

# POINCARÉ-BIRKHOFF-WITT EXPANSIONS OF THE CANONICAL ELLIPTIC DIFFERENTIAL FORM

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ABSTRACT. We study the canonical  $U(\mathfrak{n}_-)$ -valued elliptic differential form, whose projections to different Kac-Moody algebras are key ingredients of the hypergeometric integral solutions of elliptic KZ differential equations and Bethe ansatz constructions. We explicitly determine the coefficients of the projections in the simple Lie algebras  $A_r, B_r, C_r, D_r$  in a conveniently chosen Poincaré-Birkhoff-Witt basis. As an application we give a new formula for eigenfunctions of Hamiltonians of the Calogero-Moser model.

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a simple Lie algebra with Cartan decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . Let  $V$  be the tensor product  $V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$  of highest weight  $\mathfrak{g}$ -modules. The  $V$ -valued hypergeometric solutions of the elliptic Knizhnik-Zamolodchikov differential equations have the form ([FV1], cf. [SV])

$$(1) \quad I(z, \lambda, \tau) = \int_{\gamma(z, \lambda, \tau)} \Phi(t, z, \tau) \cdot \Omega^V(t, z, \lambda, \tau).$$

Here  $t = (t_1, \dots, t_N)$ ,  $z = (z_1, \dots, z_n)$ ,  $\tau \in \mathbb{C}$  with  $\text{Im } \tau > 0$ ,  $\lambda \in \mathfrak{h}$ ,  $\Phi$  is an explicit scalar-valued master function,  $\gamma$  is a suitable cycle in  $t$ -space, and  $\Omega^V$  is a  $V$ -valued differential  $k$ -form in  $t$ -space.

The same  $\Phi$  and  $\Omega^V$  have applications to the Bethe Ansatz method. It is known [RV, EK1, EK2, FV1, FV2] that the values of  $\Omega^V$  at the critical points of  $\Phi$  (with respect to  $t$ ) give eigenfunctions of the Hamiltonians of the quantum elliptic Calogero-Moser model [GH] (in the  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  case).

For every  $V = V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$ , the  $V$ -valued differential form  $\Omega^V$  can be constructed out of a single  $U(\mathfrak{n}_-)$ -valued differential form  $\Theta^{\mathfrak{g}}$ , where  $U(\ )$  denotes the universal enveloping algebra. The form  $\Theta^{\mathfrak{g}}$  is called the canonical elliptic differential form. In applications it is important to have convenient formulas for  $\Theta^{\mathfrak{g}}$ , and this is the goal of the present paper.

The form  $\Theta^{\mathfrak{g}}$  (depending also on a vector  $k \in \mathbb{N}^r$ ) is defined as the projection of the ‘canonical element’  $\Omega_k$  according to the scheme

$$\begin{array}{ccc} \Omega_k \in \mathcal{A}(k) \otimes U[k] & \rightarrow & \mathcal{A}(k) \otimes U(\mathfrak{n}_-)[k] \\ \downarrow & & \downarrow \\ W \otimes U[k] & \rightarrow & W \otimes U(\mathfrak{n}_-)[k] \ni \Theta_k^{\mathfrak{g}}. \end{array}$$

Here  $\mathcal{A}(k)$  is a homogeneous component of an Orlik-Solomon algebra;  $U[k]$  is its dual, and  $W$  is a space of certain differential forms written in terms of theta functions.

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In Section 2 we define and study  $\mathcal{A}(k) \otimes U[k]$ , and its canonical element  $\Omega_k$ . In Sections 3 and 4 we find Poincaré-Birkhoff-Witt formulas for the projection of  $\Omega_k$  in  $\mathcal{A}(k) \otimes U(\mathfrak{n}_-)[k]$ . In Section 5 we define representations of  $\mathcal{A}(k)$  in spaces  $W$  of suitable differential forms. In Section 6 we use the PBW expansion of  $\Theta_k^{sl_{r+1}}$  to find a new formula for eigenfunctions of the Hamiltonian of the Calogero-Moser model. In the Appendix we describe the image of the faithful representation  $\mathcal{A}(k) \rightarrow W$ .

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## 2. COHOMOLOGY OF THE COMPLEMENT OF THE DISCRIMINANTAL ARRANGEMENT. COMBINATORIAL CODES AND OPERATIONS.

**2.1.** An *ordered  $p$ -forest* is a graph with no cycles, with  $p$  edges, and a numbering of its edges by the numbers  $1, 2, \dots, p$ . Let  $\mathcal{A}_n^p$  be the complex vector space generated by the ordered  $p$ -forests on the vertex set  $\{z, t_1, \dots, t_n\}$  (a set of symbols now), modulo the following two kinds of relations:

**R1:**  $T_1 = -T_2$  if  $T_1$  and  $T_2$  have the same underlying graph, and the order of their edges differ by a transposition;

**R2:**

$$(2) \quad \begin{array}{c} * & & * \\ & \backslash & / \\ & a & b \\ & / & \backslash \\ * & & * \end{array} + \begin{array}{c} * & a & * \\ & \backslash & / \\ & b & \\ & / & \\ * & & * \end{array} + \begin{array}{c} * & b & * \\ & / & \backslash \\ & & a \\ & \backslash & / \\ * & & * \end{array} = 0 \quad (a, b \in \{1, \dots, p\}),$$

that is, the sum of three  $p$ -forests that locally (i.e. their subgraphs spanned by 3 vertices) differ as above, but are otherwise identical, is 0.

Let  $z$  be a complex number, and consider  $\mathbb{C}^n$  with coordinates  $t_1, \dots, t_n$ . The discriminantal arrangement  $\mathcal{C}^n = \mathcal{C}^n(z)$  in the space  $\mathbb{C}^n$  is the collection of hyperplanes  $t_i = t_j$  ( $1 \leq i < j \leq n$ ) and  $t_i = z$  ( $1 \leq i \leq n$ ). Let  $\mathcal{U}_n$  denote the complement of  $\mathcal{C}^n$ , i.e.  $\mathcal{U}_n = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{C}^n} H$ .

**Proposition 2.1.** *The cohomology group  $H^p(\mathcal{U}_n; \mathbb{C})$  is isomorphic to  $\mathcal{A}_n^p$ .*

*Proof.* The definition of  $\mathcal{A}(k)$  is a combinatorial code for the description of  $H^p(\mathcal{U}_n; \mathbb{C})$  by Arnold in [A].  $\square$

Now let us fix  $r \in \mathbb{N}$ , and  $r$  non-negative integers  $k = (k_1, k_2, \dots, k_r)$  with  $\sum k_i = |k|$ . The set of coordinates

$$(3) \quad (t_1^{(1)}, \dots, t_{k_1}^{(1)}, t_1^{(2)}, \dots, t_{k_2}^{(2)}, \dots, t_1^{(r)}, \dots, t_{k_r}^{(r)}).$$

will be denoted by  $\mathcal{T}(k)$ . We will consider  $\mathbb{C}^{|k|}$  with coordinates  $\mathcal{T}(k)$ , and the space  $\mathcal{A}_{|k|}^p$  with vertex set  $\mathcal{T}(k) \cup \{z\}$ . The group  $G_k = \prod \Sigma_{k_i}$  acts on  $\mathbb{C}^{|k|}$  (by permuting the coordinates with the same upper indices) which then induces an action of  $G_k$  on  $\mathcal{A}_{|k|}^p$ . The skew-invariant subspace (i.e. the collection of  $x$ 's for which  $\pi \cdot x = \text{sgn}(\pi)x, \forall \pi \in G_k$ ) of  $\mathcal{A}_{|k|}^p$  will be denoted by  $\mathcal{A}(k)$ .

**Example 2.2.** In the  $r \leq 2$  examples (in the whole paper) we will use  $t_j$  for  $t_j^{(1)}$  and  $s_j$  for  $t_j^{(2)}$ . The vector space  $\mathcal{A}((1, 1))$  is two-dimensional with a basis  $\{z \frac{1}{1} t_1 \frac{1}{2} s_1, z \frac{1}{1} s_1 \frac{1}{2} t_1\}$ . The vector space  $\mathcal{A}((2))$  is one-dimensional with a basis  $\{t_1 \frac{1}{1} z \frac{1}{2} t_2\}$ .

## 2.2. The star-product.

**Definition 2.3.** Let  $e_1, e_2, \dots, e_{|k|}$  be the edges (in order) of an ordered  $|k|$ -tree  $T$  on the vertex set  $\mathcal{T}(k) \cup \{z\}$ . The sign  $\varepsilon(T)$  of  $T$  is defined by

$$\bigwedge_{i=1}^{|k|} d h(e_i) = \varepsilon(T) \cdot \bigwedge_{i=1}^r \bigwedge_{j=1}^{k_i} d t_j^{(i)},$$

where  $h(e)$  is the ‘head’ of the edge  $e$ , i.e. the end vertex of  $e$  farther from the root  $z$ .

For example the signs of the three trees of Example 2.2 are  $+1, -1, +1$ , respectively.

**Definition 2.4.** For an element  $x$  in a vector space acted upon by  $G_k$ , let  $\text{ASym}_k(x) = \text{ASym}(x)$  be the anti-symmetrization of  $x$ :  $\sum_{\pi \in G_k} \text{sgn}(\pi) \pi(x)$ . Let  $\text{Sym}_k(x) = \text{Sym}(x)$  be the symmetrization of  $x$ :  $\sum_{\pi \in G_k} \pi(x)$ .

For  $k, l \in \mathbb{N}^r$  let  $T_1$  and  $T_2$  be ordered  $|k|$ - and  $|l|$ -trees on the vertex set  $\mathcal{T}(k) \cup \{z\}$  and  $\mathcal{T}(l) \cup \{z\}$  respectively. We will define their *star-product*  $T_1 * T_2$  by the following procedure. We replace the vertices  $t_j^{(i)}$  of  $T_2$  with  $t_{k_i+j}^{(i)}$ , add  $|k|$  to the numbers on the edges of  $T_2$ , and identify the  $z$ -vertices of the two trees—thus we obtain a tree  $T$  with  $|k| + |l| + 1$  vertices. The product  $T_1 * T_2$  is defined as  $\varepsilon(T_1) \varepsilon(T_2) \varepsilon(T) / \prod_i (k_i! l_i!) \cdot \text{ASym}_{k+l} T$ , i.e.  $T_1 * T_2 =$

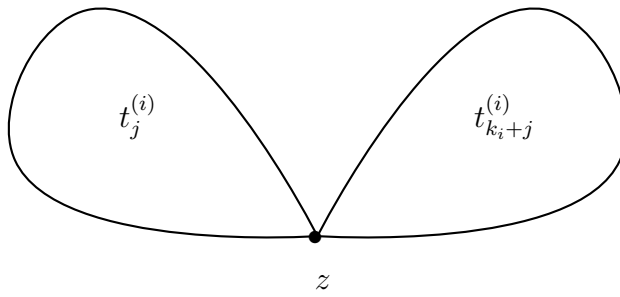
$$\frac{\varepsilon(T_1) \varepsilon(T_2) \varepsilon(T)}{\prod_{i=1}^r k_i! l_i!} \cdot \text{ASym}_{k+l}$$

$T_1$

(edges:  $1, \dots, |k|$ )

$T_2$

(edges:  $|k| + 1, \dots, |k| + |l|$ )



The star-product induces a product (that we will also call star-product)  $\mathcal{A}(k) \otimes \mathcal{A}(l) \rightarrow \mathcal{A}(k+l)$ . The vector space  $\bigoplus_{k \in \mathbb{N}^r} \mathcal{A}(k)$  with the star-product is an associative and commutative algebra.

### Example 2.5.

$$(z \xrightarrow{1} t_1) * (z \xrightarrow{1} s_1) = (t_1 \xrightarrow{1} z \xrightarrow{2} s_1) = (z \xrightarrow{1} t_1 \xrightarrow{2} s_1) - (z \xrightarrow{1} s_1 \xrightarrow{2} t_1).$$

Computation shows that

$$(z \xrightarrow{1} t_1)^{*n} = n! \text{ASym}(z \xrightarrow{1} t_1 \xrightarrow{2} t_2 \xrightarrow{3} \dots \xrightarrow{n} t_n).$$

**2.3. Residue.** Let  $T$  be a  $p$ -tree on the vertex set  $\mathcal{T}(k) \cup \{z\}$ , and choose a coordinate  $t_j^{(i)}$ . We will define the  $t_j^{(i)}$ -residue  $\text{Res}_j^{(i)} T$  of the tree  $T$  as a  $p-1$ -tree on the vertex set  $\mathcal{T}(k) \cup \{z\} \setminus \{t_j^{(i)}\}$ , constructed as follows.

- If  $t_j^{(i)}$  and  $z$  are connected by an edge with number  $a$ , then we contract that edge (i.e. delete the edge, and identify its vertices), and label this contracted vertex by  $z$ . Then, if the number of an edge is greater than  $a$ , we decrease it by 1.  $\text{Res}_j^{(i)}$  is defined as  $(-1)^{a-1}$  times this modified tree. See Figure 1.

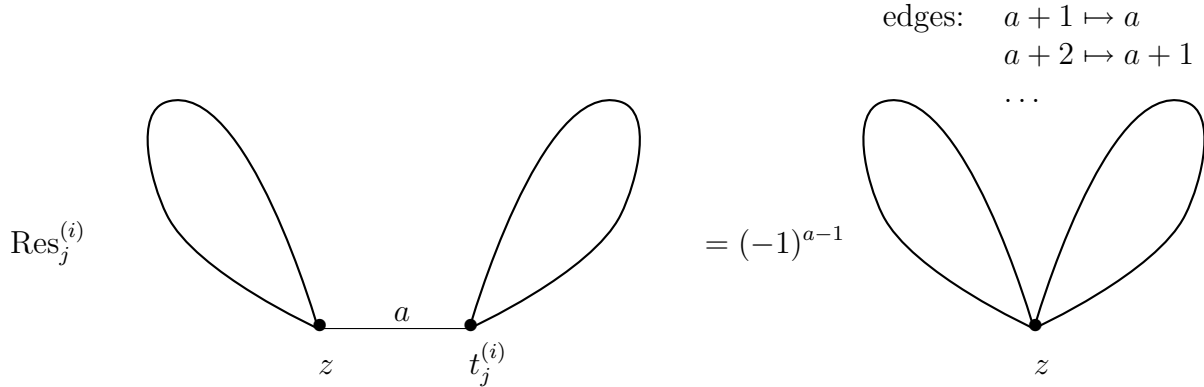


Figure 1

- If  $t_j^{(i)} \text{ --- } z$  is not an edge of  $T$ ,  $\text{Res}_j^{(i)} T$  is defined to be 0.

This operation is consistent with relations R1 and R2, hence they are defined for  $\mathcal{A}(k)$ . Observe that  $\text{Res}_{k_i}^{(i)}$  is a map  $\mathcal{A}(k) \rightarrow \mathcal{A}(k-1_i)$ . (Here, and in the sequel,  $1_i$  denotes a vector, whose coordinates are 0 except the  $i$ 'th coordinate, which is 1.)

**2.4. The dual space of the cohomology of the complement.** Let  $U_r$  denote the free associative complex algebra generated by the  $r$  symbols  $\tilde{f}_1, \dots, \tilde{f}_r$ . It is multigraded by  $\mathbb{N}^r$  ( $\deg \tilde{f}_i = 1$ ), the degree  $k$  subspace will be denoted by  $U[k]$ . In [SV]  $U[k]$  is shown to be isomorphic to the  $G_k$  skew-invariant part of the dual space of  $H^{|k|}(\mathcal{U}_{|k|}; \mathbb{C})$ . In this section we recall this isomorphism in our combinatorial language.

**Definition 2.6.** Let  $r \in \mathbb{N}$  and  $k \in \mathbb{N}^r$ .

- Maps  $J : \{1, \dots, |k|\} \rightarrow \{1, \dots, r\}$  with  $\#J^{-1}(i) = k_i$  will be called  $(k)$ -multiindices.
- For a multiindex  $J$ , let  $f_J$  be the monomial  $\tilde{f}_{J(|k|)} \tilde{f}_{J(|k|-1)} \cdots \tilde{f}_{J(1)}$  in  $U[k]$ .
- For a multiindex  $J$  let  $c$  be the unique map  $\{1, \dots, |k|\} \rightarrow \mathbb{N}$  that restricted to  $J^{-1}(i)$  is the increasing function onto  $\{1, \dots, k_i\}$ .
- The sign  $\text{sgn}(J)$  of a multiindex  $J$  is  $(-1)^m$  where  $m$  is the minimal number of transpositions to be applied to the list  $J(1), J(2), \dots, J(|k|)$  to get  $1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r$ .

Let  $T_0$  be the graph with a single vertex  $z$  and with no edge. For  $T \in \mathcal{A}(k)$  and  $J$  a  $k$ -multiindex, the expression

$$(4) \quad \text{sgn}(J) \cdot \text{Res}_{c(|k|)}^{(J(|k|))} \left( \text{Res}_{c(|k|-1)}^{(J(|k|-1))} \left( \dots \text{Res}_{c(1)}^{(J(1))} (T) \dots \right) \right)$$

is equal to a constant times  $T_0$ . Let this constant be denoted by  $\langle \tilde{f}_J, T \rangle$ .

**Theorem 2.7.** [SV, Th. 2.4] *The map  $\tilde{f}_J \mapsto \langle \tilde{f}_J, \quad \rangle$  is an isomorphism between  $U[k]$  and the dual space  $\mathcal{A}(k)^*$  of  $\mathcal{A}(k)$ .*

□

**Corollary 2.8.** *For the basis  $\{\tilde{f}_J\}$  of  $U_r$  the dual basis is*

$$\tilde{f}_J^* = \text{sgn}(J) \cdot \text{ASym} \left( z \frac{\quad}{1} t_{c(1)}^{(J(1))} \frac{\quad}{2} t_{c(2)}^{(J(2))} \frac{\quad}{3} \cdots \frac{\quad}{|k|} t_{c(|k|)}^{(J(|k|))} \right).$$

**2.5. Product on  $\sum \mathcal{A}(k)$ , coproduct on  $\sum U[k]$ .** Recall that  $U_r$  is equipped with its standard Hopf algebra structure. The comultiplication  $\Delta : U_r \rightarrow U_r \otimes U_r$  is defined by  $\Delta(x) = 1 \otimes x + x \otimes 1$  for degree 1 elements.

**Theorem 2.9.** *Under the duality between  $\sum_{k \in \mathbb{N}^r} \mathcal{A}(k)$  and  $U_r$  the coproduct  $\Delta$  corresponds to the star-multiplication.*

*Proof.* We need the following notion: a triple

$$(S_1 \subset \{1, \dots, |k+l|\}, S_2 \subset \{1, \dots, |k+l|\}, J : \{1, \dots, |k+l|\} \rightarrow \{1, \dots, r\})$$

is called a *shuffle* of  $J_1 : \{1, \dots, |k|\} \rightarrow \{1, \dots, r\}$  and  $J_2 : \{1, \dots, |l|\} \rightarrow \{1, \dots, r\}$  if

- $\#S_1 = |k|$ ,  $\#S_2 = |l|$ ,  $S_1 \cup S_2 = \{1, \dots, |k+l|\}$ ,
- for the increasing bijections  $s_1 : S_1 \rightarrow \{1, \dots, |k|\}$ ,  $s_2 : S_2 \rightarrow \{1, \dots, |l|\}$  we have

$$J(i) = \begin{cases} J_1 \circ s_1(i) & i \in S_1 \\ J_2 \circ s_2(i) & i \in S_2. \end{cases}$$

The definition of  $\Delta$  implies that  $\Delta^*(\tilde{f}_{J_1}^* \otimes \tilde{f}_{J_2}^*) = \sum \tilde{f}_J^*$ , where the sum runs over all the shuffles of  $J_1$  and  $J_2$ . On the other hand  $\tilde{f}_{J_1}^* * \tilde{f}_{J_2}^* =$

$$\pm \text{ASym} \left( z \frac{\quad}{1} t_{c_1(1)}^{(J_1(1))} \frac{\quad}{2} \cdots \frac{\quad}{|k|} t_{c_1(|k|)}^{(J_1(|k|))} \right) * \text{ASym} \left( z \frac{\quad}{1} t_{c_2(1)}^{(J_2(1))} \frac{\quad}{2} \cdots \frac{\quad}{|l|} t_{c_2(|l|)}^{(J_2(|l|))} \right).$$

Using relation (2), this is equal to the sum of terms

$$\pm \text{ASym} \left( z \frac{\quad}{1} t_{c(1)}^{(J(1))} \frac{\quad}{2} \cdots \frac{\quad}{|k+l|} t_{c(|k+l|)}^{(J(|k+l|))} \right),$$

where  $J$  is a shuffle of  $J_1$  and  $J_2$ . We illustrate this argument with the following example:

$$(\tilde{f}_2 \tilde{f}_1)^* * \tilde{f}_1^* = \left( z \frac{\quad}{1} t_1 \frac{\quad}{2} s \right) * \left( z \frac{\quad}{1} t_1 \right) =$$

$$\begin{aligned} & - \text{ASym} \left( z \frac{\quad}{1} t_1 \frac{\quad}{2} s \right) = - \text{ASym} \left( z \frac{\quad}{1} t_1 \frac{\quad}{3} t_2 + z \frac{\quad}{3} t_2 \frac{\quad}{1} t_1 \frac{\quad}{2} s \right) = \\ & - \text{ASym} \left( z \frac{\quad}{1} t_1 \frac{\quad}{3} t_2 + z \frac{\quad}{1} t_1 \frac{\quad}{3} t_2 \frac{\quad}{2} s + z \frac{\quad}{3} t_2 \frac{\quad}{1} t_1 \frac{\quad}{2} s \right) = \end{aligned}$$

$$2 \text{ASym} \left( z \frac{\quad}{1} t_1 \frac{\quad}{2} t_2 \frac{\quad}{3} s \right) - \text{ASym} \left( z \frac{\quad}{1} t_1 \frac{\quad}{2} s \frac{\quad}{3} t_2 \right) = 2(\tilde{f}_2 \tilde{f}_1^2)^* + (\tilde{f}_1 \tilde{f}_2 \tilde{f}_1)^*.$$

A careful examination of signs shows that  $\Delta^*$  and the star-product agree on the basis  $\tilde{f}_J^*$ , hence the Theorem is proved. □

**2.6. The canonical element.** We obtained that the spaces  $\mathcal{A}(k)$  and  $U[k]$  are dual vector spaces, and hence there is a canonical element in  $\mathcal{A}(k) \otimes U[k]$ . It is defined by  $\sum b_i^* \otimes b_i$  for a basis  $b_i$  in  $U[k]$  and its dual basis  $b_i^*$  in  $\mathcal{A}(k)$ . This definition does not depend on the choice of the basis. Equivalently, the canonical element is the identity in  $\mathcal{A}(k) \otimes U[k] = \text{Hom}(\mathcal{A}(k)^*, U[k]) = \text{Hom}(U[k], U[k])$ .

**Definition 2.10.** *The canonical element in  $\mathcal{A}(k) \otimes U[k]$  will be denoted by  $\Omega_k$ .*

Using Corollary 2.8 we obtain an explicit form for  $\Omega_k$ .

**Theorem 2.11.** *We have*

$$(5) \quad \Omega_k = \sum_J \text{sgn}(J) \cdot \text{ASym} \left( z \underset{1}{\text{---}} t_{c(1)}^{(J(1))} \underset{2}{\text{---}} t_{c(2)}^{(J(2))} \underset{3}{\text{---}} \cdots \underset{|k|}{\text{---}} t_{c(|k|)}^{(J(|k|))} \right) \otimes \tilde{f}_J,$$

where the summation runs over all  $k$ -multiindices.

**Example 2.12.**  $\Omega_{(1,1)} = (z \underset{1}{\text{---}} t \underset{2}{\text{---}} s) \otimes \tilde{f}_2 \tilde{f}_1 + (z \underset{1}{\text{---}} s \underset{2}{\text{---}} t) \otimes \tilde{f}_1 \tilde{f}_2$ .

$$(6) \quad \begin{aligned} \Omega_{(2,1)} = & \left( (z \underset{1}{\text{---}} t_1 \underset{2}{\text{---}} t_2 \underset{3}{\text{---}} s) - (z \underset{1}{\text{---}} t_2 \underset{2}{\text{---}} t_1 \underset{3}{\text{---}} s) \right) \otimes \tilde{f}_2 \tilde{f}_1^2 - \\ & \left( (z \underset{1}{\text{---}} t_1 \underset{2}{\text{---}} s \underset{3}{\text{---}} t_2) - (z \underset{1}{\text{---}} t_2 \underset{2}{\text{---}} s \underset{3}{\text{---}} t_1) \right) \otimes \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 + \\ & \left( (z \underset{1}{\text{---}} s \underset{2}{\text{---}} t_1 \underset{3}{\text{---}} t_2) - (z \underset{1}{\text{---}} s \underset{2}{\text{---}} t_2 \underset{3}{\text{---}} t_1) \right) \otimes \tilde{f}_1^2 \tilde{f}_2. \end{aligned}$$

### 3. CANONICAL ELEMENTS ASSOCIATED WITH SIMPLE LIE ALGEBRAS

Let  $\mathfrak{g}$  be a simple Lie algebra of rank  $r$  with Cartan decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . The  $\mathbb{N}^r$ -multigraded universal enveloping algebra  $U(\mathfrak{n}_-)$  of  $\mathfrak{n}_-$  is generated by  $r$  elements  $f_1, \dots, f_r$  (the standard Chevalley generators), subject to the Serre relations. The degree  $k$  part of  $U(\mathfrak{n}_-)$  ( $\deg f_i = 1$ ) is denoted by  $U(\mathfrak{n}_-)[k]$ . We have the quotient map  $q : U_r \rightarrow U(\mathfrak{n}_-)$  by mapping  $f_i$  to  $f_i$ .

**Definition 3.1.** *The canonical element  $\Omega_k^{\mathfrak{g}}$  associated with the simple Lie algebra  $\mathfrak{g}$  is defined as the image of the canonical element  $\Omega_k$  under the map*

$$(7) \quad \text{id} \otimes q : \mathcal{A}(k) \otimes U[k] \rightarrow \mathcal{A}(k) \otimes U(\mathfrak{n}_-)[k].$$

**3.1. PBW expansion of the canonical element.** The Lie algebra  $\mathfrak{n}_-$  is the direct sum of 1-dimensional weight spaces  $\mathfrak{n}_\beta$  labeled by the positive roots:  $\mathfrak{n}_- = \bigoplus_{\beta} \mathfrak{n}_\beta$ . Now let us make the following choices:

- (C1) a linear ordering  $\beta_1 < \dots < \beta_m$  of the positive roots (thus  $m = \dim \mathfrak{n}_-$ ),
- (C2) a generator  $F_{\beta_j}$  in  $\mathfrak{n}_{\beta_j}$ .

When considering  $F_{\beta_j}$  in  $U(\mathfrak{n}_-)$ , let its degree be  $k^{(j)} \in \mathbb{N}^r$ . Equivalently,  $k_i^{(j)}$  is the coefficient of the  $i$ 'th simple root (i.e. the simple root corresponding to  $f_i$ ) in the decomposition of  $\beta_j$ . According to the Poincaré-Birkhoff-Witt theorem, a  $\mathbb{C}$ -basis of the algebra  $U(\mathfrak{n}_-)$  is given by the collection of elements  $F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}$  where  $p = (p_1, \dots, p_m) \in \mathbb{N}^m$ .

Hence, the canonical element associated with  $\mathfrak{g}$  can be written (depending on the choices (C1,C2)) as

$$(8) \quad \Omega_k^{\mathfrak{g}} = \sum_p T_p \otimes F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m},$$

where the summation runs over those  $p \in \mathbb{N}^m$  for which  $\sum_j p_j k^{(j)} = k \in \mathbb{N}^r$ . Here  $T_p \in \mathcal{A}(k)$  is a linear combination of ordered spanning trees on the vertex set  $\mathcal{T}(k) \cup \{z\}$ .

**Example 3.2.** In the Lie algebra  $sl_3(\mathbb{C})$  of traceless  $3 \times 3$  matrices let  $f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,

$f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ . Let  $\alpha_1$  and  $\alpha_2$  be the simple roots corresponding to  $f_1$  and  $f_2$  respectively.

The positive roots are  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_1 + \alpha_2$ . Let us choose the ordering

$$\bullet \beta_1 = \alpha_1 < \beta_2 = \alpha_1 + \alpha_2 < \beta_3 = \alpha_2,$$

and the elements

$$\bullet F_{\alpha_1} = f_1, F_{\alpha_1 + \alpha_2} = [f_2, f_1], F_{\alpha_2} = f_2,$$

with  $k^{(1)} = (1, 0)$ ,  $k^{(2)} = (1, 1)$ ,  $k^{(3)} = (0, 1)$ . Then for example  $\Omega_{(1,1)}^{sl_3}$  can be written as  $T_{(1,0,1)} \otimes f_1 f_2 + T_{(0,1,0)} \otimes [f_2, f_1]$ . Using Theorem 2.11 we obtain

$$\begin{aligned} \Omega_{(1,1)}^{sl_3} &= (z \frac{\quad}{1} t \frac{\quad}{2} s) \otimes f_2 f_1 - (z \frac{\quad}{1} s \frac{\quad}{2} t) \otimes f_1 f_2 = \\ &= (z \frac{\quad}{1} t \frac{\quad}{2} s) \otimes ([f_2, f_1] + f_1 f_2) - (z \frac{\quad}{1} s \frac{\quad}{2} t) \otimes f_1 f_2 = \\ &= ((z \frac{\quad}{1} t \frac{\quad}{2} s) - (z \frac{\quad}{1} s \frac{\quad}{2} t)) \otimes f_1 f_2 + (z \frac{\quad}{1} t \frac{\quad}{2} s) \otimes [f_2, f_1]. \end{aligned}$$

Observe that  $T_{(1,0,1)}$  (the coefficient of  $f_1 f_2$ ) is equal to  $(z \frac{\quad}{1} t) * (z \frac{\quad}{1} s)$ . In general we have the following formula.

**Theorem 3.3.** (Product formula.) *There exist unique elements  $\eta_{\beta_j} \in \mathcal{A}(k^{(j)})$  ( $j = 1, \dots, m$ ) such that*

$$(9) \quad T_p = \frac{1}{\prod_i p_i!} \cdot \overbrace{\eta_{\beta_1} * \dots * \eta_{\beta_1}}^{p_1} * \overbrace{\eta_{\beta_2} * \dots * \eta_{\beta_2}}^{p_2} * \dots * \overbrace{\eta_{\beta_m} * \dots * \eta_{\beta_m}}^{p_m}.$$

*Proof.* Let  $F^p = F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}$ . The key observation is that in a PBW-basis the co-multiplication is given by

$$\Delta(F^p) = \sum_{p'+p''=p} \left( \prod_{i=1}^m \frac{p_i!}{p_i'! p_i''!} \right) F^{p'} \otimes F^{p''}.$$

Hence, for its dual we get

$$\Delta^*(F^{p'*} \otimes F^{p''*}) = \prod_{i=1}^m \frac{p_i!}{p_i'! p_i''!} \cdot F^{(p'+p'')*}.$$

Using Theorem 2.9 this is equivalent to  $T_{p'} * T_{p''} = \prod_i (p_i! / p_i'! p_i''!) \cdot T_{p'+p''}$ , from which the theorem follows (put  $\eta_{\beta_j} = T_{\beta_j}$ ).  $\square$

**Example 3.4.** The computation in Example 3.2 shows that (with the choices made there)  $\eta_{\alpha_1} = (z \frac{\quad}{1} t)$ ,  $\eta_{\alpha_1 + \alpha_2} = (z \frac{\quad}{1} t \frac{\quad}{2} s)$ ,  $\eta_{\alpha_2} = (z \frac{\quad}{1} s)$ . Then for example the coefficient of  $f_1^2 [f_2, f_1] f_2$  in  $\Omega_{(3,2)}^{sl_3}$  is

$$T_{(2,1,1)} = \frac{1}{2} \text{ASym}_{(3,2)} \left( \begin{array}{c} t_1 \quad t_2 \\ 1 \quad 2 \\ z \quad 3 \quad t_3 \quad 4 \quad s_1 \\ 5 \quad s_2 \end{array} \right).$$

### 3.2. The residues of $\eta_\beta$ .

**Lemma 3.5.** *The map  $\psi : U[k-1_i] \rightarrow U[k]$ ,  $x \mapsto (-1)^{k_1+\dots+k_i-1} x \cdot \tilde{f}_i$  is dual to the map  $\text{Res}_{k_i}^{(i)} : \mathcal{A}(k) \rightarrow \mathcal{A}(k-1_i)$ .*

*Proof.* Let  $T \in \mathcal{A}(k)$ ,  $f_J \in U[k-1_i]$ . We need to check that  $\langle (-1)^{k_1+\dots+k_i-1} \tilde{f}_J \tilde{f}_i, T \rangle = \langle \tilde{f}_J, \text{Res}_{k_i}^{(i)} T \rangle$ . Writing out definition (4) we obtain an identity.  $\square$

**Theorem 3.6.** *We have*

$$\text{Res}_{k_i}^{(i)} \Omega_k = (-1)^{k_1+\dots+k_i-1} \Omega_{k-1_i} \cdot (1 \otimes \tilde{f}_i).$$

*Proof.* Let  $\{b_u\}$  be a basis of  $U_r[k-1_i]$ , hence  $\Omega_{k-1_i} = \sum b_u^* \otimes b_u$ . Since the map  $\psi$  in Lemma 3.5 is an embedding, the  $\psi$  images of  $b_u$  can be extended to a basis  $\{\psi(b_u), c_v\}$  of  $U_r[k]$ . Then  $\Omega_k = \sum \psi(b_u)^* \otimes \psi(b_u) + \sum c_v^* \otimes c_v$ . We have  $(\text{Res}_{k_i}^{(i)} \otimes 1) \Omega_k = \sum \text{Res}_{k_i}^{(i)}(\psi(b_u)^*) \otimes \psi(b_u) + \sum \text{Res}_{k_i}^{(i)}(c_v^*) \otimes c_v$ , which, according to the lemma, is  $\sum b_u^* \otimes \psi(b_u) = (1 \otimes \psi) \Omega_{k-1_i}$ , what we wanted to prove.  $\square$

Recall that after making the choices (C1), (C2), any element in  $U(\mathfrak{n}_-)$  can be written in the basis  $\{F^p : p \in \mathbb{N}^m\}$  (recall  $F^p = F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}$ ). If the element  $x$  is expanded in this basis, let the coefficient of  $F^p$  be denoted by  $\text{coeff}(x, F^p)$ . Theorem 3.6 has the following direct corollary.

**Corollary 3.7.** *We have*

$$(-1)^{k_1^{(j)}+\dots+k_i^{(j)}-1} \text{Res}_{k_i^{(j)}}^{(i)} \eta_{\beta_j} = \sum_p \text{coeff}(F^p \cdot f_i, F_{\beta_j}^p) \cdot T_p.$$

Since the residues of  $\eta_\beta$ 's will play a crucial role in our argument below, we will make our choices so that the coefficients on the right-hand side are easily computable.

## 4. COMPUTING $\eta_\beta$ 'S

We reduced the computation of  $T_p$ 's to the computation of  $\eta_\beta$ 's for the positive roots  $\beta$ . In this section we compute the elements  $\eta_\beta$  (with suitable choices (C1), (C2)) for the simple Lie algebras of type A, B, C, D.

**Theorem 4.1.** *Let  $\mathfrak{g}$  be a simple Lie algebra of rank  $r$ , with positive roots  $\beta_1, \dots, \beta_m$  and let choices (C1), (C2) be made. Suppose  $\bar{\eta}_{\beta_j} \in \mathcal{A}(k^{(j)})$  ( $j = 1, \dots, m$ ) satisfy*

- (i):  $\langle \tilde{F}, \bar{\eta}_{\beta_j} \rangle = 0$  if  $\tilde{F} \in U[k^{(j)}]$  belongs to the ideal of Serre relations;
- (ii):  $\text{Res}_{k_i}^{(i)} \bar{\eta}_{\beta_j} = \text{Res}_{k_i}^{(i)} \eta_{\beta_j}$  for all  $i = 1, \dots, r$ .

Then  $\bar{\eta}_{\beta_j} = \eta_{\beta_j}$ .

*Proof.* Let us choose an index  $j$ . Let  $F^p \in U(\mathfrak{n}_-)[k^{(j)}]$ , and let  $\tilde{F}^p \in U[k^{(j)}]$  be a preimage of  $F^p$  under the map  $q : U_r \rightarrow U(\mathfrak{n}_-)$ . First we show that

- (iii):  $\langle \tilde{F}^{1_j}, \bar{\eta}_{\beta_j} \rangle = 1$ ;
- (iv):  $\langle \tilde{F}^p, \bar{\eta}_{\beta_j} \rangle = 0$  if  $p \neq 1_j$ .



For this let  $\bar{\Omega}_{k(j)}^{\mathfrak{g}}$  be obtained from  $\Omega_{k(j)}^{\mathfrak{g}}$  by replacing  $\eta_{\beta_j}$  with  $\bar{\eta}_{\beta_j}$ . Using (ii) and definition (4) of  $\langle, \rangle$  we have  $(\langle \tilde{F}^p, \rangle \otimes 1)\bar{\Omega}_{k(j)}^{\mathfrak{g}} = (\langle \tilde{F}^p, \rangle \otimes 1)\Omega_{k(j)}^{\mathfrak{g}}$ . This latter is equal to

$$(1 \otimes q)(\langle \tilde{F}^p, \rangle \otimes 1)\Omega_{k(j)} = (1 \otimes q)(1 \otimes \tilde{F}^p) = 1 \otimes F^p,$$

which proves (iii) and (iv).

Conditions (i), (iii), and (iv), together with Theorem 2.7 imply  $\bar{\eta}_{\beta_j} = \eta_{\beta_j}$ .  $\square$

**Notations.** Root systems will be considered in a space with an ortho-normal basis  $\{e_1, \dots, e_r\}$ . For a sequence of integers  $I = (i_1, i_2, \dots, i_n)$  let  $[f_I]$  denote the following multiple commutator  $[\dots [[f_{i_1}, f_{i_2}], f_{i_3}] \dots, f_{i_n}]$ . Let  $\text{String}(u_1, u_2, \dots, u_n)$  denote the following tree

$$z \xrightarrow{1} u_1 \xrightarrow{2} u_2 \xrightarrow{3} \dots \xrightarrow{n} u_n.$$

**Theorem 4.2.** *The elements given in the following list are the  $\eta_{\beta}$ 's of Theorem 3.3 for the simple Lie algebras of type A, B, C, D and the choices given.*

**A<sub>r-1</sub>:** For the positive roots  $e_i - e_j$  ( $1 \leq i < j \leq r$ ) and simple roots  $\alpha_i = e_i - e_{i+1}$  ( $i = 1, \dots, r-1$ ), we make the following choices.

- $e_i - e_j < e_{i'} - e_{j'}$  if either  $i < i'$  or  $i = i'$  and  $j < j'$ ;
- $F_{e_i - e_j} = [f_{(j-1, j-2, \dots, i)}]$ .

Then we have  $\eta_{e_i - e_j} = \text{String}(t_1^{(i)}, t_1^{(i+1)}, \dots, t_1^{(j-1)})$ .

**B<sub>r</sub>:** For the positive roots  $e_i$  ( $1 \leq i \leq r$ ),  $e_i - e_j$ ,  $e_i + e_j$  ( $1 \leq i < j < r$ ), and simple roots  $\alpha_i = e_i - e_{i+1}$  ( $1 \leq i < r$ ) and  $\alpha_r = e_r$ , we make the following choices.

- Let  $\beta$  be one of  $e_i$ ,  $e_i - e_j$  or  $e_i + e_j$ , and let  $\beta'$  be one of  $e_{i'}$ ,  $e_{i'} - e_{j'}$  or  $e_{i'} + e_{j'}$ . Then we set  $\beta < \beta'$  if  $i > i'$ . If  $i < j < j'$  we set  $e_i + e_j < e_i + e_{j'} < e_i < e_i - e_{j'} < e_i - e_j$ .
- $F_{e_i - e_j} = [f_{(j-1, \dots, i)}]$ ,  $F_{e_i} = [f_{(r, \dots, i)}]$ ,  $F_{e_i + e_j} = (-1)^{\binom{j-i+1}{2} + (r-j+1)} [f_{(i, \dots, r, r, \dots, j)}]$ .

Then we have  $\eta_{e_i - e_j} = \text{String}(t_1^{(j-1)}, t_1^{(j-2)}, \dots, t_1^{(i)})$ ,  $\eta_{e_i} = \text{String}(t_1^{(r)}, t_1^{(r-1)}, \dots, t_1^{(i)})$  and  $\eta_{e_i + e_j} = \text{ASym}(\text{String}(t_2^{(j)}, \dots, t_2^{(r)}, t_1^{(r)}, \dots, t_1^{(i)}))$ .

**C<sub>r</sub>:** For the positive roots  $e_i - e_j$ ,  $e_i + e_j$  ( $1 \leq i < j \leq r$ ) and  $2e_i$  ( $1 \leq i \leq r$ ) and simple roots  $\alpha_i = e_i - e_{i+1}$  ( $1 \leq i < r$ ) and  $\alpha_r = 2e_r$ , we make the following choices.

- Let  $\beta$  be one of  $e_i - e_j$ ,  $e_i + e_j$ , or  $2e_i$ , and let  $\beta'$  be one of  $e_{i'} - e_{j'}$ ,  $e_{i'} + e_{j'}$ , or  $2e_{i'}$ . Then we set  $\beta < \beta'$  if  $i > i'$ . For  $i < j < j'$  we set  $e_i + e_j < e_i + e_{j'} < 2e_i < e_i - e_{j'} < e_i - e_j$ .
- $F_{e_i - e_j} = [f_{(j-1, j-2, \dots, i)}]$ ,  $F_{e_i + e_j} = (-1)^{r-j} [f_{(i, i+1, \dots, r, r-1, \dots, j)}]$ ,  $F_{2e_i} = (-1)^{r-i}/2 \cdot [f_{(i, i+1, \dots, r, r-1, \dots, i)}]$ .

Then we have  $\eta_{e_i + e_j} = \text{String}(t_1^{(j-1)}, \dots, t_1^{(i)})$ ,  $\eta_{e_i - e_j} = \text{ASym}(\text{String}(t_2^{(j)}, \dots, t_2^{(r-1)}, t_1^{(r)}, \dots, t_1^{(i)}))$ , and

$$\eta_{2e_i} = \text{ASym} \left( z \xrightarrow{1} t_1^{(r)} \begin{array}{l} \nearrow t_1^{(r-1)} \xrightarrow{3} \dots \xrightarrow{r-i+1} t_1^{(i)} \\ \searrow t_2^{(r-1)} \xrightarrow{2r-2i} \dots \xrightarrow{r-i+2} t_2^{(i)} \end{array} \right).$$

**D<sub>r</sub>:** For the positive roots  $e_j - e_i$ ,  $e_j + e_i$  ( $1 \leq i < j \leq r$ ), and simple roots  $\alpha_1 = e_1 + e_2$ ,  $\alpha_j = e_j - e_{j-1}$  ( $1 < j \leq r$ ), we make the following choices.

- Let  $\beta$  be one of  $e_j - e_i$ ,  $e_j + e_i$ , and let  $\beta'$  be one of  $e_{j'} - e_{i'}$ ,  $e_{j'} + e_{i'}$ . Then we set  $\beta < \beta'$  if  $j < j'$ . For  $i' < i < j$  we also set  $e_j + e_i < e_j + e_{i'} < e_j - e_{i'} < e_j - e_i$ .
- $F_{e_j - e_i} = [f_{(j, \dots, i+1)}]$ ,  $F_{e_j + e_i} = [f_{(j, \dots, 3, 1)}]$ ,  $F_{e_j + e_i} = [f_{(j, \dots, 1, 3, \dots, i)}]$  ( $i > 1$ ).

Then we have  $\eta_{e_j - e_i} = \text{String}(t_1^{(i+1)}, \dots, t_1^{(j)})$ ,  $\eta_{e_j + e_i} = \text{String}(t_1^{(1)}, t_1^{(3)}, \dots, t_1^{(j)})$ , and for  $i > 1$  we have  $\eta_{e_j + e_i} = \text{ASym}(\text{String}(t_2^{(i)}, \dots, t_2^{(3)}, t_1^{(2)}, t_1^{(1)}, t_1^{(3)}, \dots, t_1^{(j)})) + \text{ASym}(\text{String}(t_2^{(i)}, \dots, t_2^{(3)}, t_1^{(1)}, t_1^{(2)}, t_1^{(3)}, \dots, t_1^{(j)}))$ .

*Proof.* Simple combinatorics shows that for our choices of  $F_\beta$  and the order of the  $\beta$ 's, Corollary 3.7 implies

$$\text{Res}_{k_i^{(j)}}^{(i)} \eta_{\beta_j} = \begin{cases} \eta_{\beta_j - \alpha_i} & \text{if } \beta_j - \alpha_i \text{ is a positive root, } \beta_j - \alpha_i > \beta_j \\ \eta_{\frac{\beta_j - \alpha_i}{2}} * \eta_{\frac{\beta_j - \alpha_i}{2}} & \text{if } \frac{\beta_j - \alpha_i}{2} \text{ is a positive root, } \frac{\beta_j - \alpha_i}{2} > \beta_j \\ 0 & \text{otherwise.} \end{cases}$$

One can check case by case that the same residue identities hold for the  $\eta_{\beta_j}$ 's given in the Theorem, hence condition (ii) of Theorem 4.1 is satisfied.

To prove property (i) of Theorem 4.1 recall that the relevant Serre relations are

- (1)  $[f_{(i,j)}]$  if  $(\alpha_i, \alpha_j) = 0$ ,
- (2)  $[f_{(i,j,j)}]$  if  $(\alpha_i, \alpha_j) = -1$ ,
- (3)  $[f_{(i,j,j,j)}]$  if  $(\alpha_i, \alpha_j) = -2$ ,  $|\alpha_i| < |\alpha_j|$ .

If  $(\alpha_i, \alpha_j) = 0$ , then the value of  $(f_I f_i f_j f_J - f_I f_j f_i f_J)$  on the  $\mathcal{A}(k)$ -elements given in the Theorem are all 0. Indeed, our sign conventions yield that in the  $C_r$  case  $f_I f_i f_j f_J(\eta_{2e_i}) = f_I f_j f_i f_J(\eta_{2e_i})$ ; and in all other cases both sides vanish. This proves that on multiples of (1) all  $\mathcal{A}(k)$ -elements given in the Theorem as  $\eta_\beta$ 's, vanish. For multiples of (2) and (3) the proof is similar.  $\square$

## 5. DIFFERENTIAL FORM REPRESENTATIONS OF $\mathcal{A}(k)$

A linear map  $\phi : \mathcal{A}(k) \rightarrow W$  to a vector space  $W$  will be called a representation of  $\mathcal{A}(k)$ . For a representation  $\phi$  we call  $(\phi \otimes 1)\Omega_k$  the canonical  $\phi$ -element, and  $(\phi \otimes 1)\Omega_k^{\mathfrak{g}} = (\phi \otimes q)\Omega_k$  the canonical  $\phi$ -element associated with the Lie algebra  $\mathfrak{g}$ . In this section we consider  $\phi$ 's for which the canonical  $\phi$ -elements associated with simple Lie algebras play an important role in hypergeometric solutions of KZ-type differential equations. Theorem 3.3 together with Theorem 4.2 give convenient forms of these canonical elements. An example of application will be shown in Section 6.

Suppose we are given a vector space  $W$ , and a vector  $\phi(T) \in W$  for every ordered  $|k|$ -tree  $T$  on the vertex set  $\mathcal{T}(k) \cup \{z\}$ . This data induces a representation  $\phi : \mathcal{A}(k) \rightarrow W$  if the assignment  $T \mapsto \phi(T)$  respects relations R1 and R2 (see Section 2.1).

**Notation.** For an edge  $e$  of a spanning tree on the vertex set  $\mathcal{T}(k) \cup \{z\}$ ,  $h(e)$  and  $t(e)$  denote the head and tail of the edge  $e$ , i.e. the vertices adjacent to  $e$ , farther resp. closer to  $z$ .

**5.1. Rational representation.** Let  $W(t_j^{(i)}, z)$  be the vector space of differential forms in the variables  $\mathcal{T}(k) \cup \{z\}$ .

**Theorem 5.1.** *Let the edges of an ordered  $|k|$ -tree be  $e_1, \dots, e_{|k|}$  (in order). The map*

$$T \mapsto \phi(T) = \bigwedge_{i=1}^{|k|} d \log(h(e_i) - t(e_i))$$

*defines a representation  $\phi_{\text{rat}} : \mathcal{A}(k) \rightarrow W(t_j^{(i)}, z)$ .*

*Proof.* We need to check the consistency of the definition with relations R1 and R2 from Section 2.1. Relation R1 follows from the antisymmetry of the  $\wedge$ -product. In view of R1 it is enough to check R2 for  $a = 1, b = 2$ . Let the three distinguished vertices be  $t, s$ , and  $u$ . Then we need to check

$$\left( \frac{d(t-u)}{t-u} \wedge \frac{d(s-t)}{s-t} + \frac{d(s-u)}{s-u} \wedge \frac{d(t-u)}{t-u} + \frac{d(t-s)}{t-s} \wedge \frac{d(s-u)}{s-u} \right) \wedge R = 0,$$

where  $R$  is the ‘rest’ of the formula (i.e.  $R = \bigwedge_{i=3}^{|k|} d \log(h(e_i) - t(e_i))$ ), but the first factor is 0.  $\square$

This representation of the canonical element (as well as the theorems corresponding to our Theorems 3.3, 4.2) is explored in [RSV]. Observe that, according to [A], the rational representation is an isomorphism onto its image. Hence, the study of the ‘canonical differential form’ of [RSV] is equivalent to the study of the canonical element of this paper.

**5.2. Theta representation.** For  $z, \tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ , the first Jacobi theta function is defined by the infinite product

$$\theta(z) = \theta(z, \tau) = ie^{\pi i(\tau/4 - z)}(x; q) \left( \frac{q}{x}; q \right) (q; q), \quad q = e^{2\pi i \tau}, x = e^{2\pi i z}, \quad (y; q) = \prod_{j=0}^{\infty} (1 - yq^j),$$

[WW]. It is an entire holomorphic function of  $z$  satisfying

$$\theta(z+1, \tau) = -\theta(z, \tau), \quad \theta(z+\tau, \tau) = -e^{-\pi i \tau - 2\pi i z} \theta(z, \tau), \quad \theta(-z, \tau) = -\theta(z, \tau).$$

By  $\theta'(z, \tau)$  we will mean the derivative in the  $z$  variable.

**Definition 5.2.**

$$\sigma_w(t) = \sigma_w(t, \tau) = \frac{\theta(w-t, \tau)}{\theta(w, \tau)\theta(t, \tau)} \cdot \theta'(0, \tau).$$

The listed properties of the theta function yield that the  $\sigma$  function—viewed as a function of  $t$ —has simple poles at the points of  $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ , as well as the properties

$$(10) \quad \sigma_w(t+1, \tau) = \sigma_w(t, \tau), \quad \sigma_w(t+\tau, \tau) = e^{2\pi i w} \sigma_w(t, \tau), \quad \text{Res}_{t=0} \sigma_w(t, \tau) = 1.$$

**Theorem 5.3.** *We have*

$$(11) \quad \sigma_{w_1+w_2}(t-u)\sigma_{w_2}(s-t) - \sigma_{w_2}(s-u)\sigma_{w_1}(t-u) + \sigma_{w_1}(t-s)\sigma_{w_1+w_2}(s-u) = 0.$$

*Proof.* Consider the left hand side of the equation as a function  $f(t)$  of  $t$ . Then  $f$  is a function on  $\mathbb{C} \setminus ((u + \Lambda_\tau) \cup (s + \Lambda_\tau))$ , satisfying  $f(t+1) = f(t)$ ,  $f(t+\tau) = e^{2\pi i w_1} f(t)$ . The above properties of  $\sigma$  also imply that

$$\begin{aligned} \text{Res}_{t=u} f(t) &= 1 \cdot \sigma_{w_2}(s-u) - \sigma_{w_2}(s-u) \cdot 1 - 0 = 0 \\ \text{Res}_{t=s} f(t) &= \sigma_{w_1+w_2}(s-u)(-1) + 0 + 1 \cdot \sigma_{w_1+w_2}(s-u) = 0. \end{aligned}$$

Therefore we have an entire function. According to  $f(t+1) = f(t)$  this can be written in the form of  $\sum d_n e^{2\pi n i t}$ . Substituting  $f(t+\tau) = e^{2\pi i w_1} f(t)$  we obtain that  $d_n = 0$  for all  $n$ .  $\square$

For  $k \in \mathbb{N}^r$  we consider variables (weights)  $w_j^{(i)}$  ( $i = 1, \dots, r, j = 1, \dots, k_i$ ), and say that the weight of the coordinate  $t_j^{(i)} \in \mathcal{T}(k)$  is  $w_j^{(i)}$ . Let  $T$  be a rooted tree (with root  $z$ ) and  $v$  a vertex of  $T$ . We define the branch  $B(v)$  of  $v$  to be the collection of those vertices  $w$  for which the unique path connecting  $w$  with  $z$  contains  $v$ . By definition  $v \in B(v)$ . The load  $L(v)$  of a vertex  $v$  in a tree  $T$  is defined to be the sum of the weights of the vertices in  $B(v)$ . Let  $W(t_j^{(i)}, z, w_j^{(i)}, \tau)$  be the vector space of differential forms in the variables  $\mathcal{T}(k) \cup \{z\}$  depending on the parameters  $\{w_j^{(i)}\}$  and  $\tau$ .

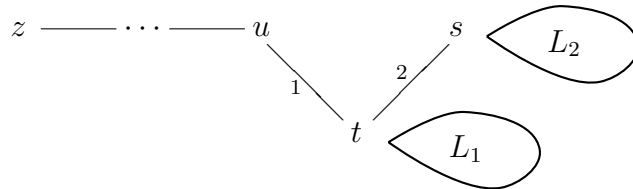
**Theorem 5.4.** *Let the edges of an ordered  $|k|$ -tree be  $e_1, \dots, e_{|k|}$  (in order). The map*

$$T \mapsto \phi(T) = \bigwedge_{i=1}^{|k|} \sigma_{L(h(e_i))}(h(e_i) - t(e_i)) d(h(e_i) - t(e_i))$$

*defines a representation  $\phi_\theta : \mathcal{A}(k) \rightarrow W(t_j^{(i)}, z, w_j^{(i)}, \tau)$ .*

*Proof.* We need to check the consistency of the definition with relations R1 and R2 from section 2.1. Relation R1 follows from the antisymmetry of the  $\wedge$ -product.

In view of R1 it is enough to check R2 for  $a = 1, b = 2$ . In R2, let the three distinguished vertices be  $t$  (bottom),  $s$  (upper-right),  $u$  (upper-left). Because of symmetry, we can assume that among  $t, s, u$  it is  $u$  that is the closest to the vertex  $z$ . Consider the first of the three graphs pictured in R2. Let  $L_1$  be the total weight of the vertices in  $B(t) \setminus B(s)$ , and let  $L_2$  be the load of  $s$ , as in the following picture.



With these notations we need to check

$$\left( \sigma_{L_1+L_2}(t-u)d(t-u) \wedge \sigma_{L_2}(s-t)d(s-t) + \sigma_{L_2}(s-u)d(s-u) \wedge \sigma_{L_1}(t-u)d(t-u) + \right. \\ \left. \sigma_{L_1}(t-s)d(t-s) \wedge \sigma_{L_1+L_2}(s-u)d(s-u) \right) \wedge R = 0,$$

where  $R$  is the ‘rest’ of the formula, i.e.  $\bigwedge_{i=3}^{|k|} \sigma_{L(h(e_i))}(h(e_i) - t(e_i)) d(h(e_i) - t(e_i))$ . Considering the path connecting  $u$  with  $z$  one finds that  $R$  has a  $du$  factor, so our equation reduces to (11). This proves the theorem.  $\square$

**Theorem 5.5.** *The representations  $\phi_{rat}$  and  $\phi_\theta$  are injective.*

*Proof.* The representation  $\phi_{rat}$  is injective by [A]. The representation  $\phi_\theta$  is injective because  $\phi_{rat}$  is obtained from  $\phi_\theta$  by the following degeneration. As  $\tau \rightarrow i\infty$ , the function  $\sigma_w(t, \tau)$  tends to  $\pi \sin(\pi(w-t))/(\sin(\pi w) \sin(\pi t))$ . This function tends to  $\pi/(e^{2\pi t} - 1)$  as  $w$  tends to  $+\infty$ . For small  $t$ , that function approximates  $2/t$ .  $\square$

**5.3. The  $\mathfrak{g}$ -theta representation.** Let  $\mathfrak{g}$  be a simple Lie algebra with Cartan decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  and simple roots  $\alpha_i \in \mathfrak{h}^*$ . Consider the theta representation from Section 5.2 with the choices  $w_j^{(i)} = \alpha_i(\lambda)$ ,  $\lambda \in \mathfrak{h}$ . Thus we obtain a representation  $\phi_{\mathfrak{g},\theta} : \mathcal{A}(k) \rightarrow W(t_j^{(i)}, z, \lambda, \tau)$  to the space  $W(t_j^{(i)}, z, \lambda, \tau)$  of differential forms in the variables  $t_j^{(i)}, z$  depending on  $\lambda \in \mathfrak{h}$  and  $\tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ . We call this the  $\mathfrak{g}$  theta-representation.

**5.4. The canonical elliptic differential form.** For the simple Lie algebra  $\mathfrak{g}$  we defined a map on both factors of  $\mathcal{A}(k) \otimes U[k]$ , namely  $\phi_{\mathfrak{g},\theta} : \mathcal{A}(k) \rightarrow W(t_j^{(i)}, z, \lambda, \tau)$  (Section 5.3), and  $q : U[k] \rightarrow U(\mathfrak{n}_-)[k]$  (Definition 3.1). Let the canonical elliptic differential form  $\Theta_k^{\mathfrak{g}} = \Theta_k^{\mathfrak{g}}(t_j^{(i)}, z, \lambda, \tau)$  be the image in  $W(t_j^{(i)}, z, \lambda, \tau) \otimes U(\mathfrak{n}_-)[k]$  of the canonical element  $\Omega_k$  under the map  $\phi_{\mathfrak{g},\theta} \otimes q$ .

**Example 5.6.** By Theorems 3.3 and 4.2 we have

$$\begin{aligned} \Theta_{(2,1)}^{sl_3} &= (\phi_{sl_3,\theta} \otimes q)\Omega_{(2,1)} = \\ &= (\phi_{sl_3,\theta} \otimes 1) \left( \text{ASym} \left( z \begin{array}{c} \swarrow t_1 \\ \searrow \\ \text{---} t_2 \text{---} \\ \searrow \\ \text{---} s_1 \end{array} \right) \otimes f_1[f_2, f_1] + \frac{1}{2} \text{ASym} \left( z \begin{array}{c} \swarrow t_1 \\ \searrow t_2 \\ \text{---} \\ \searrow \\ \text{---} s_1 \end{array} \right) \otimes f_1^2 f_2 \right) \\ &= \left( \sigma_{\alpha_1(\lambda)}(t_1 - z)dt_1 \wedge \sigma_{\alpha_1(\lambda)+\alpha_2(\lambda)}(t_2 - z)dt_2 \wedge \sigma_{\alpha_2(\lambda)}(s_1 - t_2)ds_1 - \right. \\ &\quad \left. \sigma_{\alpha_1(\lambda)}(t_2 - z)dt_2 \wedge \sigma_{\alpha_1(\lambda)+\alpha_2(\lambda)}(t_1 - z)dt_1 \wedge \sigma_{\alpha_2(\lambda)}(s_1 - t_1)ds_1 \right) \otimes f_1[f_2, f_1] \\ &\quad + \left( \sigma_{\alpha_1(\lambda)}(t_1 - z)dt_1 \wedge \sigma_{\alpha_1(\lambda)}(t_2 - z)dt_2 \wedge \sigma_{\alpha_2(\lambda)}(s_1 - z)ds_1 \right) \otimes f_1^2 f_2. \end{aligned}$$

## 6. APPLICATION: EIGENFUNCTIONS OF CALOGERO-MOSER HAMILTONIAN OPERATOR.

Let  $p$  and  $r$  be natural numbers,  $\tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ . Consider the Hamilton operator of the Calogero-Moser quantum  $r+1$ -body system

$$H = - \sum_{i=1}^{r+1} \frac{\partial^2}{\partial \lambda_i^2} - 2p(p+1) \sum_{1 \leq i < j \leq r+1} \rho'(\lambda_i - \lambda_j, \tau), \quad \rho(t, \tau) = \frac{\theta'(t, \tau)}{\theta(t, \tau)},$$

acting on scalar functions of  $\lambda_1, \dots, \lambda_{r+1}$ , see [FV1].

**Theorem 6.1.** *Let  $\alpha_j(\lambda) = \lambda_j - \lambda_{j-1}$  and  $k = (rp, \dots, 2p, p)$ . Use the notation  $t_j^{(-1)} = 0$  for any  $j$ . Let  $\xi \in \mathbb{C}^{r+1}$ , and suppose that  $t_j^{(i)}$  obey the ‘Bethe ansatz’ equations of [FV1, Th.11]. Then the function*

$$(12) \quad e^{2\pi i \sum_{j=1}^{r+1} \lambda_j \xi_j} \text{Sym}_k \left( \prod_{l=1}^p \prod_{1 \leq i \leq j \leq r} \sigma_{(\alpha_i + \dots + \alpha_j)(\lambda)}(t_{(j-i)p+l}^{(i)} - t_{(j-i+1)p+l}^{(i-1)}) \right)$$

*is an eigenfunction of  $H$ .*

*Proof.* Consider  $\mathfrak{g} = sl_{r+1}(\mathbb{C})$  with its Cartan decomposition and simple roots as in Theorem 4.2, and the corresponding  $f_i \in \mathfrak{n}_-$ . The  $p(r+1)$ ’th symmetric power of the standard representation of  $sl_{r+1}(\mathbb{C})$  is the highest weight module  $V_\Lambda$  with highest weight  $\Lambda = \sum_i \frac{(r+1-i)p}{r+1} \alpha_i$ .

The surjective map  $U(\mathfrak{n}_-) \rightarrow V_\Lambda$ ,  $x \mapsto x \cdot v_\Lambda$  induces the map

$$W(t_j^{(i)}, z, \lambda, \tau) \otimes U^{\mathfrak{g}}(\mathfrak{n}_-)[k] \rightarrow W(t_j^{(i)}, z, \lambda, \tau) \otimes V_\Lambda.$$

Let  $\Theta_k^{V_\Lambda}$  be the image of  $\Theta_k^{\mathfrak{g}}$  under this map. Theorem 11 of [FV1] states that the function

$$(13) \quad \psi_\gamma(\lambda) = e^{2\pi i \sum_{j=1}^{r+1} \lambda_j \xi_j} \cdot \Theta_k^{V_\Lambda}(t, z, \lambda, \tau) / \bigwedge_{i,j} dt_j^{(i)}$$

is an eigenfunction of the operator  $H$ , if  $z = 0$ ,  $\text{Im } \tau > 0$ , and  $t$  satisfies the Bethe ansatz equations. We claim that

$$(14) \quad \Theta_k^{V_\Lambda} = \frac{1}{(p!)^r} \phi_\theta \left( \text{String}(z, t_1^{(1)})^{*p} * \text{String}(z, t_1^{(1)}, t_1^{(2)})^{*p} * \dots \right. \\ \left. * \text{String}(z, t_1^{(1)}, \dots, t_1^{(r)})^{*p} \right) \otimes f_1^p [f_2, f_1]^p \dots [f_{(r, r-1, \dots, 1)}]^p v_\Lambda.$$

To prove this, choose the ordering of positive roots  $\beta$  and the  $F_\beta$ 's as in Theorem 4.2, and consider  $\Theta_k^{V_\Lambda}$  in the induced PBW form:

$$\sum_p T_p \otimes F_{\beta_1}^{p_1} \dots F_{\beta_m}^{p_m} v_\Lambda.$$

Observe that the  $F_\beta v_\Lambda = 0$  unless  $\beta = e_1 - e_j$ . Hence in a non-zero term the powers of the factors  $F_{r+1}, F_{r+2}, \dots$  have to be zero. Therefore there is only one non-zero term

$$\phi_\theta(T_{(p, \dots, p)}) \otimes f_1^p [f_2, f_1]^p \dots [f_{(r, r-1, \dots, 1)}]^p v_\Lambda.$$

According to Theorems 3.3 and 4.2 this term is equal to the right hand side of formula (14).

Tracing back the definitions of the star-product and  $\phi_\theta$  in (14), we obtain the theorem.  $\square$

For  $sl_2(\mathbb{C})$  the Theorem is proved in [EK1]. A different Bethe ansatz formula is given in [FV1].

We used the elliptic version of the operator  $H$ . One can consider its trigonometric limit as in [FV2]. To obtain eigenfunctions in that case we need to replace the representation  $\phi_\theta$  with  $\lim_{\tau \rightarrow i\infty}(\phi_\theta)$  in (14).

## 7. APPENDIX: NOTES ON THE THETA REPRESENTATION

**7.1. The image of the theta representation.** We give a description of the image of the representation  $\phi_\theta : \mathcal{A}(k) \rightarrow W(t_j^{(i)}, z, w_j^{(i)}, \tau)$  for  $z = 0$  and  $k = (1, \dots, 1) \in \mathbb{N}^r$  and fixed  $\tau \in \mathbb{C}$  with  $\text{Im } \tau > 0$ . The general case can be derived from this special case.

Fix  $k = (1, \dots, 1) \in \mathbb{N}^r$ ,  $z = 0$ , and  $\tau \in \mathbb{C}$  with  $\text{Im } \tau > 0$ . Let  $\mathcal{A}(k)_{(z=0)}$  denote the space  $\mathcal{A}(k)$  with the substitution  $z = 0$ . The space  $\mathcal{A}(k)_{(z=0)}$  is spanned by ordered trees on the vertex set  $\{t_1, \dots, t_r, 0\}$  (we use the notation  $t_i = t_1^{(i)}$ ). For such a tree  $T$ ,

$$\phi_\theta(T) = \bigwedge_{i=1}^r \sigma_{L(h(e_i))}(h(e_i) - t(e_i)) d(h(e_i) - t(e_i))$$

is a meromorphic differential form in the variables  $t_1, \dots, t_r$  and it depends meromorphically on the complex parameters  $w_1, \dots, w_r$ . Consider the translates of the discriminantal arrangement

$\mathcal{C}^r(0)$  by the points of the lattice  $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$  in each coordinates. Let  $M_r$  be the union of all these translates, i.e.

$$M_r = \left( \cup_{H \in \mathcal{C}^r(0)} H \right) + \{(x_1, \dots, x_r) : x_j \in \Lambda_\tau\}.$$

The differential form  $\phi_\theta(T)$  has at most simple poles at points of  $M_r$  and it is holomorphic on  $\mathbb{C}^r - M_r$ .

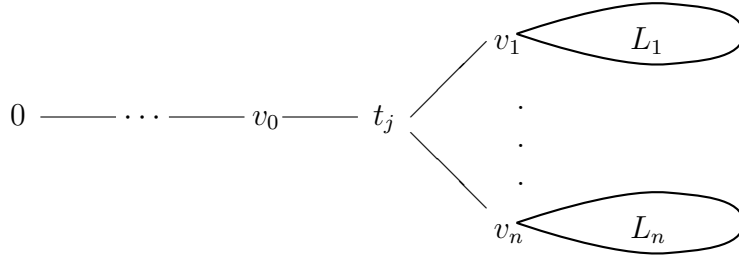
**Theorem 7.1.** *Let  $k = (1, \dots, 1) \in \mathbb{N}^r$  and fix  $\tau \in \mathbb{C}$  with  $\text{Im } \tau > 0$ . A meromorphic differential  $r$ -form  $\psi$  in the variables  $t_1, \dots, t_r$ , which depends meromorphically on the complex parameters  $w_i$ , is in the image of  $\phi_\theta : \mathcal{A}(k)_{(z=0)} \rightarrow W(t_j, 0, w_j, \tau)$  if and only if the following conditions hold.*

- (1)  $\psi(\dots, t_j + 1, \dots) = \psi(\dots, t_j, \dots)$  for all  $j = 1, \dots, r$ .
- (2)  $\psi(\dots, t_j + \tau, \dots) = e^{2\pi i w_j} \psi(\dots, t_j, \dots)$  for all  $j = 1, \dots, r$ .
- (3) For any subset  $\{u_1, \dots, u_n\} \subset \{t_1, \dots, t_r\}$ , the differential form

$$\text{Res}_{u_1=0} \left( \text{Res}_{u_2=0} \left( \dots \text{Res}_{u_n=0}(\psi) \dots \right) \right)$$

has at most simple poles at the points of  $M_{r-n}$ , and it is holomorphic on  $\mathbb{C}^{r-n} - M_{r-n}$ .

*Proof.* First we show that Properties (1)–(3) hold for  $\psi \in \text{Im}(\phi_\theta)$ . Property (1) follows from the first formula in (10). To prove Property (2) let  $T$  be an ordered spanning tree on the vertex set  $\{t_1, \dots, t_r, 0\}$ , and let  $v_0, \dots, v_n$  be the neighbors of  $t_j$ , among which  $v_0$  is the closest to 0 in  $T$ .



Let  $L_i$  be the load of  $v_i$ . The only factors of  $\phi_\theta(T)$  that contain  $t_j$  are

$$\sigma_{w_j + \sum L_i} (t_j - v_0) d(t_j - v_0) \cdot \prod_{l=1}^n \sigma_{L_l} (v_l - t_j) d(v_l - t_j).$$

According to the second formula from (10), after substituting  $t_j^{(i)} + \tau$  these factors get multiplied by

$$e^{2\pi i (w_j + \sum L_i)} \cdot \prod_{l=1}^n e^{2\pi i (-L_l)} = e^{2\pi i w_j},$$

what we needed to prove. To prove Property (3) observe that  $\text{Res}_{t_j=0} \phi_\theta(T) = \phi_\theta(\text{Res}_1^{(j)} T)$ . Here Res on the left hand side means residue of differential forms, and on the right hand side it is the one defined in Section 2.3. Therefore the multiple residue in Property (3) is the  $\phi_\theta$  image of a certain tree; therefore  $\psi$  satisfies Property (3).

We will prove the other direction by induction on  $r$ . For  $k = (1)$  let  $\psi(t)$  satisfy the properties of the Theorem. If its residue at  $t = 0$  is  $c$ , then the differential form

$$\psi - c \cdot \sigma_w(t)dt = \psi - \phi_\theta\left(c \cdot \left(0 \frac{\quad}{1} t\right)\right)$$

satisfies Properties (1), (2) and it is a holomorphic form; thus it has to be 0 (see the argument at the end of the proof of Theorem 5.3).

Now let  $\psi$  be a differential form in the variables  $t_1, \dots, t_r$  satisfying the conditions of the theorem. By the induction hypotheses we have  $\text{Res}_{t_j=0} \psi = \phi_\theta(T_j)$ . Consider

$$\psi - \phi_\theta\left(\sum_j 0 \frac{\quad}{1} t_j \text{ --- } T_j\right).$$

Here the root vertex 0 of  $T_j$  is glued to the vertex  $t_j$  (and the name  $t_j$  is kept for it); the numbers on the edges of  $T_j$  are increased by 1. This form has Properties (1), (2), and its residues at the hyperplanes  $t_j = 0$  are 0. Therefore this form is 0 (consider the new variable  $\sum t_j$ , and the argument at the end of the Proof of Theorem 5.3). Therefore  $\psi$  is in the image of  $\phi_\theta$ .  $\square$

**7.2. Differential forms depending on  $\tau$ .** We will define a modification of the representation  $\phi_\theta$  that takes into account the dependence on the modular parameter  $\tau$  more naturally. Let

$$\omega_w(t) = \sigma_w(t, \tau)dt - \frac{1}{2\pi i} \partial_w \sigma_w(t, \tau) d\tau.$$

This form is used to construct hypergeometric solutions for elliptic KZ differential equations in [FV1].

**Theorem 7.2.** *The differential form  $\omega_w(t, \tau)$  is closed in  $(t, \tau)$ -space.*

*Proof.* We need to show that

$$(15) \quad \left(\partial_\tau + \frac{1}{2\pi i} \partial_t \partial_w\right) \sigma_w(t, \tau) = 0.$$

Differentiating the functional relation  $\sigma_w(t + \tau, \tau) = \exp(2\pi i w) \sigma_w(t, \tau)$  with respect to  $\tau$  and then with respect to  $t, w$ , we see that the left-hand side  $L(t)$  of (15) obeys  $L(t + \tau) = \exp(2\pi i w) L(t)$  (the main point is that the inhomogeneous term cancels). Also trivially  $L(t + 1) = L(t)$ . The function  $t \mapsto \sigma_w(t, \tau)$  is holomorphic except for simple poles on the lattice  $\mathbb{Z} + \tau \mathbb{Z}$ . The residue at the pole of  $\sigma_w$  at  $t = 0$  is independent of  $w, \tau$  (it is 1) so  $L(t)$  is an entire function of  $t$  and must thus vanish.  $\square$

**Theorem 7.3.** *The form  $\omega$  satisfies the following functional relation*

$$\omega_{w_1+w_2}(t-u) \wedge \omega_{w_2}(s-t) + \omega_{w_2}(s-u) \wedge \omega_{w_1}(t-u) + \omega_{w_1}(t-s) \wedge \omega_{w_1+w_2}(s-u) = 0.$$

*Proof.* The left hand side of the formula is  $(f_1 dt \wedge ds + f_2 ds \wedge du + f_3 du \wedge dt) + (g_1 dt + g_2 ds + g_3 du) \wedge d\tau$ . The first term vanishes by the formula in Theorem 5.3. The vanishing of the second term reduces to the vanishing of  $\partial_{w_1}$  and  $\partial_{w_2}$  of the formula in Theorem 5.3.  $\square$

**Corollary 7.4.** *Let the edges of an ordered  $|k|$ -tree be  $e_1, \dots, e_{|k|}$  (in order). The map*

$$T \mapsto \phi(T) = \bigwedge_{i=1}^{|k|} \omega_{L(h(e_i))}(h(e_i) - t(e_i))$$



defines an injective representation of  $\mathcal{A}(k)$  to the space of closed differential forms in the variables  $\mathcal{T}(k) \cup \{z\} \cup \{\tau\}$ , depending on the parameters  $\{w_j^{(i)}\}$ .

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