THE $\mathbb{F}_p$-SELBERG INTEGRAL OF TYPE A_n

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ABSTRACT. We present an $\mathbb{F}_p$-Selberg integral formula of type A_n, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$, where p is an odd prime. The formula is motivated by analogy between multidimensional hypergeometric solutions of the KZ equations and polynomial solutions of the same equations reduced modulo p. The A_1-type formula was proved in a previous paper by the authors. The A_2-type formula is proved in this paper. We also sketch the proof of the A_n-type formula for $n > 2$.

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1. Introduction

In 1944 Atle Selberg proved the following integral formula:

$$\int_0^1 \cdots \int_0^1 \prod_{1\leqslant i<j\leqslant k} |x_i - x_j|^{2\gamma} \prod_{i=1}^{k} x_i^{\alpha-1}(1-x_i)^{\beta-1}\, dx_1 \dots dx_k = \prod_{j=1}^{k} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)}{\Gamma(\alpha+\beta+(k+j-2)\gamma)}\,, \tag{1.1}$$

where α, β, γ are complex numbers such that $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, and $\operatorname{Re}\gamma > -\min\,[(\operatorname{Re}\alpha)/(n-1)\operatorname{Re}\beta)/(n-1)]$. See [Se, AAR]. Hundreds of papers are devoted to the generalizations of the Selberg integral formula and its applications, see for example [AAR, FW] and references therein. There are q-analysis versions of the formula, the generalizations associated with Lie algebras, elliptic versions, finite field versions, see some references in [AAR, FW, As, Ha, Ka, Op, Ch, TV1, TV2, TV4, Wa1, Wa2, Sp, R, FSV, An, Ev]. In the finite field versions, one considers additive and multiplicative characters of a finite field, which map the field to the field of complex numbers, and forms an analog of equation (1.1), in which both sides are complex numbers. The simplest of such formulas is the classical relation between Jacobi and Gauss sums, see [AAR, An, Ev].

In [RV] we suggested another version of the Selberg integral formula, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$ with an odd prime number p of elements.

Our motivation in [RV] came from the theory of Knizhnik-Zamolodchikov (KZ) equations, see [KZ, EFK]. These are the systems of linear differential equations, satisfied by conformal blocks on the sphere in the WZW model of conformal field theory. The KZ equations were solved in multidimensional hypergeometric integrals in [SV1], see also [V1, V2]. The following general principle was formulated in [MuV]: if an example of the KZ-type equations has a one-dimensional space of solutions, then the corresponding multidimensional hypergeometric integral can be evaluated explicitly. As an illustration of that principle in [MuV], an example of the $\mathfrak{sl}_2$ differential KZ equations with a one-dimensional space of solutions was considered, the corresponding multidimensional hypergeometric integral was reduced to the Selberg integral and then evaluated by formula (1.1). See other illustrations in [FV, FSV, TV1, TV2, TV4, V3, RTVZ].

Recently in [SV2] the KZ equations were considered modulo a prime number p and polynomial solutions of the reduced equations were constructed, see also [SlV, V3, V4, V5, V6, V7]. The construction is analogous to the construction of the multidimensional hypergeometric solutions, and the constructed polynomial solutions were called the $\mathbb{F}_p$-hypergeometric solutions.

In [RV] we considered the reduction modulo p of the same example of the $\mathfrak{sl}_2$ differential KZ equations, that led in [MuV] to the Selberg integral. We evaluated the corresponding $\mathbb{F}_p$-hypergeometric solution by analogy with the evaluation of the Selberg integral and obtained the $\mathbb{F}_p$-Selberg integral formula in [RV, Theorem 4.1].

In [TV4, Theorem 3.3] the Selberg integral formula of type A_2 was proposed and proved,

$$\begin{aligned}(1.2)\qquad &\int_{C^{k_1,k_2}[0,1]} \prod_{i=1}^{k_1} t_i^{\alpha-1}(1-t_i)^{\beta_1-1} \prod_{j=1}^{k_2}(1-s_j)^{\beta_2-1} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2}|s_j-t_i|^{-\gamma} \\ &\qquad\times \prod_{1\leqslant i<i'\leqslant k_1}|t_i-t_{i'}|^{2\gamma} \prod_{1\leqslant j<j'\leqslant k_2}|s_j-s_{j'}|^{2\gamma}\, dt_1\dots dt_{k_1}\, ds_1\dots ds_{k_2} \\ &= \prod_{i=1}^{k_1-k_2} \frac{\Gamma(\beta_1+(i-1)\gamma)}{\Gamma(\alpha+\beta_1+(i+k_1-2)\gamma)} \\ &\times \prod_{i=1}^{k_2} \frac{\Gamma(\beta_2+(i-1)\gamma)}{\Gamma(1+\beta_2+(i+k_2-k_1-2)\gamma)}\ \frac{\Gamma(\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma)} \\ &\times \prod_{i=1}^{k_2}\frac{\Gamma(1+(i-k_1-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\ \prod_{i=1}^{k_1}\frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,.\end{aligned}$$

Here $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta_1 > 0$, $\operatorname{Re}\beta_2 > 0$, $\operatorname{Re}\gamma < 0$ and $|\operatorname{Re}\gamma|$ is sufficiently small. The integration cycle $C^{k_1,k_2}[0,1]$ is defined in [TV4, Section 3], also see its definition in [Wa1, Wa2, FW].

The starting point of this formula was an example of the joint system of the $\mathfrak{sl}_3$ trigonometric differential KZ equations and associated dynamical difference equations, an example in which the space of solutions is one-dimensional. The A_n-type Selberg integral formula for arbitrary n was obtained in [Wa1, Wa2], see also [FW].

In this paper we consider the reduction modulo p of the same example of the joint system of the $\mathfrak{sl}_{n+1}$ trigonometric differential KZ equations and associated dynamical difference equations, which led in [TV4, Wa1] to the A_n-type Selberg integral formula. Using the reduction modulo p of these differential and difference equations we obtain our A_n-type $\mathbb{F}_p$-Selberg integral formula for $n \geqslant 1$, see (3.11). For $n = 1$ the formula is proved in [RV, Theorem 4.1]. For $n = 2$ the formula is proved in Theorem 3.4 below. We sketch the proof of the formula for $n > 2$ in Section 5.4. The details of that sketch will appear elsewhere.

The paper is organized as follows. In Section 2 we collect useful facts. In Section 3 we introduce the notion of $\mathbb{F}_p$-integral and discuss the integral formula for the $\mathbb{F}_p$-beta integral. In Section 3 we define the A_n-type $\mathbb{F}_p$-Selberg integral and present its evaluation formula. Theorem 3.4 states that the formula holds for $n = 2$. In Section 3 we also prove Theorem 3.7, which is used in the transition from the A_{n-1}-type formula to the A_n-type formula, in particular, in the transition from the known A_1-type formula to the new A_2-type formula. In Section 4 we sketch the proof of formula (1.2) following [TV4]. In Section 5 we adapt this proof to prove Theorem 3.4.

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2. Preliminary remarks

In this paper p is an odd prime number.

2.1. Cancellation of factorials.

Lemma 2.1. *If a, b are non-negative integers and $a+b=p-1$, then in $\mathbb{F}_p$ we have*

$$a!\,b! \,=\, (-1)^{a+1}\,. \tag{2.1}$$

Proof. We have $a! = (-1)^a(p-1)\dots(p-a)$ and $p-a=b+1$. Hence $a!\,b! = (-1)^a(p-1)! = (-1)^{a+1}$ by Wilson's Theorem. □

2.2. Dyson's formula.

We shall use Dyson's formula

$$\text{C. T.} \prod_{1\leqslant i<j\leqslant k} (1-x_i/x_j)^c(1-x_j/x_i)^c = \frac{(kc)!}{(c!)^k}\,, \tag{2.2}$$

where C.T. denotes the constant term. See the formula in [AAR, Section 8.8].

2.3. $\mathbb{F}_p$-Integrals.

Let M be an $\mathbb{F}_p$-module. Let $P(x_1,\dots,x_k)$ be a polynomial with coefficients in M,

$$P(x_1,\dots,x_k) = \sum_d c_d\, x_1^{d_1}\dots x_k^{d_k}. \tag{2.3}$$

Let $l=(l_1,\dots,l_k)\in\mathbb{Z}^k_{>0}$. The coefficient $c_{l_1p-1,\dots,l_kp-1}$ is called the $\mathbb{F}_p$*-integral over the p-cycle* $[l_1,\dots,l_k]_p$ and is denoted by $\int_{[l_1,\dots,l_k]_p} P(x_1,\dots,x_k)\,dx_1\dots dx_k$.

Lemma 2.2. *For $i=1,\dots,k-1$ we have*

$$\begin{aligned}\int_{[l_1,\dots,l_{i+1},l_i,\dots,l_k]_p} & P(x_1,\dots,x_{i+1},x_i,\dots,x_k)dx_1\dots dx_k \\ &= \int_{[l_1,\dots,l_k]_p} P(x_1,\dots,x_k)\,dx_1\dots dx_k\,.\end{aligned} \tag{2.4}$$

□

Lemma 2.3. *For any $i=1,\dots,k$, we have*

$$\int_{[l_1,\dots,l_k]_p} \frac{\partial P}{\partial x_i}(x_1,\dots,x_k)\,dx_1\dots dx_k = 0\,.$$

□

Let $\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{>0}$ and

$$[\boldsymbol{k}]_p := [(1)_{k_1};(k_1)_{k_2};\dots;(k_{n-1})_{k_n}]_p, \tag{2.5}$$

where x_y denotes the y-tuple $(x,\dots,x)$. For example for $n=2$, $\boldsymbol{k}=(3,2)$, we have $[\boldsymbol{k}]_p = [1,1,1;3,3]_p$.

2.4. **$\mathbb{F}_p$-Beta integral.** For non-negative integers the classical beta integral formula says

$$\int_0^1 x^a(1-x)^b dx = \frac{a!\,b!}{(a+b+1)!}\,. \tag{2.6}$$

Theorem 2.4 ([V7]). *Let $a<p$, $b<p$, $p-1\leqslant a+b$. Then in $\mathbb{F}_p$ we have*

$$\int_{[1]_p} x^a(1-x)^b dx \;=\; -\,\frac{a!\,b!}{(a+b+1-p)!}\,. \tag{2.7}$$

If $a+b<p-1$, then

$$\int_{[1]_p} x^a(1-x)^b dx \;=0\,. \tag{2.8}$$

3. $\mathbb{F}_p$-Selberg integral of type A_n

3.1. **Admissible parameters.** Let $\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{>0}$ and $k_i>k_{i+1}$, $i=1,\dots,n-1$. Set $k_0=k_{n+1}=0$.

Let $a,b_1,\dots,b_n,c\in\mathbb{Z}_{>0}$. Denote $b=(b_1,\dots,b_n)$ and

(3.1)

$$R_{\boldsymbol{k}}(a,b,c)=$$
$$=\prod_{1\leqslant s\leqslant r\leqslant n}\prod_{i=1}^{k_r-k_{r+1}}\frac{(r-s+b_s+\dots+b_r+(i+s-r-1)c)!}{(r-s+1+a_s+b_s+\dots+b_r+(i+s-r+k_s-k_{s-1}-2)c-\delta_{s,1}p)!}$$
$$\times\,(-1)^{\sum_{i=1}^n k_i}\Big(\prod_{i=1}^{k_1}(a_1+(i-1)c)!\Big)\Big(\prod_{r=1}^{n}\prod_{i=1}^{k_r}\frac{(ic)!}{c!}\Big)\Big(\prod_{r=2}^{n}\prod_{i=1}^{k_r}(p+(i-k_{r-1}-1)c)!\Big),$$

where $a_1=a,\ a_2=\dots=a_n=0$; $\delta_{s,1}$ is 1 if $s=1$ and is zero otherwise.

We say that $a,b_1,\dots,b_n,c\in\mathbb{Z}_{>0}$ are *admissible* if $a+(k_1-1)c<p-1$ and for any factorial $x!$ on the right-hand side of (3.1) we have $0\leqslant x<p$. The set of all admissible (a,b,c) is denoted by $\mathcal{A}_{\boldsymbol{k}}$.

Lemma 3.1. *The set $\mathcal{A}_{\boldsymbol{k}}$ is defined in $\mathbb{Z}^{n+2}_{>0}$ by the following system of inequalities:*

$$\begin{aligned} &0\leqslant r-s+b_s+\dots+b_r+(s-r)c,\\ &r-s+b_s+\dots+b_r+(k_r-k_{r+1}+s-r-1)c\leqslant p-1, \end{aligned}\tag{3.2}$$

for $1\leqslant s\leqslant r\leqslant n$;

$$\begin{aligned} &0\leqslant r-s+1+b_s+\dots+b_r+(s-r+k_s-k_{s-1}-1)c,\\ &r-s+1+b_s+\dots+b_r+(s-r+k_r-k_{r+1}+k_s-k_{s-1}-2)c\leqslant p-1, \end{aligned}\tag{3.3}$$

for $2\leqslant s\leqslant r\leqslant n$;

$$\begin{aligned} &p\leqslant r+a+b_1+\dots+b_r+(k_1-r)c,\\ &r+a+b_1+\dots+b_r+(k_r-k_{r+1}+k_1-r-1)c<2p, \end{aligned}\tag{3.4}$$

for $1\leqslant r\leqslant n$;

$$a+(k_1-1)c<p-1,\quad b_1\geqslant p-1-(a+(k_1-1)c),\quad 0<k_1c<p\,. \tag{3.5}$$

□

Lemma 3.2. *Assume that* $(a,b,c)\in\mathcal{A}_{\boldsymbol{k}}$. *Then*

$$b_1 \geqslant p-1-(a+(k_1-1)c), \qquad b_s \geqslant (k_{s-1}-k_s+1)c-1, \quad s=2,\dots,n. \tag{3.6}$$

Proof. The inequality $b_s \geqslant (k_{s-1}-k_s+1)c-1$ for $s=2,\dots,n$ follows from the first inequality in (3.3) for $r=s$. The inequality $b_1 \geqslant p-1-(a+(k_1-1)c)$ follows from the first inequality in (3.4) for $r=1$. □

Example. Let $n=1$, $\boldsymbol{k}=(k_1)$. Then

$$R_{(k_1)}(a,b_1,c) = \prod_{i=1}^{k_1} \frac{(ic)!}{c!}\,\frac{(a+(i-1)c)!\,(b_1+(i-1)c)!}{(1+a+b_1+(i+k_1-2)c-p)!} \tag{3.7}$$

and $\mathcal{A}_{(k_1)}$ consists of $a,b,c\in\mathbb{Z}_{>0}$ such that

$$\begin{aligned} &a+(k_1-1)c<p-1, && b_1+(k_1-1)c\leqslant p-1, \quad k_1c\leqslant p-1,\\ &p-1\leqslant a+b_1+(k_1-1)c, && a+b_1+(2k_1-2)c<2p-1\,. \end{aligned} \tag{3.8}$$

3.2. **Main result.** Given $\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{>0}$ introduce $k_1+\dots+k_n$ variables

$$t=(t^{(1)},\dots,t^{(n)}), \quad \text{where} \quad t^{(i)}=(t^{(i)}_1,\dots,t^{(i)}_{k_i}), \quad i=1,\dots,n. \tag{3.9}$$

Define the *master polynomial*

$$\Phi_{\boldsymbol{k}}(t;a,b,c)=\prod_{i=1}^n\Big(\prod_{j=1}^{k_i}(t^{(i)}_j)^{a_i}(1-t^{(i)}_j)^{b_i}\prod_{1\leqslant j<j'\leqslant k_i}(t^{(i)}_j-t^{(i)}_{j'})^{2c}\Big)\prod_{i=1}^{n-1}\prod_{j=1}^{k_{i+1}}\prod_{j'=1}^{k_i}(t^{(i+1)}_j-t^{(i)}_{j'})^{p-c}\,.$$

Denote

$$S_{\boldsymbol{k}}(a,b,c)=\int_{[\boldsymbol{k}]_p}\Phi_{\boldsymbol{k}}\,dt\,. \tag{3.10}$$

The $\mathbb{F}_p$-integral $S_{\boldsymbol{k}}(a,b,c)$ is called the $\mathbb{F}_p$-*Selberg integral of type* A_n.

Conjecture 3.3. *Let* n *be a positive integer. Let* $\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{>0}$, $k_i>k_{i+1}$, $i=1,\dots,n-1$. *Then for any* $(a,b,c)\in\mathcal{A}_{\boldsymbol{k}}$ *we have the equality in* $\mathbb{F}_p$*:*

$$S_{\boldsymbol{k}}(a,b,c) \;=\; R_{\boldsymbol{k}}(a,b,c)\,. \tag{3.11}$$

For $n=1$ formula (3.11) is proved in [RV, Theorem 4.1]. For $n=2$ formula (3.11) is proved in the next theorem.

Theorem 3.4. *Let* $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2_{>0}$, $k_1>k_2$. *Then for any* $(a,b,c)\in\mathcal{A}_{\boldsymbol{k}}$ *we have the equality in* $\mathbb{F}_p$*:*

$$S_{\boldsymbol{k}}(a,b,c) \;=\; R_{\boldsymbol{k}}(a,b,c)\,. \tag{3.12}$$

Formula (3.11) for $n=2$ is deduced from formula (3.11) for $n=1$ in Section 5.

More generally, for any k formula (3.11) for $n=k$ can be deduced from formula (3.11) for $n=k-1$ similarly, see the sketch of that in Section 5.4. Details of that deduction will appear elsewhere. Because of that formula (3.11) for any n is formulated as a conjecture and not as a theorem.

Remark. Theorem 3.4 can be extended to the case of $\boldsymbol{k}$ such that $k_1 \geqslant k_2$, but the structure of inequalities in Lemma 3.1 will depend on the appearance of the equality $k_1 = k_2$ in $\boldsymbol{k}$, and the proof of Theorem 3.4 will split into different sub-cases. To shorten the exposition we restrict ourselves to $\boldsymbol{k}$ such that $k_1 > k_2$.

Example. Here is the simplest A_2-type $\mathbb{F}_p$-Selberg integral formula with $k_1 = k_2 = 1$.

Theorem 3.5. *Assume that a, b_1, b_2, c are integers such that*

$$0 \leqslant a < p, \quad 0 < c \leqslant p, \quad 0 \leqslant b_2 - c + 1 < p,$$
$$0 \leqslant b_1 + b_2 - c + 1 < p, \quad p - 1 \leqslant a + b_1 + b_2 - c + 1 < 2p - 1.$$

Then in $\mathbb{F}_p$ we have

$$\int_{[1;1]_p} t^a (1-t)^{b_1} (s-t)^{p-c} (1-s)^{b_2} dt\, ds = \frac{a!\,(b_1 + b_2 - c + 1)!}{(a + b_1 + b_2 - c + 2 - p)!}\, \frac{(p-c)!\,(b_2)!}{(b_2 - c + 1)!}\,. \tag{3.13}$$

Proof. Change variables $s = t + (1-t)v$, then the $\mathbb{F}_p$-integral becomes equal to

$$\int_{[1,1]_p} t^a (1-t)^{b_1+b_2-c+1} v^{p-c} (1-v)^{b_2} dt\, dv.$$

Applying the $\mathbb{F}_p$-beta integral formula we obtain the theorem. □

The simplest A_3-type $\mathbb{F}_p$-Selberg integral formula $k_1 = k_2 = k_3 = 1$ is given by the next theorem.

Theorem 3.6. *Let a, b_1, b_2, b_3, c be integers such that all factorials on the right-hand side of formula (3.14) are factorials of non-negative integers less than p. Then in $\mathbb{F}_p$ we have*

$$\int_{[1;1;1]_p} t^a (1-t)^{b_1} (s-t)^{p-c} (1-s)^{b_2} (u-s)^{p-c} (1-u)^{b_2} dt\, ds\, du \tag{3.14}$$
$$= -\frac{a!\,(b_1 + b_2 + b_3 - 2c + 2)!}{(a + b_1 + b_2 + b_3 - 2c + 3 - p)!}\, \frac{(p-c)!\,(b_2 + b_3 - c + 1)!}{(b_2 + b_3 - 2c + 2)!}\, \frac{(p-c)!\,(b_3)!}{(b_3 - c + 1)!}\,.$$

Proof. The proof is the same as the proof of the previous theorem. □

The versions of identities (3.13), (3.14) over complex numbers see in [MuV, Theorem 1].

3.3. Relation between the $\mathbb{F}_p$-Selberg integrals of types A_{n-1}- and A_n.

Theorem 3.7. *Let $n > 1$ and $\boldsymbol{k} = (k_1, \dots, k_n)$, $\boldsymbol{k}' = (k_1, \dots, k_{n-1})$, $b = (b_1, \dots, b_n)$, $b' = (b_1, \dots, b_{n-1})$. Assume that formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}']_p}(a, b', c)$ of type A_{n-1}. Also assume that $b_n = (k_{n-1} - k_n + 1)c - 1$. Then formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a, b, c)$ of type A_n.*

Proof. Under the assumption $b_n = (k_{n-1} - k_n + 1)c - 1$ all variables $(t_j^{(n)})$ in $\Phi_{[\boldsymbol{k}]_p}(\boldsymbol{t}; a, b, c)$ are used to reach the monomial $\prod_{j=1}^{k_n} (t_j^{(n)})^{k_{n-1}p-1}$ in the calculation of the $\mathbb{F}_p$-integral $S_{[\boldsymbol{k}]_p}(a, b, c)$. The remaining free variables $(t_j^{(i)})$ with $i < n$ all belong to the factor $\Phi_{[\boldsymbol{k}']_p}(t^{(1)}, \dots, t^{(n-1)}; a, b', c)$ of $\Phi_{[\boldsymbol{k}]_p}(\boldsymbol{t}; a, b, c)$ and are used to calculate the coefficient of $\prod_{j=1}^{k_1} (t_j^{(1)})^{p-1} \prod_{i=2}^{n-1} \prod_{j=1}^{k_i} (t_j^{(i)})^{k_{i-1}p-1}$.

More precisely, under the assumptions of the theorem we have

$$S_{\boldsymbol{k}}(a,b,c) = (-1)^{b_nk_n+ck_n(k_n-1)/2}\frac{(k_nc)!}{(c!)^{k_n}}\,S_{\boldsymbol{k}'}(a',b',c),$$

where $(-1)^{b_nk_n+ck_n(k_n-1)/2}\frac{(k_nc)!}{(c!)^{k_n}}$ is the coefficient of $\prod_{j=1}^{k_n}(t_j^{(n)})^{k_{n-1}p-1}$ in the expansion of

$$\prod_{j=1}^{k_n}\prod_{j'=1}^{k_{n-1}}(t_j^{(n)}-t_{j'}^{(n-1)})^{p-c}\prod_{1\leqslant j<j'\leqslant k_n}(t_j^{(n)}-t_{j'}^{(n)})^{2c},$$

see Dyson's formula. We have $(-1)^{b_nk_n+ck_n(k_n-1)/2}=(-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}$. Hence

$$\begin{aligned}S_{\boldsymbol{k}}(a,b,c) &= (-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}\frac{(k_nc)!}{(c!)^{k_n}}\,S_{\boldsymbol{k}'}(a,b',c)\\ &= (-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}\frac{(k_nc)!}{(c!)^{k_n}}\,R_{\boldsymbol{k}'}(a,b',c)\,,\end{aligned}$$

where $S_{\boldsymbol{k}'}(a,b',c)=R_{\boldsymbol{k}'}(a,b',c)$ holds by assumptions. To prove the theorem we need to show that

$$R_{[\boldsymbol{k}]_p}(a,(b',b_n),c) = (-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}\frac{(k_nc)!}{(c!)^{k_n}}\,R_{\boldsymbol{k}'}(a,b',c)\,.$$

Indeed we have

$$\begin{aligned}R_{\boldsymbol{k}}(a,(b',b_n),c) =\ & R_{\boldsymbol{k}'}(a,b',c)\prod_{i=1}^{k_n}\frac{(ic)!}{c!}\prod_{i=1}^{k_n}(p+(i-k_{n-1}-1)c)!\\ \times \prod_{1\leqslant s\leqslant n}\prod_{i=1}^{k_n}&\frac{(n-s+b_s+\dots+b_n+(i+s-n-1)c)!}{(n-s+1+a_s+b_s+\dots+b_n+(i+s-n+k_s-k_{s-1}-2)c-\delta_{s,1}p)!}\\ \times \prod_{1\leqslant s\leqslant n-1}\prod_{i=1}^{k_n}&\frac{(n-s+a_s+b_s+\dots+b_{n-1}+(i+s-n+k_s-k_{s-1}-3)c-\delta_{s,1}p)!}{(n-1-s+b_s+\dots+b_{n-1}+(i+s-r-1)c)!}\\ &= R_{\boldsymbol{k}'}(a,b',c)\prod_{i=1}^{k_n}\frac{(ic)!}{c!}\prod_{i=1}^{k_n}\frac{(b_n+(i-1)c)!\,(p+(i-k_{n-1}-1)c)!}{(1+b_n+(i+k_n-k_{n-1}-2)c)!}\,.\\ &= R_{\boldsymbol{k}'}(a,b',c)\prod_{i=1}^{k_n}\frac{(ic)!}{c!}\prod_{i=1}^{k_n}\frac{((i+k_{n-1}-k_n)c-1)!\,(p+(i-k_{n-1}-1)c)!}{((i-1)c))!}\,.\\ &= R_{\boldsymbol{k}'}(a,b',c)\,(-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}\frac{(k_nc)!}{(c!)^{k_n}}\,,\end{aligned}$$

where in the last step we use the cancellation Lemma 2.1. The theorem is proved. □

Corollary 3.8. *Let $n>1$, $\boldsymbol{k}=(k_1,\dots,k_n)$, and $(a,b_1,c)\in\mathcal{A}_{(k_1)}$. Let $b=(b_1,\dots,b_n)$, where $b_i=(k_{i-1}-k_i+1)c-1$ for $i=2,\dots,n$. Then formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a,b,c)$ of type A_n.*

Proof. Formula (3.11) for the $\mathbb{F}_p$-Selberg integrals of type A_1 is proved in [RV]. Hence the corollary follows from Theorem 3.7 by induction on n. □

4. The A_2-type Selberg integral over $\mathbb{C}$

In this section we formulate the A_2-type Selberg integral formula over $\mathbb{C}$, formulated and proved in [TV4], and sketch the proof of the formula, following [TV4]. In Section 5 we adapt this proof to prove the A_2-type $\mathbb{F}_p$-Selberg integral formula, that is, formula (3.11) for $n = 2$.

4.1. **The A_2-formula over $\mathbb{C}$.** For $k_1 \geqslant k_2 \geqslant 0$ let $t = (t_1, \dots, t_{k_1})$, $s = (s_1, \dots, s_{k_2})$. Define the *master function*

$$\Phi(t;s) = \prod_{i=1}^{k_1} t_i^{\alpha-1}(1-t_i)^{\beta_1-1} \prod_{j=1}^{k_2}(1-s_j)^{\beta_2-1} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2}|s_j-t_i|^{-\gamma} \times \prod_{1\leqslant i<i'\leqslant k_1}|t_i-t_{i'}|^{2\gamma} \prod_{1\leqslant j<j'\leqslant k_2}|s_j-s_{j'}|^{2\gamma} \tag{4.1}$$

and the integral

$$\tilde S(\alpha,\beta_1,\beta_2,\gamma) = \int_{C^{k_1,k_2}[0,1]} \Phi(t;s)\,dt\,ds\,, \tag{4.2}$$

where the integration cycle $C^{k_1,k_2}[0,1]$ is defined in [TV4, Section 3]. The explicit description of this cycle is of no importance in this paper.

Theorem 4.1 ([TV4, Theorem 3.3])**.** *Let $\alpha, \beta_1, \beta_2, \gamma$ be complex numbers such that* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta_1 > 0$, $\operatorname{Re}\beta_2 > 0$, $\operatorname{Re}\gamma < 0$ *and* $|\operatorname{Re}\gamma|$ *sufficiently small. Then*

$$\begin{aligned}\tilde S(\alpha,\beta_1,\beta_2,\gamma) = &\prod_{i=1}^{k_1-k_2} \frac{\Gamma(\beta_1+(i-1)\gamma)}{\Gamma(\alpha+\beta_1+(i+k_1-2)\gamma)} \\ &\times \prod_{i=1}^{k_2} \frac{\Gamma(\beta_2+(i-1)\gamma)}{\Gamma(1+\beta_2+(i+k_2-k_1-2)\gamma)}\, \frac{\Gamma(\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma)} \\ &\times \prod_{i=1}^{k_2} \frac{\Gamma(1+(i-k_1-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{i=1}^{k_1} \frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,.\end{aligned} \tag{4.3}$$

In the next Sections 4.2, 4.3 we sketch the proof of formula (4.3) following [TV4].

4.2. **Weight functions.** To evaluate $\tilde S(\alpha,\beta_1,\beta_2,\gamma)$ we introduce a collection of new integrals $J_{l_1,l_2,m}(\alpha,\beta_1,\beta_2,\gamma)$, which also can be evaluated explicitly, see [TV4].

For a function $f(t_1,\dots,t_k)$ set

$$\operatorname{Sym}_{t_1,\dots,t_k} f(t_1,\dots,t_k) \;=\; \frac{1}{k!}\sum_{\sigma\in S_k} f(t_{\sigma_1},\dots,t_{\sigma_k})\,.$$

Given $k_1 \geqslant k_2 \geqslant 0$, we say that a triple of non-negative integers (l_1, l_2, m) is *allowable* if $l_1 \leqslant k_1 - k_2 + l_2$, $l_2 \leqslant k_2$ and $m \leqslant \min(l_1, l_2)$. For any allowable triple (l_1, l_2, m) define the weight function

$$W_{l_1,l_2,m}(t_1,\dots,t_{k_1};s_1,\dots,s_{k_2}) =$$
$$= \operatorname{Sym}_{t_1,\dots,t_{k_1}} \operatorname{Sym}_{s_1,\dots,s_{k_2}} \Big(\prod_{a=1}^{l_1} t_a \prod_{a=l_1+1}^{k_1}(1-t_a)\prod_{b=1}^{m}\frac{1-s_b}{s_b-t_b}\prod_{b=l_2+1}^{k_2}\frac{1-s_b}{s_b-t_{b+k_1-k_2}}\Big)$$

and the integral

$$J_{l_1,l_2,m}(\alpha,\beta_1,\beta_2,\gamma) = \int_{C^{k_1,k_2}[0,1]} \Phi(t;s)\, W_{l_1,l_2,m}(t;s)\, dt\, ds\,.$$

In particular,

$$J_{0,k_2,0}(\alpha,\beta_1,\beta_2,\gamma) = \tilde S(\alpha,\beta_1+1,\beta_2,\gamma). \tag{4.4}$$

4.3. **Representations of $\mathfrak{sl}_3$.** Consider the complex Lie algebra $\mathfrak{sl}_3$ with standard generators $f_1, f_2, e_1, e_2, h_1, h_2$, simple roots σ_1, σ_2, fundamental weights ω_1, ω_2. Let V_{λ_1}, V_{λ_2} be the irreducible $\mathfrak{sl}_3$-modules with highest weights

$$\lambda_1 = -\frac{\alpha}{\gamma}\,\omega_1\,, \qquad \lambda_2 = -\frac{\beta_1}{\gamma}\,\omega_1 - \frac{\beta_2}{\gamma}\,\omega_2$$

and highest weight vectors v_1, v_2. For $k_1 \geqslant k_2 \geqslant 0$ consider the weight subspace $V_{\lambda_1}\otimes V_{\lambda_2}[\lambda_1+\lambda_2-k_1\sigma_1-k_2\sigma_2]$ of the tensor product $V_{\lambda_1}\otimes V_{\lambda_2}$ and the singular weight subspace $\operatorname{Sing} V_{\lambda_1}\otimes V_{\lambda_2}[\lambda_1+\lambda_2-k_1\sigma_1-k_2\sigma_2]$ consisting of the vectors $w \in V_{\lambda_1}\otimes V_{\lambda_2}[\lambda_1+\lambda_2-k_1\sigma_1-k_2\sigma_2]$ such that $e_1 w = 0$, $e_2 w = 0$. A basis of $V_{\lambda_1}\otimes V_{\lambda_2}[\lambda_1+\lambda_2-k_1\sigma_1-k_2\sigma_2]$ is formed by the vectors

$$v_{l_1,l_2,m} = \frac{f_1^{k_1-k_2-l_1+l_2}[f_1,f_2]^{k_2-l_2}v_1 \otimes f_1^{l_1-m}[f_1,f_2]^m f_2^{l_2-m}v_2}{(k_1-k_2-l_1+l_2)!\,(k_2-l_2)!\,(l_1-m)!\,m!\,(l_2-m)!}$$

labeled by allowable triples (l_1,l_2,m). It is known from the theory of KZ equations that the vector

$$J = \sum_{l_1,l_2,m} (-1)^{l_1} J_{l_1,l_2,m}(\alpha,\beta_1,\beta_2,\gamma)\, v_{l_1,l_2,m}$$

is a singular vector, see [M, Theorem 2.4], [MaV, Corollary 10.3], cf. [RSV].

The singular vector equations $e_1 J = 0$, $e_2 J = 0$ are calculated with the help of the formulas:

$$\begin{aligned}
&h_1 v_1 = -\frac{\alpha}{\gamma}\, v_1, \qquad h_2 v_1 = 0, \qquad h_1 v_2 = -\frac{\beta_1}{\gamma}\, v_2, \qquad h_2 v_2 = -\frac{\beta_2}{\gamma}\, v_2\,,\\
&[h_1,f_1] = -2f_1, \quad [h_1,f_2] = f_2, \quad [h_2,f_1] = f_1, \quad [h_2,f_2] = -2f_2,\\
&[e_1,f_1] = h_1, \qquad [e_1,f_2] = [e_2,f_1] = 0, \qquad [e_2,f_2] = h_2,
\end{aligned}$$

$$[h_1,[f_1,f_2]] = -[f_1,f_2], \quad [h_2,[f_1,f_2]] = -[f_1,f_2], \quad [e_1,[f_1,f_2]] = f_2, \quad [e_2,[f_1,f_2]] = -f_1.$$

Here are some of the singular vector relations.

Theorem 4.2 (cf. [TV4, Theorem 5.2]). *We have*

$$J_{0,l_2,0} = (-1)^{l_2} J_{0,0,0} \prod_{i=0}^{l_2-1} \frac{(k_1-k_2+i+1)\gamma}{\beta_2+i\gamma}\,. \tag{4.5}$$

Proof. We have

$$e_2 \frac{f_1^{k_1-k_2+i}[f_1,f_2]^{k_2-i}v_1\otimes f_2^i v_2}{(k_1-k_2+i)!\,(k_2-i)!\,i!} = -\ (k_1-k_2+i+1)\frac{f_1^{k_1-k_2+i+1}[f_1,f_2]^{k_2-i-1}v_1\otimes f_2^i v_2}{(k_1-k_2+i+1)!\,(k_2-i-1)!\,i!} + \ \Big(-\frac{\beta_2}{\gamma}-i+1\Big)\frac{f_1^{k_1-k_2+i}[f_1,f_2]^{k_2-i}v_1\otimes f_2^{i-1}v_2}{(k_1-k_2+i)!\,(k_2-i)!\,(i-1)!}\,.$$

Calculating the coefficient of $\frac{f_1^{k_1-k_2+i+1}[f_1,f_2]^{k_2-i-1}v_1\otimes f_2^i v_2}{(k_1-k_2+i)!\,(k_2-i)!\,i!}$ in $e_2 J = 0$ we obtain

$$(k_1-k_2+i+1)\gamma\, J_{0,i,0} + (\beta_2+i\gamma)\, J_{0,i+1,0} = 0. \tag{4.6}$$

This implies the theorem. □

Hence

$$J_{0,k_2,0}(\alpha,\beta_1,\beta_2,\gamma) = (-1)^{k_2} J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma) \prod_{i=0}^{k_2-1} \frac{(k_1-k_2+i+1)\gamma}{\beta_2+i\gamma}\,. \tag{4.7}$$

Combining (4.4) and (4.7) we observe that formula (4.3) is equivalent to the formula

$$J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma) = \prod_{i=1}^{k_1-k_2} \frac{\Gamma(1+\beta_1+(i-1)\gamma)}{\Gamma(1+\alpha+\beta_1+(i+k_1-2)\gamma)} \times \prod_{i=1}^{k_2} \frac{\Gamma(1+\beta_2+(i-1)\gamma)}{\Gamma(1+\beta_2+(i+k_2-k_1-2)\gamma)}\ \frac{\Gamma(1+\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(1+\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma)} \times \prod_{i=1}^{k_2} \frac{\Gamma((i-k_1-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\ \prod_{i=1}^{k_1} \frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,. \tag{4.8}$$

Denote by $R_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ the right-hand side of (4.8).

To prove (4.8) we use the following observation. The weight subspace $V_{\lambda_1}[\lambda_1-k_1\sigma_1-k_2\sigma_2]\subset V_{\lambda_1}$ is one-dimensional with a basis vector

$$v_{0,0,0} = \frac{f_1^{k_1-k_2}[f_1,f_2]^{k_2}v_1\otimes v_2}{(k_1-k_2)!\,(k_2)!}\,.$$

By [MaV, Theorem 5.1] the vector-valued function $J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)v_{0,0,0}$ satisfies the dynamical difference equations introduced in [TV3],

$$J_{0,0,0}(\alpha,\beta_1-1,\beta_2,\gamma)v_{0,0,0} = J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)\,\mathbb{B}_1 v_{0,0,0}\,, \tag{4.9}$$

$$J_{0,0,0}(\alpha,\beta_1,\beta_2-1,\gamma)v_{0,0,0} = J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)\,\mathbb{B}_2 v_{0,0,0}\,. \tag{4.10}$$

Here $\mathbb{B}_1$, $\mathbb{B}_2$ are certain linear operators acting on V_{λ_1} and preserving the weight decomposition of V_{λ_1}, see formulas for these operators in the example in [MaV, Section 7.1] and in [MaV, Section 3.1], also see [TV3, Formula (8)].

Written explicitly equations (4.9), (4.10) give us the difference equations for the scalar function $J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ with respect to the shift of the variables $\beta_1\to\beta_1-1$ and $\beta_2\to\beta_2-1$,

$$J_{0,0,0}(\alpha,\beta_1-1,\beta_2,\gamma)=J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)\prod_{i=1}^{k_1-k_2}\frac{\alpha+\beta_1+(i+k_1-2)\gamma}{\beta_1+(i-1)\gamma}\times\prod_{i=1}^{k_2}\frac{\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma}{\beta_1+\beta_2+(i-2)\gamma}\,, \tag{4.11}$$

$$J_{0,0,0}(\alpha,\beta_1,\beta_2-1,\gamma)=J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)\times\prod_{i=1}^{k_2}\frac{\beta_2+(i+k_2-k_1-2)\gamma}{\beta_2+(i-1)\gamma}\,\frac{\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma}{\beta_1+\beta_2+(i-2)\gamma}\,. \tag{4.12}$$

The difference equations for $J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ are the same as the difference equations for the function $R_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ with respect to the shift of the variables $\beta_1\to\beta_1-1$ and $\beta_2\to\beta_2-1$. Therefore, the functions $J_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ and $R_{0,0,0}(\alpha,\beta_1,\beta_2,\gamma)$ are proportional up to a periodic function of β_1,β_2. The periodic function can be fixed by comparing asymptotics as $\operatorname{Re}\beta_1\to\infty$, $\operatorname{Re}\beta_2\to\infty$. This finishes the proof in [TV4] of formulas (4.8) and (4.3).

5. The A_2-type Selberg integrals over $\mathbb{F}_p$

5.1. Relations between $\mathbb{F}_p$-integrals. For $\boldsymbol{k}=(k_1,k_2)$, $k_1>k_2>0$ and integers $0<a,b_1,b_2,c<p$ define the master polynomial

$$\Phi_{\boldsymbol{k}}(t;s;a,b_1,b_2,c)=\prod_{i=1}^{k_1}t_i^a(1-t_i)^{b_1}\prod_{j=1}^{k_2}(1-s_j)^{b_2}\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}(s_j-t_i)^{p-c}\times\prod_{1\leqslant i<i'\leqslant k_1}(t_i-t_{i'})^{2c}\prod_{1\leqslant j<j'\leqslant k_2}(s_j-s_{j'})^{2c} \tag{5.1}$$

and the $\mathbb{F}_p$-integral

$$S_{\boldsymbol{k}}(a,b_1,b_2,c)=\int_{[\boldsymbol{k}]_p}\Phi_{\boldsymbol{k}}(t;s;a,b_1,b_2,c)\,dt\,ds\,, \tag{5.2}$$

where the p-cycle $[\boldsymbol{k}]_p$ is defined in (2.5). This is the A_2-type $\mathbb{F}_p$-Selberg integral, see (3.10).

For an allowable triple (l_1,l_2,m) define the $\mathbb{F}_p$-integral

$$I_{l_1,l_2,m}(a,b_1,b_2,c)=\int_{[\boldsymbol{k}]_p}\Phi_{\boldsymbol{k}}(t;s;a,b_1,b_2,c)\,\frac{W_{l_1,l_2,m}(t;s)}{\prod_{i=1}^{k_1}t_i(1-t_i)\prod_{j=1}^{k_2}(1-s_j)}\,dt\,ds\,, \tag{5.3}$$

where $W_{l_1,l_2,m}(t;s)$ is the weight function defined in Section 4.2.

Clearly we have

$$I_{0,k_2,0}(a,b_1,b_2,c)=S_{\boldsymbol{k}}(a-1,b_1,b_2-1,c). \tag{5.4}$$

Denote

$$B_0(a,b_1,b_2,c) = (-1)^{k_2} \prod_{i=0}^{k_2-1} \frac{(k_1-k_2+i+1)c}{b_2+ic}\,, \tag{5.5}$$

$$B_1(a,b_1,b_2,c) = \prod_{i=1}^{k_1-k_2} \frac{a+b_1+(i+k_1-2)c}{b_1+(i-1)c} \prod_{i=1}^{k_2} \frac{a+b_1+b_2+(i+k_2-3)c}{b_1+b_2+(i-2)c}\,,$$

$$B_2(a,b_1,b_2,c) = \prod_{i=1}^{k_2} \frac{b_2+(i+k_2-k_1-2)c}{b_2+(i-1)c}\, \frac{a+b_1+b_2+(i+k_2-3)c}{b_1+b_2+(i-2)c}\,.$$

Theorem 5.1. *Assume that $k_1 < p$.*

(i) *Assume that every factor in B_0 in the numerator or denominator is a nonzero element of $\mathbb{F}_p$. Then*

$$I_{0,k_2,0}(a,b_1,b_2,c) = B_0(a,b_1,b_2,c)\, I_{0,0,0}(a,b_1,b_2,c)\,. \tag{5.6}$$

(ii) *Assume that every factor in B_1 in the numerator or denominator is a nonzero element of $\mathbb{F}_p$ and $b_1 > 1$, then*

$$I_{0,0,0}(a,b_1-1,b_2,c) \;=\; B_1(a,b_1,b_2,c)\, I_{0,0,0}(a,b_1,b_2,c). \tag{5.7}$$

(iii) *Assume that every factor in B_2 in the numerator or denominator is a nonzero element of $\mathbb{F}_p$ and $b_2 > 1$, then*

$$I_{0,0,0}(a,b_1,b_2-1,c) \;=\; B_2(a,b_1,b_2,c)\, I_{0,0,0}(a,b_1,b_2,c). \tag{5.8}$$

Proof. Equation (5.6) is an $\mathbb{F}_p$-analog of equation (4.7) and its proof is analogous to the proof of equation (4.7).

More precisely, consider the complex Lie algebra $\mathfrak{sl}_3$ with standard generators $f_1, f_2, e_1, e_2, h_1, h_2$, simple roots σ_1, σ_2, fundamental weights ω_1, ω_2. Let V_{λ_1}, V_{λ_2} be the irreducible $\mathfrak{sl}_3$-modules with highest weights

$$\lambda_1 = -\frac{a}{c+p}\,\omega_1\,, \qquad \lambda_2 = -\frac{b_1}{c+p}\,\omega_1 - \frac{b_2}{c+p}\,\omega_2$$

and highest weight vectors v_1, v_2. The module V_{λ_1} has a basis $(f_1^{r_1}[f_1,f_2]^{r_2}v_1)$ labeled by non-negative integers r_1, r_2 and the module V_{λ_2} has a basis $(f_1^{r_1}[f_1,f_2]^{r_2}f_2^{r_3}v_2)$ labeled by non-negative integers r_1, r_2, r_3. For every generator of $\mathfrak{sl}_3$ the matrix of its action on V_{λ_1} or on V_{λ_2} in these bases is a polynomial in $-\frac{a}{c+p}$ $-\frac{b_1}{c+p}$, $-\frac{b_2}{c+p}$ with integer coefficients.

Consider the Lie algebra $\mathfrak{sl}_3$ over the field $\mathbb{F}_p$. Let $V_{\lambda_1}^{\mathbb{F}_p}$ be the vector space over $\mathbb{F}_p$ with basis $(f_1^{r_1}[f_1,f_2]^{r_2}v_1)$ labeled by non-negative integers r_1, r_2 and with the action of $\mathfrak{sl}_3$ defined by the same formulas as on V_{λ_1} but reduced modulo p. Similarly we define the $\mathfrak{sl}_3$-module $V_{\lambda_2}^{\mathbb{F}_p}$.

Recall $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$. Consider the weight subspace $V_{\lambda_1}^{\mathbb{F}_p} \otimes V_{\lambda_2}^{\mathbb{F}_p}[\lambda_1+\lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ of the tensor product $V_{\lambda_1}^{\mathbb{F}_p} \otimes V_{\lambda_2}^{\mathbb{F}_p}$. This weight subspace has a basis formed by the vectors

$$v_{l_1,l_2,m} = \frac{f_1^{k_1-k_2-l_1+l_2}[f_1,f_2]^{k_2-l_2}v_1 \otimes f_1^{l_1-m}[f_1,f_2]^m f_2^{l_2-m}v_2}{(k_1-k_2-l_1+l_2)!\,(k_2-l_2)!\,(l_1-m)!\,m!\,(l_2-m)!}$$

labeled by allowable triples (l_1, l_2, m).

Lemma 5.2. *The vector*

$$I = \sum_{l_1,l_2,m} (-1)^{l_1} I_{l_1,l_2,l_m}(a, b_1, b_2, c)\, v_{l_1,l_2,m}$$

is a singular vector of $V^{\mathbb{F}_p}_{\lambda_1} \otimes V^{\mathbb{F}_p}_{\lambda_2}$*, that is,* $e_1 I = 0$*,* $e_2 I = 0$.

Proof. Equations $e_1 I = 0$, $e_2 I = 0$ are $\mathbb{F}_p$-analogs of equations $e_1 J = 0$, $e_2 J = 0$ over $\mathbb{C}$.

For $i = 1, 2$, the vector $e_i J$ is the integral of a certain differential $k_1 + k_2$-form μ_i. It is shown in [SV2, Theorems 6.16.2], [M, Theorem 2.4] that $\mu_i = \mathrm{d}\nu_i$, where ν_i is some explicitly written differential $k_1 + k_2 - 1$-form. This implies $e_i J = 0$ by Stokes' theorem.

The vector $e_i I$ is the $\mathbb{F}_p$-integral of the same μ_i reduced modulo p. It is explained in [SV2, Section 4] that the differential form ν_i also can be reduced modulo p and this implies that the $\mathbb{F}_p$-integral $e_i I$ is zero by Lemma 2.3. Cf. the proof of [SV2, Theorem 2.4]. □

Lemma 5.2 implies the equations

$$(k_1 - k_2 + i + 1)c\, I_{0,i,0} + (b_2 + ic)\, I_{0,i+1,0} = 0 \tag{5.9}$$

for $i = 0, \dots, k_2 - 1$, similarly to the proof of equations (4.6). The iterated application of equation (5.9) implies equation (5.6).

The proof of equations (5.7), (5.8) is parallel to the proof of equations (4.11), (4.12). We prove (5.7). The proof of (5.8) is similar.

Equation (4.11) follows from equation (4.9):

$$J_{0,0,0}(\alpha, \beta_1 - 1, \beta_2, \gamma) v_{0,0,0} - J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)\, \mathbb{B}_1 v_{0,0,0} \;=\; 0 \tag{5.10}$$

and equation $\mathbb{B}_1 v_{0,0,0} = B_1 v_{0,0,0}$ in V_{λ_1}. The explicit formulas for $\mathbb{B}_1$ show that under the assumptions of Theorem 5.1 the action of $\mathbb{B}_1$ on $v_{0,0,0}$ is well-defined modulo p and gives the same result $\mathbb{B}_1 v_{0,0,0} = B_1 v_{0,0,0}$ but in $V^{\mathbb{F}_p}_{\lambda_1}$.

The proof of (5.10) in [MaV] goes as follows. The left-hand side of (5.10) is a vector-valued integral of a suitable differential $k_1 + k_2$-form μ. It is shown in [M, Theorem 5.1] that $\mu = \mathrm{d}\nu$, where ν is some explicitly written differential $k_1 + k_2 - 1$-form. This implies (5.10) by Stokes' theorem.

The p-analog of the left-hand side of (5.10) is the element

$$I_{0,0,0}(a, b_1 - 1, b_2, c) v_{0,0,0} - I_{0,0,0}(a, b_1, b_2, c)\, \mathbb{B}_1 v_{0,0,0} \;\in\; V^{\mathbb{F}_p}_{\lambda_1}\,. \tag{5.11}$$

This element is the $\mathbb{F}_p$-integral of the same μ reduced modulo p. It is explained in [SV2, Section 4] that the differential form ν also can be reduced modulo p and this implies that the $\mathbb{F}_p$-integral in the left-hand side of (5.11) equals zero by Lemma 2.3. Hence equation (5.7) is proved and Theorem 5.1 is proved. □

5.2. **Proof of Theorem 3.4.** Recall the set of admissible parameters $\mathcal{A}_{\boldsymbol{k}}$ introduced in Section 3.1 for $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$.

Lemma 5.3. *Assume that* $(a, b_1 - 1, b_2, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then*

$$S_{\boldsymbol{k}}(a, b_1 - 1, b_2, c) = S_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_1-k_2} \frac{1 + a + b_1 + (i + k_1 - 2)c - p}{b_1 + (i-1)c} \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.12}$$

Assume that $(a, b_1, b_2 - 1, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then*

$$S_{\boldsymbol{k}}(a, b_1, b_2 - 1, c) = S_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_2} \frac{1 + b_2 + (i + k_2 - k_1 - 2)c}{b_2 + (i-1)c} \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.13}$$

Proof. The lemma follows from formulas (5.4) and (5.6) and Theorem 5.1. □

For $n = 2$ formula (3.1) takes the form:

$$R_{\boldsymbol{k}}(a, b_1, b_2, c) = (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(b_1 + (i-1)c)!}{(1 + a + b_1 + (i + k_1 - 2)c - p)!} \times \prod_{i=1}^{k_2} \frac{(b_2 + (i-1)c)!}{(1 + b_2 + (i + k_2 - k_1 - 2)c)!} \, \frac{(1 + b_1 + b_2 + (i-2)c)!}{(2 + a + b_1 + b_2 + (i + k_1 - 3)c - p)!} \times \prod_{i=1}^{k_1} (a + (i-1)c)! \prod_{i=1}^{k_2} (p + (i - k_1 - 1)c)! \prod_{r=1}^{2} \prod_{i=1}^{k_r} \frac{(ic)!}{c!}. \tag{5.14}$$

Lemma 5.4. *Assume that* $(a, b_1 - 1, b_2, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then*

$$R_{\boldsymbol{k}}(a, b_1 - 1, b_2, c) = R_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_1-k_2} \frac{1 + a + b_1 + (i + k_1 - 2)c - p}{b_1 + (i-1)c} \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.15}$$

Assume that $(a, b_1, b_2 - 1, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}(a, b_1, b_2, c)$. *Then*

$$R_{\boldsymbol{k}}(a, b_1, b_2 - 1, c) = R_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_2} \frac{1 + b_2 + (i + k_2 - k_1 - 2)c}{b_2 + (i-1)c} \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.16}$$

□

By Lemmas 5.3 and 5.4 the functions $S_{\boldsymbol{k}}(a,b_1,b_2,c)$ and $R_{\boldsymbol{k}}(a,b_1,b_2,c)$ defined on $\mathcal{A}_{\boldsymbol{k}}$ satisfy the same difference equations with respects to the shifts of variables $b_1 \to b_1 - 1$ and $b_2 \to b_2 - 1$.

Lemma 5.5. *Assume that a, c are positive integers such that $0 < k_1 c \leqslant p-1$, $a + (k_1-1)c < p-1$. Then the point*

$$(a,b_1,b_2,c) = (a, p-1-(a+(k_1-1)c), (k_1-k_2+1)c-1, c) \tag{5.17}$$

lies in $\mathcal{A}_{\boldsymbol{k}}$.

Proof. If (a,b_1,b_2,c) is given by (5.17), then

$$\begin{aligned} R_{\boldsymbol{k}} &= (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(p-1-(a+(k_1-i)c))!}{((i-1)c)!} \\ &\times \prod_{i=1}^{k_2} \frac{((k_1-k_2+i)c-1)!}{((i-1)c)!} \, \frac{(p-1-(a+(k_2-i)c))!}{((k_1-k_2+i-1)c)!} \\ &\times \prod_{i=1}^{k_1} (a+(i-1)c)! \prod_{i=1}^{k_2} (p+(i-k_1-1)c)! \prod_{r=1}^{2} \prod_{i=1}^{k_r} \frac{(ic)!}{c!}\,. \end{aligned}$$

This proves the lemma. □

Lemma 5.6. *Assume that $\tilde a, \tilde c$ are non-negative integers such that $0 < k_1 \tilde c \leqslant p-1$, $\tilde a + (k_1-1)\tilde c \leqslant p-1$. Denote by $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ the set of all $(a,b_1,b_2,c) \in \mathcal{A}_{\boldsymbol{k}}$ such that $a = \tilde a$, $c = \tilde c$. Then $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ consists of the pairs (b_1,b_2) of non-negative integers satisfying the inequalities*

$$p-1-(\tilde a + (k_1-1)\tilde c) \leqslant b_1, \qquad (k_1-k_2+1)\tilde c - 1 \leqslant b_2 \tag{5.18}$$

and some other inequalities of the form

$$b_1 \leqslant A_1, \qquad b_2 \leqslant A_2 \qquad b_1 + b_2 \leqslant A_{12}, \tag{5.19}$$

where A_1, A_2, A_{12} are some integers such that

$$\begin{aligned} & A_1 \geqslant p-1-(\tilde a + (k_1-1)\tilde c), \quad A_2 \geqslant (k_1-k_2+1)\tilde c - 1, \\ & A_{12} \geqslant p-1-(\tilde a + (k_1-1)\tilde c) + (k_1-k_2+1)\tilde c - 1\,. \end{aligned}$$

Proof. The lemma follows from Lemmas 3.1 and 3.2. □

Corollary 5.7. *Any point $(\tilde a, b_1, b_2, \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ can be connected with the point $(\tilde a, p-1-(\tilde a + (k_1-1)\tilde c), (k_1-k_2+1)\tilde c - 1, \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ by a piece-wise linear path in $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ consisting of the vectors $(0,-1,0,0)$ or $(0,0,-1,0)$.* □

Proof of Theorem 3.4. For $n=1$, $\boldsymbol{k} = (k_1)$ and the point $(\tilde a, p-1-(\tilde a + (k_1-1)\tilde c), \tilde c)$ formula (3.11) holds by [RV, Theorem 4.1].

For $n=2$, $\boldsymbol{k} = (k_1,k_2)$ and the point $(\tilde a, p-1-(\tilde a + (k_1-1)\tilde c), (k_1-k_2+1)\tilde c - 1, \tilde c)$ formula (3.11) holds by Lemma 5.5 and Theorem 3.7.

For $n=2$, $\boldsymbol{k} = (k_1,k_2)$ and arbitrary $(\tilde a, b_1, b_2, \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ formula (3.11) holds by Lemmas 5.3, 5.4 and Corollary 5.7. Theorem 3.4 for $n=2$ is proved. □

Corollary 5.8. *Let $n > 2$, $\boldsymbol{k} = (k_1, \dots, k_n)$, and $(a, (b_1, b_2), c) \in \mathcal{A}_{(k_1,k_2)}$. Let $b = (b_1, \dots, b_n)$, where $b_i = (k_{i-1} - k_i + 1)c - 1$ for $i = 3, \dots, n$. Then formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a, b, c)$ of type A_n.*

Proof. Formula (3.11) for the $\mathbb{F}_p$-Selberg integrals of type A_2 is proved in Theorem 3.4. Hence the corollary follows from Theorem 3.7 by induction on n. ☐

5.3. **Evaluation of $I_{0,0,0}(a, b_1, b_2, c)$.** In this section we evaluate $I_{0,0,0}(a, b_1, b_2, c)$ without using the evaluation of $S_{\boldsymbol{k}}(a, b_1, b_2, c)$.

Theorem 5.9. *Let $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$ and $a, b_1, b_2, c \in \mathbb{Z}_{>0}$. Assume that $a + (k_1 - 1)c < p$ and all factorials on the right-hand side of the next formula are factorials of the non-negative integers less than p. Then*

$$I_{0,0,0}(a, b_1, b_2, c) = (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(b_1 + (i-1)c)!}{(a + b_1 + (i + k_1 - 2)c - p)!} \tag{5.20}$$
$$\times \prod_{i=1}^{k_2} \frac{(b_2 + (i-1)c)!}{(b_2 + (i + k_2 - k_1 - 2)c)!} \frac{(b_1 + b_2 + (i-2)c)!}{(a + b_1 + b_2 + (i + k_1 - 3)c - p)!}$$
$$\times \prod_{i=1}^{k_1} (a + (i-1)c - 1)! \prod_{i=1}^{k_2} (p + (i - k_1 - 1)c - 1)! \prod_{r=1}^{2} \prod_{i=1}^{k_i} \frac{(ic)!}{c!}.$$

Proof. The proof is parallel to the proof of Theorem 3.4 for $n = 2$.

Denote by $\mathcal{A}^I_{\boldsymbol{k}}$ the set of all $a, b_1, b_2, c \in \mathbb{Z}_{>0}$ satisfying the assumptions of Theorem 5.9. Notice that if $(a, b_1, b_2, c) \in \mathcal{A}^I_{\boldsymbol{k}}$, then

$$b_1 \geqslant p - (a + (k_1 - 1)c), \qquad b_2 \geqslant (k_1 - k_2 + 1)c. \tag{5.21}$$

Lemma 5.10. *Formula (5.20) holds if $b_2 = (k_1 - k_2 + 1)c$ and $(a, b_1, (k_1 - k_2 + 1)c, c) \in \mathcal{A}^I_{\boldsymbol{k}}$.*

Proof. If $b_2 = (k_1 - k_2 + 1)c$, then all variables (s_j) in the integrand of $I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c)$ are used to reach the monomial $\prod_{j=1}^{k_n} s_j^{k_1 p - 1}$ in the calculation of the $\mathbb{F}_p$-integral $I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c)$. The remaining free variables (t_i) all belong to the factor

$$\Phi_{(k_1)}(t_1, \dots, t_{k_1}, a - 1, b_1, c) = \prod_{1 \leqslant i < i' \leqslant k_1} (t_i - t_{i'})^{2c} \prod_{i=1}^{k_1} t_i^{a-1} (1 - t_i)^{b_1}$$

of the integrand and are used to calculate the coefficient of the monomial $\prod_{j=1}^{k_1} t_i^{p-1}$.

More precisely, under the assumptions of the theorem we have

$$I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c) = (-1)^{b_2 k_2 + c k_2 (k_2 - 1)/2} \frac{(k_2 c)!}{(c!)^{k_2}} S_{(k_1)}(a - 1, b_1, c),$$

cf. the proof of Theorem 3.7. We have $(-1)^{b_2 k_2 + c k_2 (k_2 - 1)/2} = (-1)^{(k_1 k_2 - k_2 (k_2 + 1)/2)c - k_2}$.

By [RV, Theorem 4.1] we have $S_{(k_1)}(a - 1, b_1, c) = R_{(k_1)}(a - 1, b_1, c)$. Hence

$$I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c) = (-1)^{(k_1 k_2 - k_2 (k_2 + 1)/2)c} (-1)^{k_1 + k_2} \tag{5.22}$$
$$\times \frac{(k_2 c)!}{(c!)^{k_2}} \prod_{j=1}^{k_1} \frac{(jc)!}{c!} \frac{(a + (j-1)c - 1)!\,(b + (j-1)c)!}{(a + b + (k_1 + j - 2)c - p)!}.$$

Denote by $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$ the right-hand side in (5.20). We have

(5.23)

$$R^I_{\boldsymbol{k}}(a,b_1,(k_1-k_2+1)c,c) = (-1)^{k_1+k_2}\prod_{i=1}^{k_1-k_2}\frac{(b_1+(i-1)c)!}{(a+b_1+(i+k_1-2)c-p)!}$$
$$\times\prod_{i=1}^{k_2}\frac{((k_1-k_2+1)c+(i-1)c)!}{((k_1-k_2+1)c+(i+k_2-k_1-2)c)!}$$
$$\times\prod_{i=1}^{k_2}\frac{(b_1+(k_1-k_2+1)c+(i-2)c)!}{(a+b_1+(k_1-k_2+1)c+(i+k_1-3)c-p)!}$$
$$\times\prod_{i=1}^{k_1}(a+(i-1)c-1)!\prod_{i=1}^{k_2}(p+(i-k_1-1)c-1)!\prod_{r=1}^{2}\prod_{i=1}^{k_i}\frac{(ic)!}{c!}\,.$$

$$= (-1)^{k_1+k_2}(-1)^{(k_1k_2-k_2(k_2+1)/2)c}\,\frac{(k_2c)!}{(c!)^{k_2}}\prod_{j=1}^{k_1}\frac{(ic)!}{c!}\,\frac{(a+(j-1)c-1)!\,(b+(j-1)c)!}{(a+b+(k_1+j-2)c-p)!},$$

where we used the cancellation Lemma 2.1 in the last step. Hence
$I_{0,0,0}(a,b_1,(k_1-k_2+1)c,c) = R^I_{\boldsymbol{k}}(a,b_1,(k_1-k_2+1)c,c)$ and Lemma 5.10 is proved. □

Comparing equations (5.7), (5.8) and the formula for $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$, we conclude that the functions $I_{0,0,0}(a,b_1,b_2,c)$ and $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$ on $\mathcal{A}^I_{\boldsymbol{k}}$ satisfy the same difference equations with respect to the shifts of variables $b_1\to b_1-1$ and $b_2\to b_2-1$ and are equal if b_2 takes its minimal value $(k_1-k_2+1)c$. This implies Theorem 5.9, cf. Lemmas 5.5, 5.6 and Corollary 5.7. □

5.4. **Sketch of the proof of formula (3.11) for $n>2$.** The proof is parallel to the proof of Theorem 3.4.

Analogously to the proof of Theorem 5.1, consider the Lie algebra $\mathfrak{sl}_{n+1}$ and its representations $V^{\mathbb{F}_p}_{\lambda_1}$ and $V^{\mathbb{F}_p}_{\lambda_2}$ over $\mathbb{F}_p$ with highest weights

$$\lambda_1 = -\frac{a}{c+p}\,\omega_1\,, \qquad \lambda_2 = -\frac{b_1}{c+p}\,\omega_1-\dots-\frac{b_n}{c+p}\,\omega_n$$

and highest weight vectors v_1, v_2. Consider the PBW basis $\mathcal{B}=(u)$ of the weight subspace $V^{\mathbb{F}_p}_{\lambda_1}\otimes V^{\mathbb{F}_p}_{\lambda_2}[\lambda_1+\lambda_2-\sum_{i=1}^n k_i\sigma_i]$ like in the proof of Theorem 5.1. We distinguish two elements of that basis:

$$u_1 = \frac{f_1^{k_1-k_2}[f_1,f_2]^{k_2-k_3}\dots[f_1,[f_2,\dots,[f_{n-1},f_n]\dots]]^{k_n}v_1\otimes v_2}{(k_1-k_2)!\,(k_2-k_3)!\,\dots\,(k_n)!}\,,$$
$$u_2 = \frac{f_1^{k_1}v_1\otimes f_2^{k_2}\dots f_n^{k_n}v_2}{(k_1)!\,(k_2)!\,\dots\,(k_n)!}\,.$$

For $n=2$ these vectors are the vectors $v_{0,0,0}$ and $v_{0,k_2,0}$ in the proof of Theorem 5.1.

To any basis vector $u \in \mathcal{B}$ we assign the weight function $W_u(t)$ defined in [RSV, Section 6.1], here t is the collection of variables defined in (3.9). Then we consider the $\mathbb{F}_p$-integrals

$$I_u(a,b,c) = \int_{[\boldsymbol{k}]_p} \Phi(t,a,b,c) W_u(t)\, dt.$$

It follows from the formulas for the weight functions that

$$I_{u_2}(a,b,c) = S_{\boldsymbol{k}}(a-1, b_1, b_2-1, \dots, b_n-1, c),$$

cf. (5.4). It is known from the theory of KZ equations that the vector

$$I(a,b,c) = \sum_{u\in\mathcal{B}} I_u(a,b,c)\, u$$

is a singular vector in $V^{\mathbb{F}_p}_{\lambda_1} \otimes V^{\mathbb{F}_p}_{\lambda_2}[\lambda_1+\lambda_2 - \sum_{i=1}^n k_i\sigma_i]$. From the singular vector condition it follows that

$$I_{u_2}(a,b,c) = B_0(a,b,c)\, I_{u_1}(a,b,c), \tag{5.24}$$

where $B_0(a,b,c)$ is an explicit expression like in (5.6).

The vector $I_{u_1}(a,b,c)u_1$ is a generator of the one-dimensional weight subspace $V^{\mathbb{F}_p}_{\lambda_1}[\lambda_1+\lambda_2 - \sum_{i=1}^n k_i\sigma_i]$. That vector satisfies the dynamical equations defined in [TV3]. The dynamical equations take the form

$$I_{u_1}(a, b_1, \dots, b_i-1, \dots, b_n, c) = B_i(a,b,c) I_{u_1}(a,b,c), \qquad i=1,\dots,n, \tag{5.25}$$

where $B_i(a,b,c)$ are explicit products like in (5.7) and (5.8).

Equation (5.24) and difference equations (5.25) imply that the two functions $S_{\boldsymbol{k}}(a,b,c)$ and $R_{\boldsymbol{k}}(a,b,c)$, defined on the set $\mathcal{A}_{\boldsymbol{k}}$, satisfy the same difference equations with respect to the shift of variables $b_i \to b_i - 1$ for $i=1,\dots,n$. By Corollary 3.8 we also know that the two functions are equal at the distinguished point

$$(a, p-1-(a+(k_1-1)c), (k_1-k_2+1)c-1, \dots, (k_{n-1}-k_n+1)c-1, c) \;\in\; \mathcal{A}_{\boldsymbol{k}}\,.$$

This implies that the two functions are equal (cf. Corollary 5.7) and formula (3.11) holds for any n. The details of this sketch will be published elsewhere.

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