

3d Mirror Symmetry and Elliptic Stable Envelopes

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Abstract

We consider a pair of quiver varieties $(X; X')$ related by 3d mirror symmetry, where $X = T^*Gr(k, n)$ is the cotangent bundle of the Grassmannian of k -planes of n -dimensional space. We give formulas for the elliptic stable envelopes on both sides. We show an existence of an equivariant elliptic cohomology cohomology class on $X \times X'$ (the *Mother function*) whose restrictions to X and X' are the elliptic stable envelopes of those varieties. This implies, that the restriction matrices of the elliptic stable envelopes for X and X' are equal after transposition and identification of the equivariant parameters on one side with the Kähler parameters on the dual side.

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1 Introduction

1.1 Mirror symmetries

Mirror symmetry is one of the most important physics structures that enters the world of mathematics and arouses lots of attention in the last several decades. Its general philosophy is that a space X should come with a dual X' which, though usually different from and unrelated to X in the appearance, admits some deep connections with X in geometry. Mirror symmetry in 2 dimensions turns out to extremely enlightening in the study of algebraic geometry, symplectic geometry, and representation theory. In particular, originated from the 2d topological string theory, the Gromov–Witten theory has an intimate connection with 2d mirror symmetry; for an introduction, see [8, 24].

Similar types of duality also exist in 3 dimensions. More precisely, as introduced in [6, 14, 23, 7, 10, 9, 25, 15], the 3d mirror symmetry is constructed between certain pairs of 3d $\mathcal{N} = 4$ supersymmetric gauge theories, under which they exchanged their *Higgs branches* and *Coulomb branches*, as well as their FI parameters and mass parameters. In mathematics, the $\mathcal{N} = 4$ supersymmetries implies that the corresponding geometric object of our interest should admit a hyperKähler structure, or if one prefers to stay in the algebraic context, a holomorphic symplectic structure. In particular, for theories of the class as mentioned above, the Higgs branch, which is a certain branch of its moduli of vacua, can be interpreted as a holomorphic symplectic quotient in mathematics, where the prequotient and group action are determined by the data defining the physics theory. The FI parameters and mass parameters of the theory are interpreted as *Kähler parameters* and *equivariant parameters* respectively.

The Coulomb branch, however, did not have such a clear mathematical construction until recently [38, 35, 5]. In this general setting it is not a holomorphic symplectic quotient, and it is difficult to study its geometry. Nevertheless, in many special cases e.g., already appearing in the physics literature [6, 13], the

Coulomb branch might also be taken as some holomorphic symplectic quotient. Those special cases include hypertoric varieties, Hilbert schemes of points on \mathbb{C}^2 , the moduli space of instantons on the resolved A_N surfaces, and so on. For a mathematical exposition, see [3, 4], where 3d mirror symmetry is referred to as *symplectic duality*.

A typical mirror symmetry statement for a space X and its mirror X' , is to relate certain geometrically defined invariants on both sides. For example, in the application of 2d mirror symmetry to genus-zero Gromow–Witten theory, the J -function counting rational curves in X is related to the I -function, which arises from the mirror theory.

In the 3d case, instead of cohomological counting, one should consider counting in the K-theory. One of the K-theoretic enumerative theory in this setting, which we are particularly interested in, is developed by A. Okounkov and his collaborators [39, 30, 40, 1]. The 3d mirror symmetry statement in this theory looks like

$$V(X) \cong V(X'), \tag{1}$$

where X and X' is a 3d mirror pair of hypertoric or Nakajima quiver varieties, and $V(X)$, $V(X')$ are the so-called *vertex functions*, defined via equivariant K-theoretic counting of quasimaps into X and X' [39].

On both sides, the vertex functions, which depend on Kähler parameters z_i and equivariant parameters a_i , can be realized as solutions of certain geometrically defined q -difference equations. We call those solutions that are holomorphic in Kähler parameters and meromorphic in equivariant parameters the z -solutions, and those in the other way the a -solutions. In particular, vertex functions are by definition z -solutions.

Under the correspondence (1), the Kähler and equivariant parameters on X and X' are exchanged with each other, and hence z -solutions of one side should be mapped to a -solutions of the other side and vice versa. In particular, for the correspondence to make sense, (1) should involve a transition between a basis of z -solutions, and a basis of a -solutions. In [1], this transition matrix is introduced geometrically as the *elliptic stable envelope*.

1.2 Elliptic stable envelopes

The notion of stable envelopes first appear in [31] to generate a basis for Nakajima quiver varieties which admits many good properties. Their definition depends on a choice of cocharacter, or equivalently, a chamber in the Lie algebra of the torus that acting on the space X . The transition matrices between stable envelopes defined for different chambers turn out to be certain R -matrices, satisfying the Yang–Baxter equation and hence defining quantum group structures. Stable envelopes are generalized to K-theory [39, 30, 40], where they not only depend on the choice of cocharacter σ , but also depend piecewise linearly on the choice of slope s , which lives in the space of Kähler parameters.

In [1], stable envelopes are further generalized to the equivariant elliptic cohomology, where the piecewise linear dependence on the slope s is replaced by the meromorphic dependence to a Kähler parameter z . In particular, the elliptic version of the stable envelope is the most general structure, K-theoretic and cohomological stable envelopes can be considered as limits of elliptic. The elliptic stable envelopes depends on both, the equivariant and Kähler parameters which makes it a natural object for the study of 3d mirror symmetry.

In this paper, we will concentrate on a special case where $X = T^*Gr(k, n)$, the cotangent bundle of the Grassmannian of k -dimensional subspaces in \mathbb{C}^n . This variety is a simplest example of Nakajima quiver variety associated to the A_1 -quiver, with dimension vector $\mathbf{v} = k$ and framing vector $\mathbf{w} = n$. We will always assume that $n \geq 2k$ ¹. Its mirror, which we denote by X' , can also be constructed as a Nakajima quiver variety, associated to the A_{n-1} -quiver. It has dimension vector

$$\mathbf{v} = (1, 2, \dots, k-1, \underbrace{k, \dots, k}_{(n-2k+1)\text{-times}}, k-1, \dots, 2, 1)$$

and framing vector

$$\mathbf{w} = \delta_k + \delta_{n-k}.$$

For Nakajima quiver varieties, there is always a torus action induced by that on the framing spaces. Let T and T' be the tori on X and X' respectively. They both have $n!/(k!(n-k)!)$ fixed points, which admit

¹Only in the case $n \geq 2k$ the dual variety X' can also be realized as quiver variety.

very nice combinatorial descriptions. Elements in X^\top can be interpreted as k -subsets $\mathbf{p} \subset \mathbf{n} := \{1, 2, \dots, n\}$, while $(X')^{\top'}$ is the set of partitions λ that fit into a $k \times (n - k)$ rectangle. There is a natural bijection (41) between those fixed points

$$\text{bj} : (X')^{\top'} \xrightarrow{\sim} X^\top.$$

We will consider the extended equivariant elliptic cohomology of X and X' under the corresponding framing torus actions, denoted by $E_\top(X)$ and $E_{\top'}(X')$ respectively. By definition, they are certain schemes, associated with structure maps which are finite (and hence affine)

$$E_\top(X) \rightarrow \mathcal{E}_\top \times \mathcal{E}_{\text{Pic}_\top(X)}, \quad E_{\top'}(X') \rightarrow \mathcal{E}_{\top'} \times \mathcal{E}_{\text{Pic}_{\top'}(X')},$$

where $\mathcal{E}_\top \times \mathcal{E}_{\text{Pic}_\top(X)}$ and $\mathcal{E}_{\top'} \times \mathcal{E}_{\text{Pic}_{\top'}(X')}$ are powers of an elliptic curve E , the coordinates on which are the Kähler and equivariant parameters. There is a natural identification (42)

$$\kappa : \mathbb{K} \rightarrow \top', \quad \top \rightarrow \mathbb{K}'$$

between the Kähler and equivariant tori of the two sides, which induces an isomorphism between $\mathcal{E}_\top \times \mathcal{E}_{\text{Pic}_\top(X)}$ and $\mathcal{E}_{\top'} \times \mathcal{E}_{\text{Pic}_{\top'}(X')}$.

By localization theorems, the equivariant elliptic cohomology of X has the form

$$E_\top(X) = \left(\prod_{\mathbf{p} \in X^\top} \widehat{\mathcal{O}}_{\mathbf{p}} \right) / \Delta,$$

where each $\widehat{\mathcal{O}}_{\mathbf{p}}$ is isomorphic to the base $\mathcal{E}_\top \times \mathcal{E}_{\text{Pic}_\top(X)}$. The \top -action on X is good enough, in the sense that it is of the *GKM type*, which means that it admits finitely many isolated fixed points, and finitely many 1-dimensional orbits. Due to this GKM property, the gluing data Δ of X is easy to describe: it is simply the gluing of $\widehat{\mathcal{O}}_{\mathbf{p}}$ and $\widehat{\mathcal{O}}_{\mathbf{q}}$ for those fixed points \mathbf{p} and \mathbf{q} connected by 1-dimensional \top -orbits. For X' , $E_{\top'}(X')$ also has the form as above; however, the gluing data Δ' is more complicated.

By definition, the elliptic stable envelope $\text{Stab}_\sigma(\mathbf{p})$ for a given fixed point $\mathbf{p} \in X^\top$ is the section of a certain line bundle $\mathcal{T}(\mathbf{p})$. We will describe this section in terms of its components

$$T_{\mathbf{p}, \mathbf{q}} := \text{Stab}_\sigma(\mathbf{p})|_{\widehat{\mathcal{O}}_{\mathbf{q}}},$$

which are written explicitly in terms of theta functions and satisfy prescribed quasiperiodics and compatibility conditions. Similar for X' , we will describe the components

$$T'_{\lambda, \mu} := \text{Stab}'_{\sigma'}(\lambda)|_{\mu}.$$

1.3 Coincidence of stable envelopes for dual variates

Our main result is that the restriction matrices for elliptic stable envelopes on the dual varieties coincide (up to transposition and normalization by the diagonal elements):

Corollary 1. *Restriction matrices of elliptic stable envelopes for X and X' are related by:*

$$T_{\mathbf{p}, \mathbf{p}} T'_{\lambda, \mu} = T'_{\mu, \mu} T_{\mathbf{q}, \mathbf{p}} \quad (2)$$

where $\mathbf{p} = \text{bj}(\lambda)$, $\mathbf{q} = \text{bj}(\mu)$ and parameters are identified by (42).

In (2), the prefactors $T_{\mathbf{p}, \mathbf{p}}$ and $T'_{\mu, \mu}$ have very simple expressions as product of theta functions. The explicit formula for matrix elements $T'_{\lambda, \mu}$ and $T_{\mathbf{q}, \mathbf{p}}$, however, involves complicated summations.

Explicit formulas (see Theorem 3 and 4) for elliptic stable envelopes are obtained by the *abelianization technique* [44, 45, 1, 46]. In the spirit of abelianization, the formula for $T_{\mathbf{q}, \mathbf{p}}$ involves a symmetrization sum over the symmetric group \mathfrak{S}_k , the Weyl group of the gauge group $GL(k)$. However, the formula $T'_{\lambda, \mu}$ involves not only a symmetrization over $\mathfrak{S}_{n, k}$, the Weyl group of the corresponding gauge group, but also a *sum over trees*. Similar phenomenon already appear in the abelianization formula for the elliptic stable envelopes of $\text{Hilb}(\mathbb{C}^2)$ [46]. The reason for this sum over trees to occur is that in the abelianization for X' , the preimage of a point is no longer a point, as in the case of X .

As a result, the correspondence (44) we obtained here actually generates an infinite family of nontrivial identities among product of theta functions. See Section 7 and 8 for examples in the simplest cases $k = 1$ and $n = 4, k = 2$. In particular, in the $n = 4, k = 2$ case, we obtain the well-known 4-term theta identity.

Motivated by the correspondence (44) and the Fourier–Mukai philosophy, a natural guess is that the identity might actually come from a universal “Mother function” \mathfrak{m} , living on the product $X \times X'$. Consider the following diagram of embeddings

$$X = X \times \{\lambda\} \xrightarrow{i_\lambda} X \times X' \xleftarrow{i_{\mathfrak{p}}} \{\mathfrak{p}\} \times X' = X'.$$

Corollary 2 then follows directly from our main theorem:

Theorem 1. *There exists a holomorphic section \mathfrak{m} (the Mother function) of a line bundle \mathfrak{M} on the $\mathbb{T} \times \mathbb{T}'$ equivariant elliptic cohomology of $X \times X'$ such that*

$$i_\lambda^*(\mathfrak{m}) = \mathbf{Stab}(\mathfrak{p}), \quad i_{\mathfrak{p}}^*(\mathfrak{m}) = \mathbf{Stab}'(\lambda),$$

where $\mathfrak{p} = \mathbf{bj}(\lambda)$.

The existence of the Mother function was already predicted by Aganagic and Okounkov in the original paper [1]. This paper originated from our attempt to check their conjecture and construct the mother function for the simplest examples of dual quiver varieties.

1.4 Relation to $(\mathfrak{gl}_n, \mathfrak{gl}_m)$ -duality

Let $\mathbb{C}^2(u)$ be the fundamental evaluation module with evaluation parameter u of the quantum affine algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_2)$. Similarly, let $\bigwedge^k \mathbb{C}^n(a)$ be the k -th fundamental evaluation module with the evaluation parameter a of quantum affine algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_n)$. Recall that the equivariant K-theory of quiver varieties are naturally equipped with an action of quantum affine algebras [37]. In particular, for $X = T^*Gr(k, n)$ we have isomorphism of weight subspaces in $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_2)$ -modules:

$$K_{\mathbb{T}}(X) \cong \text{weight } k \text{ subspace in } \mathbb{C}^2(u_1) \otimes \cdots \otimes \mathbb{C}^2(u_n) \quad (3)$$

In geometry, the evaluation parameters u_i are identified with equivariant parameters of torus \mathbb{T} . Similarly, the dual variety X' is related to representation theory of $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_n)$:

$$K_{\mathbb{T}'}(X') \subset \bigwedge^k \mathbb{C}^n(a_1) \otimes \bigwedge^{n-k} \mathbb{C}^n(a_2) \quad (4)$$

the corresponding weight subspace is spanned by the following vectors

$$K_{\mathbb{T}'}(X') = \text{Span}\{(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes (e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}), \{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \mathbf{n}\},$$

where e_i is the canonical basis in \mathbb{C}^n . So that both spaces have dimension $n!/(k!(n-k)!)$.

Let us recall that the elliptic stable envelopes features in the representation theory as a building block for solutions of quantum Knizhnik-Zamolodchikov equations and quantum dynamical equations associated to affine quantum groups [11]. The integral solutions of these equations have the form [2, 26, 41, 28]:

$$\Psi_{p,q} \sim \int_C \prod_i dx_i \Phi_p(x_1, \dots, x_n) \mathbf{Stab}_q(x_1, \dots, x_n)$$

Here $\Psi_{p,q}$ represents the matrix of fundamental solution of these equations in some basis. The functions $\Phi_p(x_1, \dots, x_n)$ are the so called master functions and $\mathbf{Stab}_q(x_1, \dots, x_n)$ denotes the elliptic stable envelope of the fixed point (elliptic weight function). The variables of integration x_i correspond to the Chern roots of tautological bundles.

The Theorem 1 implies, in particular, that 3D mirror symmetry for the pair (X, X') identifies $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_2)$ solutions in (3) with $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}_n)$ solutions in (4). Under this identification the evaluation parameters turn to dynamical parameters of the dual side, so that the quantum Knizhnik-Zamolodchikov equations and dynamical equations change their roles. This way, our results suggest a new geometric explanation of $(\mathfrak{gl}_n, \mathfrak{gl}_m)$ and bispectral dualities [34, 33, 47].

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2 Overview of equivariant elliptic cohomology

We start with a very pedestrian exposition of the equivariant elliptic cohomology. For more detailed discussions we refer to [16, 17, 20, 22, 29, 43].

2.1 Elliptic cohomology functor

Let X be a smooth variety endowed with an action of torus $\mathbb{T} \cong (\mathbb{C}^\times)^r$. We say X is a \mathbb{T} -variety. Recall that taking spectrums of the equivariant cohomology and K-theory, $\text{Spec } H_{\mathbb{T}}^*(X)$ can be viewed as an affine scheme over the Lie algebra of the torus $\text{Spec } H_{\mathbb{T}}^*(\text{pt}) \cong \mathbb{C}^r$, and $\text{Spec } K_{\mathbb{T}}(X)$ is an affine scheme over the algebraic torus $\text{Spec } K_{\mathbb{T}}(\text{pt}) \cong (\mathbb{C}^\times)^r$. Equivariant elliptic cohomology is an elliptic analogue of this viewpoint.

Let us fix an elliptic curve

$$E = \mathbb{C}^\times / q^{\mathbb{Z}},$$

i.e., fix the modular parameter q . The equivariant elliptic cohomology is a covariant functor:

$$\text{Ell}_{\mathbb{T}} : \{\mathbb{T}\text{-varieties}\} \rightarrow \{\text{schemes}\},$$

which assigns to a \mathbb{T} -variety X certain scheme $\text{Ell}_{\mathbb{T}}(X)$. For example, the equivariant elliptic cohomology of a point is

$$\text{Ell}_{\mathbb{T}}(\text{pt}) = \mathbb{T} / q^{\text{cochar}(\mathbb{T})} \cong E^{\dim(\mathbb{T})}.$$

We denote this abelian variety by $\mathcal{E}_{\mathbb{T}} := \text{Ell}_{\mathbb{T}}(\text{pt})$. We will refer to the coordinates on $\mathcal{E}_{\mathbb{T}}$ (same as coordinates on \mathbb{T}) as *equivariant parameters*.

Let $\pi : X \rightarrow \text{pt}$ be the canonical projection to a point. The functoriality of the elliptic cohomology provides the map $\pi^* : \text{Ell}_{\mathbb{T}}(X) \rightarrow \mathcal{E}_{\mathbb{T}}$. For each point $t \in \mathcal{E}_{\mathbb{T}}$, we take a small analytic neighborhoods U_t , which is isomorphic via the exponential map to a small analytic neighborhood in \mathbb{C}^r . Consider the sheaf of algebra

$$\mathcal{H}_{U_t} := H_{\mathbb{T}}^\bullet(X^{\mathbb{T}_t}) \otimes_{H_{\mathbb{T}}^\bullet(\text{pt})} \mathcal{O}_{U_t}^{\text{an}},$$

where

$$T_t := \bigcap_{\chi \in \text{char}(\mathbb{T}), \chi(t)=0} \ker \chi \subset \mathbb{T}.$$

Those algebras glue to a sheaf \mathcal{H} over $\mathcal{E}_{\mathbb{T}}$, and we define $\text{Ell}_{\mathbb{T}}(X) := \text{Spec}_{\mathcal{E}_{\mathbb{T}}} \mathcal{H}$. The fiber of $\text{Ell}_{\mathbb{T}}(X)$ over t is obtained by setting local coordinates to 0, as described in the following diagram [1]:

$$\begin{array}{ccccc} \text{Spec } H^\bullet(X^{\mathbb{T}_t}) & \hookrightarrow & \text{Spec } H_{\mathbb{T}}^\bullet(X^{\mathbb{T}_t}) & \longleftarrow & (\pi^*)^{-1}(U_t) & \longrightarrow & \text{Ell}_{\mathbb{T}}(X) \\ \downarrow \pi^* & & \downarrow & & \downarrow & & \downarrow \pi^* \\ \{t\} & \hookrightarrow & \mathbb{C}^r & \longleftarrow & U_t & \longrightarrow & \mathcal{E}_{\mathbb{T}}. \end{array} \quad (5)$$

Example 1. Let us consider a two-dimensional vector space $V = \mathbb{C}^2$ with coordinates (z_1, z_2) , and a torus $\mathbb{T} = (\mathbb{C}^\times)^2$ acting on it by scaling the coordinates: $(z_1, z_2) \rightarrow (u_1 z_1, u_2 z_2)$. Let us set $X = \mathbb{P}(V)$. The action of \mathbb{T} on V induces a structure of \mathbb{T} -space on X . We have $\mathcal{E}_{\mathbb{T}} = E \times E$ and the equivariant parameters u_1 and

u_2 represent the coordinates on the first and the second factor. Note that for a generic point $t = (u_1, u_2) \in \mathcal{E}_\mathbb{T}$ the fixed set $X^{\mathbb{T}^t}$ consists of two points, which in homogeneous coordinates of $\mathbb{P}(V)$ are:

$$p = [1 : 0], \quad q = [0 : 1].$$

The stalk of \mathcal{H} at t is $H_\mathbb{T}^\bullet(p \cup q) \otimes_{H_\mathbb{T}^\bullet} \mathcal{O}_{\mathcal{E}_\mathbb{T}, t}$, and the fiber is $H^\bullet(p \cup q)$. We conclude, that over a general point $t \in \mathcal{E}_\mathbb{T}$ the fiber of π^* in (5) consists of two points.

At the points $t = (u_1, u_2)$ with $u_1 = u_2$ the torus \mathbb{T}_t acts trivially on X , thus locally the sheaf \mathcal{H} looks like

$$H_\mathbb{T}^\bullet(X^{\mathbb{T}^t}) = H_\mathbb{T}^\bullet(\mathbb{P}^1) = \mathbb{C}[\delta u_1, \delta u_2, z]/(z - \delta u_1)(z - \delta u_2),$$

where $\delta u_1, \delta u_2$ are local coordinates centered at x . Taking Spec, this is the gluing of two copies of \mathbb{C}^2 along the diagonal. Overall we obtain that

$$\text{Ell}_\mathbb{T}(X) = (\mathcal{O}_p \cup_\Delta \mathcal{O}_q) / \Delta,$$

where $\mathcal{O}_p \cong \mathcal{O}_q \cong \mathcal{E}_\mathbb{T}$, and $/\Delta$ denotes the gluing of these two abelian varieties along the diagonal

$$\Delta = \{(u_1, u_2) \mid u_1 = u_2\} \subset \mathcal{E}_\mathbb{T}.$$

2.2 GKM varieties

We assume further that the set of fixed points $X^\mathbb{T}$ is a finite set of isolated points. We will only encounter varieties of this type in our paper. In this case, for a generic one-parametric subgroup $\mathbb{T}_t \subset \mathbb{T}$ we have

$$X^{\mathbb{T}^t} = X^\mathbb{T}.$$

Therefore, similarly to Example 1 we conclude that $\text{Ell}_\mathbb{T}(X)$ is union of $|X^\mathbb{T}|$ copies of $\mathcal{E}_\mathbb{T}$:

$$\text{Ell}_\mathbb{T}(X) = \left(\coprod_{p \in X^\mathbb{T}} \mathcal{O}_p \right) / \Delta, \tag{6}$$

where $\mathcal{O}_p \cong \mathcal{E}_\mathbb{T}$ and $/\Delta$ denotes the gluing of these abelian varieties along the subschemes $\text{Spec } H^\bullet(X^{\mathbb{T}^t})$ corresponding to subtori \mathbb{T}_t for which the fixed sets $X^{\mathbb{T}^t}$ are larger than $X^\mathbb{T}$. We call \mathcal{O}_p the \mathbb{T} -orbit associated to the fixed point p in $\text{Ell}_\mathbb{T}(X)$.

In general, the subscheme Δ describing the intersections of orbits in the ‘‘bouquet’’ (6) can be quite involved. There is, however, a special case when it is relatively simple.

Definition 1. *We say that \mathbb{T} -variety X is a GKM variety if it satisfies the following conditions:*

- $X^\mathbb{T}$ is finite,
- for every two fixed points $p, q \in X^\mathbb{T}$ there is no more than one \mathbb{T} -equivariant curve connecting them.

Note that by definition, a GKM variety contains finitely many \mathbb{T} -equivariant compact curves (i.e., curves starting and ending at fixed points). We note also that all these curves are rational $C \cong \mathbb{P}^1$ because \mathbb{T} -action on C exists only in this case.

For a compact curve C connecting fixed points p and q , let $\chi_C \in \text{Char}(\mathbb{T}) = \text{Hom}(\mathcal{E}_\mathbb{T}, E)$ be the character of the tangent space $T_p C$. For all points t on the hyperplane $\chi_C^\perp \subset \mathcal{E}_\mathbb{T}$ we thus have $X^{\mathbb{T}^t} = X^\mathbb{T} \cup C$. This means that in (6) the \mathbb{T} -orbits \mathcal{O}_p and \mathcal{O}_q are glued along the common hyperplane

$$\mathcal{O}_p \supset \chi_C^\perp \subset \mathcal{O}_q.$$

Note that the character of $T_q C$ is $-\chi_C$ so it does not matter which end point we choose as the first. In sum, we have:

Proposition 1. *If X is a GKM variety, then*

$$\text{Ell}_{\mathbb{T}}(X) = \left(\prod_{p \in X^{\mathbb{T}}} \mathcal{O}_p \right) / \Delta,$$

where $/\Delta$ denotes the gluing of \mathbb{T} -orbits \mathcal{O}_p and \mathcal{O}_q along the hyperplanes

$$\mathcal{O}_p \supset \chi_C^{\perp} \subset \mathcal{O}_q,$$

for all p and q connected by an equivariant curve C .

Proof. Locally around $t \in \mathcal{E}_{\mathbb{T}}$, the stalk of \mathcal{H} is given by $H_{\mathbb{T}}^{\bullet}(X^{\mathbb{T}t})$. By the property of equivariant cohomology of GKM varieties [21], the variety $\text{Spec } H_{\mathbb{T}}^{\bullet}(X^{\mathbb{T}t})$ is the gluing of \mathfrak{t}_p along hyperplanes χ_C^{\perp} , where $\mathfrak{t}_p \cong \mathbb{C}^r$ are Lie algebras of the torus, associated to fixed points. \square

In particular, one can see that the intersections of orbits \mathcal{O}_p and \mathcal{O}_q are always transversal and hence $\text{Ell}_{\mathbb{T}}(X)$ is a variety with simple normal crossing singularities.

The classical examples of GKM varieties include Grassmannians or more generally, partial flag varieties. For non-GKM varieties the structure of subschemes $\text{Spec } H^{\bullet}(X^{\mathbb{T}t})$ and intersection of orbits in (6) can be quite involved.

2.3 Extended elliptic cohomology

We define

$$\mathcal{E}_{\text{Pic}(X)} := \text{Pic}(X) \otimes_{\mathbb{Z}} E \cong E^{\dim(\text{Pic}(X))}. \quad (7)$$

For Nakajima quiver varieties $\text{Pic}(X) \cong \mathbb{Z}^{|Q|}$ and thus $\mathcal{E}_{\text{Pic}(X)} \cong E^{|Q|}$, where $|Q|$ denotes the number of vertices in the quiver. We will refer to the coordinates in this abelian variety as *Kähler parameters*. We will usually denote the Kähler parameters by the symbol z_i , $i = 1, \dots, |Q|$.

The extended \mathbb{T} -orbits are defined by

$$\widehat{\mathcal{O}}_p := \mathcal{O}_p \times \mathcal{E}_{\text{Pic}(X)},$$

and the extended elliptic cohomology by

$$\text{E}_{\mathbb{T}}(X) := \text{Ell}_{\mathbb{T}}(X) \times \mathcal{E}_{\text{Pic}(X)}.$$

In particular, $\text{E}_{\mathbb{T}}(X)$ is a bouquet of extended orbits:

$$\text{E}_{\mathbb{T}}(X) = \left(\prod_{p \in X^{\mathbb{T}}} \widehat{\mathcal{O}}_p \right) / \Delta$$

where Δ denotes the same gluing of orbits as in (6), i.e., the extended orbits are glued only along the equivariant directions.

2.4 Line bundles on elliptic cohomology

We have the following description of a line bundle on the variety $\text{E}_{\mathbb{T}}(X)$.

Proposition 2.

- A line bundle \mathcal{T} on the scheme $\text{E}_{\mathbb{T}}(X)$ is a collection of line bundles \mathcal{T}_p on extended orbits $\widehat{\mathcal{O}}_p$, $p \in X^{\mathbb{T}}$, which coincide on the intersections:

$$\mathcal{T}_p|_{\widehat{\mathcal{O}}_p \cap \widehat{\mathcal{O}}_q} = \mathcal{T}_q|_{\widehat{\mathcal{O}}_p \cap \widehat{\mathcal{O}}_q},$$

- A meromorphic (holomorphic) section s of a line bundle \mathcal{T} is the collection of meromorphic (holomorphic) sections s_p of \mathcal{T}_p which agree on intersections:

$$s_p|_{\widehat{\mathcal{O}}_p \cap \widehat{\mathcal{O}}_q} = s_q|_{\widehat{\mathcal{O}}_p \cap \widehat{\mathcal{O}}_q}. \quad (8)$$

Since each orbit $\widehat{\mathcal{O}}_p$ is isomorphic to the base $\mathcal{E}_\mathbb{T} \times \mathcal{E}_{\text{Pic}(X)}$, each \mathcal{T}_p is isomorphic via the pull-back along π^* to a line bundle on the base. In practice, we often use the coordinates on the base to describe \mathcal{T}_p 's.

Example 2. Characterization of line bundles and sections is more complicated for non-GKM varieties. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, with homogeneous coordinates $([x : y], [z : w])$, and let \mathbb{C}^* acts on it by

$$t \cdot ([x : y], [z : w]) = ([x : ty], [z : tw]).$$

There are four fixed points, but infinitely many \mathbb{C}^* -invariant curves: the closure of $\{([x : y], [x : \lambda y])\}$ for any $\lambda \in \mathbb{C}^*$ is a \mathbb{C}^* -invariant curve, connecting the points $([1 : 0], [1 : 0])$ and $([0 : 1], [0 : 1])$. Locally near the identity $1 \in \mathcal{E}_{\mathbb{C}^*}$, the elliptic cohomology $\text{Ell}_{\mathbb{C}^*}(X)$ looks like

$$\text{Spec } H_{\mathbb{T}}^*(X) = \text{Spec } \mathbb{C}[H, u]/(H - u)^2(H + u)^2 \rightarrow \text{Spec } \mathbb{C}[u],$$

where u is the local coordinate near $1 \in \mathcal{E}_{\mathbb{C}^*}$. We see that locally the elliptic cohomology has two non-reduced irreducible components, each of multiplicity two.

2.5 Theta functions

By Proposition 2, to specify a line bundle \mathcal{T} on $E_\mathbb{T}(X)$ one needs to define line bundles \mathcal{T}_p on each orbit $\widehat{\mathcal{O}}_p$. As $\widehat{\mathcal{O}}_p$ is an abelian variety, to fix \mathcal{T}_p it suffices to describe the transformation properties of sections as we go around periods of $\widehat{\mathcal{O}}_p$. In other words, to define \mathcal{T}_p one needs to fix quasiperiods w_i of sections

$$s(x_i q) = w_i s(x_i),$$

for all coordinates x_i on $\widehat{\mathcal{O}}_p$, i.e., for all equivariant and Kähler parameters.

The abelian variates $\widehat{\mathcal{O}}_p$ are all some powers of E , which implies that sections of a line bundle on $E_\mathbb{T}(X)$ can be expressed explicitly through the Jacobi theta function associated with E :

$$\theta(x) := (qx)_\infty (x^{1/2} - x^{-1/2})(q/x)_\infty, \quad (x)_\infty = \prod_{i=0}^{\infty} (1 - xq^i).$$

The elementary transformation properties of this function are:

$$\theta(xq) = -\frac{1}{x\sqrt{q}}\theta(x), \quad \theta(1/x) = -\theta(x).$$

We also extend it by linearity and define

$$\Theta\left(\sum_i a_i - \sum_j b_j\right) := \frac{\prod_i \theta(a_i)}{\prod_j \theta(b_j)}.$$

By definition, the elliptic stable envelope associated with a \mathbb{T} -variety X is a section of certain line bundle on $E_\mathbb{T}(X)$ [1]. Thus, one can use theta-functions to give explicit formulas for stable envelopes, see Theorem 3 for example. It will also be convenient to introduce the following combination:

$$\phi(x, y) = \frac{\theta(xy)}{\theta(x)\theta(y)}.$$

This function has the following quasiperiods:

$$\phi(xq, y) = y^{-1}\phi(xq, y), \quad \phi(x, yq) = x^{-1}\phi(x, y).$$

These transformation properties define the so-called Poincaré line bundle on the product of dual elliptic curves $E \times E^\vee$ with coordinates x and y and $\phi(x, y)$ is a meromorphic section of this bundle.

3 Elliptic Stable Envelope for X

In this section, we discuss algebraic variety $X = T^*Gr(k, n)$ – the cotangent bundle over the Grassmannian of k -dimensional subspaces in an n -dimensional complex space.

3.1 X as a Nakajima quiver variety

We consider a Nakajima quiver variety X defined by the A_1 -quiver, with dimension $\mathbf{v} = k$ and framing $\mathbf{w} = n$. Explicitly, this variety has the following construction. Let $R = \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ be a vector space of complex $k \times n$ matrices. There is an obvious action of $GL(k)$ on this space, which extends to an Hamiltonian action on its cotangent bundle:

$$T^*R = R \oplus R^* \cong \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^k),$$

with the Hamiltonian moment map

$$\mu : T^*R \rightarrow \mathfrak{gl}(k)^*, \quad \mu(\mathbf{j}, \mathbf{i}) = \mathbf{i}\mathbf{j}.$$

Then X is defined as

$$X := \mu^{-1}(0) \cap \{\theta\text{-semistable points}\} / GL(k),$$

where $\mathbf{j} \in R$ and $\mathbf{i} \in R^*$ are $n \times k$ and $k \times n$ matrices respectively. There are two choices of stability conditions $\theta < 0$ and $\theta > 0$. In the first case the semistable points are those pairs (\mathbf{j}, \mathbf{i}) with injective \mathbf{j} :

$$\{\theta\text{-semistable points}\} = \{(\mathbf{j}, \mathbf{i}) \in T^*R \mid \text{rank}(\mathbf{j}) = k\}, \quad (9)$$

In the case $\theta > 0$ the semistable points are (\mathbf{j}, \mathbf{i}) with \mathbf{i} surjective [19]:

$$\{\theta\text{-semistable points}\} = \{(\mathbf{j}, \mathbf{i}) \in T^*R \mid \text{rank}(\mathbf{i}) = k\}.$$

The general theory assures that X is a smooth holomorphic symplectic variety. In this paper, we choose

$$\theta = (-1) \in \text{Lie}_{\mathbb{R}}(\mathbb{K}),$$

where $\mathbb{K} := U(1)$, as the stability condition defining X , in which case it is isomorphic to the cotangent bundle of the Grassmannian of complex k -dimensional vector subspaces in an n -dimensional space.

3.2 Torus action on X

Let $\mathbb{A} = (\mathbb{C}^\times)^n$ be a torus acting on \mathbb{C}^n by scaling the coordinates:

$$(z_1, \dots, z_n) \mapsto (z_1 u_1^{-1}, \dots, z_n u_n^{-1}), \quad (10)$$

which induces an action of \mathbb{A} on T^*R . We denote by \mathbb{C}_\hbar^\times the torus acting on T^*R by scaling the second component:

$$(\mathbf{j}, \mathbf{i}) \rightarrow (\mathbf{j}, \mathbf{i} \hbar^{-1})$$

We denote the whole torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_\hbar^\times$. The action of \mathbb{T} preserves semistable locus of $\mu^{-1}(0)$ and thus descends to X . Simple check shows that the action of \mathbb{A} preserves the symplectic form on X , while \mathbb{C}_\hbar^\times scales it by \hbar .

Note that the action (10) leaves invariant k -dimensional subspaces spanned by arbitrary k coordinate vectors. Thus, the set of \mathbb{T} -fixed points $X^\mathbb{T}$ consists of $n!/((n-k)!k!)$ points corresponding to k -dimensional coordinate subspaces in \mathbb{C}^n . In other word, a fixed point $\lambda \in X^\mathbb{T}$ is described by a k -subset in the set $\{1, 2, \dots, n\}$.

3.3 T-equivariant K -theory of X

Let us denote the tautological bundles on X associated to \mathbb{C}^k and \mathbb{C}^n by \mathcal{V} and \mathcal{W} respectively. The bundle \mathcal{W} is a topologically trivial rank- n vector bundle, because \mathbb{C}^n is a trivial representation of $GL(k)$. In contrast, \mathcal{V} is a nontrivial rank- k subbundle of \mathcal{W} . One can easily see that \mathcal{V} is the standard tautological bundle of k -subspaces on the Grassmannian. We assume that the tautological bundle splits in K -theory into a sum of virtual line bundles,

$$\mathcal{V} = y_1 + \cdots + y_k. \quad (11)$$

In other words, y_i denote the Chern roots of \mathcal{V} . The T-equivariant K -theory of X has the form:

$$K_{\mathbb{T}}(X) = \mathbb{C}[y_1^{\pm 1}, \dots, y_k^{\pm 1}]^{\mathfrak{S}_k} \otimes \mathbb{C}[u_i^{\pm 1}, \hbar^{\pm 1}] / I$$

where \mathfrak{S}_k is the symmetric group of k elements, and I denotes the ideal of Laurent polynomials with vanishing restrictions at the fixed points, i.e., at (12). For our choice of stability condition, the matrix \mathbf{j} representing a fixed point is of rank k , thus if \mathbf{p} is a fixed point corresponding to the k -subset $\{\mathbf{p}_1, \dots, \mathbf{p}_k\} \subset \{1, 2, \dots, n\}$, then

$$y_i|_{\mathbf{p}} = u_{\mathbf{p}_i}^{-1}, \quad i = 1, \dots, k. \quad (12)$$

This means that if a K -theory class is represented by a Laurent polynomial $f(y_i)$ then its restriction to a fixed point is given by the substitution $f(y_i)|_{\mathbf{p}} = f(u_{\mathbf{p}_i}^{-1})$.

We note that for the opposite choice of the stability parameter $\theta < 0$ the restriction would take the form $f(y_i)|_{\mathbf{p}} = f(u_{\mathbf{p}_i}^{-1} \hbar^{-1})$, where the extra factor \hbar^{-1} comes from the action of $\mathbb{C}_{\hbar}^{\times}$ on the matrix \mathbf{i} , which is of rank k for this choice of stability condition.

3.4 Tangent and polarization bundles

The definition of the elliptic stable envelope requires the choice of a polarization and a chamber [1]. The polarization $T^{1/2}X$, as a virtual bundle, is a choice of the half of the tangent space. In other words,

$$TX = T^{1/2}X + \hbar^{-1}T^{1/2}X^*.$$

We choose the polarization dual to the canonical polarization (which is defined for all Nakajima varieties, see Example 3.3.3. in [31]):

$$T^{1/2}X = \hbar^{-1}\mathcal{W}^* \otimes \mathcal{V} - \hbar^{-1}\mathcal{V}^* \otimes \mathcal{V}. \quad (13)$$

Expressing TX through the Chern roots by (11) and restricting it to a fixed point \mathbf{p} by (12), we find the T-character of the tangent space at \mathbf{p} equals:

$$T_{\mathbf{p}}X = \sum_{\substack{i \in \mathbf{p} \\ j \in \mathbf{n} \setminus \mathbf{p}}} \left(\frac{u_i}{u_j} + \hbar^{-1} \frac{u_j}{u_i} \right), \quad (14)$$

where \mathbf{p} denotes the k -subset in $\mathbf{n} = \{1, \dots, n\}$.

The definition of the stable envelope also requires the choice of a chamber, or equivalently, a cocharacter of the torus \mathbf{A} . We choose σ explicitly as

$$\sigma = (1, 2, \dots, n) \in \text{Lie}_{\mathbb{R}}(\mathbf{A}). \quad (15)$$

The choice of σ fixes the decomposition $T_{\mathbf{p}}X = N_{\mathbf{p}}^+ \oplus N_{\mathbf{p}}^-$, where $N_{\mathbf{p}}^{\pm}$ are the subspaces whose \mathbf{A} -characters take positive or negative values on σ . From (14) we obtain:

$$N_{\mathbf{p}}^- = \sum_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i < j}} \frac{u_i}{u_j} + \hbar^{-1} \sum_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i > j}} \frac{u_j}{u_i}, \quad N_{\mathbf{p}}^+ = \sum_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i > j}} \frac{u_i}{u_j} + \hbar^{-1} \sum_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i < j}} \frac{u_j}{u_i}. \quad (16)$$

3.5 Elliptic cohomology of X

Let us first note that X is a GKM variety. Two fixed points \mathbf{p}, \mathbf{q} are connected by an equivariant curve C if and only if the corresponding k -subsets differ by one index $\mathbf{p} = \mathbf{q} \setminus \{i\} \cup \{j\}$. In this case the \mathbb{T} -character of the tangent space equals:

$$T_{\mathbf{p}}C = u_i/u_j.$$

We conclude that the extended elliptic cohomology scheme equals:

$$E_{\mathbb{T}}(X) = \left(\prod_{\mathbf{p} \in X^{\mathbb{T}}} \widehat{\mathcal{O}}_{\mathbf{p}} \right) / \Delta \quad (17)$$

with $\widehat{\mathcal{O}}_{\mathbf{p}} \cong \mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\text{Pic}(X)}$ and $/\Delta$ denotes gluing of abelian varieties $\widehat{\mathcal{O}}_{\mathbf{p}}$ and $\widehat{\mathcal{O}}_{\mathbf{q}}$ with $\mathbf{p} = \mathbf{q} \setminus \{i\} \cup \{j\}$ along the hyperplanes $u_i = u_j$.

Let us consider the following functions (see Section 2.5 for the notations):

$$\mathcal{U}_{\mathbf{p}, \mathbf{q}}(X) = \Theta \left(T^{1/2}X \Big|_{\mathbf{q}} \right) \prod_{i=1}^k \frac{\phi(u_{\mathbf{p}_i}^{-1}, z^{-1})}{\phi(u_{\mathbf{q}_i}^{-1}, z^{-1})} \prod_{i=1}^n \frac{\phi(u_i, z_{u_i}^{-1} \hbar^{D_i^{\mathbf{p}}})}{\phi(u_i, z_{u_i}^{-1})}. \quad (18)$$

Here, the powers $D_i^{\mathbf{p}}$ come from the index of the polarization bundle. They are computed as follows: for our choice of the polarization (13) and chamber (15) the index of a fixed point \mathbf{p} equals:

$$\text{ind}_{\mathbf{p}} = T^{1/2}X \Big|_{\mathbf{p}, >} = \sum_{\substack{i \in \mathbf{p} \\ j \notin \mathbf{p} \\ j > i}} \frac{u_j}{u_i \hbar},$$

and the integers $D_i^{\mathbf{p}}$ are the degrees of the index bundle, i.e., the degree in variable u_i of the monomial:

$$\det \text{ind}_{\mathbf{p}} = \prod_{\substack{i \in \mathbf{p} \\ j \notin \mathbf{p} \\ j > i}} \frac{u_j}{u_i \hbar}.$$

The collection $\{\mathcal{U}_{\mathbf{p}, \mathbf{q}} \mid \mathbf{q} \in X^{\mathbb{T}}\}$ form is a meromorphic section of some line bundle which we denote by $\mathcal{T}(\mathbf{p})$. It means that the transformation properties of sections of $\mathcal{T}(\mathbf{p})|_{\widehat{\mathcal{O}}_{\mathbf{q}}}$ are given by quasiperiods of functions $\mathcal{U}_{\mathbf{p}, \mathbf{q}}(X)$.

By definition, the elliptic stable envelope $\text{Stab}_{\sigma}(\mathbf{p})$ of a fixed point \mathbf{p} (corresponding to the choice of chamber σ and polarization $T^{1/2}X$) is a section of $\mathcal{T}(\mathbf{p})$ fixed uniquely by a list of properties [1]. Alternative version of the elliptic stable envelope for cotangent bundles to partial flag varieties was defined in [42, 12]. Comparing explicit formulas for elliptic stable envelopes in the case of the variety X from [1] and from [42, 12] one observes that they differ by a multiple. The definition of [42, 12] is based on the fact that X is a GKM variety, while definition of [1] is more general and is not restricted to GKM varieties. In fact, the Nakajima varieties are almost never GKM varieties. In this paper we choose the approach of [42, 12], because GKM structure of X will simplify the computations. As we mentioned already, in the case of variety X both approaches lead to the same explicit formulas, thus there is no ambiguity in this choice.

Definition 2. *The elliptic stable envelope of a fixed point $\text{Stab}_{\sigma}(\mathbf{p})$ is the unique section of $\mathcal{T}(\mathbf{p})$, such that its components*

$$T_{\mathbf{p}, \mathbf{q}} := \text{Stab}_{\sigma}(\mathbf{p})|_{\widehat{\mathcal{O}}_{\mathbf{q}}}$$

satisfy the following properties

- 1) $T_{\mathbf{p}, \mathbf{p}} = \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i < j}} \theta\left(\frac{u_i}{u_j}\right) \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right).$
- 2) $T_{\mathbf{p}, \mathbf{q}} = f_{\mathbf{p}, \mathbf{q}} \prod_{\substack{i \in \mathbf{q}, \\ j \in \mathbf{n} \setminus \mathbf{q}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right),$ where $f_{\mathbf{p}, \mathbf{q}}$ is holomorphic in parameters u_i .

Let us note that the fact that $\text{Stab}_{\sigma}(\mathbf{p})$ is a section of $\mathcal{T}(\mathbf{p})$ implies that its restrictions $T_{\mathbf{p}, \mathbf{q}}$ are sections of line bundles on abelian varieties $\widehat{\mathcal{O}}_{\mathbf{q}}$ which have the same transformation properties in all variables as $\mathcal{U}_{\mathbf{p}, \mathbf{q}}(X)$.

3.6 Uniqueness of stable envelope for X

To justify the last definition, we need the following uniqueness theorem.

Theorem 2. [Appendix A, [12]] *The matrix $T_{\mathbf{p},\mathbf{q}}$ satisfying:*

1) *For a given fixed \mathbf{p} , the collection $\{T_{\mathbf{p},\mathbf{q}} \mid \mathbf{q} \in X^\Gamma\}$ form a section of the line bundle $\mathcal{T}(\mathbf{p})$ (as defined by (18)).*

$$2) T_{\mathbf{p},\mathbf{p}} = \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i < j}} \theta\left(\frac{u_i}{u_j}\right) \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right).$$

$$3) T_{\mathbf{p},\mathbf{q}} = f_{\mathbf{p},\mathbf{q}} \prod_{\substack{i \in \mathbf{q}, \\ j \in \mathbf{n} \setminus \mathbf{q}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right), \text{ where } f_{\mathbf{p},\mathbf{q}} \text{ is holomorphic in parameters } u_i.$$

in unique.

Proof. Assume that we have two matrices which satisfy 1),2),3) and let $\kappa_{\mathbf{p},\mathbf{q}}$ be their difference. Assume that $\kappa_{\mathbf{p},\mathbf{q}} \neq 0$ for some \mathbf{p} . Let \mathbf{q} be a maximal (in the order defined by the chamber) fixed point such that $\kappa_{\mathbf{p},\mathbf{q}} \neq 0$. By 3) we know that

$$\kappa_{\mathbf{p},\mathbf{q}} = f_{\mathbf{p},\mathbf{q}} \prod_{\substack{i \in \mathbf{q}, \\ j \in \mathbf{n} \setminus \mathbf{q}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right), \quad (19)$$

where $f_{\mathbf{p},\mathbf{q}}$ is a holomorphic function of u_i .

For $i \in \mathbf{q}$ and $j \in \mathbf{n} \setminus \mathbf{q}$ with $i < j$, consider the point $\mathbf{q}' = \mathbf{q} \setminus \{i\} \cup \{j\}$. By construction, \mathbf{q} and \mathbf{q}' are connected by an equivariant curve with character u_i/u_j . The condition 1) means:

$$\left(\kappa_{\mathbf{p},\mathbf{q}} - \kappa_{\mathbf{p},\mathbf{q}'}\right)\Big|_{u_i=u_j} = 0.$$

By construction $\mathbf{q} < \mathbf{q}'$ (in the order on fixed points) and thus $\kappa_{\mathbf{p},\mathbf{q}'} = 0$, which implies $\kappa_{\mathbf{p},\mathbf{q}}|_{u_i=u_j} = 0$. Comparing with (19) we conclude that $f_{\mathbf{p},\mathbf{q}}$ is divisible by $\theta(u_i/u_j)$. Going over all such pairs of i, j we find:

$$\kappa_{\mathbf{p},\mathbf{q}} = f'_{\mathbf{p},\mathbf{q}} \prod_{\substack{i \in \mathbf{q}, \\ j \in \mathbf{n} \setminus \mathbf{q}, \\ i < j}} \theta\left(\frac{u_i}{u_j}\right) \prod_{\substack{i \in \mathbf{q}, \\ j \in \mathbf{n} \setminus \mathbf{q}, \\ i > j}} \theta\left(\frac{u_j}{u_i} \hbar^{-1}\right) = f'_{\mathbf{p},\mathbf{q}} T_{\mathbf{q},\mathbf{q}},$$

where $f'_{\mathbf{p},\mathbf{q}}$ is holomorphic in u_i . As a holomorphic function it can be expanded as $f'_{\mathbf{p},\mathbf{q}} = \sum_{k=0}^{\infty} c_k u_i^k$ with nonzero radius of convergence.

The quasiperiods of functions $T_{\mathbf{p},\mathbf{q}}$ are the same as those of the functions $\mathcal{U}(\mathbf{p})|_{\mathbf{q}}$. In particular, for all $i \notin \mathbf{p} \cap \mathbf{q}$ from (18) we find:

$$f'_{\mathbf{p},\mathbf{q}}(u_i q) = f'_{\mathbf{p},\mathbf{q}}(u_i) z^{\pm 1} \hbar^m$$

for some integer m . We obtain:

$$\sum_{k=0}^{\infty} c_k (z^{\pm 1} \hbar^m - q^k) u_i^k = 0$$

and thus $c_k = 0$ for all k , i.e., $f'_{\mathbf{p},\mathbf{q}} = 0$. □

3.7 Existence of elliptic stable envelope for X

The following result is proven in [1, 12, 27]:

Theorem 3. *For canonical polarization (13) and chamber (15) the elliptic stable envelope of a fixed point $\mathbf{p} \in X^\Gamma$ has the following explicit form:*

$$\text{Stab}_\sigma(\mathbf{p}) = \text{Sym} \left(\frac{\prod_{l=1}^k \left(\prod_{i=1}^{\mathbf{p}_l-1} \theta(y_l u_i \hbar^{-1}) \frac{\theta(y_l u_{\mathbf{p}_l} z^{-1} \hbar^{k-n+\mathbf{p}_l-2l})}{\theta(z^{-1} \hbar^{k-n+\mathbf{p}_l-2l})} \prod_{i=\mathbf{p}_l+1}^n \theta(y_l^{-1} u_i^{-1}) \right)}{\prod_{1 \leq i < j \leq n} \theta(y_i/y_j) \theta(\hbar y_i/y_j)} \right) \quad (20)$$

where the symbol *Sym* stands for the symmetrization over all Chern roots y_1, \dots, y_k .

Note that the components $T_{\mathbf{p},\mathbf{q}}$ are defined by this explicit formula as restriction $T_{\mathbf{p},\mathbf{q}} = \text{Stab}_{\theta,\sigma}(\mathbf{p})|_{\mathbf{q}} = \text{Stab}_{\theta,\sigma}(\mathbf{p})|_{y_i = u_{\mathbf{q}_i}^{-1}}$. The proof of this theorem is by checking the properties 1)-3) from Theorem 2 explicitly, details can be found in [12].

3.8 Holomorphic normalization

Note that the stable envelope (20) has poles in the Kähler parameter z . It will be more convenient to work with a different normalization of the stable envelope in which it is holomorphic in z :

$$\mathbf{Stab}(\mathbf{p}) := \Theta_{\mathbf{p}} \text{Stab}_\sigma(\mathbf{p}), \quad (21)$$

where $\Theta_{\mathbf{p}}$ is the section of a line bundle on the Kähler part $\mathcal{E}_{\text{Pic}(X')}$ defined explicitly by

$$\Theta_{\mathbf{p}} = \prod_{m=1}^k \theta(z^{-1} \hbar^{k-n+\mathbf{p}_m-2m}).$$

(For X' and T' , see Section 4.) Similarly to Theorem 2, the stable envelope $\mathbf{Stab}(\mathbf{p})$ can be defined as a unique section of the twisted line bundle on $E_{\mathcal{T}}(X)$:

$$\mathfrak{M}(\mathbf{p}) = \mathcal{T}(\mathbf{p}) \otimes \Theta_{\mathbf{p}}. \quad (22)$$

with diagonal restrictions (Property 2 in Theorem 2) given by $T_{\mathbf{p},\mathbf{p}} \Theta_{\mathbf{p}}$. Note that the function $\Theta_{\mathbf{p}}$ only depends on Kähler variables. Thus, the twist of line bundle (22) does not affect quasiperiods of stable envelopes in the equivariant parameters.

We will see that the section $\Theta_{\mathbf{p}}$ has the following geometric meaning: it represents the elliptic Thom class of the repelling normal bundle on the dual variety X' (see (28)):

$$\Theta_{\mathbf{p}} = \Theta(N_{\lambda}'^{-}),$$

where λ is related to \mathbf{p} by (41), with parameter a_1/a_2 related to Kähler parameter z by (42).

4 Elliptic Stable Envelope for X'

4.1 X' as a Nakajima quiver variety

From now on we always assume that $n \geq 2k$. In this section we consider the variety X' which is a Nakajima quiver variety associated to the A_{n-1} quiver. This variety is defined by the framing dimension vector:

$$\mathbf{w}_i = \delta_{k,i} + \delta_{n-k,i},$$

i.e., all framing spaces are trivial except those at position k and $n-k$. Both non-trivial framing spaces are one-dimensional. The dimension vector has the form

$$\mathbf{v} = (1, 2, \dots, k-1, \underbrace{k, \dots, k}_{(n-2k+1)\text{-times}}, k-1, \dots, 2, 1).$$

By definition, this variety is given by the following symplectic reduction. Let us consider the vector space:

$$R = \bigoplus_{i=1}^{n-2} \text{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_{i+1}}) \bigoplus \text{Hom}(\mathbb{C}, \mathbb{C}^{\mathbf{v}_k}) \bigoplus \text{Hom}(\mathbb{C}^{\mathbf{v}_{n-k}}, \mathbb{C}),$$

and denote the representatives by $(\mathbf{a}_l, \mathbf{i}_k, \mathbf{j}_{n-k})$, $l = 1, \dots, n-2$. Similarly, the dual vector space:

$$R^* = \bigoplus_{i=1}^{n-2} \text{Hom}(\mathbb{C}^{\mathbf{v}_{i+1}}, \mathbb{C}^{\mathbf{v}_i}) \bigoplus \text{Hom}(\mathbb{C}^{\mathbf{v}_k}, \mathbb{C}) \bigoplus \text{Hom}(\mathbb{C}, \mathbb{C}^{\mathbf{v}_{n-k}})$$

with representatives by $(\mathbf{b}_l, \mathbf{j}_k, \mathbf{i}_{n-k})$. We consider the symplectic space $T^*R = R \oplus R^*$ and the moment map

$$\mu : T^*R \rightarrow \bigoplus_{i=1}^{n-1} \mathfrak{gl}(\mathbf{v}_i)^*.$$

Denote $\mathbf{a} = \bigoplus_i \mathbf{a}_i$, $\mathbf{b} = \bigoplus_i \mathbf{b}_i$, $\mathbf{i} = \bigoplus_i \mathbf{i}_i$ and $\mathbf{j} = \bigoplus_i \mathbf{j}_i$, then the moment takes the explicit form $\mu((\mathbf{a}, \mathbf{b}, \mathbf{i}, \mathbf{j}) = [\mathbf{b}, \mathbf{a}] + \mathbf{i} \circ \mathbf{j}$. With this notation X' is defined as the quotient:

$$X' := \mu^{-1}(0) \cap \{\theta' \text{-semistable points}\} / \prod_{i=1}^{n-1} GL(\mathbf{v}_i).$$

We will use the canonical choice of the stability parameter ²

$$\theta' = (1, 1, \dots, 1) \in \text{Lie}_{\mathbb{R}}(\mathbf{K}'),$$

where $\mathbf{K}' := U(1)^{n-1}$. The set of the θ' -semistable points in T^*R is described as follows : a point $((\mathbf{a}, \mathbf{i}_k, \mathbf{j}_{n-k}), (\mathbf{b}, \mathbf{j}_k, \mathbf{i}_{n-k})) \in \mu^{-1}(0)$ is θ' -semistable, if and only if the image of $\mathbf{i}_k \oplus \mathbf{i}_{n-k}$ under the actions of $\{\mathbf{a}_l, \mathbf{b}_l, 1 \leq l \leq n-2\}$ generate the entire space $\bigoplus_{i=1}^{n-1} \mathbb{C}^{\mathbf{v}_i}$.

4.2 Tautological bundles over X'

We denote by \mathcal{V}_i the rank \mathbf{v}_i tautological vector bundle on X' associated to $\mathbb{C}^{\mathbf{v}_i}$. It will be convenient to represent the dimension vector and associated tautological bundles using the following combinatorial description. Let us consider a rectangle $R_{n,k}$ with dimensions $k \times (n-k)$. We turn $R_{n,k}$ by 45° as in the Fig.3. We will denote by $\square = (i, j) \in R_{n,k}$ a box in $R_{n,k}$ with coordinates (i, j) , $i = 1, \dots, n-k$ and $j = 1, \dots, k$. We define a function of *diagonal number* on boxes:

$$c_{\square} = i - j + k.$$

Note that $1 \leq c_{\square} \leq n-1$. It may be convenient to visualize c_{\square} as the horizontal coordinate of a box \square as in Fig.4. The total number of boxes with $c_{\square} = i$ is $\mathbf{v}_i = \dim \mathcal{V}_i$. With a box (i, j) we associate a variable x_{ij} . It will be convenient to think about the set of x_{ij} with the same c_{\square} as Chern roots of tautological bundles, such that in K -theory we have:

$$\mathcal{V}_m = \sum_{c_{\square}=m} x_{\square}.$$

The tautological bundles \mathcal{V}_i generate the equivariant K -theory of X' . The K -theory classes are represented by Laurent polynomials in x_{\square} :

$$K_{\mathbb{T}'}(X') = \mathbb{C}[x_{ij}^{\pm 1}]^{\mathfrak{S}_{n,k}} \otimes \mathbb{C}[a_1^{\pm}, a_2^{\pm}, \hbar^{\pm}] / I,$$

²We use the same notations for stability condition as in the Maulik-Okounkov [31]. In particular, for us the stability parameter $\theta = (\theta_i)$ corresponds to a character $\chi : \prod_i GL(\mathbf{v}_i) \rightarrow \mathbb{C}^{\times}$ given by

$$\chi : (g_i) \mapsto \prod_i (\det g_i)^{\theta_i}.$$

This notation is opposite to one used in Ginzburg's lectures [19], where θ corresponds to the character $\prod_i (\det g_i)^{-\theta_i}$.

where T' is the torus described in the next subsection. These are the Laurent polynomials symmetric with respect to each group of Chern roots, i.e., invariant under the group:

$$\mathfrak{S}_{n,k} = \prod_{i=1}^{n-1} \mathfrak{S}_{v_i}. \quad (23)$$

where \mathfrak{S}_{v_i} acts by permutations on x_{ij} with $c_{\square} = l$. The ideal I is the ideal of polynomials which restricts to zero at every fixed point:

$$I = \{f(x_{i,j}) : f(x_{i,j})|_{x_{i,j}=\varphi_{i,j}^{\lambda}}, \forall \lambda \in (X')^{T'}\}.$$

4.3 Torus action on X'

Let $A' = (\mathbb{C}^{\times})^2$ be a 2-dimensional torus acting on the framing space $\mathbb{C} \oplus \mathbb{C}$ by

$$(z_1, z_2) \mapsto (z_1 a_1, z_2 a_2).$$

Let \mathbb{C}_h^{\times} be the 1-dimensional torus acting on T^*R by scaling the cotangent fiber

$$((\mathbf{a}, \mathbf{i}_k, \mathbf{j}_{n-2k}), (\mathbf{b}, \mathbf{j}_k, \mathbf{i}_{n-2k})) \mapsto ((\mathbf{a}, \mathbf{i}_k, \mathbf{j}_{n-2k}), \hbar(\mathbf{b}, \mathbf{j}_k, \mathbf{i}_{n-2k})).$$

Denote their product by $T' = A' \times \mathbb{C}_h^{\times}$. The fixed loci in X' under the A' -action admit a *tensor product* decomposition:

$$(X')^{A'} = \coprod_{\mathbf{v}^{(1)} + \mathbf{v}^{(2)} = \mathbf{v}} \mathcal{M}(\mathbf{v}^{(1)}, \delta_k) \times \mathcal{M}(\mathbf{v}^{(2)}, \delta_{n-2k}),$$

where $\mathcal{M}(\mathbf{v}^{(1)}, \delta_k)$ is the quiver variety associated with the A_{n-1} quiver with dimension vector $\mathbf{v}^{(1)}$, framing vector δ_k and the same stability condition θ' ; similar with $\mathcal{M}(\mathbf{v}^{(2)}, \delta_{n-2k})$.

We now give a combinatorial description of the quiver variety $\mathcal{M}(\mathbf{v}^{(1)}, \delta_k)$. By definition, a representative of a point in $\mathcal{M}(\mathbf{v}^{(1)}, \delta_k)$ takes the form $(\mathbf{a}, \mathbf{i}, \mathbf{b}, \mathbf{j})$. It is θ' -semistable, if and only if the image of \mathbf{i} under the actions of all \mathbf{a} and \mathbf{b} 's generate the space

$$\mathbf{V}^{(1)} := \bigoplus_{i=1}^{n-1} \mathbb{C}^{v_i^{(1)}}.$$

One can show that in this case, as an analogue of Lemma 2.8 in [36], we must have $\mathbf{j} = 0$. The moment map equation, together with $\mathbf{j}_k = 0$ implies that \mathbf{a} commutes with \mathbf{b} , as operators on $\mathbf{V}^{(1)}$. Therefore, we see that $\mathbf{V}^{(1)}$ is spanned by vectors $\mathbf{a}^i \mathbf{b}^j \mathbf{i}_k(1)$, which if nonzero, lie in $\mathbb{C}^{v_{i-j+k}^{(1)}}$. The stability condition implies that the set $\{(i, j) \in \mathbb{Z}_{>0}^2 \mid \mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k(1) \neq 0\}$ form a Young diagram, which corresponds to a partition λ .

In summary, the quiver variety $\mathcal{M}(\mathbf{v}^{(1)}, \delta_k)$ is either empty or a single point, where the latter case only happens when there exists a partition λ , whose number of boxes in the m -th diagonal is $\mathbf{v}_{m+k}^{(1)}$. The quiver variety $\mathcal{M}(\mathbf{v}^{(2)}, \delta_{n-2k})$ can be described in exactly the same way.

The restriction of Chern roots to the fixed point can be determined as follows. Consider

$$\mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k : \mathbb{C} \rightarrow \mathbf{V}_{i-j+k}.$$

The action of the group $GL(\mathbf{v}^{(1)})$ on $\mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k$ is

$$\mathbf{a} \mapsto g \mathbf{a} g^{-1}, \quad \mathbf{b} \mapsto g \mathbf{b} g^{-1}, \quad \mathbf{i}_k \mapsto g_k \mathbf{i}_k,$$

where $g = (g_1, \dots, g_{n-1}) \in \prod_i GL(v_i)$. So

$$\mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k \mapsto g \mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k,$$

and the action of A' on the framing space \mathbb{C} , $z \mapsto a_1 z$, induces the action

$$\mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k \mapsto a_1^{-1} \mathbf{a}^{i-1} \mathbf{b}^{j-1} \mathbf{i}_k.$$

Here a_1 becomes a_1^{-1} because the framing \mathbb{C} is the domain space of \mathbf{i}_k . To determine the restriction of the Chern root φ_{ij} , we need g to compensate the action of Γ' , i.e.

$$g_i = a_1, \quad \forall i.$$

So the (A' -equivariant) restriction is $\varphi_{ij} = a_1$. For the \hbar -weight, \mathbb{C}_\hbar^\times acts on \mathbf{b} directly by \hbar . So the Γ' -equivariant restriction is $\varphi_{ij} = a_1 \hbar^{q-1}$. Exactly same consideration applies to the second part $\mathcal{M}(\mathbf{v}^{(2)}, \delta_{n-2k})$.

Let us summarize the above discussion. The set of fixed points $(X')^{\Gamma'}$ is a finite set labeled by Young diagrams which fit into rectangle $R_{n,k}$. If λ is such a diagram we denote its complement in the $(n-k) \times k$ rectangle $R_{n,k}$ by $\bar{\lambda}$. It is easy to see that $\bar{\lambda}$ is also a Young diagram. The Young diagrams λ and $\bar{\lambda}$ divide the rectangle $R_{n,k}$ into two non-intersecting set of boxes. Our notations are clear from the following example.

Example 3. Let us fix $n = 8, k = 3$ and consider a Young diagram $\lambda = [3, 2]$, then $\bar{\lambda} = [4, 3, 3]$. The union of λ and $\bar{\lambda}$ is the rectangular of dimensions 5×3 :

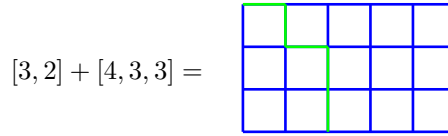


Figure 1: An example of a fixed point represented by $[3, 2] \in R_{8,3}$

It is clear that the number of fixed points is $n!/((n-k)!k!)$, i.e., the same as for X^Γ . To describe the values of the Chern roots x_{ij} at a fixed point λ , we introduce the following function:

$$\varphi_{ij}^\lambda = \begin{cases} a_1 \hbar^{j-1}, & \text{if } (i, j) \in \lambda, \\ a_2 \hbar^{n-k-i+1}, & \text{if } (i, j) \in \bar{\lambda} \end{cases} \quad (24)$$

The values of this function at the boxes are clear from the example in Fig.2:

$a_1 \hbar^2$	$a_2 \hbar^4$	$a_2 \hbar^3$	$a_2 \hbar^2$	$a_2 \hbar$
$a_1 \hbar$	$a_1 \hbar$	$a_2 \hbar^3$	$a_2 \hbar^2$	$a_2 \hbar$
a_1	a_1	$a_2 \hbar^3$	$a_2 \hbar^2$	$a_2 \hbar$

Figure 2: The values of $\varphi_{\square}^\lambda$ for $\lambda = [3, 2]$ and $n = 8, k = 3$.

If $\lambda \in (X')^{\Gamma'}$ is a fixed point, then the restriction of the Chern roots of tautological bundles are given by the formula:

$$x_{\square}|_\lambda = \varphi_{\square}^\lambda. \quad (25)$$

4.4 Tangent and polarization bundles for X'

To define the elliptic stable envelope we need to specify a polarization a chamber. We choose the canonical polarization:

$$T^{1/2} X' = a_1^{-1} \mathcal{V}_k + a_2 \mathcal{V}_{n-k}^* + \sum_{i=1}^{n-2} \mathcal{V}_{i+1} \mathcal{V}_i^* - \sum_{i=1}^{n-1} \mathcal{V}_i^* \mathcal{V}_i, \quad (26)$$

such that the virtual tangent space takes the form:

$$TX' = T^{1/2} X' + (T^{1/2} X')^* \otimes \hbar^{-1}.$$

We choose a chamber in the following form:

$$\sigma' : (0, 1) \in \text{Lie}_{\mathbb{R}}(\mathbf{A}'). \quad (27)$$

The character of the tangent space at a fixed point $\lambda \in (X')^{\Gamma}$ can be computed by restriction (24):

$$T_{\lambda}X' = TX'|_{\lambda}.$$

The tangent space at a fixed point decomposes into attracting and repelling parts:

$$T_{\lambda}X' = N_{\lambda}^{\prime+} \oplus N_{\lambda}^{\prime-},$$

where $N_{\lambda}^{\prime\pm}$ are the subspaces with \mathbf{A} -characters which take positive and negative values on the cocharacter (27) respectively. Explicitly these characters equal:

$$N_{\lambda}^{\prime-} = \sum_{m=1}^k \frac{a_1}{a_2} \hbar^{2k-n+\mathbf{p}_m-2m-1}, \quad N_{\lambda}^{\prime+} = \sum_{m=1}^k \frac{a_2}{a_1} \hbar^{-2k+n-\mathbf{p}_m+2m} \quad (28)$$

where $\mathbf{p} = \{\mathbf{p}_1, \dots, \mathbf{p}_k\} = \text{bj}(\lambda)$, for bj described in (41).

4.5 Elliptic cohomology of X'

The extended elliptic cohomology scheme of X' is a bouquet of Γ' orbits

$$\mathbf{E}_{\Gamma'}(X') := \coprod_{\lambda \in (X')^{\Gamma'}} \widehat{\mathcal{O}}_{\lambda}' / \Delta',$$

where $\widehat{\mathcal{O}}_{\lambda}' \cong \mathcal{E}_{\Gamma'} \times \mathcal{E}_{\text{Pic}(X')}$ with equivariant parameters and Kähler parameters to be coordinated in the first and second factor respectively.

Similarly to our discussion in Section 3.5 for fixed points λ, μ we consider the following functions:

$$\mathcal{U}'_{\lambda\mu} = \Theta \left(T^{1/2}X' \Big|_{\mu} \right) \prod_{\square \in \mathbf{R}_{n,k}} \frac{\phi(\varphi_{\square}^{\lambda}, z_{c_{\square}}^{-1})}{\phi(\varphi_{\square}^{\mu}, z_{c_{\square}}^{-1})} \prod_{i=1}^2 \frac{\phi(a_i, z_{a_i}^{-1} \hbar^{D_i^{\lambda}})}{\phi(a_i, z_{a_i}^{-1})}.$$

The powers D_i^{λ} are determined as follows: let us consider the index of the fixed point

$$\text{ind}_{\lambda} = T^{1/2}X' \Big|_{\lambda, >}$$

The symbol $>$ means that among the Γ' weights of $T^{1/2}X' \Big|_{\lambda, >}$ we choose only those which are positive at σ' . Let $\det(\text{ind}_{\lambda})$ denote the product of all these weights, then D_i^{λ} is a degree of this monomial in variable a_i .

For fixed λ the functions $\mathcal{U}'_{\lambda\mu}$ are components of a meromorphic section of a line bundle on $\mathbf{E}_{\Gamma'}(X')$ [1]. We denote this line bundle by $\mathcal{T}'(\lambda)$. By definition, the sections of restrictions $\mathcal{T}'(\lambda)|_{\widehat{\mathcal{O}}_{\mu}'}$ have the same quasiperiods (in all, equivariant and Kähler variables) as the function $\mathcal{U}'_{\lambda\mu}$.

The elliptic stable envelope $\text{Stab}'_{\sigma'}(\lambda)$ of a fixed point λ is a section of this line bundle, which is specified by a list of conditions similar to those of Definition 2. In this case, however, X' is not of GKM type. In particular, for $k > 1$ it may contain families of curves connecting two fixed points. This means that the subscheme Δ' over which the orbits $\widehat{\mathcal{O}}_{\lambda}'$ are glued and the condition of agreement for sections on different components are more complicated. Nevertheless, the $\text{Stab}'_{\sigma'}(\lambda)$ can be described very explicitly using the abelianization technique, see Section 5.

4.6 Holomorphic normalization

It will be convenient to work with stable envelopes which differ from one defined in [1] by normalization

$$\mathbf{Stab}'(\lambda) = \Theta'_\lambda \mathbf{Stab}'_{\sigma'}(\lambda) \quad (29)$$

with prefactor Θ'_λ given by

$$\Theta'_\lambda = \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i < j}} \theta\left(\frac{u_i}{u_j}\right) \prod_{\substack{i \in \mathbf{p}, \\ j \in \mathbf{n} \setminus \mathbf{p}, \\ i > j}} \theta\left(\hbar^{-1} \frac{u_j}{u_i}\right)$$

where $\mathbf{p} = \mathbf{bj}(\lambda)$ (see (41) below) and variables u_i are related to Kähler parameters z_i through (42). The stable envelope $\mathbf{Stab}'(\lambda)$ is a section of the twisted line bundle on $E_{\tau'}(X')$

$$\mathfrak{M}'(\lambda) = \mathcal{T}'(\lambda) \otimes \Theta'_\lambda. \quad (30)$$

As the function Θ'_λ only depends on Kähler variables this twist does not affect quasiperiods of stable envelopes in equivariant parameters. Geometrically the section Θ'_λ is related to repelling part of the normal bundles $N_{\bar{\mathbf{p}}}$ on the side X see (16):

$$\Theta'_\lambda = \Theta(N_{\bar{\mathbf{p}}}). \quad (31)$$

We will see that in this normalization the stable envelopes are *holomorphic sections* of $\mathfrak{M}'(\lambda)$.

5 Abelianization formula for elliptic stable envelope for X'

5.1 Non-Kähler part of stable envelope

Define a function in the boxes of the rectangle $\mathbf{R}_{n,k}$ by:

$$\rho_{\square}^\lambda = \begin{cases} i + j, & \text{if } \square \in \lambda \\ -i - j, & \text{if } \square \notin \lambda \end{cases}$$

The following function describes the part of elliptic stable envelope of a fixed point λ which is independent on Kähler parameters:

$$\mathbf{S}_\lambda^{n,k} = (-1)^{k(n-k)} \frac{\prod_{\substack{c_I=k \\ I \in \lambda}} \theta\left(\frac{x_I}{a_1}\right) \prod_{\substack{c_I=k \\ I \notin \lambda}} \theta\left(\frac{a_1}{x_I \hbar}\right) \prod_{c_I=n-k} \theta\left(\frac{a_2 \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J \\ \rho_I^\lambda > \rho_J^\lambda}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J \\ \rho_I^\lambda < \rho_J^\lambda}} \theta\left(\frac{x_I}{x_J}\right)}{\prod_{\substack{c_I=c_J \\ \rho_I^\lambda > \rho_J^\lambda}} \theta\left(\frac{x_I}{x_J}\right) \theta\left(\frac{x_I}{x_J \hbar}\right)} \quad (32)$$

where all products run over boxes in $\mathbf{R}_{n,k}$ which satisfy the specified conditions. For example, $\prod_{\substack{c_I=k \\ I \notin \lambda}}$ denotes a product over all boxes $I \in \lambda$ and projection $c_I = k$. Similarly, $\prod_{\substack{c_I=c_J \\ \rho_I^\lambda > \rho_J^\lambda}}$ denotes double product over all boxes $I, J \in \mathbf{R}_{n,k}$ with $c_I = c_J$ and $\rho_I^\lambda > \rho_J^\lambda$.

Example 4.

$$\begin{aligned} \mathbf{S}_{[1]}^{3,1} &= \theta\left(\frac{x_{1,1}}{a_1}\right) \theta\left(\frac{a_2\hbar}{x_{2,1}}\right) \theta\left(\frac{x_{2,1}\hbar}{x_{1,1}}\right), \\ \mathbf{S}_{[1,1]}^{4,2} &= \frac{\theta\left(\frac{x_{1,1}}{a_1}\right) \theta\left(\frac{a_2\hbar}{x_{1,1}}\right) \theta\left(\frac{a_1}{\hbar x_{2,2}}\right) \theta\left(\frac{a_2\hbar}{x_{2,2}}\right) \theta\left(\frac{x_{1,1}}{x_{2,1}}\right) \theta\left(\frac{x_{1,2}}{x_{1,1}}\right) \theta\left(\frac{\hbar x_{2,2}}{x_{1,2}}\right) \theta\left(\frac{x_{2,2}}{x_{2,1}}\right)}{\theta\left(\frac{x_{1,1}}{x_{2,2}}\right) \theta\left(\frac{x_{1,1}}{\hbar x_{2,2}}\right)}, \\ \mathbf{S}_{[2]}^{4,2} &= \frac{\theta\left(\frac{x_{1,1}}{a_1}\right) \theta\left(\frac{a_2\hbar}{x_{1,1}}\right) \theta\left(\frac{a_1}{\hbar x_{2,2}}\right) \theta\left(\frac{a_2\hbar}{x_{2,2}}\right) \theta\left(\frac{x_{2,1}\hbar}{x_{1,1}}\right) \theta\left(\frac{x_{1,1}\hbar}{x_{1,2}}\right) \theta\left(\frac{\hbar x_{2,2}}{x_{1,2}}\right) \theta\left(\frac{x_{2,2}}{x_{2,1}}\right)}{\theta\left(\frac{x_{1,1}}{x_{2,2}}\right) \theta\left(\frac{x_{1,1}}{\hbar x_{2,2}}\right)}. \end{aligned}$$

5.2 Trees in Young diagrams

Let us consider a Young diagram λ . We will say that two boxes $\square_1 = (i_1, j_1), \square_2 = (i_2, j_2) \in \lambda$ are *adjacent* if

$$i_1 = i_2, \quad |j_1 - j_2| = 1 \quad \text{or} \quad j_1 = j_2, \quad |i_1 - i_2| = 1.$$

Definition 3. A λ -tree is a rooted tree with:

- (\star) a set of vertices given by the boxes of a partition λ ,
- (\star, \star) a root at the box $r = (1, 1)$,
- (\star, \star, \star) edges connecting only the adjacent boxes.

Note that the number of λ -trees depends on the shape of λ . In particular, there is exactly one tree for ‘‘hooks’’ $\lambda = (\lambda_1, 1, \dots, 1)$.

We assume that each edge of a λ -tree is oriented in a certain way. In particular, on a set of edges we have two well-defined functions

$$h, t : \{\text{edges of a tree}\} \longrightarrow \{\text{boxes of } \lambda\},$$

which for an edge e return its head $h(e) \in \lambda$ and tail $t(e) \in \lambda$ boxes respectively. In this paper we will work with a distinguished *canonical orientation* on λ -trees.

Definition 4. We say that a λ -tree has canonical orientation if all edges are oriented from the root to the end points of the tree.

For a box $\square \in \lambda$ and a canonically oriented λ -tree \mathfrak{t} we have a well-defined canonically oriented subtree $[\square, \mathfrak{t}] \subset \mathfrak{t}$ with root at \square . In particular, $[r, \mathfrak{t}] = \mathfrak{t}$ for a root r of \mathfrak{t} .

We rotate the rectangle $\mathbb{R}_{n,k}$ by 45° as in the Fig.3, such that the horizontal coordinate of the box is equal to c_\square . The boundary of a Young diagram $\lambda \subset \mathbb{R}_{n,k}$ is a graph Γ of a piecewise linear function. We define a function on boxes in $\mathbb{R}_{n,k}$ by:

$$\beta_\lambda^{(1)}(\square) = \begin{cases} +1 & \text{if } \square \in \lambda \text{ and } \Gamma \text{ has maximum above } \square \\ -1 & \text{if } \square \in \lambda \text{ and } \Gamma \text{ has minimum above } \square \\ 0 & \text{else} \end{cases} \quad (33)$$

Note that $\beta_\lambda^{(1)}(\square) = 0$ for all $\square \in \bar{\lambda}$. For example, the Fig.3 gives the values of $\beta_\lambda^{(1)}(\square)$ for $\lambda = (4, 4, 4, 3, 3, 2) \in \mathbb{R}_{10,4}$.

We also define

$$\beta_\lambda^{(2)}(\square) = \begin{cases} +1 & \text{if } c_\square < k \\ -1 & \text{if } c_\square > n - k \\ 0 & \text{else} \end{cases}$$

and we set

$$v(\square) = \beta_\lambda^{(1)}(\square) + \beta_\lambda^{(2)}(\square). \quad (34)$$

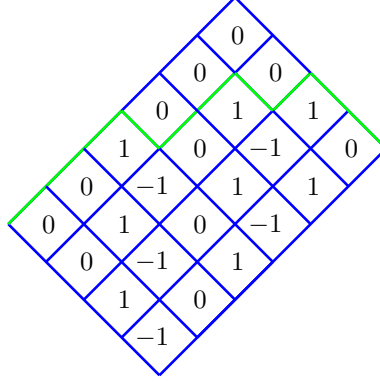


Figure 3: Values of function $\beta_\lambda^{(1)}(\square)$ for the diagram $\lambda = (4, 4, 4, 3, 3, 2) \in \mathbb{R}_{10,4}$. The boundary Γ of the Young diagram λ is denoted by green color.

5.3 Kähler part of the stable envelope

Let $\lambda \subset \mathbb{R}_{n,k}$ be a Young diagram and $\bar{\lambda} = \mathbb{R}_{n,k} \setminus \lambda$ is the complement Young diagram as above. Let $\mathfrak{t} \cup \bar{\mathfrak{t}}$ be the (disjoint) union of λ -tree \mathfrak{t} and $\bar{\lambda}$ -tree $\bar{\mathfrak{t}}$. We define a function:

$$\mathbf{W}^{Ell}(\mathfrak{t} \cup \bar{\mathfrak{t}}; x_i, z_i) := \mathbf{W}^{Ell}(\mathfrak{t}; x_i, z_i) \mathbf{W}^{Ell}(\bar{\mathfrak{t}}; x_i, z_i),$$

for the elliptic weight of a tree, where

$$\mathbf{W}^{Ell}(\mathfrak{t}; x_i, z_i) := (-1)^{\kappa(\mathfrak{t})} \phi\left(\frac{\varphi_r^\lambda}{x_r}, \prod_{\square \in [r, \mathfrak{t}]} z_{c_\square}^{-1} \hbar^{-v(\square)}\right) \prod_{e \in \mathfrak{t}} \phi\left(\frac{x_{t(e)} \varphi_{h(e)}^\lambda}{\varphi_{t(e)}^\lambda x_{h(e)}}, \prod_{\square \in [h(e), \mathfrak{t}]} z_{c_\square}^{-1} \hbar^{-v(\square)}\right), \quad (35)$$

and similar with $\mathbf{W}^{Ell}(\bar{\mathfrak{t}}; x_i, z_i)$.

Here $\square \in \mathfrak{t}$ or $e \in \mathfrak{t}$ means the box or edge belongs to the tree. The sign of a tree depends on the number $\kappa(\mathfrak{t})$ which is equal to the number of edges in the tree with wrong orientation. In other words, $\kappa(\mathfrak{t})$ is the number of edges in \mathfrak{t} directed down or to the left, while $\kappa(\bar{\mathfrak{t}})$ is the number of edges in $\bar{\mathfrak{t}}$ directed up or to the right. To avoid ambiguity, we also define $\mathbf{W}^{Ell}(\mathfrak{t}; x_i, z_i) := 1$ for a tree in the empty Young diagram.

Example 5. Let us consider a Young diagram $[2, 2] \subset \mathbb{R}_{5,2}$ with trees

By definition we have:

$$\mathbf{W}^{Ell}\left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}\right) = \mathbf{W}^{Ell}\left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}\right) \mathbf{W}^{Ell}\left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}\right).$$

In this case we have six boxes with the following characters:

$$\varphi_{11}^\lambda = a_1, \quad \varphi_{21}^\lambda = a_1, \quad \varphi_{31}^\lambda = a_2 \hbar, \quad \varphi_{12}^\lambda = a_1 \hbar, \quad \varphi_{22}^\lambda = a_1 \hbar, \quad \varphi_{32}^\lambda = a_2 \hbar.$$

Similarly for the \hbar -weights of boxes (34) we obtain:

$$\begin{aligned} \beta(1, 1) &= \beta^{(1)}(1, 1) + \beta^{(2)}(1, 1) = 1 + 0 = 1, \\ \beta(1, 2) &= \beta^{(1)}(1, 2) + \beta^{(2)}(1, 2) = 0 + 1 = 1, \\ \beta(2, 1) &= \beta^{(1)}(2, 1) + \beta^{(2)}(2, 1) = 0 + 0 = 0, \\ \beta(2, 2) &= \beta^{(1)}(2, 2) + \beta^{(2)}(2, 2) = 1 + 0 = 1, \\ \beta(3, 1) &= \beta^{(1)}(3, 1) + \beta^{(2)}(3, 1) = 0 - 1 = -1, \\ \beta(3, 2) &= \beta^{(1)}(3, 2) + \beta^{(2)}(3, 2) = 0 + 0 = 0. \end{aligned}$$

First, let us consider $\mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|} \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \end{array} \right)$. In this case we have a tree with the root at $r = (1, 1)$ and three edges with the following heard and tails:

$$t(e_1) = (1, 1), \quad h(e_1) = (1, 2), \quad t(e_2) = (1, 1), \quad h(e_2) = (2, 1), \quad t(e_3) = (1, 2), \quad h(e_3) = (2, 2).$$

For the first factor in (35) we obtain:

$$\phi \left(\frac{\varphi_r^\lambda}{x_r}, \prod_{\square \in [r, \dagger]} z_{c_\square}^{-1} \hbar^{-v(\square)} \right) = \phi \left(\frac{a_1}{x_{1,1}}, z_1^{-1} z_2^{-2} z_3^{-1} \hbar^{-3} \right)$$

For the edges in the product (35) we obtain:

$$\phi \left(\frac{x_{t(e_1)} \varphi_{h(e_1)}^\lambda}{\varphi_{t(e_1)}^\lambda x_{h(e_1)}}, \prod_{\square \in [h(e_1), \dagger]} z_{c_\square}^{-1} \hbar^{-v(\square)} \right) = \phi \left(\frac{x_{11} \hbar}{x_{12}}, z_1^{-1} z_2^{-1} \hbar^{-2} \right),$$

$$\phi \left(\frac{x_{t(e_2)} \varphi_{h(e_2)}^\lambda}{\varphi_{t(e_2)}^\lambda x_{h(e_2)}}, \prod_{\square \in [h(e_2), \dagger]} z_{c_\square}^{-1} \hbar^{-v(\square)} \right) = \phi \left(\frac{x_{11}}{x_{21}}, z_3^{-1} \right),$$

$$\phi \left(\frac{x_{t(e_3)} \varphi_{h(e_3)}^\lambda}{\varphi_{t(e_3)}^\lambda x_{h(e_3)}}, \prod_{\square \in [h(e_3), \dagger]} z_{c_\square}^{-1} \hbar^{-v(\square)} \right) = \phi \left(\frac{x_{12}}{x_{22}}, z_2^{-1} \hbar^{-1} \right).$$

Thus, overall we obtain:

$$\mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|} \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \end{array} \right) = \phi \left(\frac{a_1}{x_{1,1}}, \frac{1}{z_1 z_2^2 z_3 \hbar^3} \right) \phi \left(\frac{x_{11} \hbar}{x_{12}}, \frac{1}{z_1 z_2 \hbar^2} \right) \phi \left(\frac{x_{11}}{x_{21}}, \frac{1}{z_3} \right) \phi \left(\frac{x_{12}}{x_{22}}, \frac{1}{z_2 \hbar} \right).$$

Similarly, for the second multiple we obtain:

$$\mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \color{red}{\square} \\ \hline \square & \square & \square \\ \hline \end{array} \right) = \phi \left(\frac{a_2 \hbar}{x_{32}}, \frac{\hbar}{z_3 z_4} \right) \phi \left(\frac{x_{32}}{x_{31}}, \frac{\hbar}{z_4} \right).$$

5.4 Formula for elliptic stable envelope

Definition 5. The skeleton Γ_λ of a partition λ is the graph, whose vertices are given by the set of boxes of λ and whose edges connect all adjacent boxes in λ .

Definition 6. A \lrcorner -shaped subgraph in λ is a subgraph $\gamma \subset \Gamma_\lambda$ consisting of two edges $\gamma = \{\delta_1, \delta_2\}$ with the following end boxes:

$$\delta_{1,1} = (i, j), \quad \delta_{2,1} = \delta_{1,2} = (i+1, j), \quad \delta_{2,2} = (i+1, j+1). \quad (36)$$

It is easy to see that the total number of \lrcorner -shaped subgraphs in λ is equal to

$$m = \sum_{l \in \mathbb{Z}} (d_l(\lambda) - 1), \quad (37)$$

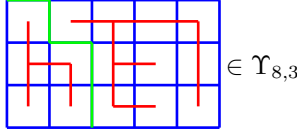
where $d_l(\lambda)$ is the number of boxes in the l -diagonal of λ

$$d_l(\lambda) = \#\{\square \in \lambda \mid c_\square = l\}. \quad (38)$$

There is a special set of λ -trees, constructed as follows. For each \lrcorner -shaped subgraph γ_i in λ we choose one of its two edges. We have 2^m of such choices. For each such choice the set of edges $\Gamma_\lambda \setminus \{\delta_i\}$ is a λ -tree. We denote the set of 2^m λ -trees which appear this way by Υ_λ .

Now let us define $\Upsilon_{n,k} = \Upsilon_\lambda \times \Upsilon_{\bar{\lambda}}$, whose elements are pairs of trees $(\mathfrak{t}, \bar{\mathfrak{t}})$, where \mathfrak{t} is a λ -tree with root $(1, 1)$, $\bar{\mathfrak{t}}$ is a $\bar{\lambda}$ -tree with root $(n-k, k)$. Both trees are constructed in the way described as above, and they are disjoint, i.e., do not have common vertices.

Example 6. Let us consider $\lambda = [3, 2] \in \mathbb{R}_{8,3}$ and $\bar{\lambda} = [4, 3, 3]$. A typical element of $\Upsilon_{8,3}$ looks like:



The following theorem can be proved using the same arguments as in [46].

Theorem 4. The elliptic stable envelope of a fixed point λ for the chamber σ' defined by (27) and polarization (26) has the following form:

$$\text{Stab}'_{\sigma'}(\lambda) = \text{Sym}_{\mathfrak{S}_{n,k}} \left(\mathbf{S}_\lambda^{n,k} \sum_{(\mathfrak{t}, \bar{\mathfrak{t}}) \in \Upsilon_{n,k}} \mathbf{W}^{Ell}(\mathfrak{t} \cup \bar{\mathfrak{t}}) \right) \quad (39)$$

where the symbol $\text{Sym}_{\mathfrak{S}_{n,k}}$ denotes a sum over all permutations in the group (23).

5.5 Refined formula

In this subsection, we prove a refined version of formula (23), in the sense that when restricted to another fixed point μ , the summation will be rewritten as depending on the trees $\bar{\mathfrak{t}}$ only, but not on the trees \mathfrak{t} . The refined formula will be of crucial use to us in the proof of the main theorem.

Given a fixed point λ , the original unrefined formula (39) has the following structure (for simplicity we omit the chamber subscript σ'):

$$\text{Stab}'(\lambda) = \sum_{\sigma \in \mathfrak{S}_{n,k}, \mathfrak{t}, \bar{\mathfrak{t}}} \frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma} \mathcal{R}^\sigma(\mathfrak{t}, \bar{\mathfrak{t}}) \mathcal{W}^\sigma(\mathfrak{t}, \bar{\mathfrak{t}}),$$

where we denote

$$\begin{aligned} \mathcal{N} &:= (-1)^{k(n-k)-1} \prod_{\substack{c_I=k \\ I \in \lambda \\ I \neq (1,1)}} \theta\left(\frac{x_I}{a_1}\right) \prod_{\substack{c_I=k \\ I \notin \lambda}} \theta\left(\frac{a_1}{x_I \hbar}\right) \prod_{\substack{c_I=n-k \\ I \neq (n-k,k)}} \theta\left(\frac{a_2 \hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J \\ \rho_I^\lambda > \rho_J^\lambda \\ (I \leftrightarrow J) \notin \Gamma_\lambda \cup \Gamma_{\bar{\lambda}}}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J \\ \rho_I^\lambda < \rho_J^\lambda \\ (I \leftrightarrow J) \notin \Gamma_\lambda \cup \Gamma_{\bar{\lambda}}}} \theta\left(\frac{x_I}{x_J}\right), \\ \mathcal{D} &:= \prod_{c_I=c_J, \rho_I^\lambda > \rho_J^\lambda} \theta\left(\frac{x_I}{x_J}\right) \prod_{c_I=c_J, \rho_I^\lambda > \rho_J^\lambda + 2} \theta\left(\frac{x_I}{x_J \hbar}\right), \\ \mathcal{R}(\mathfrak{t}, \bar{\mathfrak{t}}) &:= \frac{\prod_{\substack{c_I+1=c_J, \rho_I^\lambda = \rho_J^\lambda + 1 \\ (I \leftrightarrow J) \in \Gamma_\lambda \setminus \mathfrak{t} \cup \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J, \rho_I^\lambda + 1 = \rho_J^\lambda \\ (I \leftrightarrow J) \in \Gamma_\lambda \setminus \mathfrak{t} \cup \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_I}{x_J}\right)}{\prod_{c_I=c_J, \rho_i^\lambda = \rho_j^\lambda + 2} \theta\left(\frac{x_I}{x_J \hbar}\right)}, \\ \mathcal{W}(\mathfrak{t}, \bar{\mathfrak{t}}) &:= \frac{\theta\left(\frac{a_1}{x_r}\right) \prod_{I \in [r, \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}}{\theta\left(\prod_{I \in [r, \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)} \prod_{e \in \mathfrak{t}} \frac{\theta\left(\frac{x_{\mathfrak{t}(e)} \varphi_{\hbar}^\lambda(e)}{x_{\hbar(e)} \varphi_{\mathfrak{t}}^\lambda(e)}\right) \prod_{I \in [h(e), \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}}{\theta\left(\prod_{I \in [h(e), \mathfrak{t}]} z_{c_I}^{-1} \hbar^{-v(I)}\right)} \end{aligned}$$

$$\frac{\theta\left(\frac{a_2\hbar}{x_{\bar{r}}}\prod_{I\in[\bar{r},\bar{t}]}z_{c_I}^{-1}\hbar^{-\nu(I)}\right)}{\theta\left(\prod_{I\in[\bar{r},\bar{t}]}z_{c_I}^{-1}\hbar^{-\nu(I)}\right)}\prod_{e\in\bar{t}}\frac{\theta\left(\frac{x_{t(e)}\varphi_{\hbar(e)}^\lambda}{x_{h(e)}\varphi_{t(e)}^\lambda}\prod_{I\in[h(e),\bar{t}]}z_{c_I}^{-1}\hbar^{-\nu(I)}\right)}{\theta\left(\prod_{I\in[h(e),\bar{t}]}z_{c_I}^{-1}\hbar^{-\nu(I)}\right)},$$

and \mathcal{N}^σ , \mathcal{D}^σ , $\mathcal{R}^\sigma(t, \bar{t})$, $\mathcal{W}^\sigma(t, \bar{t})$ are the functions obtained by permuting x_i 's via $\sigma \in \mathfrak{S}_{n,k}$ in \mathcal{N} , \mathcal{D} , \mathcal{R} , \mathcal{W} .

We would like to consider its restriction to a fixed point $\nu \supset \lambda$; in other words, to evaluate $x_I = \varphi_I^\nu$. The symmetrization ensures that $\text{Stab}'(\lambda)$ does not have poles for those values of x_I 's, and hence $\text{Stab}'(\lambda)|_\nu$ is well-defined.

For an individual term such as

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}\mathcal{R}^\sigma(t, \bar{t})\mathcal{W}^\sigma(t, \bar{t}),$$

however, its restriction to ν is not well-defined; in other words, it may depend on the order we approach the limit $x_I = \varphi_I^\nu$. We discuss these properties in more details here.

Lemma 1. The restriction to ν of

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}$$

is well-defined, i.e., does not depend on the ordering of evaluation.

Proof. The proof is the same as Proposition 9 of [46]. \square

Lemma 2. If

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}\Big|_\nu \neq 0,$$

then σ fixes every box in $\bar{\nu}$.

Proof. Suppose that $\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}\Big|_\nu \neq 0$. Then by Lemma 1, \mathcal{N}^σ contains no factors that vanish when restricted to ν . First note that \mathcal{N}^σ contains

$$\prod_{c_i=n-k, i\neq(n-k,k)}\theta\left(\frac{a_2\hbar}{x_{\sigma(i)}}\right),$$

which vanishes unless $\sigma(i) \neq (n-k, k)$ for any $i \neq (n-k, k)$. Hence $\sigma(n-k, k) = (n-k, k)$.

We proceed by induction on the ρ -values of boxes in $\bar{\nu}$. Assume that σ fixes every box with $\rho \leq \rho_0$. Consider a box (a, b) with $\rho(a, b) = \rho_0 + 2$. Then either $(a+1, b)$ or $(a, b+1)$ lies in $\bar{\nu}$, and both of them have $\rho = \rho_0$. Suppose $\sigma^{-1}(a, b) \neq (a, b)$, then it is adjacent to neither $(a+1, b)$ nor $(a, b+1)$, and by induction hypothesis, $\rho_{\sigma^{-1}(a,b)} > \rho_{a+1,b}, \rho_{a,b+1}$. We see that \mathcal{N}^σ contains the factor

$$\theta\left(\frac{x_{\sigma(a+1,b)}}{x_{\sigma^{-1}(a,b)}}\right) = \theta\left(\frac{x_{a+1,b}}{x_{ab}}\right) \quad \text{or} \quad \theta\left(\frac{x_{\sigma^{-1}(a,b)}}{x_{\sigma(a,b+1)\hbar}}\right) = \theta\left(\frac{x_{ab}}{x_{a,b+1}\hbar}\right),$$

which vanishes at ν . Hence σ must fix (a, b) and the lemma holds. \square

Lemma 3. If

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma}\Big|_\nu \neq 0$$

then σ preserves the set of boxes of λ .

Proof. We proceed by induction on the diagonals. For the initial step, we need to show that the box with least content in λ , denoted by $(1, b)$, is fixed by σ . If $(1, b+1) \in \bar{\nu}$, then $(2, b+1) \in \bar{\nu}$, and σ fixes $(1, b)$ by Lemma 2. Now assume that $(1, b+1) \in \nu \setminus \lambda$. Let $X_1 = (1, b+1), X_2, \dots$ be the boxes in the diagonal of $\nu \setminus \lambda$ with one less content than $(1, b)$. Since $\rho_{X_i} < 0 < \rho_{1,b}$, by Lemma 2 we always have in \mathcal{N}^σ the factor

$$\prod_{m\geq 1}\theta\left(\frac{x_{\sigma(X_m)}}{x_{\sigma(1,b)}}\right) = \prod_{m\geq 1}\theta\left(\frac{x_{X_m}}{x_{\sigma(1,b)}}\right),$$

which vanishes at ν unless $\sigma(1, b)$ has no box to the left of it. This implies $\sigma(1, b) = (1, b)$.

Now assume that σ preserves the l -th diagonal of λ . Consider the $(l+1)$ -th diagonal. There are several cases.

- Both the l -th and $(l+1)$ -th diagonals of $\nu \setminus \lambda$ are empty. The lemma holds trivially for $l+1$.
- $\nu \setminus \lambda$ is empty in the l -th diagonal, but has one box X_1^{l+1} in the $(l+1)$ -th diagonal. In this case, let Y_1^l, Y_2^l, \dots be boxes in the l -th diagonal of λ . In \mathcal{N}^σ , there is the theta factor

$$\prod_{m \geq 1} \theta\left(\frac{x_{\sigma(Y_m^l)}}{x_{\sigma(X_1^{l+1})} \hbar}\right) = \prod_{m \geq 1} \theta\left(\frac{x_{Y_m^l}}{x_{\sigma(X_1^{l+1})} \hbar}\right),$$

which vanishes at ν unless $\sigma(X_1^{l+1}) = X_1^{l+1}$. Hence σ preserves the $(l+1)$ -th diagonal of λ .

- The l -th diagonal of $\nu \setminus \lambda$ is nonempty.

In this case, let X_1^l, X_2^l, \dots be the boxes in the l -th diagonal of $\nu \setminus \lambda$, and consider a general box Y in the $(l+1)$ -th diagonal of λ . We have in \mathcal{N}^σ the factor

$$\prod_{m \geq 1} \theta\left(\frac{x_{\sigma(X_m^l)}}{x_{\sigma(Y)}}\right) = \prod_{m \geq 1} \theta\left(\frac{x_{X_m^l}}{x_{\sigma(Y)}}\right).$$

If $\sigma(Y) \notin \lambda$, then it must be in $\nu \setminus \lambda$. Let Z be the box to the left of $\sigma(Y)$, which must either also lie in $\nu \setminus \lambda$ and has to be one of those X_i^l 's, or lie in λ . In the former case the product vanishes at ν ; in the latter case we have another factor $\theta\left(\frac{x_Z}{x_{\sigma(Y)}}\right)$, which also vanishes at ν .

The lemma holds by induction. □

Consider the subgroups in $\mathfrak{S}_{n,k}$ defined as

$$\mathfrak{S}_{\nu \setminus \lambda} := \{\sigma \mid \sigma \text{ fixes each box in } \lambda \cup \bar{\nu}\}, \quad \mathfrak{S}_{\bar{\lambda}} := \{\sigma \mid \sigma \text{ fixes each box in } \lambda\}.$$

Lemma 4. If

$$\left. \frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma} \right|_{\nu} \neq 0,$$

then $\sigma \in \mathfrak{S}_{\mu \setminus \lambda}$.

Proof. The proof is exactly the same as Lemma 2, by induction on the ρ -values of boxes. □

Now we would like to restricted the formula to the fixed point ν , in a specific choice of limit. We call the following the *row limit* for λ : first take

$$x_I = x_J$$

for each pair of boxes $I, J \in \lambda$; then take any limit $x_I \rightarrow \varphi_I^\nu$ of the remaining variables.

By previous lemmas, we see that only $\sigma \in \mathfrak{S}_{\nu \setminus \lambda}$ survives. Moreover, under the row limit, one can see that only one tree \mathfrak{t} (which contains all rows of λ) survives, and one can write all terms independent of trees in λ :

$$\mathcal{R}^\sigma(\mathfrak{t}, \bar{\mathfrak{t}}) = (-1)^{m(\lambda)} \mathcal{R}^\sigma(\bar{\mathfrak{t}}), \quad \mathcal{W}^\sigma(\mathfrak{t}, \bar{\mathfrak{t}}) = \mathcal{W}^\sigma(\bar{\mathfrak{t}}),$$

where $m(\lambda) = \sum_{I \in \mathbb{Z}} (d_I(\lambda) - 1)$, and

$$\begin{aligned} \mathcal{R}(\bar{\mathfrak{t}}) &:= \frac{\prod_{\substack{c_I+1=c_J, \rho_I=\rho_J+1 \\ (I \leftrightarrow J) \in \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_I+1=c_J, \rho_I+1=\rho_J \\ (I \leftrightarrow J) \in \Gamma_{\bar{\lambda}} \setminus \bar{\mathfrak{t}}}} \theta\left(\frac{x_I}{x_J}\right)}{\prod_{\substack{c_I=c_J, \rho_I=\rho_J+2 \\ I, J \in \lambda}} \theta\left(\frac{x_I}{x_J \hbar}\right)}, \\ \mathcal{W}(\bar{\mathfrak{t}}) &:= \frac{\theta\left(\frac{a_2 \hbar}{x_{\bar{r}}}\right) \prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-\nu(I)}}{\theta\left(\prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-\nu(I)}\right)} \prod_{e \in \bar{\mathfrak{t}}} \frac{\theta\left(\frac{x_{\mathfrak{t}(e)} \varphi_{\hbar}^\lambda}{x_{\mathfrak{h}(e)} \varphi_{\mathfrak{t}(e)}^\lambda}\right) \prod_{I \in [\mathfrak{h}(e), \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-\nu(I)}}{\theta\left(\prod_{I \in [\mathfrak{h}(e), \bar{\mathfrak{t}}]} z_{c_I}^{-1} \hbar^{-\nu(I)}\right)}. \end{aligned}$$

For \mathcal{N}^σ , \mathcal{D}^σ and $\sigma \in \mathfrak{S}_{\bar{\lambda}}$, we have the factorization

$$\frac{\mathcal{N}^\sigma}{\mathcal{D}^\sigma} = \frac{\mathcal{N}_\lambda}{\mathcal{D}_\lambda} \cdot \tilde{\mathcal{N}}_{\lambda'}^{-} \cdot \frac{\mathcal{N}_{\bar{\lambda}}^\sigma}{\mathcal{D}_{\bar{\lambda}}^\sigma},$$

where

$$\begin{aligned} \Theta(\tilde{\mathcal{N}}_{\lambda'}^{-}) &= (-1)^{k(n-k)-1} \frac{\prod_{\substack{c_I=k \\ I \notin \lambda}} \theta\left(\frac{a_1}{x_I \hbar}\right) \prod_{\substack{c_I=n-k \\ I \in \lambda}} \theta\left(\frac{a_2 \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J \\ I \in \lambda, J \in \lambda}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J \\ I \in \bar{\lambda}, J \in \bar{\lambda}}} \theta\left(\frac{x_I}{x_J}\right)}{\prod_{\substack{c_I=c_J \\ I \in \lambda, J \in \bar{\lambda}}} \theta\left(\frac{x_I}{x_J}\right) \theta\left(\frac{x_I}{x_J \hbar}\right)}, \\ \mathcal{N}_{\bar{\lambda}}^\sigma &= \prod_{\substack{c_I=n-k, i \in \bar{\lambda} \\ i \neq (n-k, k)}} \theta\left(\frac{a_2 \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J, \rho_I > \rho_J \\ (I \leftrightarrow J) \notin \Gamma_{\bar{\lambda}}, I, J \in \bar{\lambda}}} \theta\left(\frac{x_{\sigma(J) \hbar}}{x_{\sigma(I)}}\right) \prod_{\substack{c_{I+1}=c_J, \rho_I < \rho_J \\ (I \leftrightarrow J) \notin \Gamma_{\bar{\lambda}}, I, J \in \bar{\lambda}}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right) \\ \mathcal{N}_\lambda &= \prod_{\substack{c_I=k, I \in \lambda \\ I \neq (1, 1)}} \theta\left(\frac{x_I}{a_1}\right) \prod_{\substack{c_{I+1}=c_J, \rho_I > \rho_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} \theta\left(\frac{x_J \hbar}{x_I}\right) \prod_{\substack{c_{I+1}=c_J, \rho_I < \rho_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} \theta\left(\frac{x_I}{x_J}\right), \\ \mathcal{D}_{\bar{\lambda}}^\sigma &= \prod_{\substack{c_I=c_J, \rho_I > \rho_J \\ I, J \in \bar{\lambda}}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right) \prod_{\substack{c_I=c_J, \rho_I > \rho_{J+2} \\ I, J \in \bar{\lambda}}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J) \hbar}}\right), \quad \mathcal{D}_\lambda = \prod_{\substack{c_I=c_J, \rho_I > \rho_J \\ I, J \in \lambda}} \theta\left(\frac{x_I}{x_J}\right) \prod_{\substack{c_I=c_J, \rho_I > \rho_{J+2} \\ I, J \in \lambda}} \theta\left(\frac{x_I}{x_J \hbar}\right). \end{aligned}$$

In summary, we have the following refined formula:

Proposition 3. For any choice of limit $x_i \rightarrow \varphi'_i$ for $i \in \bar{\lambda}$, we have

$$\text{Stab}'(\lambda)|_\nu = \epsilon(\lambda) \Theta(\tilde{\mathcal{N}}_{\lambda'}^{-})|_\nu \cdot \sum_{\sigma \in \mathfrak{S}_{\nu \setminus \lambda, \bar{\mathfrak{t}}}} \frac{\mathcal{N}_\lambda^\sigma}{\mathcal{D}_\lambda^\sigma} \mathcal{R}^\sigma(\bar{\mathfrak{t}}) \mathcal{W}^\sigma(\bar{\mathfrak{t}})|_\nu,$$

where

$$\epsilon(\lambda) := (-1)^{m(\lambda)} \prod_{\substack{c_{I+1}=c_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} (-1).$$

As a corollary, we have the following identity in elliptic cohomology:

$$\boxed{\text{Stab}'(\lambda) = \epsilon(\lambda) \Theta(\tilde{\mathcal{N}}_{\lambda'}^{-}) \sum_{\sigma \in \mathfrak{S}_{\bar{\lambda}, \bar{\mathfrak{t}}}} \frac{\mathcal{N}_\lambda^\sigma}{\mathcal{D}_\lambda^\sigma} \mathcal{R}^\sigma(\bar{\mathfrak{t}}) \mathcal{W}^\sigma(\bar{\mathfrak{t}}).} \quad (40)$$

Proof. Computations above show that

$$\text{Stab}'(\lambda)|_\nu = (-1)^{m(\lambda)} \frac{\mathcal{N}_\lambda}{\mathcal{D}_\lambda}|_\nu \cdot \Theta(\tilde{\mathcal{N}}_{\lambda'}^{-})|_\nu \cdot \sum_{\sigma \in \mathfrak{S}_{\nu \setminus \lambda, \bar{\mathfrak{t}}}} \frac{\mathcal{N}_\lambda^\sigma}{\mathcal{D}_\lambda^\sigma} \mathcal{R}^\sigma(\bar{\mathfrak{t}}) \mathcal{W}^\sigma(\bar{\mathfrak{t}})|_\nu.$$

The refined formula is proved by the following lemma. □

Lemma 5.

$$\frac{\mathcal{N}_\lambda}{\mathcal{D}_\lambda}|_\nu = \prod_{\substack{c_{I+1}=c_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} (-1).$$

Proof. Let $t_1 = \hbar^{-1}$, $t_2 = 1$, $x_I \mapsto x_I/a_1$ in Proposition 10 of [46]. We have

$$\frac{\mathcal{N}_\lambda}{\mathcal{D}_\lambda}|_\nu = \frac{\mathcal{N}_\lambda}{\mathcal{D}_\lambda}|_\lambda = \prod_{\substack{c_{I+1}=c_J, \rho_I < \rho_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} (-1) \cdot \prod_{\substack{c_{I+1}=c_J, \rho_I > \rho_J \\ (I \leftrightarrow J) \notin \Gamma_\lambda, I, J \in \lambda}} (-1).$$

□

6 The Mother function

6.1 Bijection on fixed points

Recall that the set X^\top consists of $n!/((n-k)!k!)$ fixed points corresponding to k -subsets $\mathbf{p} = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ in the set $\mathbf{n} = \{1, 2, \dots, n\}$. On the dual side, the set $(X')^{\top'}$ consists of the same number of fixed points, labeled by Young diagrams λ which fit into the rectangle $\mathbb{R}_{n,k}$ with dimensions $(n-k) \times k$. There is a natural bijection

$$\text{bj} : (X')^{\top'} \xrightarrow{\sim} X^\top \quad (41)$$

defined in the following way.

Let $\lambda \in (X')^{\top'}$ be a fixed point. The boundary of the Young diagram λ is the graph of a piecewise linear function with exactly n -segments. Clearly, we have exactly k -segments where this graph has slope -1 . This way we obtain a k -subset in $\mathbf{p} \subset \{1, 2, \dots, n\}$ which defines a fixed point in X^\top . For example, consider a Young diagram $\lambda = [4, 4, 4, 3, 3, 2]$ in $\mathbb{R}_{10,4}$ as in the Fig.4. Clearly, the boundary of λ has negative slope at segments 4, 7, 9, 10, thus $\mathbf{p} = \{4, 7, 9, 10\}$.

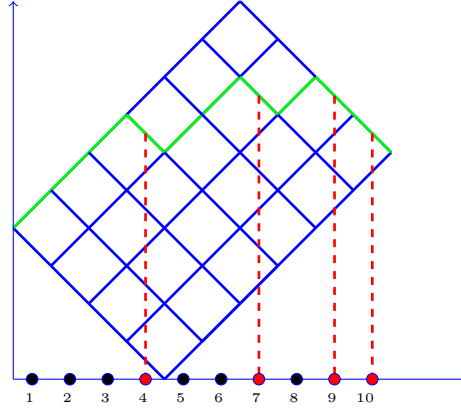


Figure 4: The point $\lambda = [4, 4, 4, 3, 3, 2] \in \mathbb{R}_{10,4}$ corresponds to $\mathbf{p} = \{4, 7, 9, 10\} \subset \{1, 2, \dots, 10\}$.

We note that this bijection preserves the standard dominant ordering on the set of fixed points. For instance in the case $n = 4, k = 2$ the fixed points on X are labeled by 2-subsets in $\{1, 2, 3, 4\}$, which are ordered as:

$$X^\top = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

The fixed points on X' correspond to Young diagrams which fit into 2×2 rectangle. The bijection above gives the following ordered list of fixed points in X' :

$$(X')^{\top'} = \{\emptyset, [1], [1, 1], [2], [2, 1], [2, 2]\}.$$

6.2 Identification of equivariant and Kähler parameters

Recall that the coordinates on the abelian variety $\widehat{\mathcal{O}}_{\mathbf{p}} = \mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\text{Pic}(X)}$ are the equivariant parameters $u_i/u_{i+1}, \hbar$ and the Kähler parameter z . The coordinates on $\widehat{\mathcal{O}}_{\lambda} = \mathcal{E}_{\mathbb{T}'} \times \mathcal{E}_{\text{Pic}(X')}$ are the equivariant parameters $a_1/a_2, \hbar$ and Kähler parameters z_1, \dots, z_{n-1} . Let us consider an isomorphism identifying the equivariant and Kähler tori on the dual sides

$$\kappa : \mathbb{T}' \rightarrow \mathbb{K}, \quad \mathbb{K}' \rightarrow \mathbb{T},$$

defined explicitly by

$$\frac{a_1}{a_2} \mapsto z\hbar^{k-1}, \quad \hbar \mapsto \hbar^{-1}, \quad z_i \mapsto \begin{cases} \frac{u_i \hbar}{u_{i+1}}, & i < k, \\ \frac{u_i}{u_{i+1}}, & k \leq i \leq n-k, \\ \frac{u_{i+1}}{u_{i+1} \hbar}, & i > n-k. \end{cases} \quad (42)$$

Recall that the stability and chamber parameters for X are defined by the following vectors:

$$\sigma = (1, 2, \dots, n) \in \text{Lie}_{\mathbb{R}}(\mathbf{A}), \quad \theta = (-1) \in \text{Lie}_{\mathbb{R}}(\mathbf{K}).$$

Using the map (42) we find that:

$$d\kappa^{-1}(\sigma) = (-1, \dots, -1) = -\theta', \quad d\kappa^{-1}(\theta) = (1) = -\sigma'.$$

We see that the isomorphism κ is chosen such that the stability parameters are matched to chamber parameters on the dual side.

6.3 Mother function and 3d mirror symmetry

For the $(\mathbb{T} \times \mathbb{T}')$ -variety $X \times X'$ we consider equivariant embeddings defined by fixed points:

$$X = X \times \{\lambda\} \xrightarrow{i_\lambda} X \times X' \xleftarrow{i_{\mathbf{p}}} \{\mathbf{p}\} \times X' = X' \quad (43)$$

We consider $X \times \{\lambda\}$ as a $\mathbb{T} \times \mathbb{T}'$ variety with trivial action on the second component. This gives

$$\text{Ell}_{\mathbb{T} \times \mathbb{T}'}(X \times \{\lambda\}) = \text{Ell}_{\mathbb{T}}(X) \times \mathcal{E}_{\mathbb{T}'} = \mathbf{E}_{\mathbb{T}}(X),$$

where in the last equality we used the isomorphism κ to identify $\mathcal{E}_{\text{Pic}(X)} = \mathcal{E}_{\mathbb{T}'}$. Similarly,

$$\text{Ell}_{\mathbb{T}'}(\{\mathbf{p}\} \times X') = \mathbf{E}_{\mathbb{T}'}(X').$$

We conclude that $\mathbb{T} \times \mathbb{T}'$ -equivariant embeddings (43) induce the following maps of extended elliptic cohomologies:

$$\mathbf{E}_{\mathbb{T}}(X) \xrightarrow{i_\lambda^*} \text{Ell}_{\mathbb{T} \times \mathbb{T}'}(X \times X') \xleftarrow{i_{\mathbf{p}}^*} \mathbf{E}_{\mathbb{T}'}(X').$$

Here is our main result.

Theorem 5.

- There exists a line bundle \mathfrak{M} on $\text{Ell}_{\mathbb{T} \times \mathbb{T}'}(X \times X')$ such that

$$(i_\lambda^*)^*(\mathfrak{M}) = \mathfrak{M}(\mathbf{p}), \quad (i_{\mathbf{p}}^*)^*(\mathfrak{M}) = \mathfrak{M}'(\lambda).$$

where $\mathbf{p} = \text{bj}(\lambda)$.

- There exists a holomorphic section \mathfrak{m} (the Mother function) of \mathfrak{M} , such that

$$(i_\lambda^*)^*(\mathfrak{m}) = \mathbf{Stab}(\mathbf{p}), \quad (i_{\mathbf{p}}^*)^*(\mathfrak{m}) = \mathbf{Stab}'(\lambda),$$

where $\mathbf{p} = \text{bj}(\lambda)$

We will prove this theorem in Section 9. This theorem implies that (up to normalization by diagonal elements) the restriction matrices of elliptic stable envelopes of X and X' are related by transposition:

(Similarly with notations in Definition 2, we denote $T'_{\lambda, \mu} := \text{Stab}'(\lambda)|_{\widehat{\mathcal{O}}'_\mu}$; we also use the simplified notation $(-)|_{\mathbf{p}}$ for $(-)|_{\widehat{\mathcal{O}}_{\mathbf{p}}}$.)

Corollary 2. *Restriction matrices of elliptic stable envelopes for X and X' are related by:*

$$T_{\mathbf{p},\mathbf{p}}T'_{\lambda,\mu} = T'_{\mu,\mu}T_{\mathbf{q},\mathbf{p}} \quad (44)$$

where $\mathbf{p} = \text{bj}(\lambda)$, $\mathbf{q} = \text{bj}(\mu)$ and parameters are identified by (42).

Proof. For fixed points $\lambda, \mu \in (X')^\top$, let $\mathbf{p} = \text{bj}(\lambda)$, $\mathbf{q} = \text{bj}(\mu)$ denote the corresponding fixed points in X^\top . Note that $(X \times X')^{\top \times \top'} = X^\top \times (X')^{\top'}$. Let us consider the point (\mathbf{p}, μ) from this set. By Theorem 5 we have

$$\mathbf{Stab}(\mathbf{q})|_{\mathbf{p}} = \mathfrak{m}|_{(\mathbf{p},\mu)} = \mathbf{Stab}'(\lambda)|_{\mu}$$

By definition (21) (29) we have $\mathbf{Stab}'(\lambda)|_{\mu} = \Theta'_\lambda T'_{\lambda,\mu}$, $\mathbf{Stab}(\mathbf{q})|_{\mathbf{p}} = \Theta_{\mathbf{q}} T_{\mathbf{q},\mathbf{p}}$. In the standard normalization of elliptic stable envelope, the diagonal elements of the restriction matrix are given by normal bundles of repelling part of the normal bundles:

$$T_{\mathbf{p},\mathbf{p}} = \Theta(N_{\mathbf{p}}^-), \quad T'_{\lambda,\lambda} = \Theta(N_{\lambda}^{\prime-}),$$

with $N_{\mathbf{p}}^-$ and $N_{\lambda}^{\prime-}$ as in (16), (28). We see that $\Theta'_\lambda = T_{\mathbf{p},\mathbf{p}}$, $\Theta_{\mathbf{q}} = T'_{\mu,\mu}$. \square

As we will see in Section 8, the equality (44) encodes certain infinite family of highly nontrivial identities for theta function.

7 The Mother function in case $k = 1$

Before we prove the Theorem 5 in general, it might be very instructive to check its prediction in the case $k = 1$. In this case the formulas for stable envelopes for X and X' are simple enough to compute the Mother function explicitly.

7.1 Explicit formula for the mother function

In the case $k = 1$ both X and X' are hypertoric, $X = T^*\mathbb{P}^{n-1}$ and X' is isomorphic to the A_{n-1} surface (resolution of singularity $\mathbb{C}^2/\mathbb{Z}_n$). The map κ has the following form:

$$\kappa : \quad \hbar \mapsto \frac{1}{\hbar}, \quad \frac{a_1}{a_2} \mapsto z, \quad z_1 \mapsto \frac{u_1}{u_2}, \quad \dots, \quad z_{n-1} \mapsto \frac{u_{n-1}}{u_n}. \quad (45)$$

We denote by $y = y_1$ the Chern root of the tautological bundle on X and by $x_i = x_{i,1}$, $i = 1, \dots, n-1$ the Chern roots of tautological bundles on X' . For symmetry, we also denote by $x_0 = a_1$ and $x_n = a_2$. In these notations we have:

Theorem 6. *In the case $k = 1$, the Mother function equals:*

$$\mathfrak{m} = \prod_{i=0}^n \theta\left(\frac{x_i \hbar}{x_{i-1}} u_i y\right). \quad (46)$$

7.2 Stable envelope for X'

First, let us consider the elliptic stable envelopes of the fixed points in X' . In the case $k = 1$ the fixed points on the variety X' are labeled by Young diagrams inside the $1 \times (n-1)$ rectangle. There are exactly n such Young diagrams $\lambda_m = \underbrace{[1, 1, \dots, 1]}_{m-1}$ with $m = 0, \dots, n-1$. To compute the stable envelope of λ_m , we need

to consider trees in λ_m and $\bar{\lambda}_m$. Obviously, there is only one possible tree in this case, see Fig.5: For (32) we obtain:

$$\mathbf{S}_{\lambda_m}^{n,1} = (-1)^{n-1} \theta\left(\frac{x_1}{a_1}\right) \theta\left(\frac{a_2 \hbar}{x_{n-1}}\right) \prod_{i=1}^{m-2} \theta\left(\frac{x_i}{x_{i+1}}\right) \times \theta\left(\frac{x_m \hbar}{x_{m-1}}\right) \times \prod_{i=m}^{n-2} \theta\left(\frac{x_{i+1} \hbar}{x_i}\right).$$

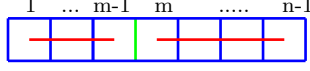


Figure 5: The tree for the fixed point representing $[1^{m-1}] \subset \mathbb{R}_{n,1}$

To compute the Kähler part of the stable envelope (35) we note that $\beta_{\lambda_m}^{(2)} = 0$ for all boxes of $\mathbb{R}_{n,1}$ and $\beta_{\lambda}^{(1)}$ is equal to zero for all boxes except the box $(m-1, 1)$ where it is equal to 1. Thus $\beta_{\lambda}((i, 1)) = \delta_{i, m-1}$.

$$\begin{aligned}
& \mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} \right) \\
= & \mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & & \\ \hline \end{array} \right) \times \mathbf{W}^{Ell} \left(\begin{array}{|c|c|c|c|c|} \hline & & & \text{---} & \text{---} \\ \hline \end{array} \right) \\
= & \phi \left(\frac{a_1}{x_1}, \hbar^{-1} \prod_{i=1}^{m-1} z_i^{-1} \right) \prod_{i=1}^{m-2} \phi \left(\frac{x_i}{x_{i+1}}, \hbar^{-1} \prod_{j=i+1}^{m-1} z_j^{-1} \right) \times \phi \left(\frac{a_2 \hbar}{x_{n-1}}, \prod_{i=m}^{n-1} z_i^{-1} \right) \prod_{i=m}^{n-2} \phi \left(\frac{x_{i+1} \hbar}{x_i}, \prod_{j=i}^m z_j^{-1} \right).
\end{aligned}$$

We conclude that:

$$\text{Stab}'(\lambda_m) = \mathbf{S}_{\lambda_m}^{n,1} \mathbf{W}_{\lambda_m}^{Ell} = (-1)^n \frac{\prod_{i=1}^{m-1} \theta \left(\frac{x_i}{x_{i-1}} \hbar \prod_{j=i}^{m-1} z_j \right) \times \theta \left(\frac{x_m}{x_{m-1}} \hbar \right) \times \prod_{i=m+1}^n \theta \left(\frac{x_i}{x_{i-1}} \hbar \prod_{j=m}^{i-1} z_j^{-1} \right)}{\prod_{i=1}^{m-1} \theta \left(\hbar \prod_{j=i}^{m-1} z_j \right) \times \prod_{i=m+1}^n \theta \left(\prod_{j=m}^{i-1} z_j^{-1} \right)} \quad (47)$$

where we denote $x_0 = a_1$ and $x_n = a_2$. The restriction of stable envelope to fixed points is given by evaluation of Chern roots (24). In this case the restriction to m -th fixed point is given by:

$$\{x_1 = a_1, \quad \dots, \quad x_{m-1} = a_1, \quad x_m = a_2 \hbar^{n-m}, \quad \dots, \quad x_{n-1} = a_2 \hbar\} \quad (48)$$

Thus, for the diagonal matrix elements of restriction matrix we obtain:

$$T'_{\lambda_m, \lambda_m} = \text{Stab}'(\lambda_m)|_{\lambda_m} = (-1)^n \theta \left(\frac{a_2 \hbar^{n-m+1}}{a_1} \right).$$

Finally, the stable envelope written in terms of parameters of X , i.e., all with the parameters substituted by (45), equals:

$$\text{Stab}'(\lambda_m) = (-1)^n \frac{\prod_{i=1}^n \theta \left(\frac{x_i}{x_{i-1} \hbar} \frac{u_i}{u_m} \right)}{\prod_{i=1}^{m-1} \theta \left(\frac{u_i}{u_m \hbar} \right) \prod_{i=m+1}^n \theta \left(\frac{u_i}{u_m} \right)}, \quad (49)$$

with diagonal elements of the restriction matrix:

$$T'_{\lambda_m, \lambda_m} = (-1)^n \theta(z^{-1} \hbar^{-n+m-1}). \quad (50)$$

7.3 Stable envelope for X

Under the bijection of fixed points we have $\text{bj}(\lambda_m) = \{m\} \subset \mathbf{n}$. From (20) for the stable envelope of X in the case $k = 1$ we obtain:

$$\text{Stab}(m) = (-1)^{n-m} \prod_{i=1}^{m-1} \theta \left(\frac{y u_i}{\hbar} \right) \times \frac{\theta(y u_m z^{-1} \hbar^{-n+m-1})}{\theta(z^{-1} \hbar^{-n+m-1})} \times \prod_{i=m+1}^n \theta(y u_i). \quad (51)$$

The restriction to the m -th fixed point is given by substitution $y = u_m^{-1}$. Thus, for diagonal of restriction matrix we obtain:

$$T_{m,m} = (-1)^{n-m} \text{Stab}(m)|_m = (-1)^{n-m} \prod_{i=1}^{m-1} \theta \left(\frac{u_i}{u_m \hbar} \right) \prod_{i=m+1}^n \theta \left(\frac{u_i}{u_m} \right) \quad (52)$$

7.4 Stable envelopes are restrictions of the Mother functions

We are now ready to check Theorem 6 in the $k = 1$ case. Note that (52) gives exactly the denominator of (49) and we obtain:

$$\mathbf{Stab}'(\lambda_m) = T_{m,m} \mathbf{Stab}'(\lambda_m) = (-1)^m \prod_{i=1}^n \theta\left(\frac{x_i}{x_{i-1}\hbar} \frac{u_i}{u_m}\right) = (-1)^m \mathbf{m}|_m$$

where \mathbf{m} is defined by (46) by $\mathbf{m}|_m$ we denotes the restriction of this class to the m -th fixed point on X , i.e. the evaluation $y = u_m^{-1}$. Similarly, we note that (50) is exactly the denominator of (51) and we obtain:

$$\mathbf{Stab}(m) = T'_{\lambda_m, \lambda_m} \mathbf{Stab}(m) = (-1)^m \prod_{i=1}^{m-1} \theta\left(\frac{y u_i}{\hbar}\right) \times \theta(y u_m z^{-1} \hbar^{-n+m-1}) \times \prod_{i=m+1}^n \theta(y u_i) = (-1)^m \mathbf{m}|_{\lambda_m}$$

where $\mathbf{m}|_{\lambda_m}$ denoted the restriction to λ_m on X' , i.e. the substitution (48) (one should not forget to substitute $\hbar \rightarrow \hbar^{-1}$ in (48), as all formulas written in terms of the parameters of X). Theorem 6 for $k = 1$ is proven.

8 Simplest non-abelian case $n = 4, k = 2$

8.1 Identification of parameters and fixed points

In the case $k = 1$ considered in the previous section, the matrix elements of restriction matrices $T'_{\lambda, \mu}$ and $T_{\mathbf{p}, \mathbf{q}}$ factorize into a product of theta functions and Theorem 5 can be proved by explicit computation. In contrast, when $k \geq 2$ the matrix elements are much more complicated. In particular, Theorem 5 (and Corollary 2) gives a set of very non-trivial identities satisfied by the theta functions. In this section we consider the simplest example with $n = 4$ and $k = 2$. In this case the fixed points on X are labeled by 2-subsets in $\{1, 2, 3, 4\}$. We consider the basis ordered as:

$$X^T = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

The fixed points on X' correspond to Young diagrams which fit into a 2×2 square. The bijection on the fixed points described in the Section 6.1 gives the corresponding points on X' (in the same order):

$$(X')^T = \{\emptyset, [1], [1, 1], [2], [2, 1], [2, 2]\}.$$

The identification of Kähler and equivariant parameters (42) in this case reads:

$$\kappa : \quad \frac{a_1}{a_2} \mapsto z\hbar, \quad \hbar \mapsto \hbar^{-1}, \quad z_1 \mapsto \frac{u_1\hbar}{u_2}, \quad z_2 \mapsto \frac{u_2}{u_3}, \quad z_3 \mapsto \frac{u_3}{u_4\hbar}. \quad (53)$$

We will denote a fixed point simply by its number $m = 1, \dots, 6$. For example, $T_{2,3}$ will denote the coefficient of the restriction matrix for X given by $T_{\{1,3\}, \{1,4\}}$. Similarly, $T'_{1,4}$ denotes $T'_{\emptyset, [2]}$ on the dual side X' .

8.2 Explicit expressions for stable envelopes

Using (20),(21),(39), (29) one can compute explicit expressions for stable envelopes. We list two of them here for example (after applying κ (53)):

$$\begin{aligned} \mathbf{Stab}(6) &= \frac{\theta\left(\frac{y_1 u_1}{\hbar}\right) \theta\left(\frac{y_1 u_2}{\hbar}\right) \theta\left(\frac{y_1 u_3}{z\hbar}\right) \theta(y_1^{-1} u_4^{-1}) \theta\left(\frac{y_2 u_1}{\hbar}\right) \theta\left(\frac{y_2 u_2}{\hbar}\right) \theta\left(\frac{y_2 u_3}{\hbar}\right) \theta\left(\frac{y_2 u_4}{z\hbar^2}\right)}{\theta\left(\frac{y_1}{y_2}\right) \theta\left(\frac{y_1 \hbar}{y_2}\right)} + (y_1 \leftrightarrow y_2) \\ \mathbf{Stab}'(1) &= \frac{\theta\left(\frac{u_1}{u_4}\right) \theta\left(\frac{u_2}{u_4}\right) \theta\left(\frac{x_m u_3 u_4}{\hbar x_{2,2} u_2 u_1}\right) \theta\left(\frac{x_{2,2} u_3}{x_{1,2} \hbar u_1}\right) \theta\left(\frac{x_{2,2} u_4}{x_{2,1} u_3}\right) \theta\left(\frac{x_{1,2} u_3}{x_{1,1} u_2}\right) \theta\left(\frac{x_{2,1}}{x_{1,1} \hbar}\right) \theta\left(\frac{\hbar x_0}{x_{1,1}}\right) \theta\left(\frac{\hbar x_0}{x_{2,2}}\right) \theta\left(\frac{x_m}{x_{1,1} \hbar}\right)}{\theta\left(\frac{u_3 u_4}{u_1 u_2}\right) \theta\left(\frac{u_4}{u_3}\right) \theta\left(\frac{x_{1,1}}{x_{2,2}}\right) \theta\left(\frac{\hbar x_{1,1}}{x_{2,2}}\right)} \end{aligned}$$

$$\frac{\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{x_m u_3 u_4}{\hbar x_{2,2} u_1 u_2}\right)\theta\left(\frac{x_{2,2} u_4}{x_{1,2} \hbar u_1}\right)\theta\left(\frac{x_{1,2} u_4}{x_{1,1} u_2}\right)\theta\left(\frac{x_{1,1} u_4 \hbar}{x_{2,1} u_3}\right)\theta\left(\frac{x_{2,2}}{x_{2,1}}\right)\theta\left(\frac{\hbar x_0}{x_{1,1}}\right)\theta\left(\frac{\hbar x_0}{x_{2,2}}\right)\theta\left(\frac{x_m}{\hbar x_{1,1}}\right)}{\theta\left(\frac{u_3 u_4}{u_1 u_2}\right)\theta\left(\frac{u_4}{u_3}\right)\theta\left(\frac{x_{1,1}}{x_{2,2}}\right)\theta\left(\frac{\hbar x_{1,1}}{x_{2,2}}\right)}$$

+ $(x_{1,1} \leftrightarrow x_{2,2})$,

where we denote $x_0 = a_1, x_m = a_2$.

8.3 Theorem 5 in case $n = 4, k = 2$

The Corollary 1 means that the functions above are related by the following identities:

$$\mathbf{Stab}(a)|_b = \mathbf{Stab}'(b)|_a,$$

where the restriction to the fixed points on X is given by substitution of variables y_i (12). The restrictions to the fixed points on X' are defined by (24) (together with identification of parameters (53)!). We only compute non-zero restrictions and only those $\mathbf{Stab}(a)|_b$ with $a \neq b$ (the case $a = b$ is trivial).

For example:
Case $a = 2, b = 1$:

$$\begin{aligned}\mathbf{Stab}'(1)|_2 &= \theta(z\hbar^3)\theta(\hbar)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{zu_2\hbar^3}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right), \\ \mathbf{Stab}(2)|_1 &= \theta(z\hbar^3)\theta(\hbar)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{zu_2\hbar^3}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right).\end{aligned}$$

We see that for $(a, b) = (2, 1)$ the two are trivially equal as product of theta functions, which also happens in cases $(a, b) = (3, 2), (4, 2), (5, 2), (5, 3), (4, 3), (6, 3), (5, 4), (6, 5)$. However, the identity is nontrivial for the remaining cases $(a, b) = (3, 1), (4, 1), (5, 1), (6, 1), (6, 2), (6, 4)$.

Case $a = 3, b = 1$:

$$\begin{aligned}\mathbf{Stab}'(1)|_3 &= \frac{\theta(\hbar)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_1}{u_3}\right)\left(\theta\left(\frac{zu_3\hbar^2}{u_4}\right)\theta\left(\frac{zu_2\hbar^3}{u_3}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar) - \theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta(z\hbar^2)\theta\left(\frac{zu_2\hbar^3}{u_4}\right)\right)}{\theta\left(\frac{u_3}{u_4}\right)}, \\ \mathbf{Stab}(3)|_1 &= \theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta(\hbar)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2}{u_4}\right)\theta(z\hbar^3).\end{aligned}$$

Case $a = 4, b = 1$:

$$\begin{aligned}\mathbf{Stab}'(1)|_4 &= \theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta(z\hbar^3)\theta\left(\frac{z\hbar^2 u_1}{u_3}\right), \\ \mathbf{Stab}(4)|_1 &= \frac{\theta(\hbar)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\left(\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_1}{u_2}\right)\theta(\hbar)\theta\left(\frac{zu_2\hbar^3}{u_3}\right) - \theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{zu_1\hbar^3}{u_3}\right)\theta(z\hbar^2)\right)}{\theta\left(\frac{u_1}{u_2}\right)}.\end{aligned}$$

Case $a = 5, b = 1$:

$$\begin{aligned}\mathbf{Stab}'(1)|_5 &= \frac{\theta(\hbar)\left(-\theta\left(\frac{z\hbar^2 u_1}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta(z\hbar^2)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2}{u_3}\right) + \theta\left(\frac{z\hbar^2 u_1}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{zu_3\hbar^2}{u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)\right)}{\theta\left(\frac{u_3}{u_4}\right)}, \\ \mathbf{Stab}(5)|_1 &= \frac{\theta(\hbar)\left(\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{z\hbar^2 u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2}{u_4}\right)\theta(\hbar) - \theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_1}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta(z\hbar^2)\right)}{\theta\left(\frac{u_1}{u_2}\right)}.\end{aligned}$$

Case $a = 6, b = 1$:

$$\mathbf{Stab}'(1)|_6 = \frac{1}{\theta\left(\frac{u_1 u_2}{u_3 u_4}\right)\theta\left(\frac{u_3}{u_4}\right)}\left(\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{u_3 \hbar}{u_4}\right)\theta(z\hbar^2)\theta\left(\frac{zu_1 u_2}{u_3 u_4}\right)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta(\hbar)\right)$$

$$\begin{aligned}
& -\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta(z\hbar^2)\theta(\hbar)\theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_2}{u_3}\right) \\
& -\theta\left(\frac{u_2}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right)\theta(\hbar^2)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{u_3}{u_4}\right)\theta(z\hbar)\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right), \\
\text{Stab}(6)|_1 = & \frac{\theta(\hbar)^2\left(\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{z\hbar u_1}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2}{u_4}\right) - \theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_1}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\right)}{\theta\left(\frac{u_1}{u_2}\right)}.
\end{aligned}$$

Case $a = 6, b = 2$:

$$\begin{aligned}
\text{Stab}'(2)|_6 = & \theta\left(\frac{u_3 \hbar}{u_2}\right)\theta(\hbar)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{z\hbar u_1}{u_4}\right)\theta(z\hbar^2)\theta\left(\frac{u_3 \hbar}{u_4}\right), \\
\text{Stab}(6)|_2 = & \frac{\theta(\hbar)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{u_3 \hbar}{u_2}\right)\left(\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{z\hbar u_1}{u_3}\right)\theta\left(\frac{z u_3 \hbar^2}{u_4}\right)\theta(\hbar) - \theta(z\hbar)\theta\left(\frac{z\hbar^2 u_1}{u_4}\right)\theta\left(\frac{u_3 \hbar}{u_1}\right)\theta\left(\frac{u_3}{u_4}\right)\right)}{\theta\left(\frac{u_1}{u_3}\right)}.
\end{aligned}$$

Case $a = 6, b = 4$:

$$\begin{aligned}
\text{Stab}'(4)|_6 = & \theta\left(\frac{u_3 \hbar}{u_1}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{u_3 \hbar}{u_4}\right)\theta(\hbar)\theta(z\hbar^2), \\
\text{Stab}(6)|_4 = & \frac{\theta(\hbar)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{u_3 \hbar}{u_1}\right)\left(\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{z u_3 \hbar^2}{u_4}\right)\theta(\hbar) - \theta(z\hbar)\theta\left(\frac{z\hbar^2 u_2}{u_4}\right)\theta\left(\frac{u_3 \hbar}{u_2}\right)\theta\left(\frac{u_3}{u_4}\right)\right)}{\theta\left(\frac{u_2}{u_3}\right)}.
\end{aligned}$$

8.4 Identities for theta functions

In all these nontrivial cases the identity follows from the well-known 3-term identity

$$\theta\left(\frac{a y_1}{x}\right)\theta\left(\frac{h y_2}{x}\right)\theta\left(\frac{h y_1}{y_2}\right)\theta(a) = \theta\left(\frac{a h y_2}{x}\right)\theta\left(\frac{y_2}{x}\right)\theta\left(\frac{y_1}{y_2}\right)\theta\left(\frac{a}{h}\right) + \theta\left(\frac{h y_2}{x}\right)\theta\left(\frac{a y_2}{x}\right)\theta(h)\theta\left(\frac{a y_1}{y_2}\right), \quad (54)$$

and 4-term identity for theta functions:

$$\begin{aligned}
& \theta(h)\theta\left(\frac{y_1}{y_2}\right)\theta\left(\frac{h y_1}{x_1}\right)\theta\left(\frac{a_2 h y_2}{x_1}\right)\theta\left(\frac{a_1 a_2 h y_1}{x_2}\right)\theta\left(\frac{x_2}{y_2}\right)\theta\left(\frac{a_1 x_2}{x_1}\right) \\
& -\theta\left(\frac{a_1 a_2 h y_1}{x_1}\right)\theta\left(\frac{x_1}{y_2}\right)\theta\left(\frac{h y_1}{x_2}\right)\theta\left(\frac{a_2 h y_2}{x_2}\right)\theta\left(\frac{h x_2}{x_1}\right)\theta\left(\frac{y_1}{y_2}\right)\theta(a_1) \\
& = -\theta(h)\theta\left(\frac{x_1}{x_2}\right)\theta\left(\frac{a_1 a_2 h y_2}{x_1}\right)\theta\left(\frac{a_2 h y_1}{x_2}\right)\theta\left(\frac{a_1 y_1}{y_2}\right)\theta\left(\frac{h y_1}{x_1}\right)\theta\left(\frac{x_2}{y_2}\right) \\
& +\theta\left(\frac{h y_2}{x_1}\right)\theta\left(\frac{x_2}{y_1}\right)\theta\left(\frac{x_1}{x_2}\right)\theta\left(\frac{h y_1}{y_2}\right)\theta\left(\frac{a_1 a_2 h y_1}{x_1}\right)\theta\left(\frac{a_2 h y_2}{x_2}\right)\theta(a_1)
\end{aligned} \quad (55)$$

Let us show the identity for the most complicated case $a = 6, b = 1$. The other cases are analyzed in the same manner. First, we specialize the parameters in the 4-term relation (55) to the following values:

$$\{a_1 = \hbar^{-1}, \quad a_2 = z\hbar, \quad x_1 = u_3, \quad x_2 = u_4, \quad y_1 = u_2, \quad y_2 = u_1, \quad h = \hbar\}.$$

After this substitution the above 4-term (up to a common multiple $\theta(\hbar)$) takes the form:

$$\begin{aligned}
& -\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{\hbar u_3}{u_4}\right)\theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{z u_1 \hbar^2}{u_3}\right)\theta\left(\frac{z\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_3}\right) + \theta\left(\frac{u_1}{u_2}\right)\theta\left(\frac{\hbar u_2}{u_4}\right)\theta\left(\frac{\hbar u_4}{u_3}\right)\theta\left(\frac{u_1}{u_3}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{z u_1 \hbar^2}{u_4}\right) \\
& = -\theta\left(\frac{u_1}{u_4}\right)\theta\left(\frac{u_3}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_2}\right)\theta\left(\frac{z\hbar u_1}{u_3}\right)\theta\left(\frac{z\hbar^2 u_2}{u_4}\right)\theta\left(\frac{\hbar u_2}{u_3}\right) + \theta\left(\frac{u_3}{u_4}\right)\theta\left(\frac{u_2}{u_4}\right)\theta\left(\frac{\hbar u_1}{u_3}\right)\theta\left(\frac{\hbar u_2}{u_1}\right)\theta\left(\frac{z\hbar u_2}{u_3}\right)\theta\left(\frac{z u_1 \hbar^2}{u_4}\right)
\end{aligned} \quad (56)$$

Now, the identity for $a = 6, b = 1$ has the form:

$$A_1 + A_2 + A_3 = B_1 + B_2$$

where the terms have the following explicit form (after clearing the denominators):

$$\begin{aligned}
A_1 &= \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{\hbar u_1}{u_3}\right) \theta\left(\frac{\hbar u_3}{u_4}\right) \theta(z\hbar^2) \theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \theta\left(\frac{u_2}{u_4}\right) \theta(\hbar) \\
A_2 &= -\theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_1}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta(z\hbar^2) \theta(\hbar) \theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{u_2}{u_3}\right) \\
A_3 &= -\theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{u_2}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta(\hbar^2) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{u_3}{u_4}\right) \theta(z\hbar) \theta\left(\frac{u_1}{u_4}\right) \theta\left(\frac{u_2}{u_4}\right) \\
B_1 &= \theta(\hbar)^2 \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{u_3}{u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \theta\left(\frac{\hbar u_1}{u_2}\right) \theta\left(\frac{z\hbar u_1}{u_3}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2}{u_4}\right) \\
B_2 &= -\theta(\hbar)^2 \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{u_3}{u_4}\right) \theta\left(\frac{u_2}{u_4}\right) \theta\left(\frac{z\hbar u_2}{u_3}\right) \theta\left(\frac{\hbar u_1}{u_3}\right) \theta\left(\frac{z u_1 \hbar^2}{u_4}\right) \theta\left(\frac{\hbar u_2}{u_1}\right)
\end{aligned}$$

For some values of the parameters the three term relation (54) can be written in the form:

$$\theta(z\hbar^2) \theta\left(\frac{u_2}{u_4}\right) \theta\left(\frac{\hbar u_1}{u_3}\right) \theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right) = -\theta\left(\frac{\hbar u_4}{u_2}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta\left(\frac{u_1}{u_3}\right) \theta(z\hbar) + \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta(\hbar) \theta\left(\frac{z\hbar u_2}{u_4}\right) \theta\left(\frac{z u_1 \hbar^2}{u_3}\right)$$

and thus for A_1 we can write:

$$\begin{aligned}
A_1 &= -\theta\left(\frac{\hbar u_4}{u_2}\right) \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{\hbar u_3}{u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(\hbar) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta(z\hbar) \\
&\quad + \theta\left(\frac{z u_1 \hbar^2}{u_3}\right) \theta\left(\frac{z\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(\hbar)^2 \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{\hbar u_3}{u_4}\right).
\end{aligned}$$

Similarly we can write the 3-term relation as:

$$\theta(z\hbar^2) \theta\left(\frac{u_2}{u_3}\right) \theta\left(\frac{\hbar u_1}{u_4}\right) \theta\left(\frac{z\hbar u_1 u_2}{u_3 u_4}\right) = -\theta\left(\frac{\hbar u_3}{u_2}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(z\hbar) + \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta(\hbar) \theta\left(\frac{z\hbar u_2}{u_3}\right) \theta\left(\frac{z u_1 \hbar^2}{u_4}\right)$$

and thus:

$$\begin{aligned}
A_2 &= \theta\left(\frac{\hbar u_3}{u_2}\right) \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(\hbar) \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta(z\hbar) \\
&\quad - \theta\left(\frac{u_1}{u_2}\right) \theta(\hbar)^2 \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \theta\left(\frac{z\hbar u_2}{u_3}\right) \theta\left(\frac{z u_1 \hbar^2}{u_4}\right)
\end{aligned}$$

Finally,

$$\theta(\hbar^2) \theta\left(\frac{u_2}{u_3}\right) \theta\left(\frac{u_2}{u_4}\right) \theta\left(\frac{u_3}{u_4}\right) = \theta(\hbar) \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta\left(\frac{\hbar u_3}{u_2}\right) - \theta\left(\frac{\hbar u_4}{u_2}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta(\hbar) \theta\left(\frac{\hbar u_3}{u_4}\right)$$

which gives:

$$\begin{aligned}
A_3 &= \theta\left(\frac{\hbar u_4}{u_2}\right) \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{\hbar u_3}{u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(\hbar) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta(z\hbar) \\
&\quad - \theta\left(\frac{\hbar u_3}{u_2}\right) \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{u_1}{u_4}\right) \theta(\hbar) \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta\left(\frac{u_1}{u_3}\right) \theta\left(\frac{z\hbar^2 u_2 u_1}{u_3 u_4}\right) \theta(z\hbar).
\end{aligned}$$

Several terms in the sum $A_1 + A_2 + A_3$ cancels and we obtain:

$$\begin{aligned}
A_1 + A_2 + A_3 &= \theta(\hbar)^2 \theta\left(\frac{u_1}{u_2}\right) \theta\left(\frac{u_1 u_2}{u_3 u_4}\right) \left(\theta\left(\frac{z u_1 \hbar^2}{u_3}\right) \theta\left(\frac{z\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_2}{u_3}\right) \theta\left(\frac{\hbar u_3}{u_4}\right) \theta\left(\frac{u_1}{u_4}\right) \right. \\
&\quad \left. - \theta\left(\frac{\hbar u_2}{u_4}\right) \theta\left(\frac{\hbar u_4}{u_3}\right) \theta\left(\frac{z\hbar u_2}{u_3}\right) \theta\left(\frac{z u_1 \hbar^2}{u_4}\right) \theta\left(\frac{u_1}{u_3}\right) \right)
\end{aligned}$$

Now, modulo a common multiple $\theta(\hbar)^2 \theta\left(\frac{u_1 u_2}{u_3 u_4}\right)$ the relation $A_1 + A_2 + A_3 = B_1 + B_2$ is exactly the 4-term relation (56).

9 Proof of Theorem 5

Let us first discuss the idea of the proof. We denote the restriction matrices for the elliptic stable envelopes in (holomorphic normalization) by:

$$\mathbf{T}_{\mathbf{q},\mathbf{p}} = \mathbf{Stab}(\mathbf{q})|_{\widehat{\mathcal{O}}_{\mathbf{p}}}, \quad \mathbf{T}'_{\lambda,\mu} = \mathbf{Stab}'(\lambda)|_{\widehat{\mathcal{O}}'_{\mu}}.$$

Recall that the isomorphism κ induces an isomorphism of extended orbits $\widehat{\mathcal{O}}'_{\mu} \cong \widehat{\mathcal{O}}_{\mathbf{p}} \cong \mathcal{E}_{\mathbb{T} \times \mathbb{T}'}$. First, we show that under this isomorphism we have the following identity

$$\mathbf{T}'_{\lambda,\mu} = \mathbf{T}_{\mathbf{q},\mathbf{p}}, \quad \text{for } \mathbf{p} = \mathbf{bj}(\lambda), \quad \mathbf{q} = \mathbf{bj}(\mu). \quad (57)$$

By Theorem 2, to prove this identity it is enough to check that the matrix elements $\mathbf{T}'_{\lambda,\mu}$ satisfies the conditions 1),2), 3).

The condition 1) says that for fixed μ the set of functions $\mathbf{T}'_{\lambda,\mu}$ is a section of the line bundle $\mathfrak{M}(\mathbf{q})$ see (22). By Proposition 1, to check this property it is enough to show that $\mathbf{T}'_{\lambda,\mu}$ has the same quasiperiods in equivariant and Kähler variables as sections of $\mathfrak{M}(\mathbf{q})|_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ and that it satisfies the GKM conditions:

$$\mathbf{T}'_{\lambda,\mu}|_{u_i=u_j} = \mathbf{T}'_{\nu,\mu}|_{u_i=u_j}, \quad (58)$$

if the fixed points $\mathbf{p} = \mathbf{bj}(\lambda)$ and $\mathbf{s} = \mathbf{bj}(\nu)$ are connected by equivariant curve, i.e., if $\mathbf{p} = \mathbf{s} \setminus \{i\} \cup \{j\}$ as k -sets. We prove it in the next Subsection 9.2.

The condition 2) is trivial and follows from our choice of holomorphic normalization.

The condition 3) says that $\mathbf{T}'_{\lambda,\mu}$ must be divisible by some explicit product of theta functions and the result of division is a holomorphic function in variables u_i . We will refer to these properties as *divisibility* and *holomorphicity*. These properties of the matrix $\mathbf{T}'_{\lambda,\mu}$ will be proven in Subsections 9.3 and 9.4 respectively.

Let us consider the following scheme:

$$\mathbf{S}(X, X') := \mathcal{E}_{\mathbb{T} \times \mathbb{T}'} \times S^k(E) \times \prod_{i=1}^{n-1} S^{v_i}(E). \quad (59)$$

Here $S^k(E)$ denotes k -th symmetric power of the elliptic curve E . We assume that coordinates on $S^k(E)$ are given by symmetric functions on Chern roots of tautological bundle on X . Similarly, $S^{v_i}(E)$ denotes the scheme with coordinates given by Chern roots of i -th tautological bundle on X' , i.e., symmetric functions in x_{\square} with $c_{\square} = i$, see Section 4.2 for the notations.

Recall that the stable envelopes $\mathbf{Stab}(\mathbf{q})$ and $\mathbf{Stab}'(\lambda)$ are defined explicitly by (20) and (39). In particular, they are symmetric functions in the Chern roots of tautological bundles. This means that the function defined by

$$\tilde{\mathbf{m}} := \sum_{\mathbf{p} \in X^{\mathbb{T}}} \sum_{\lambda \in (X')^{\mathbb{T}'}} \mathbf{T}_{\mathbf{q},\mathbf{p}}^{-1} \mathbf{Stab}(\mathbf{q}) \mathbf{Stab}'(\lambda). \quad (60)$$

can be considered as a meromorphic section of certain line bundle on $\mathbf{S}(X, X')$. We denote the corresponding line bundle by $\tilde{\mathfrak{M}}$.³

Let us consider the map

$$\tilde{\mathbf{c}} : \text{Ell}_{\mathbb{T} \times \mathbb{T}'}(X \times X') \rightarrow \mathbf{S}(X, X')$$

which is defined as follows: the component of $\tilde{\mathbf{c}}$ mapping to the first factor of (59) is the projection to the base. The components of the map $\tilde{\mathbf{c}}$ to $S^k(E)$ and to $S^{v_i}(E)$ are given by the elliptic Chern classes of the corresponding tautological classes. For the definition of elliptic Chern classes see Section 1.8 in [18] or Section 5 in [16]. It is known that $\tilde{\mathbf{c}}$ is an embedding [32], see also Section 2.4 in [1] for discussion.

Finally, the line bundle and the section of the Theorem 5 can be defined as $\mathfrak{M} = \tilde{\mathbf{c}}^* \tilde{\mathfrak{M}}$ and $\mathbf{m} = \tilde{\mathbf{c}}^*(\tilde{\mathbf{m}})$. Indeed, from the very definition (60) and (57) it is obvious that

$$(i_{\lambda}^*)^*(\mathbf{m}) = \tilde{\mathbf{m}}|_{\lambda} = \mathbf{Stab}(\mathbf{p}), \quad (i_{\mathbf{p}}^*)^*(\mathbf{m}) = \tilde{\mathbf{m}}|_{\mathbf{p}} = \mathbf{Stab}'(\lambda).$$

i.e., the section \mathbf{m} is the Mother function.

³Note that $\mathbf{T}_{\mathbf{q},\mathbf{p}}$ is triangular matrix with non-vanishing diagonal, thus it is invertible and the sum in (60) is well defined.

9.1 Cancellation of trees

Before checking Conditions 1)-3), we need a key lemma which describes that, under specialization of some u_i parameters, contributions from trees will cancel with each other and simply the summation dramatically.

Define the *boundary* of $\bar{\lambda}$ to be the set

$$\{(i, j) \in \bar{\lambda} \mid (i-1, j-1) \notin \bar{\lambda}\}.$$

Define the *upper boundary* of $\bar{\lambda}$ to be the set

$$U := \{(i, j) \in \bar{\lambda} \mid j = k\}.$$

Consider a 2×2 square in $\bar{\lambda}$, consisting of $(c, d), (c+1, d), (c, d+1), (c+1, d+1)$, where $(c+1, d)$ is in the a -th diagonal. Let $\bar{\mathfrak{t}}$ be a tree, which contains the edge $(c+1, d+1) \rightarrow (c+1, d)$.

The *involution* of $\bar{\mathfrak{t}}$ at the box $(c+1, d)$ is defined to be the tree $\text{inv}(\bar{\mathfrak{t}}, (c+1, d))$ obtained by removing $(c+1, d+1) \rightarrow (c+1, j)$ from $\bar{\mathfrak{t}}$ and adding the edge $(c, d) \rightarrow (c+1, d)$. We abbreviate the notation as $\text{inv}(\bar{\mathfrak{t}})$ if there's no confusion. Define $\text{inv}(\text{inv}(\bar{\mathfrak{t}})) = \bar{\mathfrak{t}}$. Involution is a well-defined operation on all trees at all boxes that are not in U or the boundary of $\bar{\lambda}$.

Let $\bar{\mathfrak{s}}$ be the subtree

$$\bar{\mathfrak{s}} := [(c+1, d), \bar{\mathfrak{t}}] = [(c+1, d), \text{inv}(\bar{\mathfrak{t}})].$$

The u -parameter contributed from $\bar{\mathfrak{s}}$ is

$$u(\bar{\mathfrak{s}}) := \prod_{I \in \bar{\mathfrak{s}}} \frac{u_{c_I+1}}{u_{c_I}}.$$

Lemma 6 (Cancellation lemma).

$$\left. \frac{\mathcal{R}(\bar{\mathfrak{t}})\mathcal{W}(\bar{\mathfrak{t}})}{\mathcal{R}(\text{inv}(\bar{\mathfrak{t}}))\mathcal{W}(\text{inv}(\bar{\mathfrak{t}}))} \right|_{u(\bar{\mathfrak{s}})=1} = -1.$$

As a corollary,

$$\sum_{\sigma \in \mathfrak{S}_{\bar{\lambda}}} \frac{\mathcal{N}_{\bar{\lambda}}^{\sigma}}{\mathcal{D}_{\bar{\lambda}}^{\sigma}} \mathcal{R}^{\sigma}(\bar{\mathfrak{t}})\mathcal{W}^{\sigma}(\bar{\mathfrak{t}}) = - \sum_{\sigma \in \mathfrak{S}_{\bar{\lambda}}} \frac{\mathcal{N}_{\bar{\lambda}}^{\sigma}}{\mathcal{D}_{\bar{\lambda}}^{\sigma}} \mathcal{R}^{\sigma}(\text{inv}(\bar{\mathfrak{t}}))\mathcal{W}^{\sigma}(\text{inv}(\bar{\mathfrak{t}})).$$

Proof. Direct computation shows that

$$\frac{\mathcal{R}(\bar{\mathfrak{t}})}{\mathcal{R}(\text{inv}(\bar{\mathfrak{t}}))} = \frac{\theta\left(\frac{x_{c+1,d}\bar{h}}{x_{c,d}}\right)}{\theta\left(\frac{x_{c+1,d+1}}{x_{c+1,d}}\right)}. \quad (61)$$

The quotient $\mathcal{W}(\bar{\mathfrak{t}})/\mathcal{W}(\text{inv}(\bar{\mathfrak{t}}))$ has contribution from an edge e if the subtree $[h(e), \bar{\mathfrak{t}}]$ or $[h(e), \text{inv}(\bar{\mathfrak{t}})]$ contains $(c+1, d+1)$ or (c, d) . Those contributions are all of the form

$$\frac{\theta\left(\frac{x_{t(e)}\varphi_{h(e)}^{\lambda}}{x_{h(e)}\varphi_{t(e)}^{\lambda}} \prod_{I \in [h(e), \text{inv}(\bar{\mathfrak{t}})]} \frac{u_{c_I+1}}{u_{c_I}} \cdot u(\bar{\mathfrak{s}})\right)}{\theta\left(\frac{x_{t(e)}\varphi_{h(e)}^{\lambda}}{x_{h(e)}\varphi_{t(e)}^{\lambda}} \prod_{I \in [h(e), \text{inv}(\bar{\mathfrak{t}})]} \frac{u_{c_I+1}}{u_{c_I}}\right)} \quad \text{or} \quad \frac{\theta\left(\frac{x_{t(e)}\varphi_{h(e)}^{\lambda}}{x_{h(e)}\varphi_{t(e)}^{\lambda}} \prod_{I \in [h(e), \bar{\mathfrak{t}}]} \frac{u_{c_I+1}}{u_{c_I}}\right)}{\theta\left(\frac{x_{t(e)}\varphi_{h(e)}^{\lambda}}{x_{h(e)}\varphi_{t(e)}^{\lambda}} \prod_{I \in [h(e), \bar{\mathfrak{t}}]} \frac{u_{c_I+1}u(\bar{\mathfrak{s}})}{u_{c_I}}\right)}$$

which are both 1 under $u(\bar{\mathfrak{s}}) = 1$, and the only remaining factor comes from the edges $(c+1, d+1) \rightarrow (c+1, d)$ and $(c, d) \rightarrow (c+1, d)$:

$$\left. \frac{\theta\left(\frac{x_{c+1,d+1}}{x_{c+1,d}} \frac{\varphi_{c+1,d}^{\lambda}}{\varphi_{c+1,d+1}^{\lambda}} u(\bar{\mathfrak{s}})\right)}{\theta\left(\frac{x_{c,d}}{x_{c+1,d}} \frac{\varphi_{c+1,d}^{\lambda}}{\varphi_{c,d}^{\lambda}} u(\bar{\mathfrak{s}})\right)} \right|_{u(\bar{\mathfrak{s}})=1} = \frac{\theta\left(\frac{x_{c+1,d+1}}{x_{c+1,d}}\right)}{\theta\left(\frac{x_{c,d}}{x_{c+1,d}} \bar{h}^{-1}\right)}.$$

The lemma follows. \square

9.2 GKM conditions

The goal of this section is to prove that the elliptic stable envelopes $\mathbf{Stab}'(\lambda)$ satisfy the GKM condition (58). For simplicity we assume that $(1, k) \in \bar{\lambda}$; in other words, $\bar{\lambda}$ starts with diagonal 1. The general case can be easily reduced to this.

A subtree of $\bar{\mathfrak{t}}$ is called a *strip* if it contains at most one box in each diagonal. We will also abuse the name *strip* for a connected subset in a partition that contains at most one box in each diagonal. We call a strip that starts from diagonal i to $j - 1$ an (i, j) -strip.

Let λ and μ be two partitions, and $\mathbf{p} = \mathbf{bj}(\lambda)$, $\mathbf{q} = \mathbf{bj}(\mu)$. Suppose that as fixed points in X , \mathbf{p} and \mathbf{q} are connected by a torus-invariant curve, which means that

$$\mathbf{q} = \mathbf{p} \setminus \{i\} \cup \{j\},$$

for some $1 \leq i, j \leq n$ (assume $i < j$). On the dual side, that means $\mu \supset \lambda$, and $\mu \setminus \lambda$ is an (i, j) -strip, lying the boundary of $\bar{\lambda}$.

Recall the GKM condition:

Proposition 4. For partitions λ and μ as above,

$$\mathbf{Stab}'(\lambda)|_{u_i=u_j} = \mathbf{Stab}'(\mu)|_{u_i=u_j}.$$

By localization and the triangular property of stable envelopes, it suffices to show that for any partition $\nu \supset \lambda$,

$$\mathbf{Stab}'(\lambda)|_{\nu, u_i=u_j} = \mathbf{Stab}'(\mu)|_{\nu, u_i=u_j}.$$

Before proving the GKM condition, we need some analysis on the specialization of the stable envelopes under $u_i = u_j$.

9.2.1 Specialization of $\mathbf{Stab}'(\lambda)$ under $u_i = u_j$

Recall that $\mathbf{p} \subset \mathbf{n}$ and $i \in \mathbf{p}$, $j \notin \mathbf{p}$, $i < j$. We would like to study the specialization $u_i = u_j$.

By definition

$$\mathbf{Stab}'(\lambda) = T_{\mathbf{p}, \mathbf{p}} \cdot \mathbf{Stab}'(\lambda),$$

where

$$T_{\mathbf{p}, \mathbf{p}} = \prod_{i \in \mathbf{p}, j \in \mathbf{n} \setminus \mathbf{p}, i < j} \theta\left(\frac{u_i}{u_j}\right) \prod_{i \in \mathbf{p}, j \in \mathbf{n} \setminus \mathbf{p}, i > j} \theta\left(\frac{u_j}{u_i \hbar}\right).$$

In particular, $T_{\mathbf{p}, \mathbf{p}}$ contains a factor $\theta\left(\frac{u_i}{u_j}\right)$.

For any tree $\bar{\mathfrak{t}}$ in $\bar{\lambda}$, consider all subtrees of $\bar{\mathfrak{t}}$ that are (i, j) -strips

$$B = \{B_i, B_{i+1}, \dots, B_{j-1}\},$$

where B_l is the box in the l -th diagonal. We define $B(\bar{\mathfrak{t}}, i, j)$ to be one whose B_i has the *smallest height*. If $\bar{\mathfrak{t}}$ does not contain any (i, j) -strips as subtrees, define $B(\bar{\mathfrak{t}}, i, j) = \emptyset$. A tree $\bar{\mathfrak{t}}$ in $\bar{\lambda}$ is called *distinguished*, if its strip $B(\bar{\mathfrak{t}}, i, j) \neq \emptyset$, and lies in the boundary of $\bar{\lambda}$.

A simple observation is that, for the contribution from $\bar{\mathfrak{t}}$ to $\mathbf{Stab}'(\lambda)$ to be nonzero under $u_i = u_j$, $B(\bar{\mathfrak{t}}, i, j)$ has to be nonempty.

Lemma 7. Let B be an (i, j) -strip in $\bar{\mathfrak{t}}$ which is a subtree. Let B_U be the box in $B \cap U$ with largest content. We have

- if $B_i \notin U$, then B_i is the root of B ;
- if $B_i \in U$, then B_U is the root of B .

Proof. If $B_i \notin U$, and the root of B is some box other than B_i . Then the unique path from B_i to U has a box \square in its interior with local maximal content. \square must be connected to both the boxes to the left and above it, which is not allowed.

If $B_i \in U$, then every box in B from B_i to B_U is in U . It is clear that the root of B is B_U . \square

Lemma 8. Let B be an (i, j) -strip in $\bar{\mathfrak{t}}$ which is a subtree. If B_i lies in the boundary of $\bar{\lambda}$, then B lies entirely in the boundary of $\bar{\lambda}$; in other words, $B(\bar{\mathfrak{t}}, i, j) = B$.

Proof. Suppose B_i lies in the boundary, but B does not. Then there exists a box in the boundary of $\bar{\lambda}$, not in B , but in a diagonal less than $j - 1$. Since $\bar{\mathfrak{t}}$ is a tree, there is a unique path from that box to some box in U . This path would contain a box with local maximal content in its interior. Contradiction. \square

Lemma 9. Under the specialization $u_i = u_j$,

$$\mathbf{Stab}'(\lambda)|_{u_i=u_j} = T_{\mathbf{p}, \mathbf{p}} \cdot \epsilon(\lambda) \Theta(\tilde{N}'_\lambda, -) \sum_{\substack{\sigma \in \mathfrak{S}_{\bar{\lambda}} \\ \bar{\mathfrak{t}} \text{ distinguished}}} \frac{\mathcal{N}_{\bar{\lambda}}^\sigma}{\mathcal{D}_{\bar{\lambda}}^\sigma} \mathcal{R}^\sigma(\bar{\mathfrak{t}}) \mathcal{W}^\sigma(\bar{\mathfrak{t}}) \Big|_{u_a=u_b}.$$

Proof. Let $B = B(\bar{\mathfrak{t}}, i, j)$. Since $T_{\mathbf{p}, \mathbf{p}}$ contains a zero u_i/u_j , if $B = \emptyset$, it is clear that the stable envelope will vanish. Now assume $B \neq \emptyset$.

If $i = 1$, then $B_i = (1, k)$. By Lemma 8 B lies in the boundary and $\bar{\mathfrak{t}}$ is distinguished.

If $i \neq 1$, it is easy to see that $B_i \notin U$ (otherwise as a subtree B must contain $(1, k)$). If moreover B_i is not in the boundary, then one can construct its involution $\text{inv}(\bar{\mathfrak{t}})$. By Lemma 6, the contributions from $\bar{\mathfrak{t}}$ and $\text{inv}(\bar{\mathfrak{t}})$ cancel with each other. Therefore, in the summation over trees, we are left with those $\bar{\mathfrak{t}}$ whose B_i lies in the boundary of $\bar{\lambda}$, which by Lemma 8 are distinguished. \square

Fix a distinguished tree $\bar{\mathfrak{t}}$, and $B = B(\bar{\mathfrak{t}}, i, j)$. Let's consider the restriction of $\mathbf{Stab}'(\lambda)$ to a certain fixed point $\nu \supset \lambda$. For an individual contribution from given $\bar{\mathfrak{t}}$ and σ , we take the following limit, called *B-column limit* for $\nu \setminus \lambda$: first, for each pair of $I, J \in \nu \setminus \lambda$ such that I is above J and $I, J \notin B$, take

$$x_I = x_J \hbar;$$

for any $I \in B$, take $x_I = \varphi_I'$; finally take any well-defined evaluation of the remaining variables. Note that this limit only depends on the partition λ and the pair i, j , and does not depend on $\bar{\mathfrak{t}}$.

Lemma 10. The restriction

$$\frac{\mathcal{N}_{\bar{\lambda}}^\sigma}{\mathcal{D}_{\bar{\lambda}}^\sigma} \mathcal{R}^\sigma(\bar{\mathfrak{t}}) \mathcal{W}^\sigma(\bar{\mathfrak{t}}) \Big|_\nu$$

under the B -column limit vanishes unless σ fixes B .

Proof. Suppose that the restriction does not vanish under the chosen limit. Recall that B_l , $i \leq l \leq j - 1$ is the box in the l -th diagonal of B . We use induction on l , from $j - 1$ to i . Recall that by the refined formula, σ lies in $\mathfrak{S}_{\nu \setminus \lambda}$.

First we show that B_{j-1} is fixed by σ . Let Y_1, Y_2, \dots be the boxes in the j -th diagonal of $\nu \setminus \lambda$, such that the heights of Y_m 's are increasing. Since $j \notin \mathbf{p}$, Y_1 is the box to the right of B_{j-1} . Hence we have the theta factors

$$\prod_{m \geq 1} \theta\left(\frac{x_{\sigma(B_{j-1})}}{x_{Y_m} \hbar}\right),$$

as $\rho_{B_{j-1}} > \rho_{Y_m}$ and B_{j-1} is not connected to Y_1 . Under the B -column limit for $\nu \setminus \lambda$, this product vanishes unless $\sigma(B_{j-1})$ has no box below it, which implies $\sigma(B_{j-1}) = B_{j-1}$.

Next, suppose that B_{l+1} is fixed by σ , consider B_l . Let e be the edge connecting B_l and B_{l+1} . Let $X_1 = B_l, X_2, \dots$ and $Y_1 = B_{l+1}, Y_2, \dots$ be respectively the boxes in the l -th and $(l + 1)$ -th diagonals of $\nu \setminus \lambda$.

If e is *horizontal*, then we have factors

$$\prod_{m \geq 2} \theta\left(\frac{x_{\sigma(X_m)}}{x_{B_{l+1}}}\right),$$

since we know $\rho_{X_m} < \rho_{B_{l+1}}$ and X_2 is not connected to B_{l+1} . If $\sigma(X_m) = X_1 = B_l$ for some $m \neq 1$, then the factor $\theta\left(\frac{x_{\sigma(X_m)}}{x_{B_{l+1}}}\right) = \theta\left(\frac{x_{B_l}}{x_{B_{l+1}}}\right)$ vanishes under the B -column ordering. Hence $\sigma(B_l) = B_l$.

If e is vertical, then we have factors

$$\prod_{m \geq 2} \theta\left(\frac{x_{\sigma(B_l)}}{x_{\sigma(Y_m)} \hbar}\right) = \prod_{m \geq 2} \theta\left(\frac{x_{\sigma(B_l)}}{x_{Y_m} \hbar}\right),$$

since we know $\rho_{B_l} > \rho_{Y_m}$ and B_l is not connected to Y_2 . If $\sigma(B_l) = X_m$ for some $m \neq 1$, then the factor $\theta\left(\frac{x_{\sigma(B_l)}}{x_{Y_m} \hbar}\right) = \theta\left(\frac{x_{X_m}}{x_{Y_m} \hbar}\right)$ vanishes under the B -column ordering, since X_m is the box above Y_m and they are not in B for $m \geq 2$. Hence σ fixes B_l . \square

In summary, after restriction to ν in the B -column limit for $\nu \setminus \lambda$, only contributions from distinguished $\bar{\mathfrak{t}}$ and permutations σ that fix B survive. We are now ready to prove Proposition 4.

9.2.2 Proof of Proposition 4: μ is not contained in ν

In this case, the strip $\mu \setminus \lambda$ is not entirely contained in $\nu \setminus \lambda$. Clearly we have $\mathbf{Stab}'(\mu)|_{\nu} = 0$.

Lemma 11.

$$\mathbf{Stab}'(\lambda)|_{\nu, u_i = u_j} = 0.$$

Proof. By Lemma 9, only distinguished trees $\bar{\mathfrak{t}}$, with strip $B = B(\bar{\mathfrak{t}}, i, j)$ contributes. Let B_i, \dots, B_{j-1} be boxes in B , and X be the first box in B that does not lie in $\nu \setminus \lambda$. For restriction to ν of an individual contribution by given $\bar{\mathfrak{t}}$ and σ , we take the *column limit* for $\bar{\nu}$, i.e. first let $x_I = x_J$ for any $I, J \in \bar{\nu}$ in the same column, and then take any limit for the remaining variables.

If $X \neq B_i$, then there's a box Y above it, which also lies in $\bar{\nu}$. Since $Y \notin B$, the edge connecting X and Y is not in $\bar{\mathfrak{t}}$. The contribution from $\bar{\mathfrak{t}}$ then contains a factor $\theta(X/Y)$, which vanishes under the column limit.

If $X = B_i$, then either $i \neq 1$, or $i = 1$, and the entire $B \cap U$, and in particular B_U , lie in $\bar{\nu}$. By Lemma 7 we know the root $r_B = B_i$ or B_U respectively. Denote the box not in B and connected to r by C . The factor in $\mathcal{W}^{\sigma=1}(\bar{\mathfrak{t}})$ that contributes the pole u_i/u_j is

$$\frac{\theta\left(\frac{x_C \varphi_{r_B}^{\lambda} u_j}{x_{r_B} \varphi_C^{\lambda} u_i}\right)|_{\nu}}{\theta\left(\frac{u_j}{u_i}\right)} = 1.$$

$\mathbf{Stab}'(\lambda)|_{\nu} = 0$ under $u_i = u_j$ because of the zero u_i/u_j in $T_{\mathbf{p}, \mathbf{p}}$. \square

9.2.3 Proof of Proposition 4: $\mu \subset \nu$

In this case B is contained entirely in $\nu \setminus \lambda$; in other words, $\lambda \subset \mu \subset \nu$. Let r_B be the root of B , which if $i = 1$, is B_U ; and if $i \neq 1$, is B_a .

If $(n-k, k) \notin B$, let $C \in \bar{\mathfrak{t}} \setminus B$ be the box connected to r_B . C could be in or not in $\nu \setminus \lambda$. If $(n-k, k) \in B$, we denote by convention that $x_C/\varphi_C^{\lambda} = 1$. Then

$$\begin{aligned} & \theta\left(\frac{u_j}{u_i}\right) \mathbf{Stab}'(\lambda)|_{\nu, u_i = u_j} \\ = & \theta\left(\frac{u_j}{u_i}\right) \epsilon(\lambda) \Theta(\tilde{N}'_{\lambda}, -)|_{\nu} \cdot \sum_{\sigma \in \mathfrak{S}_{\nu \setminus \mu}, \bar{\mathfrak{t}} \cap \bar{\mu}} \frac{\mathcal{N}_{\bar{\mu}}^{\sigma}}{D_{\bar{\mu}}^{\sigma}} \mathcal{R}^{\sigma}(\bar{\mathfrak{t}} \cap \bar{\mu}) \mathcal{W}^{\sigma}(\bar{\mathfrak{t}} \cap \bar{\mu})|_{\nu} \cdot \prod_{\substack{c_I = n-k, I \neq (n-k, k) \\ I \in \mu \setminus \lambda}} \theta\left(\frac{a_2 \hbar}{x_I}\right)|_{\nu} \\ & \prod_{\substack{c_I + 1 = c_J, \rho_I > \rho_J \\ (I \leftrightarrow J) \notin \bar{\mathfrak{t}}, I \text{ or } J \in \mu \setminus \lambda}} \theta\left(\frac{x_{\sigma(J)} \hbar}{x_{\sigma(I)}}\right)|_{\nu} \prod_{\substack{c_I + 1 = c_J, \rho_I < \rho_J \\ (I \leftrightarrow J) \notin \bar{\mathfrak{t}}, I \text{ or } J \in \mu \setminus \lambda}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right)|_{\nu} \cdot \prod_{\substack{c_I = c_J, \rho_I > \rho_J \\ I \text{ or } J \in \mu \setminus \lambda}} \theta\left(\frac{x_{\sigma(I)}}{x_{\sigma(J)}}\right)^{-1} \theta\left(\frac{x_{\sigma(I)} \hbar}{x_{\sigma(J)} \hbar}\right)^{-1}|_{\nu} \\ & \cdot \frac{\theta\left(\frac{x_C \varphi_{r_B}^{\lambda} u_j}{x_{r_B} \varphi_C^{\lambda} u_i}\right)|_{\nu}}{\theta\left(\frac{u_j}{u_i}\right)} \prod_{e \in B \setminus U} \frac{\theta\left(\frac{x_{t(e)} \varphi_{h(e)}^{\lambda} u_j}{x_{h(e)} \varphi_{t(e)}^{\lambda} u_{h(e)}}\right)|_{\nu}}{\theta\left(\frac{u_j}{u_{h(e)}}\right)} \prod_{e \in B \cap U} \frac{\theta\left(\frac{x_{t(e)} \varphi_{h(e)}^{\lambda} u_{t(e)}}{x_{h(e)} \varphi_{t(e)}^{\lambda} u_i}\right)|_{\nu}}{\theta\left(\frac{u_{t(e)}}{u_i}\right)}, \end{aligned}$$

which can be compared with $\text{Stab}'(\mu)$. Direct computation shows that

$$\begin{aligned} \theta\left(\frac{u_j}{u_i}\right) \frac{\text{Stab}'(\lambda)}{\text{Stab}'(\mu)} \Big|_{\nu, u_i=u_j} &= (-1)^{j-i} \theta(\hbar^{-1}) \prod_{e \in B \setminus U} \frac{\theta\left(\frac{x_{t(e)} \varphi_{\hbar(e)}^\lambda u_j}{x_{h(e)} \varphi_{t(e)}^\lambda u_{h(e)}}\right) \Big|_{\nu, u_i=u_j}}{\theta\left(\frac{u_j}{u_{h(e)}}\right)} \prod_{e \in B \cap U} \frac{\theta\left(\frac{x_{t(e)} \varphi_{\hbar(e)}^\lambda u_{t(e)}}{x_{h(e)} \varphi_{t(e)}^\lambda u_i}\right) \Big|_{\nu, u_i=u_j}}{\theta\left(\frac{u_{t(e)}}{u_i}\right)} \\ &= (-1)^{j-i} \theta(\hbar^{-1}) \prod_{\substack{i < m < j \\ m \in \mathbf{p}}} \frac{\theta\left(\frac{\hbar u_j}{u_m}\right)}{\theta\left(\frac{u_j}{u_m}\right)} \prod_{\substack{i < m < j \\ m \in \mathbf{n} \setminus \mathbf{p}}} \frac{\theta\left(\frac{u_j}{u_m \hbar}\right)}{\theta\left(\frac{u_j}{u_m}\right)}, \end{aligned}$$

where the last equality is because for $e \in B \setminus U$,

$$\frac{\theta\left(\frac{x_{t(e)} \varphi_{\hbar(e)}^\lambda u_j}{x_{h(e)} \varphi_{t(e)}^\lambda u_{h(e)}}\right) \Big|_{\nu}}{\theta\left(\frac{u_j}{u_{h(e)}}\right)} = \begin{cases} \frac{\theta\left(\frac{\hbar u_j}{u_{h(e)}}\right)}{\theta\left(\frac{u_j}{u_{h(e)}}\right)}, & h(e) \in \mathbf{p}, e \notin U \\ \frac{\theta\left(\hbar^{-1} \frac{u_j}{u_{h(e)}}\right)}{\theta\left(\frac{u_j}{u_{h(e)}}\right)}, & h(e) \notin \mathbf{p}, e \notin U \end{cases}$$

and for $e \in B \cap U$,

$$\frac{\theta\left(\frac{x_{t(e)} \varphi_{\hbar(e)}^\lambda u_{t(e)}}{x_{h(e)} \varphi_{t(e)}^\lambda u_i}\right) \Big|_{\nu, u_i=u_j}}{\theta\left(\frac{u_{t(e)}}{u_i}\right)} = \frac{\theta\left(\frac{\hbar u_{t(e)}}{u_j}\right)}{\theta\left(\frac{u_{t(e)}}{u_j}\right)}.$$

The proposition is proved by making the change of variable $\hbar \mapsto \hbar^{-1}$ in the above result, and compare with the following lemma.

Lemma 12.

$$\theta\left(\frac{u_i}{u_j}\right)^{-1} \frac{T_{\mathbf{p}, \mathbf{p}}}{T_{\mathbf{q}, \mathbf{q}}} \Big|_{u_i=u_j} = \theta(\hbar^{-1})^{-1} \prod_{\substack{i < m < j \\ m \in \mathbf{n} \setminus \mathbf{p}}} \theta\left(\frac{u_j}{u_m}\right) \prod_{\substack{i < m < j \\ m \in \mathbf{p}}} \theta\left(\frac{u_m}{u_j}\right) \prod_{\substack{i < m < j \\ m \in \mathbf{n} \setminus \mathbf{p}}} \theta\left(\frac{u_m}{u_j \hbar}\right)^{-1} \prod_{\substack{i < m < j \\ m \in \mathbf{p}}} \theta\left(\frac{u_j}{u_m \hbar}\right)^{-1}$$

Proof. Straightforward computation. □

9.3 Divisibility

In this subsection we aim to prove the following divisibility result. Let $\mathbf{p} = \text{bj}(\lambda)$, $\mathbf{q} = \text{bj}(\mu) \in X^T$ be two fixed points.

Proposition 5. The function $\frac{T_{\mathbf{p}, \mathbf{p}}}{T'_{\mu, \mu}} \cdot T'_{\lambda, \mu}$ is of the form

$$f_{\mu, \lambda} \cdot \prod_{\substack{i \in \mathbf{p}, j \in \mathbf{n} \setminus \mathbf{p} \\ i > j}} \theta\left(\frac{u_j}{u_i \hbar}\right),$$

where $f_{\mu, \lambda}$ is holomorphic in parameters u_i .

Proof. Recall that

$$T_{\mathbf{p}, \mathbf{p}} = \prod_{i \in \mathbf{p}, j \in \mathbf{n} \setminus \mathbf{p}, i < j} \theta\left(\frac{u_i}{u_j}\right) \prod_{i \in \mathbf{p}, j \in \mathbf{n} \setminus \mathbf{p}, i > j} \theta\left(\frac{u_j}{u_i \hbar}\right),$$

and $T'_{\mu, \mu}$ does not depend on u_i 's. By formula (40), we can see that all possible poles of $T'_{\mu, \lambda}$ take the form u_i/u_j . Therefore, all possible poles of the function $f_{\mu, \lambda}$ in the proposition are of the form u_i/u_j . Moreover, by the proof of holomorphicity (Proposition 9) below, they have no poles at u_i/u_j . We conclude that $f_{\mu, \lambda}$ is holomorphic in u_i . □

9.4 Holomorphicity

In this subsection we will prove the holomorphicity, i.e., the normalized restriction matrices of stable envelopes on X' are holomorphic in u_i 's. The idea is to apply general results for q -difference equations associated to Nakajima quiver varieties.

9.4.1 Quantum differential equations

Let X be a Nakajima variety. For the cone of effective curves in $H_2(X, \mathbb{Z})$ we consider the semigroup algebra which is spanned by monomials z^d with $d \in H_2(X, \mathbb{Z})_{\text{eff}}$. It has a natural completion which we denote by $\mathbb{C}[[z^d]]$. The cup product in the equivariant cohomology $H_{\mathbb{T}}^{\bullet}(X)$ has a natural commutative deformation, parametrized by z :

$$\alpha \star \beta = \alpha \cup \beta + O(z) \quad (62)$$

known as the quantum product.

The quantum multiplication defines a remarkable flat connection on the trivial $H_{\mathbb{T}}^{\bullet}(X)$ -bundle over $\text{Spec}(\mathbb{C}[[z^d]])$. Flat sections $\Psi(z)$ of this connection, considered as $H_{\mathbb{T}}^{\bullet}(X)$ -valued functions, are defined by the following system of differential equations (known as the quantum differential equation or Dubrovin connection):

$$\varepsilon \frac{d}{d\lambda} \Psi(z) = \lambda \star \Psi(z), \quad \Psi(z) \in H_{\mathbb{T}}^{\bullet}(X)[[z]],$$

where $\lambda \in H^2(X, \mathbb{C})$ and the differential operator is defined by

$$\frac{d}{d\lambda} z^d = (\lambda, d) z^d. \quad (63)$$

9.4.2 Quantum multiplication by divisor

The equivariant cohomology of Nakajima varieties are equipped with a natural action of certain Yangian $Y_{\hbar}(\mathfrak{g}_X)$ [48]. In the case of Nakajima varieties associated to quivers of ADE type this algebra coincides with the Yangian of the corresponding Lie algebra (but in general can be substantially larger).

The Lie algebra \mathfrak{g}_X has a root decomposition:

$$\mathfrak{g}_X = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

in which $\mathfrak{h} = H^2(X, \mathbb{C}) \oplus \text{center}$, and $\alpha \in H_2(X, \mathbb{Z})_{\text{eff}}$. All root subspaces \mathfrak{g}_{α} are finite dimensional and $\mathfrak{g}_{-\alpha} = \mathfrak{g}_{\alpha}^*$ with respect to the symmetric nondegenerate invariant form.

The quantum multiplication (62) for Nakajima varieties can be universally described in terms of the corresponding Yangians:

Theorem 7 (Theorem 10.2.1 in [31]). *The quantum multiplication by a class $\lambda \in H^2(X)$ is given by:*

$$\lambda \star = \lambda \cup + \hbar \sum_{(\theta, \alpha) > 0} \alpha(\lambda) \frac{z^{\alpha}}{1 - z^{\alpha}} e_{\alpha} e_{-\alpha} + \dots \quad (64)$$

where $\theta \in H^2(X, \mathbb{R})$ is a vector in the ample cone (i.e., in the summation, θ selects the effective representative from each $\pm\alpha$ pair) and \dots denotes a diagonal term, which can be fixed by the condition $\lambda \star 1 = \lambda$.

Let z_i with $i = 1, \dots, n-1$ denote the Kähler parameters of the Nakajima variety X' from Section 4.

Corollary 3. *The quantum connection associated with the Nakajima variety X' is a connection with regular singularities supported on the hyperplanes*

$$z_i z_{i+1} \dots z_j = 1, \quad 1 \leq i < j \leq n-1.$$

Proof. The variety X' is a Nakajima quiver variety associated with the A_{n-1} -quiver. Thus the corresponding Lie algebra $\mathfrak{g}_X \cong \mathfrak{sl}_n$. The Kähler parameters z_i associated to the tautological line bundles on X' correspond to the simple roots of this algebra. In other words, in the notation of (63) they correspond to $z_i = z^{\alpha_i}$, where $\alpha_i, i = 1, \dots, n-1$ are the simple roots of \mathfrak{sl}_n (more precisely, simple roots with respect to positive Weyl chamber $(\theta', \alpha_i) > 0$ where θ' is the choice stability parameters for X').

By (64), the singularities of quantum differential equation of X' are located at

$$z^\alpha = 1$$

for positive roots α . All positive roots of \mathfrak{sl}_n are of the form $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$ with $1 \leq i < j \leq n-1$. Thus, the singularities are at

$$z^\alpha = z_i z_{i+1} \dots z_j = 1.$$

□

9.4.3 Quantum difference equation

In the equivariant K-theory, the differential equation is substituted by its q -difference version:

$$\Psi(zq^\mathcal{L})\mathcal{L} = \mathbf{M}_\mathcal{L}(z)\Psi(z) \quad (65)$$

where $\mathcal{L} \in \text{Pic}(X)$ is a line bundle and $q = e^\varepsilon$ and $\Psi(z) \in K_\mathbb{T}(X)[[z]]$. The theory of quantum difference equations for Nakajima varieties was developed in [40]. In particular, the operators $\mathbf{M}_\mathcal{L}(z) \in \text{End}(K_\mathbb{T}(X))$ were constructed for an arbitrary line bundle \mathcal{L} . These operators are the q -deformations of (64), i.e., in the cohomological limit they behave as:

$$\mathbf{M}_\mathcal{L}(z) = 1 + \lambda \star + \dots$$

where \dots stands for the terms vanishing in the cohomological limit and $\lambda = c_1(\mathcal{L})$.

In K-theory the sum over roots in (64) is substituted by a product:

$$\mathbf{M}_\mathcal{L}(z) = \mathcal{L} \prod_w \mathbf{B}_w(z)$$

over certain set of affine root hyperplanes of an affine algebra $\widehat{\mathfrak{g}}_X$.

The singularities of the quantum difference equations, i.e., the singularities of matrix $\mathbf{M}_\mathcal{L}(z)$ are located in the union of singularities of $\mathbf{B}_w(z)$. The wall crossing operators $\mathbf{B}_w(z)$ are constructed in Section 5.3 of [40]. In particular, if $z^\alpha = 1$ are the singularities of the quantum differential equation in cohomology then the singularities of (65) can only be located at $z^\alpha q^p \hbar^s = 1$ for some integral p, s . This, together with Corollary 3 gives:

Proposition 6. *The singularities of the quantum difference equation associated with the Nakajima variety X' are located at*

$$z_i z_{i+1} \dots z_j q^p \hbar^s = 1, \quad 1 \leq i < j \leq n, \quad p, s \in \mathbb{Z}.$$

9.4.4 Pole subtraction theorem

The elliptic stable envelope is closely related to q -difference equations (65). It describes the monodromy of q -difference equations. More precisely, the q -difference equation (65) has two distinguished bases of solutions, known as vertex functions, see Section 6.1 of [1]. The z -solutions are represented by functions Ψ^z which are holomorphic in Kähler parameters in the neighborhood $|z_i| < 1$. Similarly a -solutions Ψ^a are solutions which are holomorphic in equivariant parameters in some neighborhood of zero.

From a general theory of q -difference equations, every two bases of solutions must be related by a q -periodic transition matrix

$$\Psi^a = W(z)\Psi^z, \quad (66)$$

known as the monodromy matrix from solutions Ψ^z to solutions Ψ^a . The central result of [1] (in the case when $X^\mathbb{T}$ is finite) is the following.

Theorem 8 (Theorem 5 in [1]). *Let X be a Nakajima variety and let*

$$T_{\lambda,\mu}(z) = \text{Stab}(\lambda)|_{\mu}, \quad \lambda, \mu \in X^{\Gamma}$$

be the restriction matrix for elliptic stable envelope in the basis of fixed points. Then, the q -periodic matrix $W(z)$ from (66) in this basis equals:

$$W(z)_{\lambda,\mu} = \frac{T_{\lambda,\mu}(z)}{\Theta(T^{1/2}X)_{\mu}}$$

where $T^{1/2}X$ is the polarization bundle for X .

The singularities of solutions Ψ^a and Ψ^z are supported on the singularities of the corresponding q -difference equation. It implies that the transition matrix also may have only these singularities (if $W(z)$ is singular on a hyperplane h , which is not a singularity of qde then, by (66) Ψ^a is also singular along h which is not possible).

In particular, combining the last Theorem with Proposition 6 we obtain:

Corollary 4. *Let $T'_{\lambda,\mu}$ be the restriction matrix of the elliptic stable envelope for the Nakajima variety X' in the basis of fixed points. Then, the singularities of $T'_{\lambda,\mu}$ are supported to the set of hyperplanes:*

$$z_i z_{i+1} \cdots z_j q^p \hbar^s = 1, \quad 1 \leq i < j \leq n, \quad p, s \in \mathbb{Z}.$$

Note 1. The last corollary implies that all poles of the the restriction matrix $T'_{\lambda,\mu}$ in the coordinates u_i related to Kähler variables (42) are of the form:

$$\frac{u_i}{u_j} \hbar^{s_{ij}}, \quad i \neq j, \quad (67)$$

where s_{ij} are some integers.

9.4.5 Holomorphicity of stable envelope

Let us return to the Nakajima varieties X and X' defined in Sections 3 and 4 respectively. We identify the fixed points as in Section 6.1, and identify the equivariant and Kähler parameters by (42). Let $T'_{\lambda,\mu}$ and $T_{\mathbf{p},\mathbf{q}}$ be the restriction matrices of the elliptic stable envelopes for the Nakajima varieties X' and X respectively.

Theorem 9. *The functions*

$$T_{\mathbf{p},\mathbf{p}} T'_{\lambda,\mu}$$

are holomorphic in parameters u_i .

Proof. By Corollary 4 and Note 1, we need to show that the denominators of functions $T_{\mathbf{p},\mathbf{p}} T'_{\lambda,\mu}$ do not contain factors of the form

$$\prod_{i \neq j} \theta\left(\frac{u_i}{u_j} \hbar^{s_{ij}}\right).$$

On the other hand, by Proposition 3, the explicit formula for the elliptic stable envelope on X' has the form:

$$T'_{\lambda,\mu} = \epsilon(\lambda) \Theta(\tilde{N}'_{\lambda,-})|_{\mu} \cdot \sum_{\sigma \in \mathfrak{S}_{\mu \setminus \lambda, \bar{\mathfrak{t}}}} \frac{\mathcal{N}_{\lambda}^{\sigma}}{\mathcal{D}_{\lambda}^{\sigma}} \mathcal{R}^{\sigma}(\bar{\mathfrak{t}}) \mathcal{W}^{\sigma}(\bar{\mathfrak{t}})|_{\mu},$$

where $\Theta(\tilde{N}'_{\lambda,-})|_{\mu}$, $\mathcal{N}_{\lambda}^{\sigma}$, $\mathcal{D}_{\lambda}^{\sigma}$, $\mathcal{R}^{\sigma}(\bar{\mathfrak{t}})$ are independent of u_i , and

$$\mathcal{W}^{\sigma}(\bar{\mathfrak{t}}) = \frac{\theta\left(\frac{a_2 \hbar}{x_{\sigma(\bar{r})}} \prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} \frac{u_{c_I}}{u_{c_I+1}}\right)}{\theta\left(\prod_{I \in [\bar{r}, \bar{\mathfrak{t}}]} \frac{u_{c_I}}{u_{c_I+1}}\right)} \prod_{e \in \bar{\mathfrak{t}}} \frac{\theta\left(\frac{x_{\sigma(t(e))} \varphi_{h(e)}^{\lambda}}{x_{\sigma(h(e))} \varphi_{t(e)}^{\lambda}} \prod_{I \in [h(e), \bar{\mathfrak{t}}]} \frac{u_{c_I}}{u_{c_I+1}}\right)}{\theta\left(\prod_{I \in [h(e), \bar{\mathfrak{t}}]} \frac{u_{c_I}}{u_{c_I+1}}\right)}.$$

Therefore, we conclude that among (67) only factors with $s_{ij} = 0$ may appear. To show that those are actually not poles, it suffices to prove that

$$\theta\left(\frac{u_i}{u_j}\right) T_{\mathbf{p}, \mathbf{p}} T'_{\lambda, \mu} \Big|_{u_i=u_j} = 0.$$

As discussed before, the only possible nontrivial terms of the LHS come from trees $\bar{\mathfrak{t}}$ which contains some (i, j) -strip B .

If $j \in \mathbf{p}$, one can see that $\bar{\lambda} \setminus B$ contains a path in $\bar{\mathfrak{t}}$ admitting a box with local maximal content, which is not allowed. In other words, contributions from all $\bar{\mathfrak{t}}$ are zero in this case.

If $i \in \mathbf{n} \setminus \mathbf{p}$, then the boxes above and to the left of the root of B both lie in $\bar{\lambda}$, and the involution $\text{inv}(\bar{\mathfrak{t}})$ is also a tree in $\bar{\lambda}$. By the cancellation Lemma 6, contribution from $\bar{\mathfrak{t}}$ cancels with that from $\text{inv}(\bar{\mathfrak{t}})$. Sum over all $\bar{\mathfrak{t}}$ gives 0.

If $i \in \mathbf{p}$ and $j \in \mathbf{n} \setminus \mathbf{p}$, then $T_{\mathbf{p}, \mathbf{p}}$ contains a factor $\theta\left(\frac{u_i}{u_j}\right)$, and nontrivial terms come from trees $\bar{\mathfrak{t}}$ that contains at least two (i, j) -strips, e.g., B_1, B_2 . At least one of them, say B_1 , is not contained in the boundary of $\bar{\lambda}$ and hence the involution of $\bar{\mathfrak{t}}$ with respect to B_1 is well-defined. Contribution from $\bar{\mathfrak{t}}$ then cancels with that from $\text{inv}(\bar{\mathfrak{t}})$.

Therefore, we exclude all possible poles $\theta(u_i/u_j)$, and $T_{\mathbf{p}, \mathbf{p}} T'_{\lambda, \mu}$ is holomorphic in u_i . \square

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