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Chapter 1

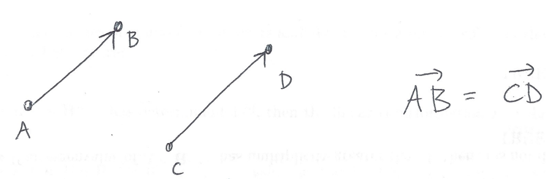
Affine and Euclidean Geometry

1.1 Points and vectors

We recall coordinate plane geometry from Calculus.

The set $\mathbb{R}^2$ will be called the plane. Elements of $\mathbb{R}^2$, that is ordered pairs (x, y) of real numbers, are called points.

Consider directed segments (also called "arrows") between points of the plane. We allow the start point and the end point of an arrow to coincide. Arrows up to translation are called (plane) vectors. That is, the arrow from $A = (1, 3)$ to $B = (5, 6)$ represents the same vector as the arrow from $C = (-4, -4)$ to $D = (0, -1)$. We write $\vec{AB} = \vec{CD}$. The vector $\vec{AA}$ is called the zero-vector and denoted by $\mathbf{0}$.



We can represent a vector by an ordered pair of real numbers as well: the vector $\vec{AB}$ where $A = (a_1, a_2)$ and $B = (b_1, b_2)$ will be represented by $\langle b_1 - a_1, b_2 - a_2 \rangle$. This is a fair definition, because if $\vec{AB} = \vec{CD}$ then $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$. The vector $\vec{AB}$ of the paragraph above is $\langle 4, 3 \rangle$.

1.2 Linear operations on vectors

The sum and difference of two vectors are defined geometrically in Figure 1.1. In this context real numbers will also be called scalars. A scalar multiple of a vector is defined in Figure 1.2.

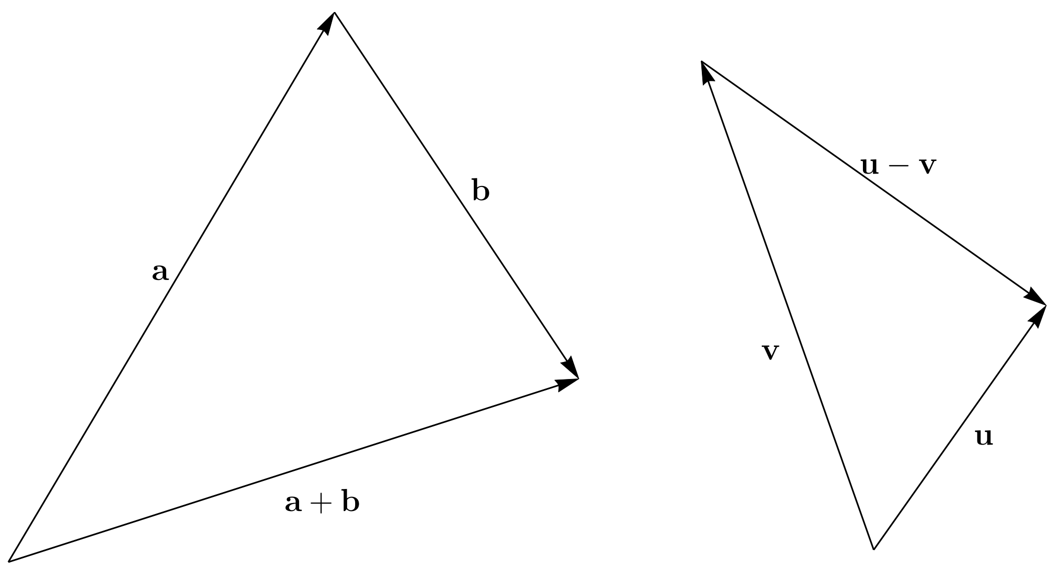


Figure 1.1: Sum and difference

The operations above (addition, subtraction, multiplication by a scalar) are called the linear operations on vectors. The geometric definitions above translate to the following algebraic expressions.

- $\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$
- $\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$
- $\lambda \cdot \langle a_1, a_2 \rangle = \langle \lambda a_1, \lambda a_2 \rangle$

Proposition 1.2.1 (Vector space "axioms")**.** *The linear operations on vectors satisfy the following properties.*

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

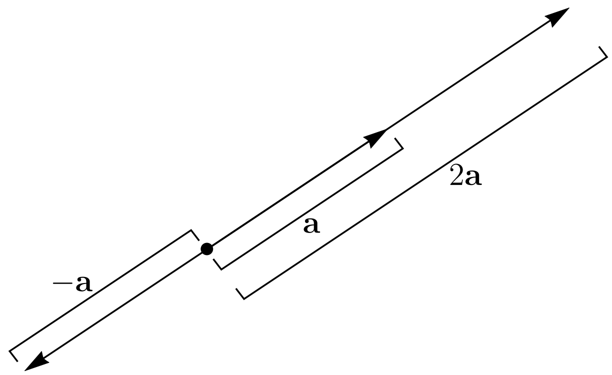


Figure 1.2: Scalar multiple

- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
- $\lambda \cdot (\mathbf{a} + \mathbf{b}) = \lambda \cdot \mathbf{a} + \lambda \cdot \mathbf{b}$
- $(\lambda + \mu) \cdot \mathbf{a} = \lambda \cdot \mathbf{a} + \mu \cdot \mathbf{a}$
- $\lambda \cdot (\mu \cdot \mathbf{a}) = (\lambda\mu) \cdot \mathbf{a}$
- $1 \cdot \mathbf{a} = \mathbf{a}$

Proof. The properties follow from the algebraic expressions for the linear operations. $\square$

Proposition 1.2.2 (2-dimensionality)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-parallel vectors (algebraically* $a_1b_2 - a_2b_1 \neq 0$*). For a vector* $\mathbf{c}$ *there are unique* λ, μ *real numbers such that* $\mathbf{c} = \lambda \cdot \mathbf{a} + \mu \cdot \mathbf{b}$.

1.3 Convention on identifying points with vectors

To a point $A \in \mathbb{R}^2$ we can associate its “position vector” $\vec{OA}$ where $O = (0, 0)$ is the origin. To a vector $\mathbf{v}$ we can associate a point P by considering an arrow $\vec{OP}$ representing $\mathbf{v}$.

The above two associations are inverses of each other, they define a one-to-one correspondence between points and vectors. Algebraically this one-to-one correspondence is $(a, b) \leftrightarrow \langle a, b\rangle$.

Throughout this text we will build in this identification in our notation, without further explanation. For example, if A is a point, and we write $5A$ then we really mean either the vector $5\vec{OA}$ or its endpoint. Or, if we say $A/2 + B/2$ is the midpoint of the segment AB then here is how to read it precisely: the midpoint of the segment AB is the endpoint of the vector $\frac{1}{2}\vec{OA} + \frac{1}{2}\vec{OB}$.

1.4 Algebraic conditions expressing collinearity

The word "collinear" is a shorthand expression for "on the same line". The word "concurrent" is a shorthand expression for "intersecting in one point".

Proposition 1.4.1. *Let A and B be two different points. Point C is on the line through A and B if and only if there is a real number t such that*

$$C = (1-t)A + tB. \tag{1.1}$$

Moreover, t and C uniquely determine each other (i.e. for any C on the AB line there is a unique real number t, and for any real number t there is a unique point C on the AB line satisfying (1.1).)

Proof. To obtain the position vector of a point C on the line AB we need to add the vector $\vec{A}$ and a multiple of $\vec{AB}$, see Figure 1.3.

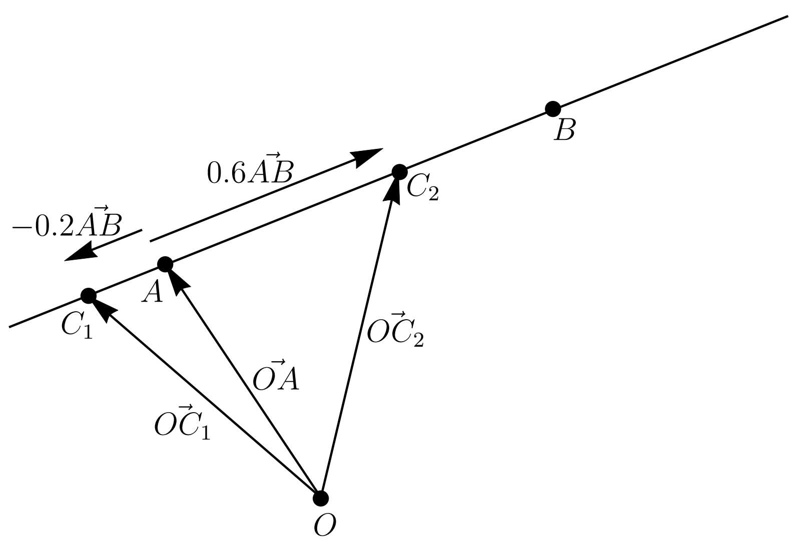


Figure 1.3: C_1 and C_2 on the line through A and B

Conversely, the end point of any such vector is obviously on the line AB. Observe that $\vec{A} + t \cdot \vec{AB} = (1-t)A + tB$, which proves the proposition.

The proof that C and t determine each other is left as an exercise. $\square$

If $A = B$ then for any t the point $(1-t)A + tB$ obviously coincides with A and B.

A useful rephrasing of Proposition 1.4.1 is that if $A \neq B$ then C is on their line if and only if there exist numbers x and y such that

$$C = xA + yB, \qquad x + y = 1.$$

Proposition 1.4.2. *The points A, B, C are collinear if and only if there exist real numbers x, y, z not all 0, such that*

$$xA + yB + zC = \mathbf{0}, \qquad\qquad x + y + z = 0.$$

Proof. Suppose A, B, C are collinear.

If A and B are different points, then C is on their line. According to Proposition 1.4.1 then there is a t such that $C = (1-t)A + tB$. After rearrangement we obtain $(1-t)A + tB + (-1)C = 0$ and hence $1-t, t, -1$ serve as x, y, z.

If $A = B$ then $x = 1, y = -1, z = 0$ satisfies the requirements.

To prove the opposite direction let us now assume that $xA+yB+zC = \mathbf{0}$, $x+y+z = 0$, and not all x, y, z are 0. Then pick one non-zero among x, y, z. Without loss of generality we may assume it is z. Rearrangement gives $C = (-x/z)A + (-y/z)B$. The condition $x + y + z = 0$ translates to $(-x/z) + (-y/z) = 1$. If A and B are different then Proposition 1.4.1 implies that C is on the AB line. If A and B coincide then the remark after Proposition 1.4.1 implies that all A, B, C are the same point, so they are collinear. $\square$

An important logical consequence of Proposition 1.4.2 is the following Corollary.

Corollary 1.4.3. *If A, B, C are not collinear (i.e. they form a triangle), and x, y, z are real numbers with*

$$xA + yB + zC = \mathbf{0}, \qquad\qquad x + y + z = 0,$$

then $x = y = z = 0$.

1.5 The ratio $(\vec{AC} : \vec{CB})$ for collinear points $A \neq B, C$

Let $A, B \neq C$ be collinear points. We will write that $(\vec{AC} : \vec{CB}) = \lambda$ if $\vec{AC} = \lambda\vec{CB}$. Such a λ exists (and is unique) since $\vec{AC}$ and $\vec{CB}$ are collinear vectors with $\vec{CB} \neq \mathbf{0}$. Sometimes it is useful to extend this notion to the case when $B = C$ (but A is not equal to them): in this case we define $(\vec{AC} : \vec{CB}) = \infty$.

Lemma 1.5.1. *Let A, B, C be collinear, $A \neq B$, and write $C = xA + yB$ with $x + y = 1$ (cf. Proposition 1.4.1). Then we have $(\vec{AC} : \vec{CB}) = y/x$.*

Proof. From $C = xA + yB$ we obtain $\vec{AC} = y\vec{AB}$ and $\vec{CB} = x\vec{AB}$. Hence $(\vec{AC} : \vec{CB}) = y/x$ (note the $(\vec{AC} : \vec{CB}) = \infty$ convention if $C = B$, that is, if $x = 0$). □

In fact we can interpret the ratio $(\vec{AC} : \vec{CB})$ without mentioning vectors. It is the ratio of the length of the segment AC over the length of the segment CB, *with a sign convention.* The sign convention is that if C is in between A and B, then $(\vec{AC} : \vec{CB})$ is positive, and if C is outside of the segment AB then $(\vec{AC} : \vec{CB})$ is negative.

Proposition 1.5.2. *Let $A \neq B$ be fixed. The ratio $(\vec{AC} : \vec{CB})$ uniquely determines C.*

Remark 1.5.3. We may be sloppy in notation and decide to write $\vec{AC}/\vec{CB}$ instead of $(\vec{AC} : \vec{CB})$, but we must be careful that this ratio is only defined in the very special situation where A, B, C are collinear (and some coincidences do not happen). In general there is no such operation where we divide a plane vector by another plane vector!

1.6 First applications

A quadrilateral $ABCD$ is a parallelogram if $\vec{AB} = \vec{DC}$. This condition can be phrased as $B - A = C - D$, or rearranged to $D - A = C - B$, which means $\vec{AD} = \vec{BC}$ also holds.

Proposition 1.6.1. *The diagonals of a parallelogram bisect each other.*

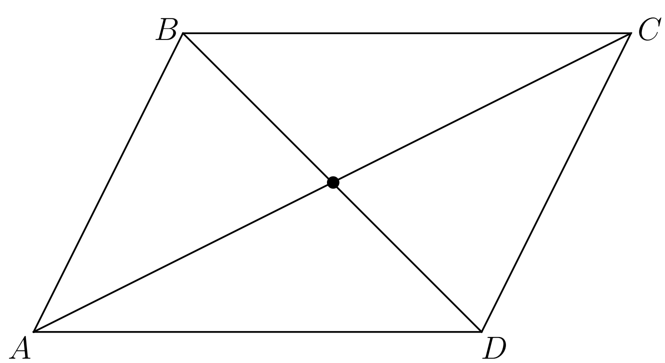


Figure 1.4: The diagonals of a parallelograph bisect each other.

Proof. Let $ABCD$ be a parallelogram. Since it is a parallelogram, we have $\vec{AB} = \vec{DC}$ (denote this vector by $\mathbf{x}$), $\vec{AD} = \vec{BC}$. These equalities imply that $(A + C)/2 = (B + D)/2$. Indeed, $B = A + \mathbf{x}$, $D = C - \mathbf{x}$, and hence $(B + D)/2 = ((A + \mathbf{x}) + (C - \mathbf{x}))/2$.

Now consider the point $P = (A+C)/2 = (B+D)/2$. The first defining expression implies that P is the midpoint of A and C. The second expression implies that P is the midpoint of B and D. Since they agree, P is the intersection of AC and BD, and it bisects both diagonals. □

The median of a triangle is a segment connecting a vertex to the midpoint of the opposite side. A triangle has three medians.

Proposition 1.6.2. *The medians of a triangle are concurrent. Moreover they divide each other by 2:1.*

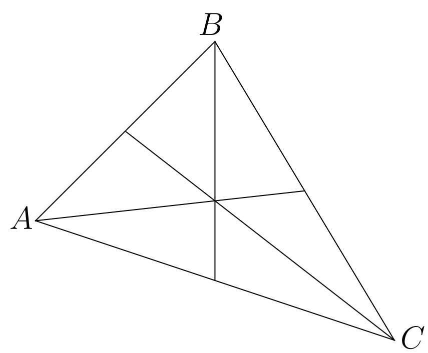


Figure 1.5: The medians of a triangle are concurrent.

Proof. Let ABC be a triangle. Consider the point $P = (A+B+C)/3$, and its equivalent expressions

$$P = \frac{2}{3} \cdot \frac{A+B}{2} + \frac{1}{3} \cdot C = \frac{2}{3} \cdot \frac{B+C}{2} + \frac{1}{3} \cdot A = \frac{2}{3} \cdot \frac{C+A}{2} + \frac{1}{3} \cdot B.$$

The first expression claims that P is on the segment connecting the midpoint of A and B with C, that is, on the median corresponding to C. The second expression claims that P is on the median corresponding to A, and the third expression claims that P is on the median corresponding to B. Since they are all equal, there is a point, namely P, that in on all three medians; and we proved that the medians are concurrent.

A byproduct of the above argument is that the intersection P of the three medians is expressed as $2/3$ of the midpoint of a side plus $1/3$ the opposite vertex. According to Section 1.5 this proves that P cuts the median $2:1$. □

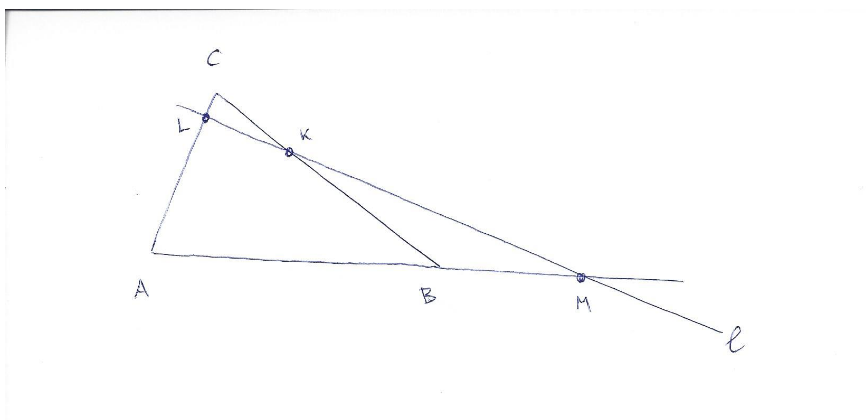
Remark 1.6.3. In the above two propositions we needed to make arguments about the intersections of certain lines. In our proofs we used a trick: we did not "compute" the intersections, but rather we "named" a point and then proved that this point is on the lines, and hence this point must be the intersection. You will find this trick useful when solving exercises.

PROJECT 1. *Invent and prove the* $3D, 4D, \ldots$ *versions of Proposition 1.6.2.*

1.7 Menelaus' theorem

Theorem 1.7.1 (Menelaus' theorem)**.** *Let* ABC *be a triangle and let a transversal line* ℓ *intersect the lines of the sides* AB, BC, CA *in* M, K, L, *respectively. We assume that none of* K, L, M *coincide with* $A, B,$ *or* C. *Then*

$$(\vec{AM} : \vec{MB}) \cdot (\vec{BK} : \vec{KC}) \cdot (\vec{CL} : \vec{LA}) = -1.$$



Proof. We have $K = xB + x'C$, $L = yC + y'A$, $M = zA + z'B$ with $x + x' = y + y' = z + z' = 1$. Since K, L, M are collinear, according to Proposition 1.4.2 we know that there are real numbers p, q, r not all 0 such that $p + q + r = 0$ and $pK + qL + rM = 0$. We have

$$p(xB + x'C) + q(yC + y'A) + r(zA + z'B) = 0,$$

or rearranged, we have

$$(rz + qy')A + (px + rz')B + (qy + px')C = 0. \tag{1.2}$$

The sum of the coefficients in this last expression is

$$(rz + qy') + (px + rz') + (qy + px') = p(x + x') + q(y + y') + r(z + z') = p + q + r = 0. \tag{1.3}$$

According to Corollary 1.4.3 (1.2) and (1.3) can only hold if all three

$$rz + qy' = px + rz' = qy + px' = 0.$$

Therefore we have

$$y'/z = -r/q, \qquad z'/x = -p/r, \qquad x'/y = -q/p,$$

and hence

$$(\vec{AM} : \vec{MB}) \cdot (\vec{BK} : \vec{KC}) \cdot (\vec{CL} : \vec{LA}) = \frac{z'}{z} \cdot \frac{x'}{x} \cdot \frac{y'}{y} = \frac{z'}{x} \cdot \frac{x'}{y} \cdot \frac{y'}{z} = \frac{-p}{r} \cdot \frac{-q}{p} \cdot \frac{-r}{q} = -1$$

what we wanted to prove. □

Theorem 1.7.2 (reverse Menelaus' theorem)**.** *Let ABC be a triangle and let M, K, L be points on the lines AB, BC, CA such that*

$$(\vec{AM} : \vec{MB}) \cdot (\vec{BK} : \vec{KC}) \cdot (\vec{CL} : \vec{LA}) = -1. \tag{1.4}$$

Then K, L, M are collinear.

Proof. Let ℓ be the line connecting L and K. In Problem 15 (Section 1.21) you will prove that ℓ intersects the AB line. Let the intersection point be M'. According to Menelaus' theorem we have

$$(\vec{AM'} : \vec{M'B}) \cdot (\vec{BK} : \vec{KC}) \cdot (\vec{CL} : \vec{LA}) = -1.$$

Comparing this with the assumption (1.4) on K, L, M we conclude that

$$(\vec{AM'} : \vec{M'B}) = (\vec{AM} : \vec{MB}).$$

Proposition 1.5.2 then implies that $M = M'$, hence the fact that K, L, M are collinear. □

1.8 Barycentric coordinates

Theorem 1.8.1. *Let A, B and C be non-collinear points in the plane. For any point P we may write*

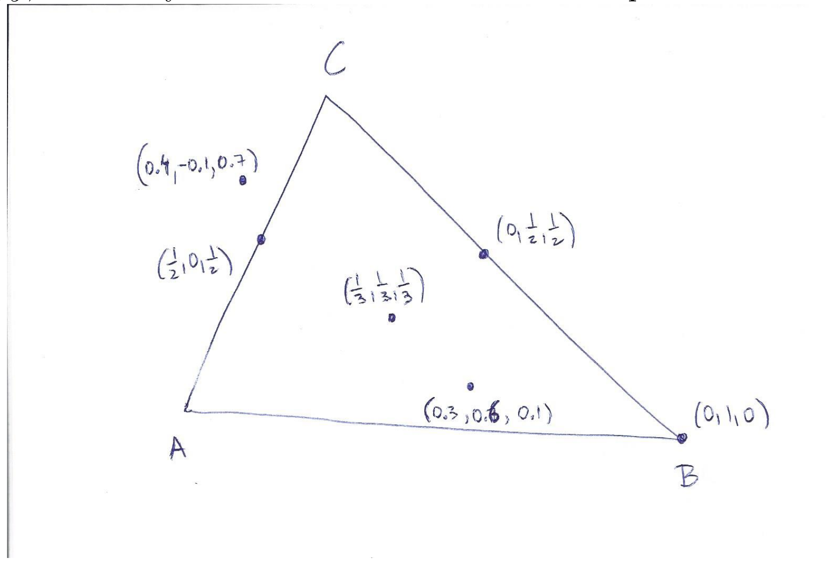
$$P = xA + yB + zC$$

where the real coefficients x, y, z satisfy

$$x + y + z = 1.$$

Moreover, x, y, z are uniquely determined by the point P.

We call x, y, z the barycentric coordinates of P with respect to the triangle ABC.



Proof. The vectors $\vec{AB}$ and $\vec{AC}$ are not parallel. Hence any vector can be written as a linear combination of them, for example $\vec{AP} = p\vec{AB} + q\vec{AC}$. Using that $\vec{AP} = P - A$, $\vec{AB} = B - A$, $\vec{AC} = C - A$ we can rearrange it to

$$P = (1 - p - q)A + pB + qC,$$

and hence $x = 1 - p - q, y = p, z = q$ satisfy the requirements.

To prove the uniqueness of barycentric coordinates assume that x, y, z and x', y', z' are such that

$$P = xA + yB + zC, \qquad x + y + z = 1,$$
$$P = x'A + y'B + z'C, \qquad x' + y' + z' = 1.$$

Then we have

$$\mathbf{0} = (x - x')A + (y - y')B + (z - z')C, \qquad (x - x') + (y - y') + (z - z') = 0.$$

Corollary 1.4.3 implies that $x - x' = y - y' = z - z' = 0$ which proves uniqueness. □

PROJECT 2. *Observe the similarity between Proposition 1.4.1 and Theorem 1.8.1. They are the 1D and 2D cases of a general n-dimensional theorem. If you learned linear algebra (specifically the notions of linear independence, generating set, basis) then find and prove this general n-dimensional theorem.*

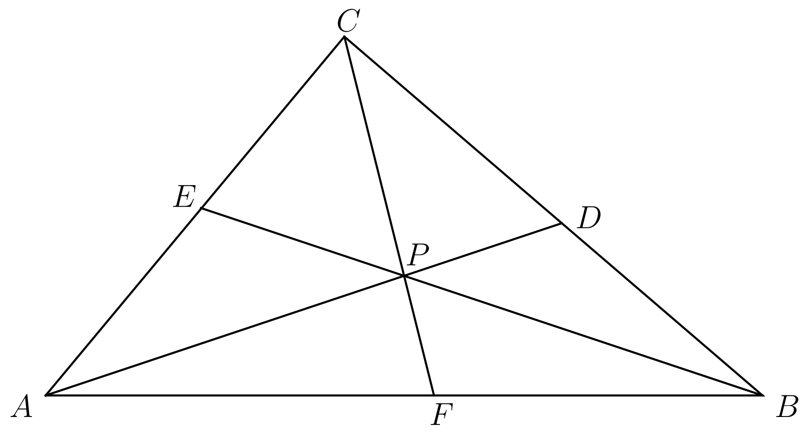


Figure 1.6: The Ceva configuration

1.9 Ceva's theorem

Theorem 1.9.1 (Ceva's Theorem[1]). *Let ABC be a triangle and let P be a point in the plane which does not lie on any of the sides of $\triangle ABC$. Suppose the lines AP, BP and CP meet the opposite sides of ABC at D, E and F, respectively. Then*

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = 1.$$

Note that P does not need to lie inside the triangle.

Proof. Using barycentric coordinates, we write P as

$$P = xA + yB + zC,$$

where $x + y + z = 1$. Let us consider the point

$$V = \frac{x}{x+y}A + \frac{y}{x+y}B. \tag{1.5}$$

This expression implies that V is on the line AB. Calculation shows that

$$V = \frac{1}{x+y}P + \frac{-z}{x+y}C,$$

[1]Geometer Giovanni Ceva (1647–1734) is credited with this theorem

and the sum of the coefficients $1/(x+y)+(-z)/(x+y) = (1-z)/(x+y) = (x+y)/(x+y) = 1$. Hence V is also on the line CP. We conclude that the point V is the intersection of AB and CP, hence $V = F$. Moreover, from 1.5 we obtain that

$$(\vec{AF} : \vec{FB}) = \frac{\frac{y}{x+y}}{\frac{x}{x+y}} = \frac{y}{x}.$$

Similarly, we find that

$$(\vec{BD} : \vec{DC}) = \frac{z}{y}, \tag{1.6}$$

$$(\vec{CE} : \vec{EA}) = \frac{x}{z}. \tag{1.7}$$

Hence

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = \frac{y}{x}\frac{z}{y}\frac{x}{z} = 1.$$

□

Theorem 1.9.2 (Reverse Ceva's theorem). *Suppose ABC is a triangle, D, E, F are points on the lines of the sides (but none of them coincide with a vertex) such that*

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = 1.$$

Then AD, BE, CF are either concurrent, or they are pairwise parallel.

Proof. If AD, BE, CF are pairwise parallel, then the theorem is proved. Assume that two of these three lines intersect. Without loss of generality we assume that it is AD and BE. Let $P = AD \cap BE$, and assume that CF intersects AB is the point F'. By Ceva's Theorem 1.9.1 we have

$$(\vec{AF'} : \vec{F'B}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = 1.$$

Comparing this with the condition in the Theorem we obtain that

$$(\vec{AF'} : \vec{F'B}) = (\vec{AF} : \vec{FB}).$$

Proposition 1.5.2 then implies that $F = F'$, hence the fact that AD, BE, CF are concurrent. □

For fun, let us include here another "high-school style" proof of Ceva's Theorem 1.9.1. This proof does not use vectors at all. Instead it uses the notion of area, and the obvious fact that the area of a triangle is half the product of base and height.

Proof. For simplicity let D, E, F be on the sides (and not outside) of the triangle ABC, and let $P = AB \cap CF = BC \cap AD = CA \cap BE$. The triangles AFC and FBC have "bases" AF and FB and they share the same height m_C. Hence the ratio of their areas equals the ratio of their bases:

$$\frac{\text{Area}(AFC)}{\text{Area}(FBC)} = \frac{AF}{FB}.$$

Similar argument for the triangles AFP and FBP gives

$$\frac{\text{Area}(AFP)}{\text{Area}(FBP)} = \frac{AF}{FB}.$$

From the two equations above simple algebra implies

$$\frac{\text{Area}(AFC) - \text{Area}(AFP)}{\text{Area}(FBC) - \text{Area}(FBP)} = \frac{AF}{FB}.$$

The difference of the triangles AFC and AFP is the triangle CAP. The difference of the triangles FBC and FBP is the triangle BCP. Hence we obtained

$$\frac{\text{Area}(CAP)}{\text{Area}(BCP)} = \frac{AF}{FB}. \tag{1.8}$$

We obtained (1.8) by considering the AB side of the triangle the "base". Repeating the same argument but now considering the BC side or the CA side to be the "base" we obtain the equations.

$$\frac{\text{Area}(ABP)}{\text{Area}(CAP)} = \frac{BD}{DC},$$

$$\frac{\text{Area}(BCP)}{\text{Area}(ABP)} = \frac{CE}{EA}.$$

From the last three equations we obtain

$$(\vec{AF} : \vec{FB}) \cdot (\vec{BD} : \vec{DC}) \cdot (\vec{CE} : \vec{EA}) = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} =$$

$$= \frac{\text{Area}(CAP)}{\text{Area}(BCP)} \cdot \frac{\text{Area}(ABP)}{\text{Area}(CAP)} \cdot \frac{\text{Area}(BCP)}{\text{Area}(ABP)} = 1,$$

which proves Ceva's theorem in the case when P is inside the triangle. Similar arguments work when P is outside. □

1.10 Desargues' theorem—a few affine versions

Desargues' theorem is a remarkable theorem on incidences of certain lines and points involving two triangles. The key notions are as follows.

- For $ABC\triangle$ and $A'B'C'\triangle$ we may consider the three lines connecting the corresponding vertexes: AA', BB', and CC'. We will consider the condition that these three lines are concurrent (or are pairwise parallel). If concurrent, we call the intersection point the center of perspectivity.
- For $ABC\triangle$ and $A'B'C'\triangle$ we may consider the intersections of the corresponding sides $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$. We will consider the condition that these three points (exist and) are collinear—or none of the three exist. If collinear, we call the obtained line the axis of perspectivity.

Theorem 1.10.1. *Let ABC and $A'B'C'$ be triangles such that $AB||A'B'$, $AC||A'C'$, $BC||B'C'$. Then the three lines AA', BB', CC' are either concurrent or pairwise parallel.*

Proof. Because of the conditions on parallel lines we can write

$$B - C = k_1(B' - C'), \qquad C - A = k_2(C' - A'), \qquad A - B = k_3(A' - B'), \tag{1.9}$$

for some real numbers k_1, k_2, k_3. Adding together these three equalities (and rearranging the right hand side) we obtain

$$\mathbf{0} = (k_2 - k_3)A' + (k_3 - k_1)B' + (k_1 - k_2)C'. \tag{1.10}$$

The coefficients of (1.10) add up to 0, hence Corollary 1.4.3 implies that

$$k_2 - k_3 = k_3 - k_1 = k_1 - k_2 = 0, \qquad \text{and hence} \qquad k_1 = k_2 = k_3.$$

Let k be the common value of k_1, k_2 and k_3. Then from (1.9) we can deduce

$$A - kA' = B - kB' = C - kC'. \tag{1.11}$$

We can consider two cases. If $k = 1$ then (1.11) implies that the lines AA', BB', CC' are pairwise parallel. If $k \neq 1$ then (1.11) can be rearranged to

$$P = \frac{1}{1-k}A + \frac{-k}{1-k}A' = \frac{1}{1-k}B + \frac{-k}{1-k}B' = \frac{1}{1-k}C + \frac{-k}{1-k}C'$$

showing that the point P is on all three lines AA', BB', CC'—proving that AA', BB', CC' are concurrent. □

Theorem 1.10.2. *Let ABC and $A'B'C'$ be triangles such that AA', BB', CC' are concurrent. Assume that the point $K = AB \cap A'B$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist. Then the points K, L, M are collinear.*

Proof. The intersection point on AA', BB', CC' can be written as

$$k_1 A + (1-k_1)A' = k_2 B + (1-k_2)B' = k_3 C + (1-k_3)C'. \tag{1.12}$$

Rearranging, say, the first equality we obtain

$$k_1 A - k_2 B = (1-k_2)B' - (1-k_1)A'.$$

If $k_1 = k_2$ then this formula says $k_1(A-B) = (1-k_1)(B'-A')$, meaning that $AB||A'B'$ which is not the case. So we know that $k_1 \neq k_2$. Hence, we may divide by $k_1 - k_2$ and write

$$\frac{k_1}{k_1-k_2}A + \frac{-k_2}{k_1-k_2}B = \frac{1-k_2}{k_1-k_2}B' + \frac{-(1-k_1)}{k_1-k_2}A'.$$

The sum of the coefficients on the left hand side is 1, and the sum of the coefficients on the right hand side is also 1 (check it!). Therefore the left hand side expression is a point on the AB line, and the right hand side expression is a point on the $A'B'$ line. Hence the common value must be the intersection $AB \cap A'B'$. We obtained that

$$K = \frac{k_1}{k_1-k_2}A + \frac{-k_2}{k_1-k_2}B,$$

equivalently

$$(k_1 - k_2)K = k_1 A - k_2 B. \tag{1.13}$$

We deduced (1.13) from the fact that the first expression and the second expression in (1.12) are equal. Similarly, the fact that the second and third, as well as the first and third expressions in (1.12) are equal we obtain

$$(k_2 - k_3)L = k_2 B - k_3 C, \qquad (k_3 - k_1)M = k_3 C - k_1 A. \tag{1.14}$$

Adding together all three equalities in (1.13) and (1.14) we get

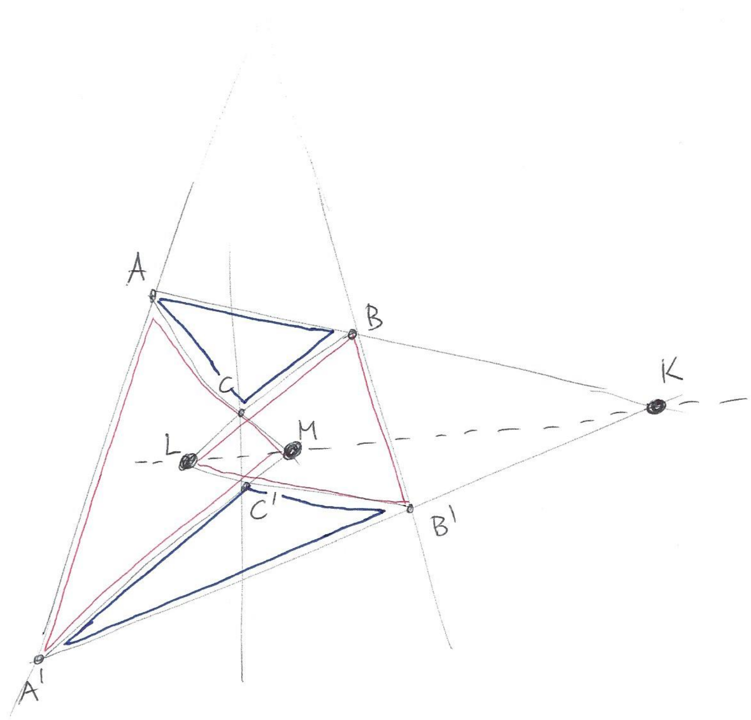
$$\mathbf{0} = (k_1 - k_2)K + (k_2 - k_3)L + (k_3 - k_1)M.$$

Observe that none of the three coefficients are 0, and they add up to 0. According to Proposition 1.4.2 this means that K, L, M are collinear, what we wanted to prove. □

Theorem 1.10.3. *Let ABC and $A'B'C'$ be triangles such that $K = AB \cap A'B'$, $L = BC \cap B'C'$, $M = CA \cap C'A'$ exist and are collinear. Then AA', BB', CC' are either concurrent or are pairwise parallel.*

It is possible to prove this theorem with the techniques we used in the last two proofs—and it may be a good practice for students to write down such a proof. However, for a change we are going to prove it by reduction to Theorem 1.10.2—showing that in some sense Desargues' theorem and its reverse are the same, in other words, Desargues' theorem is "self-dual".

Proof. If AA', BB', CC' are pairwise parallel, then we are done. If not, then two of them intersect, say, AA' intersects BB'.



Consider the triangles $AA'M$ and $BB'L$. By looking at the picture one can see that the lines AB, $A'B'$, ML connecting the corresponding vertexes are concurrent. Theorem 1.10.2 can be applied to the triangles $AA'M$ and $BB'L$, and we obtain that $C = AM \cap BL$, $C' = MA' \cap LB'$, and $AA' \cap BB'$ are collinear. That is, CC' passes through the intersection of $AA' \cap BB'$, and hence AA', BB', and CC' are concurrent. $\square$

In a later chapter we will see a simple and elegant way of phrasing Desargues' theorem—

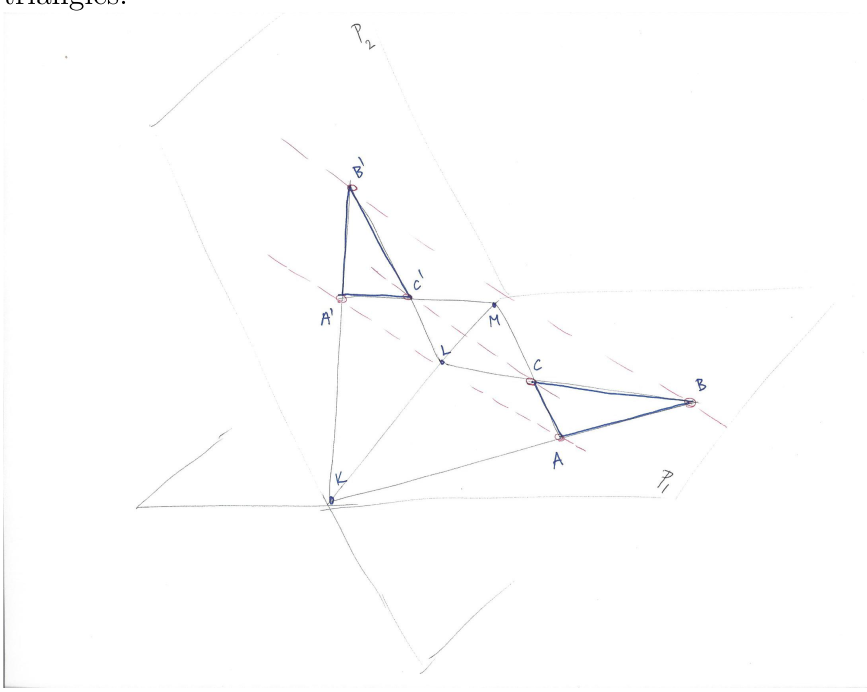
in projective geometry. All of the three theorems above (and more) are some special cases of that projective Desargues' theorem.

PROJECT 3. *We can connect two points A and B of the plane if we have a straightedge. Now suppose that B is "hidden", it is only given by portions of two intersecting lines, but we cannot go close to the intersection point B; for example it is outside of the margin of our paper. How can we connect A and B with a straightedge? How to connect two hidden points?*

1.11 Desargues triangles in intersecting planes

In this section we go out of our way again and show an interesting "high-school style" argument in relation with Desargues' theorem.

Consider two planes P_1 and P_2 in 3 dimensions, intersecting in the line ℓ. Let $ABC\triangle$ be in P_1 and let $A'B'C'\triangle$ be in P_2. We will analyse the conditions and the claim of Theorem 1.10.3 for these two triangles.

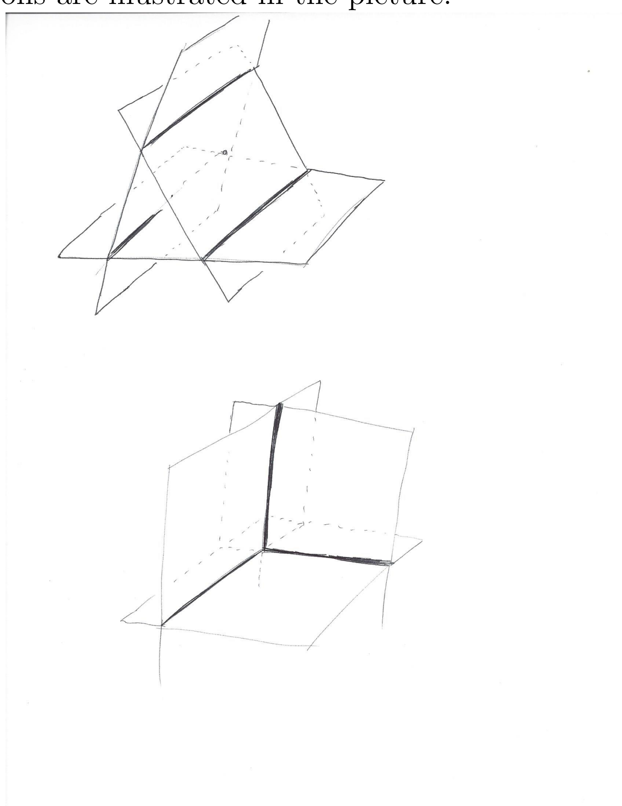


The condition is about the three points $K = AB \cap A'B$, $L = BC \cap B'C'$, $M = CA \cap C'A'$. Observe that $AB \subset P_1$, $A'B' \subset P_2$, hence $K \in P_1 \cap P_2 = \ell$. Similarly, L and M must also lie on ℓ. So the assumption of Theorem 1.10.3 that K, L, M are collinear does not even have to be assumed! It automatically holds.

The statement of Theorem 1.10.3 is about the three lines AA', BB', CC'. It will be useful to consider three more planes. Let S_K be the plane containing the intersecting lines AB, $A'B'$. Let S_L be the plane containing the intersecting lines BC, $B'C'$. Let S_M be the plane containing the intersecting lines AC, $A'C'$.

Observe that both A and A' are contained in S_K and in S_M. If two point are contained in two planes, then their connecting line must be the intersection of the two planes. We have $AA' = S_K \cap S_M$. Similarly $BB' = S_K \cap S_L$, and $CC' = S_L \cap S_M$.

We obtained that the three lines AA', BB', CC' are the pairwise intersections of three planes in space. Let's see how can three (pairwise intersecting) planes look like in three space. There are two possibilities: (i) either the third one is parallel with the intersection line of the first two, or (ii) the third one intersects the intersection line of the first two. Theses two configurations are illustrated in the picture.



In the first case the three intersection lines are pairwise parallel: AA', BB', CC' are pairwise parallel. In the second case the three intersection lines are concurrent: AA', BB', CC' are concurrent.

What we found is that the 3D version of Desargues theorem 1.10.3 is a tautology.

PROJECT 4. *Analyses 3D versions of the other two versions of Desargues' theorem above.*

PROJECT 5. *Find a no-calculation proof of e.g. Theorem 1.10.3, by first moving one of the triangles out of plane into 3D.*

1.12 Dot product: algebra and geometry

Let us recall the notion of dot product from Calculus. The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a number denoted by $\mathbf{a} \cdot \mathbf{b}$ or $\mathbf{ab}$.

Geometrically $\mathbf{ab} = |\mathbf{a}||\mathbf{b}| \cos \phi$, where $|\mathbf{x}|$ denotes the length of a vector $\mathbf{x}$ and ϕ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Especially, $\mathbf{ab} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Algebraically $(a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2$.
Problem: recall from calculus why the above geomteric and algebraic definition agree.

The following properties are easily verified from the algebraic definition.

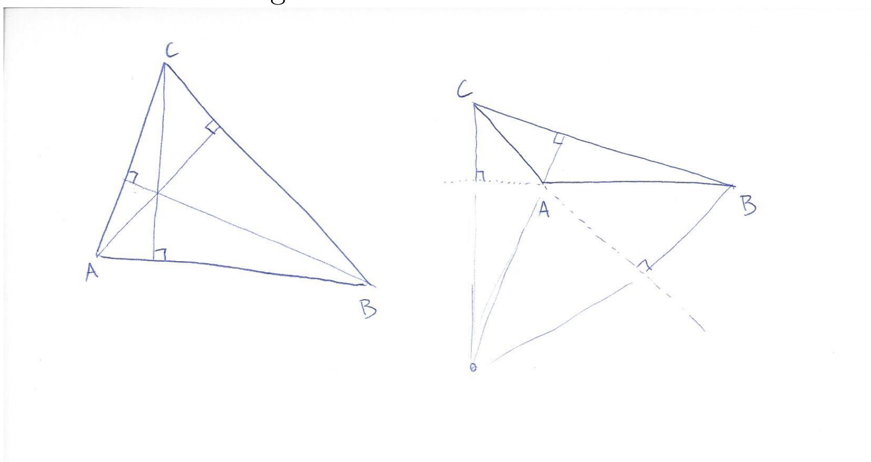
- $\mathbf{ab} = \mathbf{ba}$
- $(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}$
- $(\lambda \mathbf{a})\mathbf{b} = \lambda(\mathbf{ab})$
- $\mathbf{aa} \geq 0$, and $\mathbf{aa} = 0$ if and only if $\mathbf{a} = \mathbf{0}$
- $\mathbf{ab} = 0$ for all $\mathbf{b}$ implies that $\mathbf{a} = 0$.

The power of dot product that we will repeatedly use in geometry is the duality: its clear geometric meaning and its simple algebraic properties. (What we will not use any further is the $a_1 b_1 + a_2 b_2$ expression.)

Definition 1.12.1. The length of a vector $\mathbf{a}$ is defined to be $|\mathbf{a}| = \sqrt{\mathbf{aa}}$. (The square root makes sense because of the the non-negativity property above. Also this definition is consistent with the geometric interpretation of dot product above.) The length of a segment AB is defined to be $d(AB) = |\vec{AB}|$. The distance of two sets $P, Q \subset \mathbb{R}^2$ is defined to be $\inf\{d(A, B) : A \in P, B \in Q\}$.

1.13 Altitudes of a triangle are concurrent

Let ABC be a triangle. A line passing through the vertex and perpendicular to the opposite side is called an altitude. A triangle has three altitudes.



Theorem 1.13.1. *The three altitudes of a triangle are concurrent.*

Proof. Let D be the intersection of the altitudes containing the vertexes A and B. Then $AD \perp BC$ and $BD \perp AC$. Hence we have

$$(D-A)(B-C)=0, \qquad (D-B)(C-A)=0.$$

Adding these two equations together, and using the algebraic properties of dot product we obtain

$$0=(D-A)(B-C)+(D-B)(C-A)=...=-(D-C)(A-B).$$

Therefore $D-C \perp A-B$, that is, the line DC is the altitude containing C. All three altitudes pass through D. □

The intersection of the three altitudes is called the orthocenter of the triangle.

Consider the vertexes and the orthocenter. It is remarkable that each of theses four points is the orthocenter of the triangle formed by the other three points. Such a set of four points will be called an orthocentric tetrad.

1.14 Feuerbach circle

Lemma 1.14.1. *If A,B,C,D is an orthocentric tetrad then*

$$(A+B-C-D)^2=(A-B+C-D)^2=(A-B-C+D)^2=$$

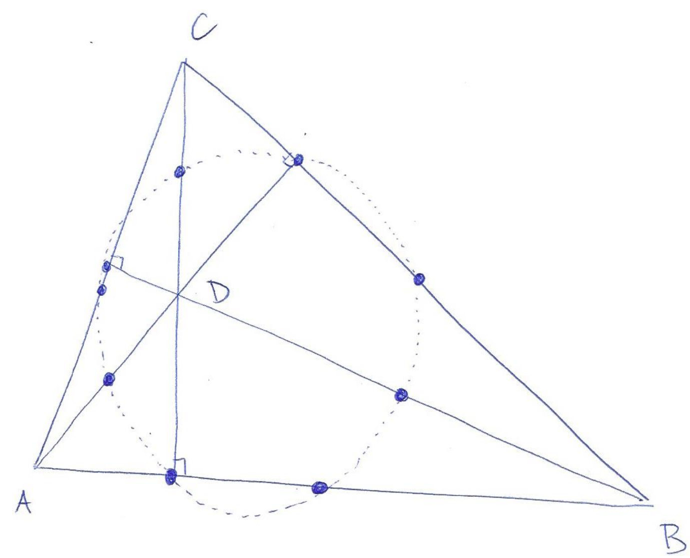
$$(-A+B+C-D)^2 = (-A+B-C+D)^2 = (-A-B+C+D)^2.$$

Proof. The six numbers above are in fact three pairs, e.g. $(A+B-C-D)^2$ and $(-A-B+C+D)^2$ are clearly equal, because they are length squares of a vector and its opposite vector. What we need to prove is that two numbers not in the same pair are also equal. Without loss of generality let us choose the first two. Calculation shows that

$$(A+B-C-D)^2 - (A-B+C-D)^2 = 4(A-D)(B-C).$$

Since A, B, C, D is an orthocentric tetrad $A-D$ is orthogonal to $B-C$, and hence $(A-D)(B-C) = 0$, showing that $(A+B-C-D)^2 = (A-B+C-D)^2$. □

Theorem 1.14.2 (Feuerbach circle)**.** *Let D be the orthocenter of the ABC triangle. Consider the following nine points (a) the midpoints of the sides, (b) the midpoints of the segments connecting vertexes to the orthocenter, (c) the feet of the altitudes. These nine points are on one circle.*



Proof. We will only prove that points (a) and (b) are on one circle. The fact that points (c) are also on the same circle is left as an exercise.

Observe that the six points in (a) and (b) are the midpoints of the six segments connecting two of A, B, C, D where A, B, C, D form an orthocentric tetrad. Hence these points are

$$(A+B)/2, (A+C)/2, (A+D)/2, (B+C)/2, (B+D)/2, (C+D)/2.$$

Let $N = (A + B + C + D)/4$. The vectors connecting N to the six points are

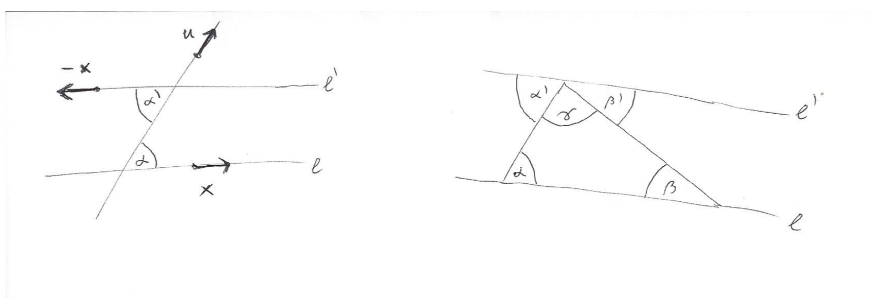
$$(-A - B + C + D)/4, (-A + B - C + D)/4, (-A + B + C - D)/4,$$

$$(A - B - C + D)/4, (A - B + C - D)/4, (A + B - C - D)/4.$$

These six vectors have the same length because of Lemma 1.14.1. Therefore all six points are of the same distance from N: they are on one circle. □

1.15 Angle sum of a triangle

Lemma 1.15.1. *Let the line m intersect a pair of parallel lines ℓ, ℓ'; and let α and α' be the angles obtained as in Figure ? (a). Then $\alpha = \alpha'$.*



Proof. Take unit vectors $\mathbf{x}$, $-\mathbf{x}$ and $\mathbf{u}$ in the lines ℓ, ℓ', and m as in the picture. Then

$$\alpha = \arccos(x \cdot u), \qquad \alpha' = \arccos((-x) \cdot (-u)),$$

so they obviously agree. □

Theorem 1.15.2. *The sum of the angles of a triangle is π.*

Proof. Let the line ℓ' be parallel to AB and pass through the point C, see Picture ? (b). According to Lemma 1.15.1 $\alpha = \alpha'$ and $\beta = \beta'$ hence we have

$$\alpha + \beta + \gamma = \alpha' + \beta' + \gamma = \pi.$$

□

1.16 Law of cosines, law of sines

For a triangle $ABC\triangle$ the sides opposite to A, B, C will be denoted by a, b, c respectively, and the angles at A, B, C will be called α, β, γ respectively.

Theorem 1.16.1 (Law of Cosines)**.** *We have*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Proof. We have $c^2 = \mathbf{c}^2 = (\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{ab} = a^2 + b^2 - 2ab\cos\gamma$. □

Corollary 1.16.2 (Pythagorean theorem)**.** *In a right triangle with hypothenuse c we have $c^2 = a^2 + b^2$.*

Proof. This is the Law of Cosines for $\gamma = \pi/2$. □

Corollary 1.16.3 (Triangle inequality)**.** *For three points A, B, C in the plane we have*

$$d(AB) \leq d(BC) + d(CA).$$

The proof follows from the Law of Cosines, details are left as an exercise.

Theorem 1.16.4 (Law of Sines)**.** *We have*

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

We will give two proves: the first one shows that the Law of Sines is a formal consequence of the Law of Cosines. The second proof is geometric.

Proof. Proof1. From the Law of Cosines we get $\cos\gamma = (a^2 + b^2 - c^2)/(2ab)$. Using $\sin\gamma = \sqrt{1 - \cos^2\gamma}$ we have

$$\sin\gamma = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} =$$

$$\sqrt{\frac{4a^2b^2 - (a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2)}{4a^2b^2}} = \frac{\sqrt{-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2}}{2ab}.$$

Dividing both sides by c we obtain

$$\frac{\sin\gamma}{c} = \text{an expression symmetric in } a, b, c.$$

Therefore we will get the *same* expression on the right hand side, if we start with α or β, not γ. This proves that $\sin\gamma/c = \sin\alpha/a = \sin\beta/b$. □

Proof. Proof2. The altitude m passing through vertex C is a side of two right triangles (See picture) yielding the two expressions: $m = b\sin\alpha$, $m = a\sin\beta$. Putting the right hand sides equal and rearranging gives $\sin\alpha/a = \sin\beta/b$. □

1.17 Angle bisectors, perpendicular bisectors

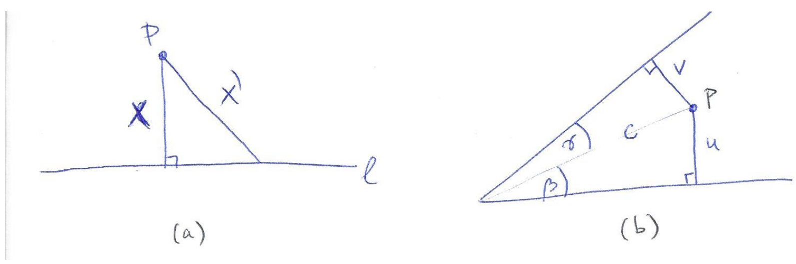
Consider an angle α less than π. The ray inside the angle that cuts α into two angles of measure $\alpha/2$ is called the angle bisector.

The distance $d(P, \ell)$ of a point P to a line ℓ is the the infimum of the distances between P and A where $A \in \ell$.

Lemma 1.17.1. *The distance of a point to a line is obtained on the perpendicular segment dropped from the point to the line.*

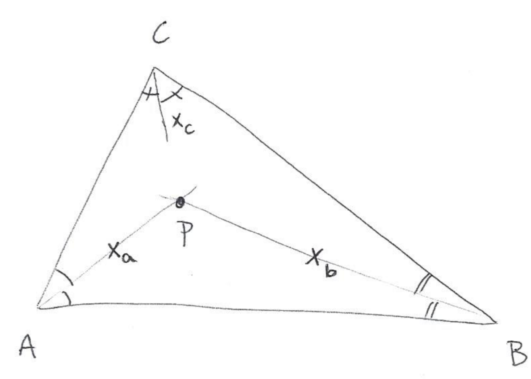
Proof. The Pythagorean theorem proves that $x < x'$ on picture ? (a). $\square$

Lemma 1.17.2. *The points in an angle that are of the same distance from the two rays of the angle are exactly the points of the angle bisector.*



Proof. For a point P as in Picture ? (b) its distance to the two sides is u and v according to Lemma 1.17.1. We have $u = c\sin(\beta)$, $v = c\sin(\gamma)$. Hence $u = v$ holds if and only if $\sin(\beta) = \sin(\gamma)$. Well known properties of the sin function imply that this holds if and only if $\beta = \gamma$. $\square$

Theorem 1.17.3. *The three angle bisectors of a triangle are concurrent.*

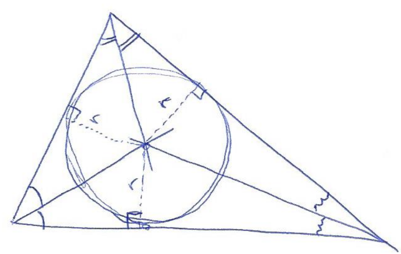


Proof. Let x_a, x_b, x_c be the angle bisectors through the vertexes A, B, C. Let P be the intersection of the x_a and x_b. Then

$$\left.\begin{array}{lcl} P \in x_a & \Rightarrow & d(P,b) = d(P,c) \\ P \in x_b & \Rightarrow & d(P,a) = d(P,c) \end{array}\right\} \Rightarrow d(P,a) = d(P,b) \Rightarrow P \in x_c,$$

where three of the four $\Rightarrow$ implications above use Lemma 1.17.2. Since the intersection of x_a and x_b is on x_c, we have that x_a, x_b, x_c are concurrent. □

A byproduct of the theorem is that the intersection of the angle bisectors has the same distance to the sides. In other words there is a circle with this center that touches the sides of the triangle: the so-called circle inscribed in the triangle.



For a segment AB, the line passing through the midpoint of AB and perpendicular to AB is called the perpendicular bisector.

Theorem 1.17.4. *The three perpendicular bisectors of the sides of a triangle are concurrent.*

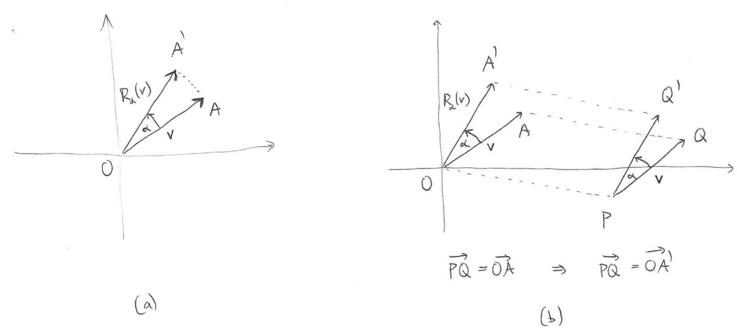
The proof is obtained by solving the first two of the following problems:
1. Find and prove a lemma analoguos to Lemma 1.17.2 but it it about the perpendicular bisector of a segment.
2. Using your lemma from Problem 1 find a proof of Theorem 1.17.4 (logically similar to the proof of Theorem 1.17.3.
3. Find a byproduct of your proof in Problem 2, analogous to the byproduct of the proof of Theorem 1.17.3.

1.18 Rotation, applications

For α an angle and $\mathbf{v} = \vec{OA}$ a plane vector let $R_\alpha(\mathbf{v})$ denote the vector obtained from rotating $\mathbf{v} = \vec{OA}$ around the origin in the counterclockwise direction, see Picture ? (a). Thus R_α is a map from vectors to vectors. Calculation shows that algebraically

$$R_\alpha : \langle x, y\rangle \mapsto \langle x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha\rangle.$$

If a vector is given by an arrow $\mathbf{v} = \vec{PQ}$ where P is not the origin, then to get $R_\alpha(\mathbf{v})$ we formally need to translate $\vec{PQ}$ to $\vec{OA}$, then rotate this $\vec{OA}$ by α. Picture ? (b) shows that this procedure is not necessary: $R_\alpha(\mathbf{v})$ is also obtained by rotating $\vec{PQ}$ around P by α.

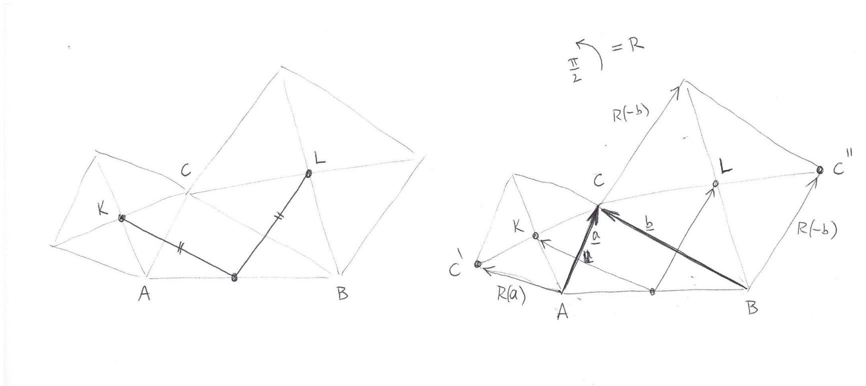


Proposition 1.18.1. *The rotation operator on vectors is consistent with vector operations as follows,*

- $R_\alpha(\mathbf{a} + \mathbf{b}) = R_\alpha(\mathbf{a}) + R_\alpha(\mathbf{b})$,
- $R_\alpha(\mathbf{a} - \mathbf{b}) = R_\alpha(\mathbf{a}) - R_\alpha(\mathbf{b})$,
- $R_\alpha(\lambda\mathbf{a}) = \lambda R_\alpha(\mathbf{a})$,
- $R_\alpha(\mathbf{a})R_\alpha(\mathbf{b}) = \mathbf{ab}$,
- $R_\beta(R_\alpha(\mathbf{a})) = R_{\alpha+\beta}(\mathbf{a})$.

Proof. All can be calculated from the algebraic description of the operations. □

Theorem 1.18.2. *Let $ABC\triangle$ be a triangle, and let $T1$ and $T2$ squares on the sides AC and BC outside the triangle. Let K and L be the centers of $T1$ and $T2$, and X is the midpoint of AB. Then XK and XL have the same length.*



Proof. Let $\mathbf{a} = \vec{AC}$, $\mathbf{b} = \vec{BC}$, and let $R = R_{\pi/2}$ be the rotation by $\pi/2$ operator. Observe that we can express all relevant vectors in our picture using $\mathbf{a}, \mathbf{b}, R$: for example $\vec{AC'} = R(\mathbf{a})$, $\vec{BC''} = -R(\mathbf{b})$.

We have

$$\vec{XK} = \vec{XA} + \vec{AK} = \frac{\vec{BA}}{2} + \frac{\vec{AC} + \vec{AC'}}{2} = \frac{-\mathbf{a} + \mathbf{b}}{2} + \frac{\mathbf{a} + R(\mathbf{a})}{2} = \frac{R(\mathbf{a}) + \mathbf{b}}{2},$$

$$\vec{XL} = \vec{XB} + \vec{BL} = \frac{\vec{AB}}{2} + \frac{\vec{BC} + \vec{BC''}}{2} = \frac{-\mathbf{b} + \mathbf{a}}{2} + \frac{\mathbf{b} - R(\mathbf{b})}{2} = \frac{\mathbf{a} - R(\mathbf{b})}{2}.$$

The idea of the proof is that we suspect that not only XK and XL are of the same length but one is the $\pi/2$ rotation of the other. Hence we calculate

$$R(\vec{XL}) = R\left(\frac{\mathbf{a} - R(\mathbf{b})}{2}\right) = \frac{R(\mathbf{a}) - R(R(\mathbf{b}))}{2}.$$

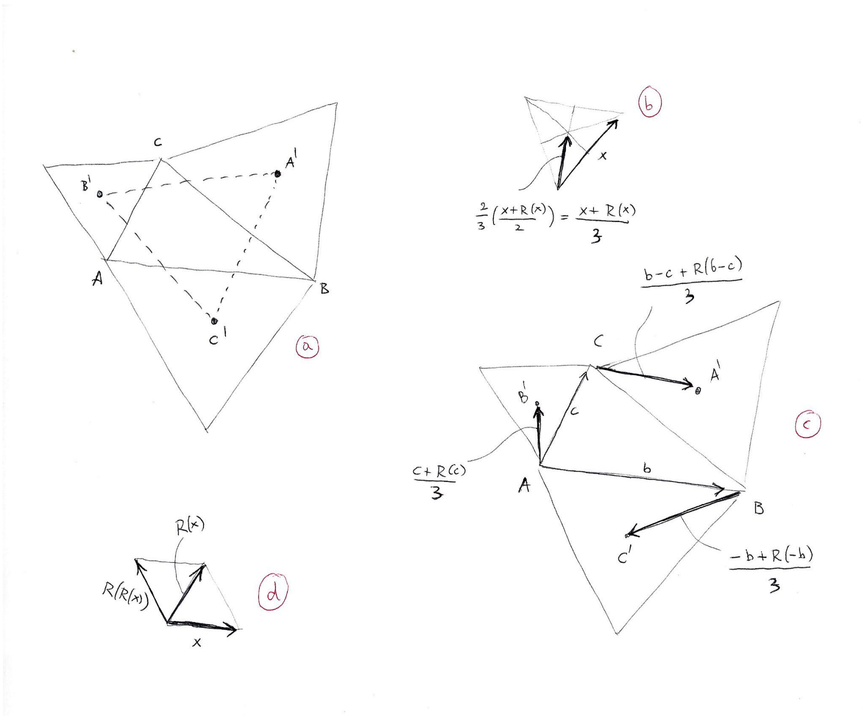
Observe that applying R twice on a vector is the same as multiplication by -1. Indeed, $R_{\pi/2}R_{\pi/2} = R_\pi$ =multiplication by (-1). Hence

$$R(\vec{XL}) = \frac{R(\mathbf{a}) + \mathbf{b}}{2} = \vec{XK},$$

what we wanted to prove. $\square$

The proof above is not the shortest or most elegant proof, but illustrates the main point: naming sufficient vectors and operations (but not more) that determine the picture we can express any other vectors in terms of the named ones. Then we can make comparisons among any two. A more "elegant" version of the same proof will be given in the exercises.

Theorem 1.18.3 (Napoleon Bonaparte[2]). *Let $ABC\triangle$ be an arbitrary triangle and let T_a, T_b, T_c by regular (a.k.a. equilateral) triangles on the sides of a, b, c, outside of ABC. Let A', B', C' be the centers of T_a, T_b, T_c. Then $A'B'C'\triangle$ is a regular triangle.*



Proof. Let $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$, and let $R = R_{\pi/3}$ be the rotation by $\pi/3$ operator. Our goal is to express relevant vectors, namely $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ in terms of these.

First observe that if $\mathbf{x}$ is a side vector of a regular triangle then the vector pointing from a vertex to its center as drawn in Picture ? (b) is $(\mathbf{x} + R(\mathbf{x}))/3$. Therefore we have

$$\overrightarrow{AB'} = \frac{\mathbf{c} + R(\mathbf{c})}{3}, \qquad \overrightarrow{CA'} = \frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3}, \qquad \overrightarrow{BC'} = \frac{-\mathbf{b} + R(-\mathbf{b})}{3},$$

see Picture ? (c). Now we can express

$$\overrightarrow{A'B'} = -\frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3} - \mathbf{c} + \frac{\mathbf{c} + R(\mathbf{c})}{3} = \frac{-1}{3}\mathbf{b} - \frac{1}{3}\mathbf{c} - \frac{1}{3}R(\mathbf{b}) + \frac{2}{3}R(\mathbf{c}),$$

$$\overrightarrow{A'C'} = -\frac{\mathbf{b} - \mathbf{c} + R(\mathbf{b} - \mathbf{c})}{3} - \mathbf{c} + \mathbf{b} + \frac{-\mathbf{b} + R(-\mathbf{b})}{3} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c} - \frac{2}{3}R(\mathbf{b}) + \frac{1}{3}R(\mathbf{c}).$$

[2] yes, him

What we want to prove is $R(\vec{A'B'}) = \vec{A'C'}$ so let us calculate

$$R(\vec{A'B'}) = R\left(\frac{-1}{3}\mathbf{b} - \frac{1}{3}\mathbf{c} - \frac{1}{3}R(\mathbf{b}) + \frac{2}{3}R(\mathbf{c})\right) = \frac{-1}{3}R(\mathbf{b}) - \frac{1}{3}R(\mathbf{c}) - \frac{1}{3}R(R(\mathbf{b})) + \frac{2}{3}R(R(\mathbf{c})).$$

Looking at Picture ? (d) we see that $R(\mathbf{x}) = \mathbf{x} + R(R(\mathbf{x}))$, and hence $R(R(\mathbf{x})) = R(\mathbf{x}) - \mathbf{x}$. We further have

$$R(\vec{A'B'}) = \frac{-1}{3}R(\mathbf{b}) - \frac{1}{3}R(\mathbf{c}) - \frac{1}{3}(R(\mathbf{b}) - \mathbf{b}) + \frac{2}{3}(R(\mathbf{c}) - \mathbf{c}) = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{c} - \frac{2}{3}R(\mathbf{b}) + \frac{1}{3}R(\mathbf{c}).$$

This last expression is the same as the expression for $\vec{A'C'}$ above, hence $R(\vec{A'B'}) = \vec{A'C'}$ what we wanted to prove. □

1.19 Wedge product of two plane vectors

In Calculus we learn the geometry and the algebra of the notion of cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ in 3-space. Geometrically $\mathbf{a} \times \mathbf{b}$ has length $|\mathbf{a}||\mathbf{b}|\sin\theta$ (where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$), it lies in the line orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ and its direction satisfies the right-hand rule. Algebraically

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

Plane vectors $\langle a_1, a_2\rangle$ can be considered space vectors by $\langle a_1, a_2, 0\rangle$. If we take the cross product of two such plane-space vectors then we obtain a vector of the form $\langle 0, 0, *\rangle$. We do not want to keep carrying the $(0, 0)$-part, hence we give a new definition capturing only the third coordinate.

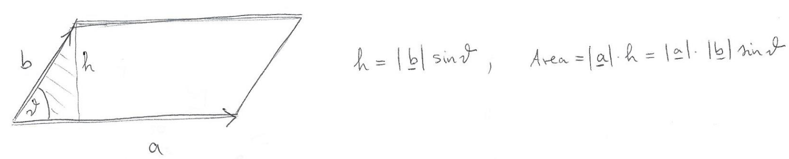
Definition 1.19.1. The wedge product $\mathbf{a} \wedge \mathbf{b}$ of two plane vectors $\mathbf{a} = \langle a_1, a_2\rangle, \mathbf{b} = \langle b_1, b_1\rangle$ is the third coordinate of the cross product $\langle a_1, a_2, 0\rangle \times \langle b_1, b_1, 0\rangle$.

From the arguments above we obtain that

- (algebra) $\langle a_1, a_2\rangle \wedge \langle b_1, b_1\rangle = a_1b_2 - a_2b_1$,
- (geometry) $\mathbf{a} \wedge \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}|\sin\theta$. Since $|\mathbf{a}||\mathbf{b}|\sin\theta$ is the area of a the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ (see Picture ?), we have

  $$\mathbf{a} \wedge \mathbf{b} = \pm\,\text{Area(parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}).$$

  Analysing the right-hand rule mentioned above we can determine wether + or - stand in the formula above: if the direction of $\mathbf{b}$ is obtained from the direction of $\mathbf{a}$ by a counterclockwise rotation by not more than π then the sign is positive, otherwise negative.



The algebraic interpretation easily proves the following properties.

- (antysymmetry) $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, $\qquad \mathbf{a} \wedge \mathbf{a} = 0$.
- (bilinearity) $(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}$, $(\lambda \cdot \mathbf{a}) \wedge \mathbf{b} = \lambda \cdot \mathbf{a} \wedge \mathbf{b}$.
- $\mathbf{a} \wedge \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel. (*)

Again, the power of this operation is the duality between the properties just listed (proved by algebra) and the geometric interpretation. Here is an application of the the wedge product.

Proposition 1.19.2. *The points A, B, C are collinear if and only if $A \wedge B + B \wedge C + C \wedge A = 0$.*

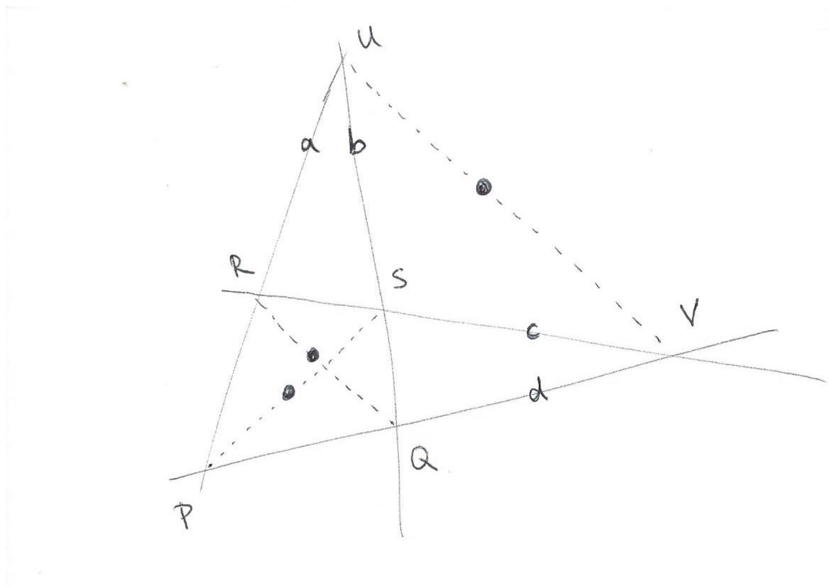
Proof. The points A, B, C are collinear if and only if the vectors $\vec{BA}$ and $\vec{CB}$ are in one line. They are in one line if and only if their spanned parallelogram degenerates to a segment, i.e. has area 0. Hence A, B, C are in one line if and only if

$$0 = (A - B) \wedge (B - C) = A \wedge B - A \wedge C - B \wedge B + B \wedge C = A \wedge B + B \wedge C + C \wedge A.$$

□

Before Proposition 1.19.2 our algebraic interpretations of collinearity were Propositions 1.4.1 and 1.4.2. All the theorems proved using those two propositions have alternative proofs using our new algebraic interpretation Proposition 1.19.2. We will not reprove earlier theorems though (students may find it a good exercise), but rather give some new incidence theorem, and as a change we will prove them using Proposition 1.19.2.

Theorem 1.19.3 (Newton-Gauss line)**.** *Let a, b, c, d be four pairwise intersecting lines (this configuration is called a complete quadrilateral). The pairwise intersections are six points. Three pairs of these six points are not connected by the lines a, b, c, d, these are called diagonals (PS, RQ, UV in the picture). The midpoints of the three diagonals are collinear.*



Proof. Using the notation of the picture consider the sum of the following twelve terms

$$\begin{array}{cccc} P\wedge R & P\wedge Q & S\wedge R & S\wedge Q \\ R\wedge U & Q\wedge V & R\wedge V & Q\wedge U \\ U\wedge P & V\wedge P & V\wedge S & U\wedge S \end{array}$$

We will view this sum in two different ways.

First: The sum of the terms in each column is zero, because the triples of points (P,R,U), (P,Q,V), (S,R,V), (S,Q,U) are collinear, see Proposition 1.19.2. Hence the total sum is zero.

Second: The first row is 4 times $(P+S)/2\wedge(R+Q)/2$, the second row is 4 times $(R+Q)/2\wedge(U+V)/2$. The third row is 4 times $(U+V)/2\wedge(P+S)/2$.

We conclude that

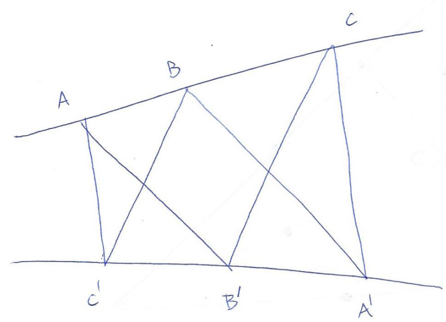
$$\frac{P+S}{2}\wedge\frac{R+Q}{2}+\frac{R+Q}{2}\wedge\frac{U+V}{2}+\frac{U+V}{2}\wedge\frac{P+S}{2}=0,$$

and hence according to Proposition 1.19.2 $(P+S)/2$, $(R+Q)/2$, $(U+V)/2$ are collinear. □

Theorem 1.19.4 (parallel case of Pappus' theorem)**.** *Let A,B,C be collinear points and let A',B',C' be collinear. Then two of*

$$AB'||A'B, \qquad BC'||B'C, \qquad AC'||A'C$$

imply the third.



Proof. Consider the following three numbers

$$(A - B') \wedge (A' - B), \qquad (B - C') \wedge (B' - C), \qquad (C - A') \wedge (C' - A).$$

The vanishing of these three numbers is equivalent to the three parallelity conditions of the theorem, according to (*) above. Hm...better way of referencing there is needed.

However, one can distribute the *sum* of these three numbers, use the antisymmetry property of $\wedge$ and conclude that the total sum is 0. Hence the vanishing of two of them indeed implies the vanishing of the third one. $\square$

We will learn more on Pappus' theorem in Section **??**.

1.20 A 3D view of plane geometry, triple product

In Calculus we learned the geometry and algebra of the triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ of three *space* vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. It will be convenient to use the following notation $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, and we may call it the triple product, or triple wedge product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. In particular the following hold

- $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$;
- $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is plus or minus the volume of the parallelepiped spanned by the space vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$;
- $$(a_1, a_2, a_3) \wedge (b_1, b_2, b_3) \wedge (c_1, c_2, c_3) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix};$$

- (3-linearity) $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is linear in each of the variables. Linearity in the first variable means

$$(\mathbf{a} \pm \mathbf{a}') \wedge \mathbf{b} \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \pm \mathbf{a}' \wedge \mathbf{b} \wedge \mathbf{c},$$

$$(\lambda \mathbf{a}) \wedge \mathbf{b} \wedge \mathbf{c} = \lambda \cdot \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c},$$

  and linearity in the second and third variables are similar;

- (antisymmetry) $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = \mathbf{b}\wedge\mathbf{c}\wedge\mathbf{a} = \mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b} = -\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{b} = -\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a} = -\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c}$.

In view of the second property above it makes no sense of considering the triple product of vectors lying in the $(x, y, 0)$ plane. A useful tool, however is considering our plane as the $z = 1$ plane in 3-space. That is, if a point was (a_1, a_2) earlier, now we consider it as $(a_1, a_2, 1)$.

What we gained is a new operation: we can form the triple product $A \wedge B \wedge C$ for three points. What we lost is that we partially lost our earlier operations: for example $A+B$ does not make sense any more since $(a_1, a_2, 1) + (b_1, b_2, 1) = (a_1 + b_1, a_2 + b_2, 2)$ is not in the $z = 1$ plane any more. However, for example $xA + yB$ makes sense if $x + y = 1$.

Proposition 1.20.1. *For a triangle $ABC\triangle$ in the $z = 1$ plane we have*

$$\text{Area}(ABC\triangle) = \pm\frac{1}{2} A \wedge B \wedge C,$$

and the sign is positive if and only if going around the triangle in the order A, B, C is counterclockwise.

The proof is left as an exercise.

Corollary 1.20.2. *The three points A, B, C of the $z = 1$ plane are collinear if and only if $A \wedge B \wedge C = 0$.*

Proof. Both conditions are equivalent to the condition that the volume of the parallelepiped spanned by the space vectors A, B, C is zero. □

Corollary 1.20.2 is now our 4th algebraic interpretation of collinearity of three points in the plane—however this in new settings. Again, all theorems that were proved using any of the earlier three interpretations (Propositions 1.4.1, 1.4.2, 1.19.2) can be proved with this new one too—just we need to be careful using operation that make sense in the $z = 1$ plane. Let us illustrate this with the following new proof Menelaus' theorem 1.7.1.

Proof. Consider our triangle in the $z = 1$ plane. We have $K = xB + x'C$, $L = yC + y'A$, $M = zA + z'B$ with $x + x' = y + y' = z + z' = 1$. According to Corollary 1.4.3 we have

$$0 = K \wedge L \wedge M = (xB + x'C) \wedge (yC + y'A) \wedge (zA + z'B).$$

Using 3-linearity and antisymmetry of the triple product we can distribute the above expression and obtain

$$0 = (xyz + x'y'z')A \wedge B \wedge C.$$

Since ABC is a triangle (with non-zero area) we have $A \wedge B \wedge C \neq 0$. Therefore $xyz + x'y'z' = 0$ which is a rearrangement of Menelaus' theorem. □

1.21 Problems

1. *Prove Proposition 1.2.2.*

2. *Prove Proposition 1.5.2.*

3. *In the formula $C = (1 - t)A + tB$, trace the position of C on the line as t varies from $-\infty$ to ∞.*

4. *In $ABC\triangle$ let U be the midpoint of AB and let V be the midpoint of AC. Prove that UV is parallel to BC and has half the length.*

5. *In a quadrilateral let U and V be the midpoints of two opposite sides. Prove that the segment UV and the segment connecting the midpoints of the diagonals bisect each other.*

6. *Let S be the centroid of the $ABC\triangle$. Calculate $\vec{SA} + \vec{SB} + \vec{SC}$.*

7. *Let R be an arbitrary point in the plane and $ABCD$ a parallelogram. Prove that $\vec{RA} + \vec{RC} = \vec{RB} + \vec{RD}$.*

8. *In the $ABCD$ quadrilateral let $\vec{AB} = \mathbf{a}$, $\vec{DC} = \mathbf{b}$. Let the points A, X_1, X_2, X_3, D divide the AD side into four equal parts. Let the points B, Y_1, Y_2, Y_3, C divide the BC segment into four equal parts. Express $\vec{X_1Y_1}$, $\vec{X_2Y_2}$, $\vec{X_3Y_3}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.*

9. *Let $ABCDA'B'C'D'$ be a cube ($ABCD$ is a square and $A'B'C'D'$ is a translated copy of $ABCD$). Let $\mathbf{a} = \vec{AB}, \mathbf{b} = \vec{AD}, \mathbf{c} = \vec{AA'}$. Let P be the midpoint of $C'D'$. Let Q be the center of the $BCC'B'$ square. Express $\vec{AP}$, $\vec{AQ}$, $\vec{AD'}$, $\vec{BD}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.*

10. *Let $ABCD$ be a parallelogram. Let the points A, X_1, X_2, B divide AB into three equal parts. Let C, Y_1, Y_2, D divide CD into three equal parts. Let X_2, U, V, Y_2 divide X_2Y_2 into three equal parts. Express $\vec{AV}$ in terms of $\vec{AB}$ and $\vec{AD}$.*

11. *Let $P_1, \ldots, P_n$ be points, and $\mu_1, \ldots, \mu_n$ be real numbers with $\sum_{i=1}^n \mu_i = 1$. For a point O consider $\mathbf{v} = \sum_{i=1}^n \mu_i \vec{OP_i}$. Let S be the end point of the vector $\mathbf{v}$ if it is measured from O. Prove that the point S does not depend on the choice of O. (Hint: choose two different O_1 and O_2 and calculate the vector between the obtained "two" S points. You should get $\mathbf{0}$.)*

12. *Let D be a point on the line of BC, and let E be a point on the line of AC of the $ABC\triangle$ (but let D, E be distinct from the vertices). Assume the lines AD and BE intersect in a point P. Prove that if $\frac{\vec{BD}}{\vec{DC}} \cdot \frac{\vec{CE}}{\vec{EA}} = -1$ then CP is parallel with AB.*

13. *Let the reflection of the point A over the point B be C. Express C in terms of A and B.*

14. *(cont.) Let $ABC\triangle$ be a triangle. The reflection of A over B is A'. The reflection of B over C is B'. The reflection of C over A is C'. Prove that the centroid of $ABC\triangle$ and the centroid of $A'B'C'\triangle$ coincide.*

15. *In the proof of the reverse Menelaus' theorem we claimed that ℓ intersects the line of AB (that is, ℓ and AB are not parallel). Prove this statement.*

16 (P)**.** *Consider the $ABC\triangle$ and non-zero real numbers k_1, k_2, k_3. Let P_{AB} and P'_{AB} be on the AB line, the first in the AB segment, the second outside of the AB segment, such that*

$$\frac{|AP_{AB}|}{|P_{AB}B|} = \frac{|AP'_{AB}|}{|P'_{AB}B|} = \frac{k_1}{k_2}.$$

Define P_{BC} and P'_{BC} on the BC line similarly with the ratio k_2/k_3; and define P_{CA} and P'_{CA} on the CA line similarly with the ratio k_3/k_1. Show that the lines AP_{BC}, BP_{CA}, CP_{AB} are concurrent.

17 (P)**.** *(cont.) Prove that the line $P_{CA}P_{AB}$ contains the point P'_{BC}.*

18 (P)**.** *(cont.) Prove that P'_{AB}, P'_{BC}, and P'_{CA} are collinear.*

19 (P)**.** *Let P be different from the vertices of the triangle $ABC\triangle$. Let $PBLC$, $PCMA$, $PANB$ be parallelograms. Prove that the segments AL, BM, CN bisect each other.*

20 (P)**.** *Points P, Q, R lie on the sides of the $ABC\triangle$ and are such that*

$$(\vec{BP} : \vec{PC}) = (\vec{CQ} : \vec{QA}) = (\vec{AR} : \vec{RB}).$$

Prove that the centroids of the triangles PQR and ABC coincide.

21 (P)**.** *The triangles $A_1B_1C_1\triangle$, $A_2B_2C_2\triangle$, and $A_3B_3C_3\triangle$ have their corresponding sides parallel. Hence each pair of triangles has a center of perspectivity (assume that these centers of perspectivities exist). Prove that the three centers of perspectivities are collinear.*

22 (P)**.** *A line drawn through the vertex A of a parallelogram $ABCD$ cuts CB in P and CD in Q. A line through C cuts AB in R and AD in S. Prove that PR and QS are parallel.*

23. *Prove that for a parallelogram the sum of squares of the sides is equal to the sum of the squares of the diagonals.*

24. *Let O be the center of the $ABCDEF$ regular hexagon whose side length is 1. Find*

$$\vec{AB} \cdot \vec{AO}, \vec{AB} \cdot \vec{AC}, \vec{BC} \cdot \vec{EF}, \vec{FC} \cdot \vec{BD}, \vec{FC} \cdot \vec{EF}.$$

25 (P)**.** *Let A, B, C, D be points in the plane, let A' be the midpoint of BC, let B' be the midpoint of CA, and let C' be the midpoint of AB. Prove that*

$$(D - A')(C - B) + (D - B')(A - C) + (D - C')(B - A) = 0.$$

26 (P)**.** *Suppose that the segment connecting the midpoints of AB with CD, and the segment connecting the midpoints of BC with DA are of the same length. Prove that AC is perpendicular to BD.*

27. *Prove Corollary 1.16.3, the triangle inequality.*

28. *(Thales' theorem) Let O be the midpoint of AC. Prove that $\angle(ABC) = \pi/2$ if and only if $d(AO) = d(BO)$. (That is $\angle(ABC) = \pi/2$ if and only if B is on the circle with center O and radius $d(AO)$.)*

29. *Finish the proof of Feuerbach's Theorem 1.14.2, i.e. prove that points (c) are also on the same circle as points (a) and (b). [Hint: Thales' theorem is useful.]*

30. *Let $\mathcal{C}$ be a circle in the plane, and let λ be a number. Prove that $\lambda\mathcal{C} = \{\lambda x \in \mathbb{R}^2 : x \in \mathcal{C}\}$ is also a circle.*

31. *Reflect the orthocenter of $ABC\triangle$ over the midpoints of the sides. Prove that the obtained three points are on the circumscribed circle.*

32. *Solve problems 1, 2, 3 after Theorem 1.17.4.*

33. *Prove that a triangle is equilateral if two of its circumcenter, centroid, and orthocenter coincide.*

34. *Let O be the center of circumscribed circle of $ABC\triangle$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vectors pointing from O to the vertexes. Let M be the endpoint of $\mathbf{a}+\mathbf{b}+\mathbf{c}$ measured from O. Prove that M is the orthocenter of $ABC\triangle$.*

35. *(cont.) Let O be the center of the circumscribed circle of $ABC\triangle$. Let M be the orthocenter, and let S be the centroid of $ABC\triangle$. Prove that O, M, S are collinear. (The obtained line is called the Euler-line of $ABC\triangle$.) Find $\vec{OS}/\vec{SM}$.*

36. *Let D be a point on the side BC of $ABC\triangle$. Prove that*

$$\frac{BD}{DC} = \frac{|AB| \sin(DAB\angle)}{|AC| \sin(DAC\angle)}.$$

37. *(cont.) Prove the Angle Bisector Theorem: If the angle bisector from A intersects BC in D, then*

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}.$$

38. *(cont.) Reprove that angle bisectors are concurrent by using the Angle Bisector Theorem and Ceva's theorem.*

39. *Let $AB = 6$, $BC = 10$ in the $ABC\triangle$. The angle bisector at B intersects AC in D. Connect D with the midpoint of AB, let the intersection point with BC be E. What is the length $|BE|$?*

40. *Let the lengths of the sides of $ABC\triangle$ be a, b, c (a is opposite of A, etc). If O is the center of the inscribed circle then prove that*

$$O = \frac{aA + bB + cC}{a + b + c}.$$

41. *Let P be an interior point of $ABC\triangle$. The lines connecting P with the vertices cut $ABC\triangle$ into six smaller triangles. We color every second of these six triangles with red, the rest with blue. Prove that the product of the areas of the red triangles is the same as the product of the areas of the blue triangles.*

42. *(cont.) In the problem above replace "area" with "radius of the circumscribed circle".*

43 (P)**.** *Let P, Q, R, S be the centers of the squares that are described externally on the sides of a quadrilateral (in this order). Prove that PR and QS are of the same length, and are perpendicular to each other.*

44 (P)**.** *If A', B', C' are the midpoints of BC, CA, AB respectively, then show that*

$$4A' \wedge B' \wedge C' = A \wedge B \wedge C.$$

Deduce that $\mathrm{Area}(A'B'C'\triangle) = \frac{1}{4}\,\mathrm{Area}(ABC\triangle)$.

45 (P)**.** *Let the side lengths of the $ABC\triangle$ be a, b, c (a is opposite with A etc), and let the angles be α, β, γ. Let the foot of the altitude from A be D. Prove that*

$$aD = (b\cos\gamma)B + (c\cos\beta)C.$$

Deduce that the area of the triangle formed by the feet of the altitudes is

$$2\cos\alpha\cos\beta\cos\gamma \cdot \mathrm{Area}(ABC\triangle).$$

46 (P)**.** *Let D, E, F be points on the sides AB, BC, CA of a triangle, dividing the sides in the ratios $k_1 : 1$, $k_2 : 1$, $k_3 : 1$. Show that*

$$\frac{\mathrm{Area}(DEF\triangle)}{\mathrm{Area}(ABC\triangle)} = \frac{1 + k_1k_2k_3}{(1+k_1)(1+k_2)(1+k_3)}.$$

47. *Give a proof of Proposition 1.20.1. [Hint: Let T be the tetrahedron with vertexes $(0,0,0), A, B, C$. Use the geometric interpretation of the triple product to conclude that the volume of T is plus or minus one sixth of the triple product ABC. Finish the proof by observing that the volume of T is one third of the area of the $ABC\triangle$.]*

48. *Reprove Ceva's theorem, using the triple wedge operation. Hint: Try to rephrase the "high-school style" proof from the end of Section 1.9.*