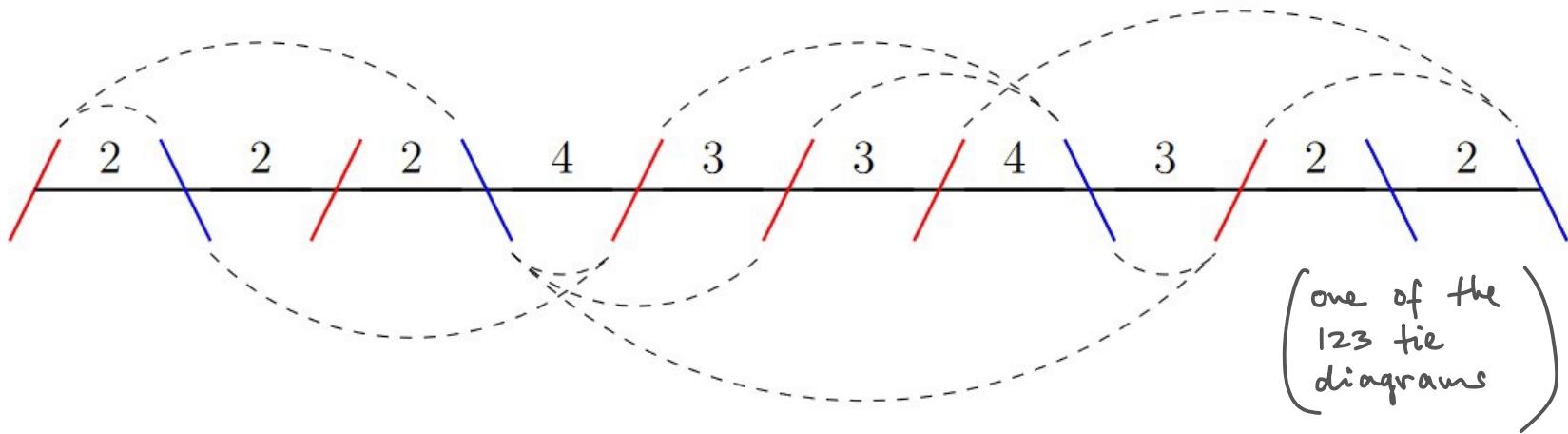
which weight space of the representation (3)



which representation (2)

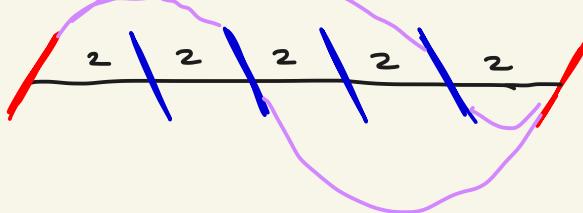
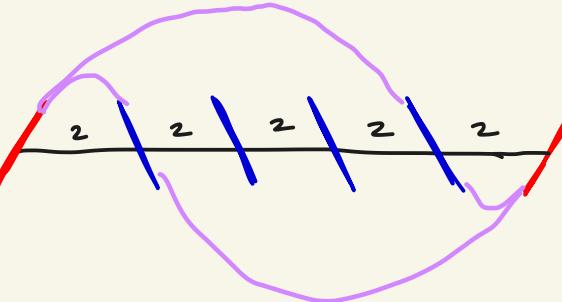
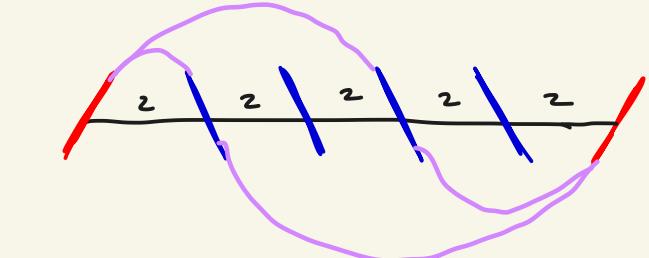
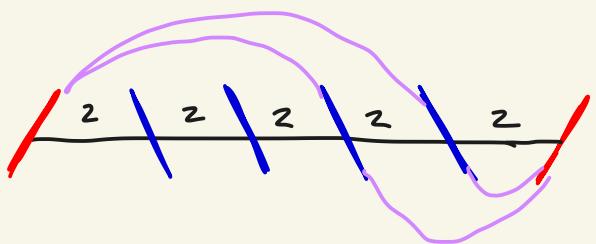
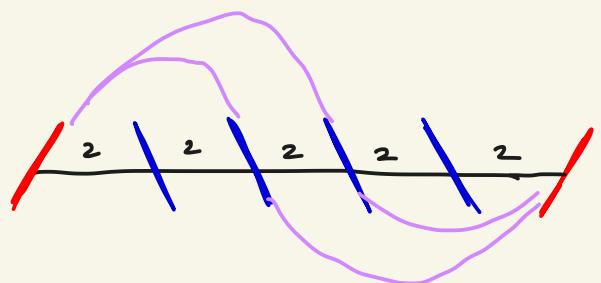
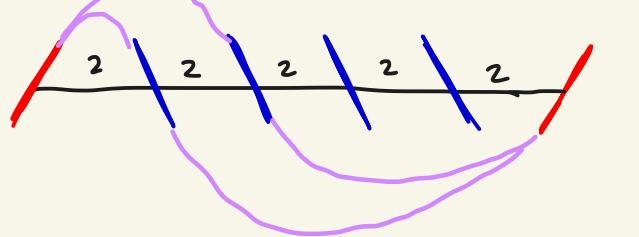
size : which quantum group (1)

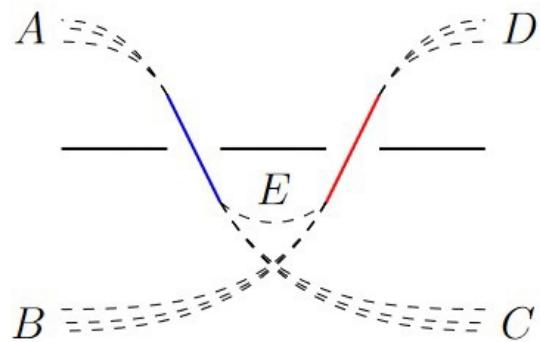
fixed points $\overset{1:1}{\leftrightarrow}$ tie diagrams



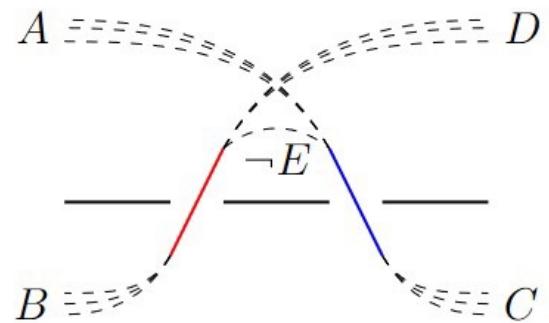
- a tie must connect 5-branes of different kinds
- each D3 brane to be covered as many times as its multiplicity

fixed points of $T^* \text{Gr}_2 \mathbb{C}^4$:

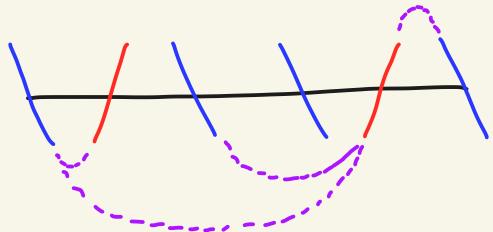




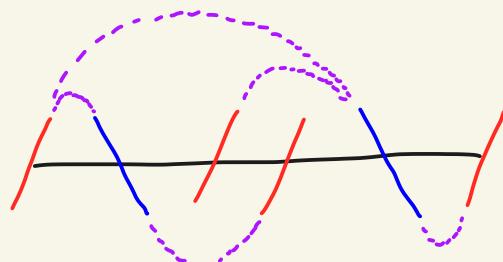
HW transition
on fixpoints



R-III



3d mirror
on fixedpoints
horizontal
reflection



$$\begin{array}{c} / \quad 2 \quad \backslash \quad 2 \quad / \quad 2 \quad \backslash \quad 4 \quad / \quad 3 \quad / \quad 3 \quad / \quad 4 \quad \backslash \quad 3 \quad / \quad 2 \quad \backslash \quad 2 \quad \backslash \\ \text{---} \quad \text{---} \end{array}$$

binary contingency tables

BCT : 0-1-matrix
with row &
column sums
the charge vectors

Thm

fix pts \longleftrightarrow BCT's

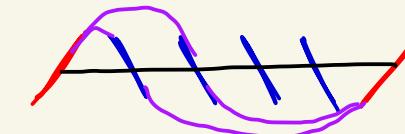
one of the 123 BCTs

	5	2	2	0	2
2	1	1	0	0	0
1	1	0	0	0	0
1	0	0	1	0	0
2	1	0	1	0	0
3	1	1	0	0	1
2	1	0	0	0	1

$\text{Gr}_2 \mathbb{C}^4$

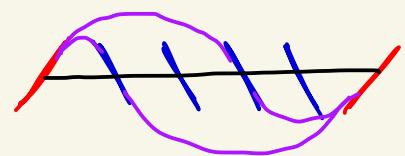
0

$\{1,2\}$



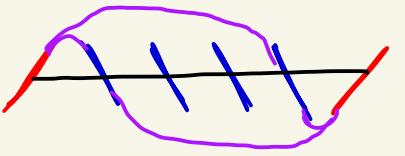
□

$\{1,3\}$



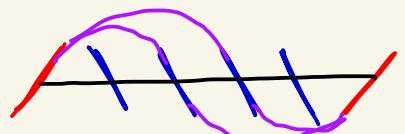
田

$\{1,4\}$



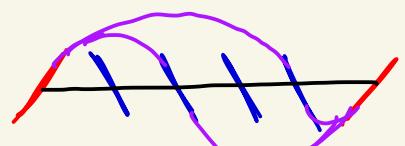
日

$\{2,3\}$



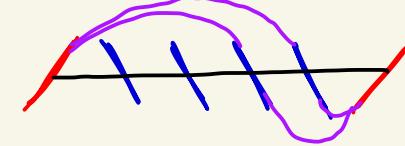
田

$\{2,4\}$



窗

$\{3,4\}$



1	1	1	1	1
2	1	1	0	0
2	0	0	1	1

1	1	1	1	1
2	1	0	1	0
2	0	1	0	1

1	1	1	1	1
2	1	0	0	1
2	0	1	1	0

1	1	1	1	1
2	0	1	1	0
2	1	0	0	1

1	1	1	1	1
2	0	1	0	1
2	1	0	1	0

1	1	1	1	1
2	0	0	1	1
2	1	1	0	0

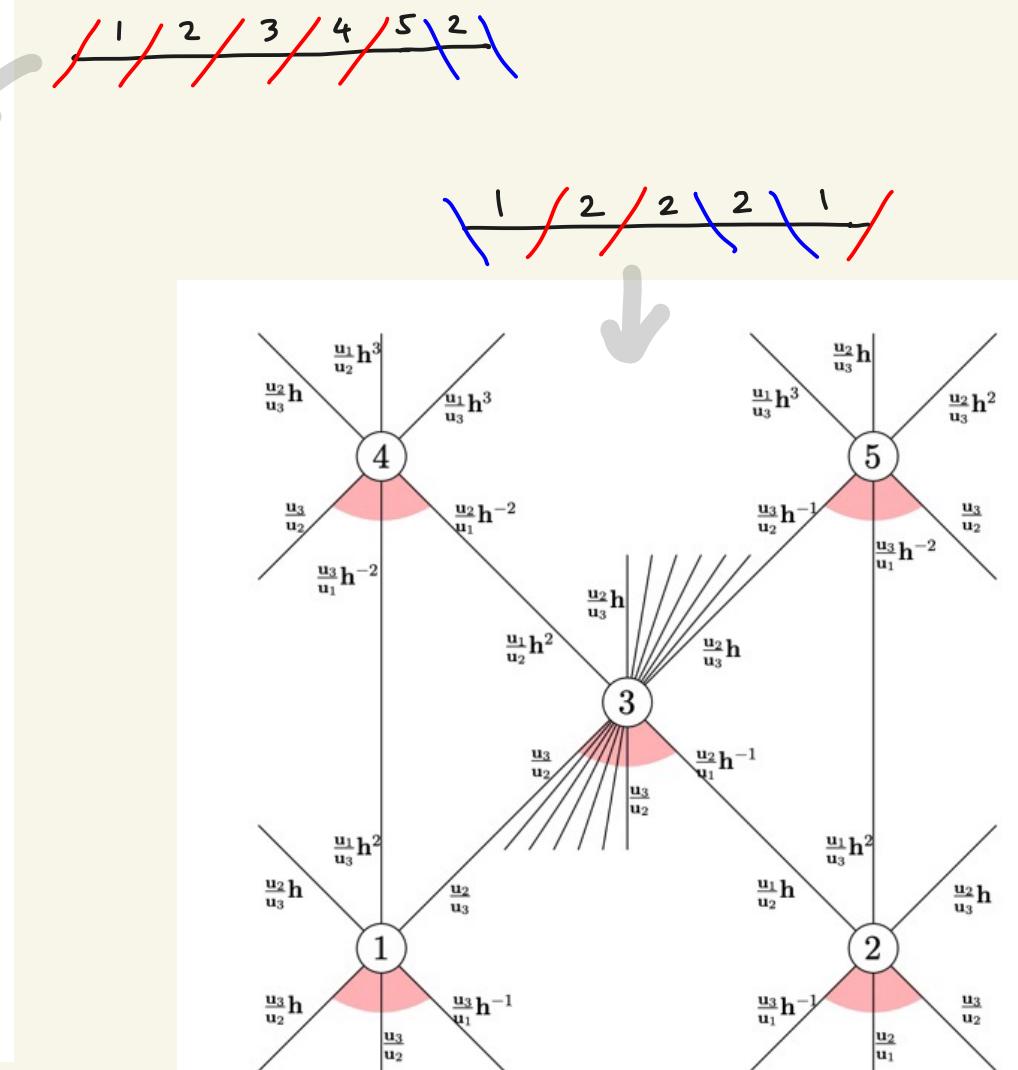
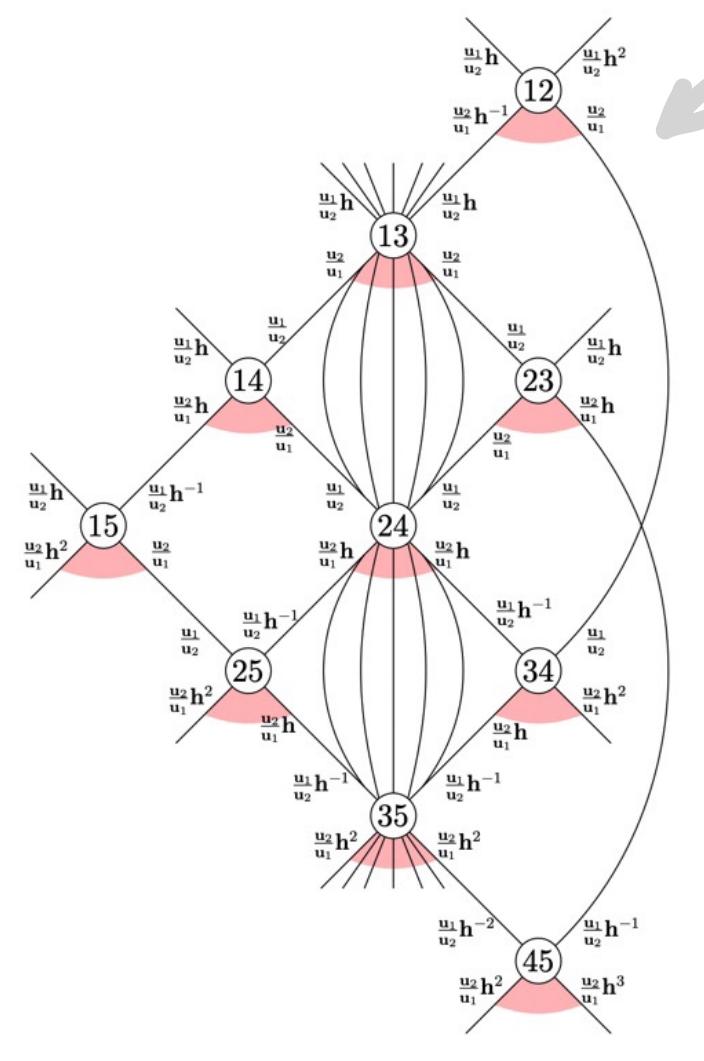
fixed points
invariant curves
(with weight)

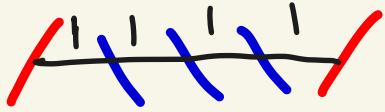


moment
graph



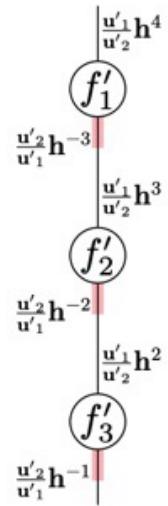
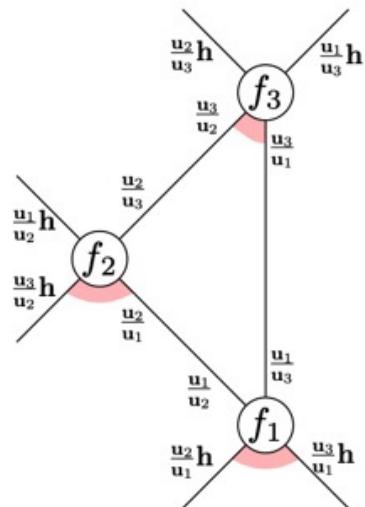
Stab_P



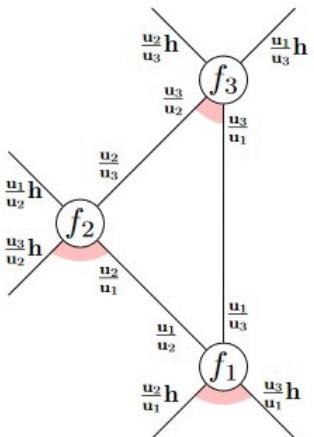


$T^* \mathbb{P}^2$

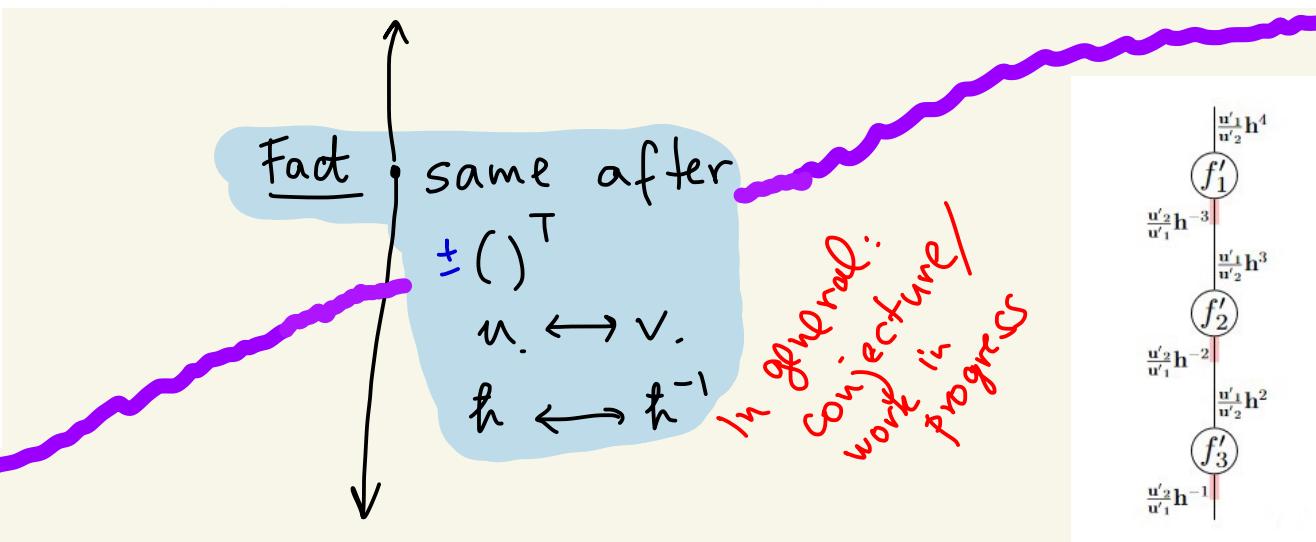
3d mirror $(T^* \mathbb{P}^2)$



$c(\text{---} \backslash \backslash \backslash \backslash \backslash \backslash)$



	f_1	f_2	f_3
f_1	$\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^4\right)$	0	0
f_2	$\theta(h)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2 v_2}{u_1 v_1}h^3\right)$	$\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{v_2}{v_1}h^3\right)$	0
f_3	$\theta(h)\theta\left(\frac{u_2}{u_1}h\right)\theta\left(\frac{u_3 v_2}{u_1 v_1}h^2\right)$	$\theta(h)\theta\left(\frac{u_1}{u_2}h\right)\theta\left(\frac{u_3 v_2}{u_2 v_1}h^2\right)$	$\theta\left(\frac{u_2}{u_3}h\right)\theta\left(\frac{u_1}{u_3}h\right)\theta\left(\frac{v_2}{v_1}h^2\right)$



	f'_1	f'_2	f'_3
f'_1	$\theta\left(\frac{u'_1}{u'_2}h^4\right)\theta\left(\frac{v'_2}{v'_1}\right)\theta\left(\frac{v'_3}{v'_1}\right)$	$\theta(h)\theta\left(\frac{v'_3}{v'_1}\right)\theta\left(\frac{v'_2 u'_2}{v'_1 u'_1}h^{-3}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h^{-1}\right)\theta\left(\frac{v'_3 u'_2}{v'_1 u'_1}h^{-2}\right)$
f'_2	0	$\theta\left(\frac{u'_1}{u'_2}h^3\right)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3}{v'_2}\right)$	$\theta(h)\theta\left(\frac{v'_2}{v'_1}h\right)\theta\left(\frac{v'_3 u'_2}{v'_2 u'_1}h^{-2}\right)$
f'_3	0	0	$\theta\left(\frac{u'_1}{u'_2}h^2\right)\theta\left(\frac{v'_3}{v'_2}h\right)\theta\left(\frac{v'_3}{v'_1}h\right)$

$c(\text{---} / \backslash \backslash \backslash \backslash \backslash)$

3d mirror symmetry for stable envelopes
is proved in special cases

- $T^* \text{Gr}_k \mathbb{C}^n$ \leftrightarrow its dual [R-Smirnov-Varchenko-Zhou]
- $T^* G/B$ \leftrightarrow $T^* G^L/B^L$ $\begin{cases} \text{type A [RSVZ]} \\ \text{general [R-Weber]} \end{cases}$
- hypertoric \leftrightarrow dual hypertoric [Smirnov-Zhou]
- finitely many other cases [R-Shou]

Summary

- $X \hookrightarrow T$

Stab_p

enumerative geometry

representations of
quantum groups

differential equations
difference equations

- $X \xleftrightarrow{\text{3d mirror}} X^!$

$\text{Stab} \sim \text{Stab}^!$

- bow varieties : closed for 3d mirror symmetry
easy combinatorics

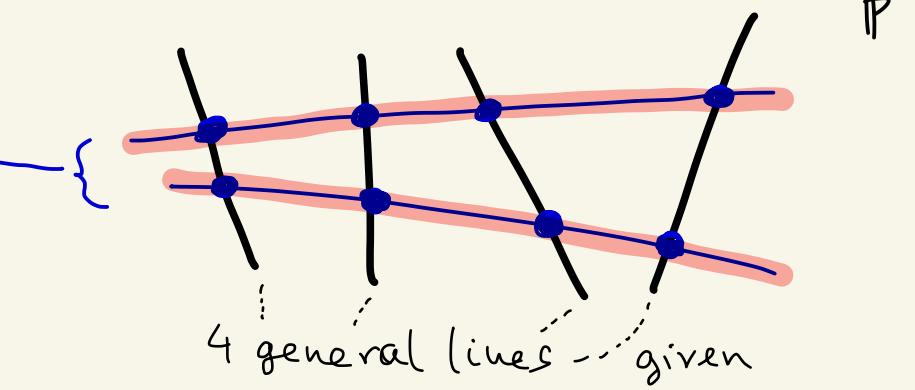
The End

BACKUP SLIDES ↴

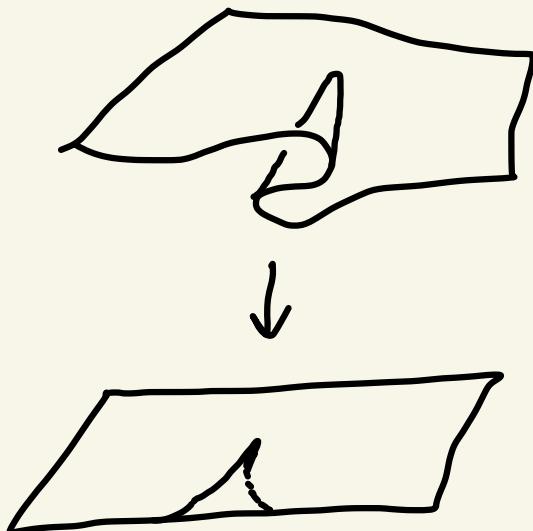
$$GL_4 \times T^4 \subset \left(\sum_{\substack{\parallel \\ \{rk \leq 2\}}} \subset M \right) \underset{\parallel}{\underset{\text{Hom}(\mathbb{C}^4, \mathbb{C}^4)}{\longrightarrow}}$$

$$[\Sigma] \in H_{GL_4 \times T^4}^*(\underbrace{\text{Hom}(\mathbb{C}^4, \mathbb{C}^4)}_{\sim pt}) = \mathbb{Z} [c_1 c_2 c_3 c_4 d_1 d_2 d_3 d_4]$$

$$= \dots + 2d_1 d_2 d_3 d_4 + \dots$$



$$GL_2 \times GL_2 \subset (\Sigma \parallel M) \quad \{ \text{those } \sim (x,y) \mapsto (x^3 + xy, y) \} \quad \{ (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \text{ holomorphic germs} \}$$



$$[\bar{\Sigma}] \in H^*_{GL_2 \times GL_2} (pt)$$

$$\parallel \\ C_1^2 + C_2$$

Cor generic map
 $\mathbb{R}\mathbb{P}^2 \rightarrow \mathbb{R}^2$
has an odd number of cusps

$$GL_3 \times D^9 \subset \left(\sum_{\parallel} \subset M \right)$$

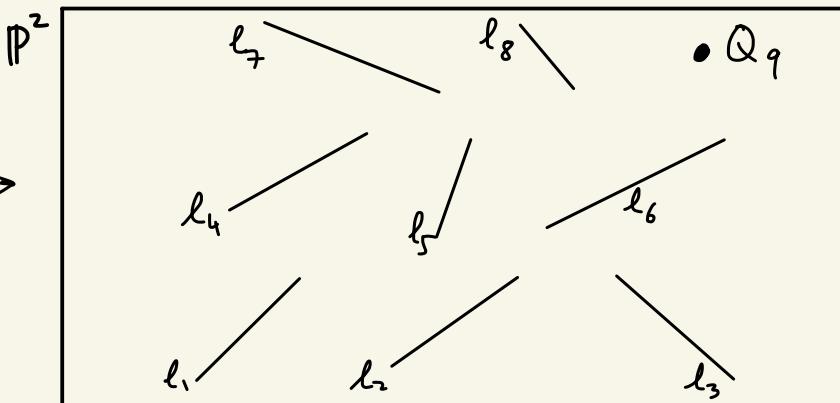
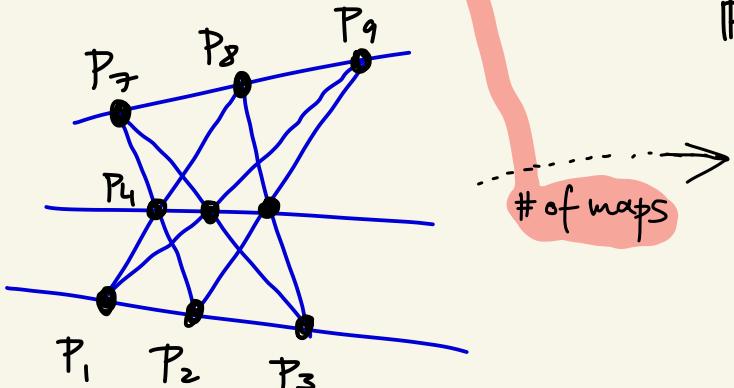
$\left\{ \begin{array}{c} \text{Diagram showing 9 points in a 3x3 grid with many blue lines connecting them} \\ \end{array} \right\}$

$\text{Hom}(\mathbb{C}^9, \mathbb{C}^3)$

$$[\Sigma] \in H_{GL_3 \times D^9}^*(pt) = \mathbb{Z}[c_1, c_2, c_3, d_1, d_2, \dots, d_9]$$

↓

$$= \dots + 5d_1d_2d_3d_4d_5d_6d_7d_8d_9 + \dots$$



X

Kähler parameters
keeping track of
deg curve



- curve counting in X : $V \in K_T(X)[\underline{v}]$ vertex function

- $p \in X^T$ $V_p \in K(\text{point})[\underline{v}]$

equivariant
parameters \underline{u}
(mero-)

Kähler parameters \underline{v}
(holo-)

- V_p satisfy difference equations

in \underline{u} variables

diff eqns
switched
for $X^!$

in \underline{v} variables

- $V_q^! := \sum \text{Stab}_{q|p} \cdot V_p$ vertex function on $X^!$