

# Pontrjagin-Thom-type Construction for Maps with Singularities

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## Introduction

The Pontrjagin-Thom construction establishes an isomorphism between the cobordism groups of embedded submanifolds in a euclidean space and the homotopy groups of the Thom space. Here we extend this construction to (stable) maps submitted to arbitrary local and global restrictions, or in other words, we prove the existence of a Thom-type construction for maps of a given stable type. Before explaining what this really means we summarize some earlier results.

An example of maps with *local* restriction is the class of immersions. The well-known Smale-Hirsch-Gromov theory reduces the investigation of immersions to algebraic topology. Using this theory Wells extended the Pontrjagin-Thom construction to immersions ([26]).

The computation of the cobordism groups of singular maps of any given type seems to be a more difficult problem. Eliashberg developed a technique of surgery of fold maps in non-positive codimensions ([2]). Koschorke established some useful exact sequences for the cobordisms of maps with singularities. This way he computed for any  $r$  the ranks of the cobordism groups of maps of corank  $\leq r$  ([6]). In 1989 Arnold and Vasiliev published a surprising isomorphism between the cobordism groups of fold maps of codimension 0 and the homotopy groups of the space of functions having only mild singularities ([1], [23]).

However, none of these results can be considered as the proper extension of the Pontrjagin-Thom construction to singular maps. (The analogue of the Thom space was not constructed in them.) This was done for the simplest singular maps by the second author in 1979 (and later he extended the construction for arbitrary corank 1 maps).

The original Pontrjagin-Thom construction is based on the fact that there is a universal codimension  $k$  embedding (namely the embedding  $BO(k) \subset MO(k)$ ), from which any other such embedding can be *pulled back*.

Now let  $\tau$  be some set of simple (multi)germs of stable codimension  $k$  maps. Call a smooth map  $f$  a  $\tau$ -map if for any point  $y$  of its target manifold the (multi-)germ of  $f$  at the (finite) set  $f^{-1}(y)$  belongs to  $\tau$ . Our main theorem gives a *universal  $\tau$ -map* from which any other  $\tau$ -map can be pulled back. This universal  $\tau$  map can be described very concretely as soon as the maximal compact symmetry groups of the germs occurring in  $\tau$  are understood. We give also an algorithm for finding these maximal compact subgroups. Finally we give a list of differential topological applications.

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Let the non-negative integer  $k$  be fixed throughout this paper. The singularity theoretic notions used in this paper can be found in any introduction to singularity theory text book, e.g. in [3] or [24]

*Definition.* Let  $A_1, \dots, A_s$  be a set (maybe with multiplicities) of local algebras corresponding to some simple stable map germs

$$\eta_i : (\mathbb{R}^{n_i}, 0) \longrightarrow (\mathbb{R}^{n_i+k}, 0).$$

The set of stable germs

$$f : (\mathbb{R}^{n_f}, \{P_1, \dots, P_s\}) \longrightarrow (\mathbb{R}^{n_f+k}, 0)$$

whose local algebras at  $P_1, \dots, P_s$  are isomorphic to  $A_1, \dots, A_s$  will be denoted by  $S(A_1, \dots, A_s)$ . We will call this set the (multi)singularity type corresponding to  $A_1, \dots, A_s$ . The number  $s$  is called the multiplicity of the singularity type.

If we divide the set of all simple stable germs  $(\mathbb{R}^*, \text{finite set}) \longrightarrow (\mathbb{R}^{*+k}, 0)$  by the equivalence relation generated by right-left ( $\mathcal{A}$ -)equivalence and suspension  $((\Sigma\eta)(u, t) = (\eta(u), t))$ , then the equivalence classes are exactly the multisingularity types just defined. If  $\eta \in S(A_1, \dots, A_s)$  has the smallest source and target dimension in its type then every other  $\zeta \in S(A_1, \dots, A_s)$  is  $\mathcal{A}$ -equivalent to a suspension of  $\eta$ . Then we will call  $\eta$  (defined up to  $\mathcal{A}$ -equivalence) the root of its type, and its singularity type will be denoted by  $[\eta]$ .

We will use the following notations:

$$Emb(k) := S(\mathbb{R}) \quad Imm^r(k) := S(\mathbb{R}, \dots, \mathbb{R}) \quad (\mathbb{R} \text{ occurs } r \text{ times})$$

$$\Sigma^{1r}(k) := S(\mathbb{R}[[x]]/(x^{r+1})) \quad III_{2,2}(k) := S(\mathbb{R}[[x, y]]/(x^2, y^2, xy)).$$

There is a hierarchy of multi-singularity types:  $S(A_1, \dots, A_s)$  is said to be *under*  $S(B_1, \dots, B_r)$  if for a (and therefore for every) representative  $f : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$  from  $S(A_1, \dots, A_s)$  there is a germ from  $S(B_1, \dots, B_r)$  arbitrary close to  $f$ , in the sense that there are points  $y$  arbitrary close to 0 in  $f(\mathbb{R}^n)$  such that the germ of  $f$  at  $f^{-1}(y)$  is from  $S(B_1, \dots, B_r)$ . The top element of this hierarchy is  $Emb(k)$ , the set of germs of  $k$ -codimensional embeddings.

From now on let  $\tau$  be an ascending set of multisingularity types.

*Definition.* A smooth map  $f : N \longrightarrow P$  is called a  $\tau$ -map if for every  $y \in f(N)$  the type of the germ of  $f$  at  $f^{-1}(y)$  is from  $\tau$ . If  $N$  is a manifold with boundary then we also suppose that  $f$  behaves nicely near  $\partial N$ , i. e.  $f(\partial N) \subset \partial P$  and for a collar  $C$  of  $\partial N$ :  $f|_C = \Sigma(f|_{\partial N})$ .

*Examples.* If  $\tau = \{Emb(k)\}$  then  $\tau$ -maps are the  $k$ -codimensional embeddings. If

$$\tau = \{Emb(1), Imm^2(1), Imm^3(1), \Sigma^{1_1}(1)\}$$

then  $\tau$ -maps are dense among the maps  $N^2 \rightarrow P^3$ . In general, if  $m$  and  $k$  are fixed and the pair  $(m, m+k)$  is *nice* (see [8]) then there is a finite  $\tau$  containing multisingularities for which  $\tau$ -maps are dense among the maps  $N^m \rightarrow P^{m+k}$ .

*Definition.* The  $\tau$ -maps  $f_1 : N_1^m \rightarrow P^{m+k}$  and  $f_2 : N_2^m \rightarrow P^{m+k}$  ( $N_1$  and  $N_2$  are closed) are called  $\tau$ -cobordant if there is a manifold  $W$  with boundary the disjoint union of  $N_1$  and  $N_2$ , and a  $\tau$ -map  $f : W \rightarrow P \times [0, 1]$  such that  $f|_{N_1} = f_1$ ,  $f|_{N_2} = f_2$ .

$\tau$ -cobordism between  $\tau$ -maps to  $P$  is an equivalence relation, its equivalence classes are called  $(\tau)$ -cobordism classes, their set is denoted by  $Cob_m(P^{m+k}; \tau)$ . If  $P = S^{m+k}$  then we can define addition on it by “remote disjoint union”, which makes it an Abelian group. The evidently defined oriented version of  $Cob_m(P; \tau)$  is denoted by  $Cob_m^{SO}(P; \tau)$  (the  $m$ -manifolds, their cobordisms and  $P$  are oriented).

## The main theorem

In this section we will deal with “submanifolds” in topological spaces. By such a submanifold in  $X$  we mean a subspace  $K$  which has a neighbourhood  $U$  in  $X$  with a fixed homeomorphism to the total space of a vector bundle over  $K$ . This definition allows one to define the transversality of a map  $P \rightarrow X$  ( $P$  is a manifold) to the submanifold  $K$ .

*Example.* The zero section  $BO(k)$  in  $EO(k)$  or in  $MO(k)$  is a submanifold.

*Definition.* A commutative diagram of topological spaces and continuous maps

$$\begin{array}{ccc} P & \xrightarrow{g} & X \\ \uparrow \bar{f} & & \uparrow f \\ N & \xrightarrow{h} & Y \end{array}$$

is called a pull-back diagram or a pull-back square if  $N$  is homeomorphic to the subspace  $\{(p, y) \in P \times Y \mid g(p) = f(y)\}$  of  $P \times Y$ , and  $\bar{f}$  and  $h$  are the projections to  $P$  and  $Y$ .

Observe that if here  $Y$  is a submanifold of the topological space  $X$  ( $f$  the inclusion),  $P$  is a manifold and  $g$  is transversal to  $f$  then  $N$  is a submanifold of  $P$ . The following theorem asserts that for any  $\tau$  there exists a “universal  $\tau$ -map”, from which any other  $\tau$ -map can be pulled back.

**THEOREM 1.** *Let  $\tau$  be as above. Then there exist topological spaces  $X\tau$  and  $Y\tau$  and a continuous map  $f\tau : Y\tau \rightarrow X\tau$  as well as submanifolds*

$$K_S \subset X\tau, \quad \bar{K}_S \subset Y\tau \quad (\text{for every } S \in \tau), \quad \text{for which}$$

$$Y\tau = \bigcup_{S \in \tau} \bar{K}_S, \quad f\tau(Y\tau) = \bigcup_{S \in \tau} K_S \quad (\text{disjoint unions}),$$

and  $f\tau|_{\bar{K}_S} : \bar{K}_S \rightarrow K_S$  is an  $r$ -fold covering — where  $r$  is the multiplicity of  $S$ .

We say that a map from a manifold to  $X\tau$  is transversal to  $f\tau$  if it is transversal to all  $K_S$ . The map  $f\tau : Y\tau \rightarrow X\tau$  will also have the following two properties:

- (A) If  $g$  is a map from a manifold  $P^{m+k}$  (possibly with boundary) to  $X\tau$  which — as well as its restriction to  $\partial P$  — is transversal to  $f\tau$  and the following square is a pull-back square

$$\begin{array}{ccccc} \partial P & \subset & P & \xrightarrow{g} & X\tau \\ \uparrow \bar{f} & & & & \uparrow f\tau \\ M^{m-1} & \longrightarrow & \xrightarrow{h} & \longrightarrow & Y\tau \end{array}$$

then there is a manifold  $N^m$  with boundary  $M$ , an extension  $\bar{g}$  of  $h$  to  $N$  and a  $\tau$ -map  $f : N \rightarrow P$  which makes the diagram

$$\begin{array}{ccccc} \partial P & \subset & P & \xrightarrow{g} & X\tau \\ \uparrow \bar{f} & & \uparrow f & & \uparrow f\tau \\ M & \subset & N & \xrightarrow{\bar{g}} & Y\tau \end{array}$$

commutative, the right hand square a pull-back square.

- (B) If  $f : N^m \rightarrow P^{m+k}$  is a  $\tau$ -map between manifolds with boundary, and a pull-back square

$$\begin{array}{ccc} \partial P & \xrightarrow{h} & X\tau \\ \uparrow f|_{\partial N} & & \uparrow f\tau \\ \partial N & \xrightarrow{\bar{h}} & Y\tau \end{array}$$

is given, where  $h$  is transversal to  $f\tau$ , then  $h$  and  $\bar{h}$  extend to maps  $g$  (transversal to  $f\tau$ ) and  $\bar{g}$ , making the following diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & X\tau \\ \uparrow f & & \uparrow f\tau \\ N & \xrightarrow{\bar{g}} & Y\tau, \end{array}$$

a pull-back square.

Before proving Theorem 1 we prove its most important corollary.

**MAIN THEOREM.** *The space  $X\tau$  is a classifying space for  $\tau$ -maps in the following sense. For any closed manifold  $P$  there is a bijection between*

$$Cob_m(P^{m+k}; \tau) \quad \text{and} \quad [P^{m+k}, X\tau]$$

( $[ , ]$  means the set of homotopy classes).

*Proof of the Main Theorem.* Let  $N$  and  $P$  be closed manifolds and let  $f : N^m \rightarrow P^{m+k}$  be a  $\tau$ -map. By part (B) of the theorem there exists a map  $\phi(f) : P \rightarrow X\tau$  (and also a map  $\psi(f) : N \rightarrow Y\tau$ ). We will prove that  $[f] \mapsto [\phi(f)]$  defines a bijection between  $Cob_m(P; \tau)$  and  $[P, X\tau]$ . First we prove it is well defined

(i.e. neither depends on the choice of  $\phi(f)$  nor on the representative  $f$  of  $[f]$ ): if  $f_1 : N_1 \rightarrow P$  is  $\tau$ -cobordant to  $f$  then the cobordism between them is a  $\tau$ -map  $F : W \rightarrow P \times [0, 1]$  satisfying the conditions of part (B) with the maps  $\phi(f) \cup \phi(f_1) : P \cup P \rightarrow X\tau$  and  $\psi(f) \cup \psi(f_1) : N \cup N_1 \rightarrow Y\tau$ . Then the map  $G : P \times [0, 1] \rightarrow X\tau$  assigned to these data in part (B) is a homotopy between  $\phi(f)$  and  $\phi(f_1)$ .

To prove surjectivity take a map  $g' : P \rightarrow X\tau$  and approximate<sup>1</sup> it with another one  $g$  which is transversal to  $f\tau$ . Part (A) of the theorem assigns to  $g$  a  $\tau$ -map  $f : N \rightarrow P$  and a map  $\bar{g} : N \rightarrow Y\tau$ . We prove that  $\phi(f)$  is homotopic to  $g$  by applying part (B) to the  $\tau$ -map  $f \times id : N \times [0, 1] \rightarrow P \times [0, 1]$  and the maps  $g \cup \phi(f) : P \cup P \rightarrow X\tau$  and  $\bar{g} \cup \psi(f) : N \cup N \rightarrow Y\tau$ .

To prove injectivity suppose we have  $\tau$ -maps  $f : N \rightarrow P$  and  $f_1 : N_1 \rightarrow P$  with  $\phi(f)$  homotopic to  $\phi(f_1)$ . Let the homotopy be given by  $G : P \times [0, 1] \rightarrow X\tau$  and suppose it is transversal to  $f\tau$ . Then the map assigned to the maps  $G$ ,  $\phi(f) \cup \phi(f_1) : P \cup P \rightarrow X\tau$  and  $\psi(f) \cup \psi(f_1) : N \cup N \rightarrow Y\tau$  is a cobordism between  $f$  and  $f_1$ .  $\square$

The rest of this section is the proof of Theorem 1.

*Proof of Theorem 1.* The proof will proceed by induction on  $\tau$ . The starting point of the induction can be the classical Thom construction ([22]):  $\tau = \{Emb(k)\}$ ,  $X\tau = MO(k)$ ,  $Y\tau = BO(k)$  and the map  $f\tau$  is the embedding. In fact, we can start the induction even earlier with  $\tau = \emptyset$ ,  $X\tau = \text{one-point-space}$ ,  $Y\tau = \emptyset$  — this way we get a proof for the Thom construction.

Now suppose we know the theorem for  $\tau'$  and we want to prove it for  $\tau = \tau' \cup \{\eta\}$  where  $\tau$  and  $\tau'$  are ascending sets in the hierarchy of singularity types and  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$  is the root of its type. For simplicity suppose also that the multiplicity of  $\eta$  is 1. For higher multiplicities the proof goes along the same line.

To define  $f\tau : Y\tau \rightarrow X\tau$  we need some knowledge of

$$Aut_{\mathcal{A}}\eta := \{ (\psi, \phi) \in \mathcal{A} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^{n+k}, 0) \mid \phi \circ \eta \circ \psi^{-1} = \eta \}.$$

Although there is no convenient topology on this group, after appropriate definitions we will see that it shares some properties with Lie groups.

*Definition.* The subgroup  $G \leq Aut_{\mathcal{A}}\eta$  is called compact if it is conjugate in  $\mathcal{A}$  with a compact linear group.

**THEOREM 2.** ([5], [25]) *Every compact subgroup of  $Aut_{\mathcal{A}}\eta$  is contained in a maximal such one and any two maximal compact subgroups are conjugate in  $Aut_{\mathcal{A}}\eta$ .  $\square$*

Let  $G$  denote the maximal compact subgroup of  $Aut_{\mathcal{A}}\eta$  with the representations  $\lambda_1$  and  $\lambda_2$  on the source and the target spaces. By possibly choosing another representative in the  $\mathcal{A}$ -equivalence class of  $\eta$ , we can assure that  $\lambda_1$  and  $\lambda_2$  are linear and orthogonal. The vector bundles associated to the universal principal  $G$ -bundle  $EG \rightarrow BG$  using these representations will be called  $\bar{\xi}_\eta$  and  $\xi_\eta$  respectively.

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<sup>1</sup>by approximation we mean taking a homotopic one. This approximation is possible according to the transversality theorems for stratified sets. Although  $g$  can be chosen close to  $g'$  in some sense, we will not need this fact.

There is also a well defined fibrewise map  $f_\eta : E\bar{\xi}_\eta \rightarrow E\xi_\eta$  near the zero sections which is  $\mathcal{A}$ -equivalent to  $\eta$  in each fibre (see also [19]). Since  $\tau$  is ascending, the restriction of  $f_\eta$  to the boundary of a small disc-bundle  $D\bar{\xi}_\eta$  is a  $\tau'$ -map so by the induction hypothesis there are maps  $\rho, \bar{\rho}$  making the diagram

$$\begin{array}{ccc} & \partial(D\xi_\eta) & \xrightarrow{\rho} & X\tau' \\ f_\eta|_{\partial(D\bar{\xi}_\eta)} & \uparrow & & \uparrow f\tau' \\ & \partial(D\bar{\xi}_\eta) & \xrightarrow{\bar{\rho}} & Y\tau' \end{array}$$

commutative (and even a pull-back square).<sup>2</sup> We will prove that the spaces

$$X\tau := X\tau' \cup_\rho D\xi_\eta,$$

$$Y\tau := Y\tau' \cup_{\bar{\rho}} D\bar{\xi}_\eta$$

and the map

$$f\tau := f\tau' \cup f_\eta,$$

satisfy the conditions of the theorem,<sup>3</sup> where the stratifications  $K_{[\zeta]} \subset X\tau$ ,  $\bar{K}_{[\zeta]} \subset Y\tau$  are defined as follows. The submanifolds  $K_{[\eta]}$  and  $\bar{K}_{[\eta]}$  are defined as the zero sections in  $E\xi_\eta \subset X\tau$  and  $E\bar{\xi}_\eta \subset Y\tau$ . For the other  $[\zeta]$ 's in  $\tau$  we will extend  $K'_{[\zeta]} \subset X\tau'$  and  $\bar{K}'_{[\zeta]} \subset Y\tau'$  to  $K_{[\zeta]} \subset X\tau$  and  $\bar{K}_{[\zeta]} \subset Y\tau$  by adding a point  $x \in D\xi_\eta$  to  $K'_{[\zeta]}$  if and only if the germ at  $f_\eta^{-1}(x)$  of  $f_\eta$  restricted to a fibre of  $D\bar{\xi}_\eta$  is from  $[\zeta]$ . Let  $\bar{K}_{[\zeta]} := f\tau^{-1}(K_{[\zeta]})$ .

*Remark.* Another characterization of this stratification is that  $x \in K_{[\zeta]}$  if and only if  $f\tau$  near  $f\tau^{-1}(x)$  is a (possibly infinite) suspension of  $\zeta$ .

*Proof of part (A).*

The space  $BG$  is in  $D\xi_\eta$  and in  $D\bar{\xi}_\eta$  (as the zero sections), so it is in  $X\tau$  and in  $Y\tau$ , too. Moreover the map  $f\tau$  maps  $BG \subset Y\tau$  homeomorphically onto  $BG \subset X\tau$ . Because of the conditions of the theorem  $g, g|_{\partial P}$  and  $h$  are transversal to  $BG$  in  $X\tau$  and in  $Y\tau$ . We also have that  $h^{-1}(BG)$  is mapped by  $\bar{f}$  onto  $g|_{\partial P}^{-1}(BG)$  diffeomorphically (because of the pull-back property of the diagram). If no confusion arises we will denote both of these manifolds by  $L$ . The submanifold  $K := g^{-1}(BG)$  in

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<sup>2</sup>In fact, one has to be a bit careful when choosing the small disc-bundles. First we choose a small ball  $D$  about 0 in the target space of  $\eta$  such that it is transversal to all the strata of  $\eta$ . Then taking its pre-image  $\bar{D}$  under  $\eta$  will be a manifold diffeomorphic to a ball about 0 in the source space of  $\eta$  (if  $D$  is small enough). Then we can take  $\bar{D}$  and  $D$  in every fibre of  $E\bar{\xi}_\eta \rightarrow BG$  and  $E\xi_\eta \rightarrow BG$  (as they are invariant under the actions of  $\lambda_1(G)$  and  $\lambda_2(G)$ ) to get  $D\bar{\xi}_\eta$  and  $D\xi_\eta$ . For these disc-bundles the restriction of  $f_\eta : \partial D\bar{\xi}_\eta \rightarrow \partial D\xi_\eta$  is a  $\tau'$ -map, provided  $BG$  is a compact manifold. In fact,  $BG$  is only the limit of compact manifolds — so we define the map  $\bar{\rho}$  and  $\rho$  first over a finite dimensional approximation  $(BG)_1$ , then over a closed tubular neighbourhood  $U$  of  $(BG)_1$  in  $(BG)_2$  (a bigger finite dimensional approximation) by suspension, and then over the closure of  $(BG)_2 - U$  using the induction hypothesis for bounded manifolds. Iterating this process will give  $\bar{\rho}$  and  $\rho$  over the union of the finite dimensional approximations, so over the whole  $BG$ .

<sup>3</sup>The definition of  $f\tau$  makes sense, since  $f\tau'$  and  $f_\eta$  coincide on  $\partial(D\bar{\xi}_\eta)$ , i. e.  $\rho \circ f_\eta|_{\partial(D\bar{\xi}_\eta)} = f\tau' \circ \bar{\rho}$  by definition.

$P$  has boundary  $L$ . Let  $U$  denote the closure of  $g^{-1}(\text{int } D\xi_\eta)$ . We may suppose that  $U$  is a tubular neighbourhood of  $K$ . Therefore  $U$  can be regarded as the disc bundle of  $g|_K^* \xi_\eta$ . The boundary of  $U$  is the union of a sphere bundle over  $K$  and a disc bundle over  $L$ . Let the latter be called  $U(L)$ . Let  $P' := \overline{P - U}$  and  $R' := \overline{\partial P - U(L)}$ .

The following diagram may help the reader to follow the proof.

$$\begin{array}{ccccccc}
\partial P & = & R' & \cup & U(L) & \supset & L \\
& & \cap & & \cap & & \cap \\
P & = & P' & \cup & U & \supset & K \\
\downarrow g & & \downarrow g|_{P'} & & \downarrow g|_U & & \downarrow g|_K \\
X_\tau & = & X_{\tau'} & \cup_\rho & D\xi_\eta & \supset & BG \\
\uparrow f_\tau & & \uparrow f_{\tau'} & & \uparrow f_\eta & & \parallel \\
Y_\tau & = & Y_{\tau'} & \cup_{\bar{\rho}} & D\bar{\xi}_\eta & \supset & BG \\
& & \uparrow \bar{g}_{N'} & & \uparrow \bar{g}_V & & \uparrow g|_K \\
N & = & N' & \cup & V & \supset & K \\
& & \cup & & \cup & & \cup \\
M & = & S' & \cup & V(L) & \supset & L
\end{array}$$

Let  $V$  be the disc bundle of  $g|_K^* \bar{\xi}_\eta$  (this defines the map  $\bar{g}_V : V \rightarrow D\bar{\xi}_\eta$ ). The boundary of  $V$  consists of a sphere bundle over  $K$  and a disc bundle over  $L$ . Observe that this disc bundle over  $L$  can be identified with a tubular neighbourhood  $V(L)$  of  $L$  in  $M$ . Then glue  $V$  and  $M$  together along it. Let  $S'$  denote  $M - V(L)$ .

The map  $f_\eta : D\bar{\xi}_\eta \rightarrow D\xi_\eta$  induces a map  $\bar{f}_\eta : V \rightarrow U$  between the bull-back bundles and the restriction of  $\bar{f}_\eta$  to  $V(L)$  coincides with the restriction of  $\bar{f}$  (by definition). Because of the definitions of  $\xi_\eta$ ,  $\bar{\xi}_\eta$  and the transversality assumptions  $\bar{f}_\eta$  is a  $\tau$ -map.

Now  $P'$  is a manifold with boundary  $R' \cup g|_K^* \partial(D\xi_\eta)$ . The closed manifold  $S' \cup g|_K^* \partial(D\bar{\xi}_\eta)$  is  $\tau'$ -mapped to  $\partial P'$ , and a map of it to  $Y_{\tau'}$  is defined ( $\bar{g}_V \cup h$ ). The relevant transversality and pull-back properties hold, so we can apply the induction hypothesis for these data, yielding to a manifold  $N'$  with boundary  $S' \cup g|_K^* \partial(D\bar{\xi}_\eta)$  and two maps: a  $\tau$ -map  $f_{N'} : N' \rightarrow P'$  and a continuous map  $\bar{g}_{N'} : N' \rightarrow Y_{\tau'}$ .<sup>4</sup> Now the diagram above is commutative.

Form the union of  $N'$  and  $V$  along  $g|_K^* \partial(D\bar{\xi}_\eta)$  to get  $N$ . As a result we see that

$$N, \quad \bar{g} := \bar{g}_V \cup \bar{g}_{N'} \quad \text{and} \quad f := \bar{f}_\eta \cup f_{N'}$$

satisfy the conditions of the theorem.  $\square$

<sup>4</sup>In fact  $P'$  is a manifold with corners (having points at the boundary diffeomorphic to a quarter of a Euclidean space), and  $S' \cup g|_K^* \partial(D\bar{\xi}_\eta)$  is not a manifold *a priori*; first we have to “fold them out”, then use the induction hypothesis and after this fold them back. What makes all this possible is the fact that a) the germ of a  $\tau$ -map at a boundary point is just a suspension, b) the map  $g$  can be supposed to be constant on short normal lines of  $\partial P$  in  $P$ , and c) since  $\partial D\bar{\xi}_\eta$  and  $\partial D\xi_\eta$  are transversal to all strata of  $f_\eta$  the restriction of  $f_\eta : D\bar{\xi}_\eta \rightarrow D\xi_\eta$  is a  $\tau$ -map, i.e. it satisfies the “suspension” condition on its boundary.

Before turning to the proof of part (B) we have to study the group  $Aut_{\mathcal{A}}\eta$ , i.e. we have to be able to reduce the structure group of “generalized vector bundles” from  $Aut_{\mathcal{A}}\eta$  to its maximal compact subgroup. The property which will help us is the “generalized contractibility” of the space  $Aut_{\mathcal{A}}\eta / G$  (Lemma 3 below). This statement, just like Theorem 2, asserts that  $Aut_{\mathcal{A}}\eta$  shares properties with Lie groups. Although Lemma 3 is a key step in the proof of Theorem 1, the singularity theoretic techniques used in its proof are completely different from the differential topological tools we are using otherwise. On the other hand analogous problems have been studied and solved extensively since the original paper of Jänich [5], see e.g. [11], [7], [12]. Therefore the proof of Lemma 3 will not be given here, a detailed proof can be found in [13].

Let  $M$  be a smooth manifold with boundary and let

$$G < H < \mathcal{A}.$$

Call a map  $q : M \rightarrow H/G$  smooth if  $M$  can be covered by open sets  $U$ , on which  $q$  can be represented by pairs of local diffeomorphisms  $(U \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^n$  and  $U \times \mathbb{R}^{n+k} \rightarrow U \times \mathbb{R}^{n+k})$  (in fact germs at the zero section), which map all the fibres  $u \times \mathbb{R}^n$  and  $u \times \mathbb{R}^{n+k}$  into themselves.

*Definition.* Let  $G$  be a subgroup of  $Aut_{\mathcal{A}}\eta$ . We call  $Aut_{\mathcal{A}}\eta/G$  contractible if for every smooth manifold  $M$  with boundary any smooth map  $q : \partial M \rightarrow Aut_{\mathcal{A}}\eta/G$  can be extended to a smooth map  $M \rightarrow Aut_{\mathcal{A}}\eta/G$ .

LEMMA 3. *If  $\eta$  is finitely determined and  $G \leq Aut_{\mathcal{A}}\eta$  is a maximal compact subgroup then  $Aut_{\mathcal{A}}\eta/G$  is contractible. There is also a section  $\sigma : Aut_{\mathcal{A}}\eta/G \rightarrow Aut_{\mathcal{A}}\eta$  such that any smooth map  $M \rightarrow Aut_{\mathcal{A}}\eta/G$  composed with  $\sigma$  is also smooth.  $\square$*

Let us fix a maximal compact subgroup  $G$  of  $Aut_{\mathcal{A}}\eta$ , where  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$  is a stable germ. If  $E$  is a set,  $B$  is a smooth manifold then a map  $p : E \rightarrow B$  is called a bundle with fibre  $Aut_{\mathcal{A}}\eta/G$  provided there is given an open cover  $\{U_i\}$  of  $B$  such that  $p|_{p^{-1}(U_i)}$  is the projection  $U_i \times Aut_{\mathcal{A}}\eta/G \rightarrow U_i$ , and the transition maps (along which these product spaces are glued together in  $E$ ) are smooth maps  $U_i \cap U_j \rightarrow Aut_{\mathcal{A}}\eta$  (remember that  $Aut_{\mathcal{A}}\eta$  acts on  $Aut_{\mathcal{A}}\eta/G$ ). A smooth section of such a bundle is a section  $s : B \rightarrow E$  satisfying that  $pr_{Aut_{\mathcal{A}}\eta/G} \circ s|_{V_i} : V_i \rightarrow Aut_{\mathcal{A}}\eta/G$  are smooth maps for some open cover  $\{V_i\}$  which is a refinement of  $\{U_i\}$ .

According to Lemma 3 all bundles with fibre  $Aut_{\mathcal{A}}\eta/G$  have a smooth section. Indeed, a section which is almost everywhere smooth can be constructed by skeleton induction: the induction step is exactly what we have claimed in Lemma 3. This section might not be smooth where the cells of  $B$  meet. However the standard smoothing procedure of that kind of sections (see e.g. [4] 2.2.11) works here with no change.

In what follows we will consider bundle germs with fibre  $\mathbb{R}^l$  whose structure group is a subgroup of  $Diff(\mathbb{R}^l)$ . Note that this kind of bundle germs over smooth base spaces can be defined even if there is no topology on  $Diff(\mathbb{R}^l)$ , because the smoothness of the transition maps is defined. The usual notion of *equivalence of bundles* also extends to these generalized bundles.



Now consider two bundle germs  $\xi_1 : E_1 \rightarrow B$  and  $\xi_2 : E_2 \rightarrow B$  with fibres  $\mathbb{R}^n$  and  $\mathbb{R}^{n+k}$  respectively. Let the structure group of  $\xi_i$  be  $pr_i(Aut_{\mathcal{A}}\eta)$  (remember that  $Aut_{\mathcal{A}}\eta \subset Diff(\mathbb{R}^n) \times Diff(\mathbb{R}^{n+k})$ , and  $pr_1, pr_2$  are projections to the 1st and 2nd factor). Also suppose that  $\xi_1$  and  $\xi_2$  are “associated to each other” in the following sense. There is an open cover  $\{U_i\}$  of  $B$  whose elements are trivializing neighbourhoods of both  $\xi_1$  and  $\xi_2$  for which the transition maps  $\phi_{ij}^1 : U_i \cap U_j \rightarrow Diff(\mathbb{R}^n)$  and  $\phi_{ij}^2 : U_i \cap U_j \rightarrow Diff(\mathbb{R}^{n+k})$  have the form  $\phi_{ij}^1 = pr_1 \circ \phi_{ij}$  and  $\phi_{ij}^2 = pr_2 \circ \phi_{ij}$  for some smooth  $\phi_{ij} : U_i \cap U_j \rightarrow Aut_{\mathcal{A}}\eta$ .

Our goal is to reduce the structure group from  $Aut_{\mathcal{A}}\eta$  to its maximal compact subgroup, which is a Lie group, so the bundles become vector bundles.

LEMMA 4. *There exist  $\xi'_1 : E'_1 \rightarrow B$  and  $\xi'_2 : E'_2 \rightarrow B$  bundle germs which*

- *are equivalent to  $\xi_1$  and  $\xi_2$ ,*
- *have structure groups  $pr_1(G)$  and  $pr_2(G)$ , and are associated to each other.*

*Proof.* First associate to  $\xi_1$  and  $\xi_2$  a “bundle”  $\xi' : E' \rightarrow B$  with fibre  $Aut_{\mathcal{A}}\eta/G$ : if the  $U_i$ ’s are trivializing neighbourhoods of  $\xi_1$  and  $\xi_2$  (with transition maps  $\phi_{ij}^1, \phi_{ij}^2$  as above) then glue  $U_i \times Aut_{\mathcal{A}}\eta/G$ ’s together by  $\phi_{ij}$ . Denote by  $p_i$  the projection of  $U_i \times Aut_{\mathcal{A}}\eta/G$  to the second factor. Take a smooth section  $s$  of  $\xi'$ . Recall from Lemma 3 that exists a section  $\sigma : Aut_{\mathcal{A}}\eta/G \rightarrow Aut_{\mathcal{A}}\eta$  of the “fibration”  $\pi : Aut_{\mathcal{A}}\eta \rightarrow Aut_{\mathcal{A}}\eta/G$ . Now let  $\lambda_i = \sigma \circ p_i \circ s : U_i \rightarrow Aut_{\mathcal{A}}\eta$  and  $\bar{\phi}_{ji} = \lambda_j^{-1} \phi_{ji} \lambda_i$ . Using the new transition maps  $pr_1 \circ \bar{\phi}_{ji}$  and  $pr_2 \circ \bar{\phi}_{ji}$  we can construct bundle germs  $\xi'_1$  and  $\xi'_2$ . From the form of  $\bar{\phi}_{ij}$  it is clear that  $\xi_1$  and  $\xi_2$  are equivalent to  $\xi'_1$  and  $\xi'_2$ , and  $\bar{\phi}_{ij}(u) \in G$  because if  $\pi : Aut_{\mathcal{A}}\eta \rightarrow Aut_{\mathcal{A}}\eta/G$  is the natural projection then

$$\pi(\bar{\phi}_{ji}(u)) = \pi(\lambda_j(u)^{-1} \phi_{ji}(u) \lambda_i(u)) = \lambda_j(u)^{-1} \phi_{ji}(u) \cdot \pi(\lambda_i(u)) =$$

$$\lambda_j(u)^{-1} \phi_{ji}(u) \cdot p_i s(u) = \lambda_j(u)^{-1} p_j s(u) = \lambda_j(u)^{-1} \pi(\lambda_j(u)) = \text{the coset of } G.$$

□

*Proof of part (B).* Suppose that the statement is true for  $\tau'$  and prove it for  $\tau = \tau' \cup \{[\eta]\}$  where we assume  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$  to be a singularity with multiplicity 1,  $\eta$  is taken to be a root of  $[\eta]$ . (The proof for singularities with higher multiplicities goes along the same line.)

Let  $K \subset P$  be the submanifold of  $y$ ’s for which the germ  $f : (N, f^{-1}(y)) \rightarrow (P, y)$  is from  $[\eta]$ , and let  $\bar{K} := f^{-1}(K)$ . (Remark that  $f|_{\bar{K}}$  is a diffeomorphism.) To understand the situation we note that the restriction of  $f$  maps a transversal slice of  $\bar{K}$  to a transversal slice of  $K$ , and this restriction is  $\mathcal{A}$ -equivalent to  $\eta$ .

Take tubular neighbourhoods  $\bar{U}$  and  $U$  of  $\bar{K}$  and  $K$  in  $N$  and  $P$  respectively, satisfying  $f(\bar{U}) \subset U$ ,  $f(\partial\bar{U}) \subset \partial U$ . The subspace of  $U$  ( $\bar{U}$ ) containing the fibres over  $\partial K$  ( $\partial\bar{K}$ ) will be denoted by  $U(\partial K)$  ( $\bar{U}(\partial\bar{K})$ ). The projection maps

$$\bar{U} \rightarrow \bar{K}, \quad U \rightarrow K$$

are bundles with fibres  $\mathbb{R}^n$  and  $\mathbb{R}^{n+k}$  and have structure groups  $pr_1(Aut_{\mathcal{A}}\eta)$  and  $pr_2(Aut_{\mathcal{A}}\eta)$  (see also [19]). Further, they are “associated to each other” in the sense used in the discussion before the proof. Now Lemma 3 states that the structure groups can be reduced to  $pr_i(MC Aut_{\mathcal{A}}\eta)$ ,  $i = 1, 2$  ( $MC$  here and in what follows

will always mean “maximal compact subgroup of”). This means that the bundles  $\bar{U} \rightarrow K$  and  $U \rightarrow K$  are pull-back bundles of  $\bar{\xi}_\eta \rightarrow BG$  and  $\xi_\eta \rightarrow BG$  by some maps  $\bar{k} : \bar{K} \rightarrow BG$  and  $k : K \rightarrow BG$ . The following diagram is commutative.

$$\begin{array}{ccccccc}
& & & \rightarrow & \xrightarrow{f_\eta} & \rightarrow & \\
& & \nearrow & & & & \searrow \\
D\bar{\xi}_\eta & \rightarrow & BG & = & BG & \leftarrow & D\xi_\eta \\
& \uparrow \bar{g}_{\bar{U}} & \uparrow \bar{k} & & \uparrow k & & \uparrow g_U \\
\bar{U} & \rightarrow & \bar{K} & \xrightarrow{f|_{\bar{K}}} & K & \leftarrow & U \\
\swarrow & \cup & \cup & & \cup & & \cup & \searrow \\
\bar{U}(\partial\bar{K}) & \rightarrow & \partial\bar{K} & \rightarrow & \partial K & \leftarrow & U(\partial K) & \\
\swarrow & & & \rightarrow & \xrightarrow{f|_{\bar{U}}} & \rightarrow & \rightarrow & \searrow \\
& \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & 
\end{array}$$

Now let  $N'$  be the closure of  $N - \bar{U}$ ,  $S'$  the closure of  $\partial N - \bar{U}(\partial\bar{K})$ ,  $P'$  the closure of  $P - U$  and  $R'$  the closure of  $\partial P - U(\partial K)$ . It can be seen that

$$\partial N' = \bar{k}^*(\partial D\bar{\xi}_\eta) \cup S'$$

and

$$\partial P' = k^*(\partial D\xi_\eta) \cup R'.$$

The restriction of  $f$  to  $N'$  is a  $\tau'$ -map and it goes into  $P'$ . Further, there are maps  $h_1 : \partial P' \rightarrow X\tau'$  and  $\bar{h}_1 : \partial N' \rightarrow Y\tau'$  defined as follows:  $h_1 := h$  on  $R'$ ,  $h_1 := \rho \circ g_U$  on  $k^*(\partial D\xi_\eta)$ ;  $\bar{h}_1 := \bar{h}$  on  $S'$ ,  $\bar{h}_1 := \bar{\rho} \circ \bar{g}_{\bar{U}}$  on  $\bar{k}^*(\partial D\bar{\xi}_\eta)$ .

Since the relevant transversality and pull-back properties hold we can use the induction hypotheses, i.e. statement (B) for  $\tau'$ , for the spaces  $N'$ ,  $P'$ ,  $\partial N'$ ,  $\partial P'$  and the maps  $h_1, \bar{h}_1$ . This gives maps  $g_{P'} : P' \rightarrow X\tau'$  and  $\bar{g}_{N'} : N' \rightarrow Y\tau'$  for which the diagram

$$\begin{array}{ccc}
P' & \xrightarrow{g_{P'}} & X\tau' \\
\uparrow f|_{N'} & & \uparrow f\tau' \\
N' & \xrightarrow{\bar{g}_{N'}} & Y\tau'
\end{array}$$

is commutative and

$$\begin{aligned}
\bar{g}_{N'}|_{\partial\bar{U}} &= \bar{\rho} \circ \bar{g}_{\bar{U}}|_{\partial\bar{U}}, \\
g_{P'}|_{\partial U} &= \rho \circ g_U|_{\partial U}, \\
\bar{g}_{N'}|_{S'} &= \bar{h}, & g_{P'}|_{R'} &= h.
\end{aligned}$$

This means that there are maps

$$\begin{aligned}
g &= g_U \cup g_{P'} : P \rightarrow X\tau', \\
\bar{g} &= \bar{g}_{\bar{U}} \cup \bar{g}_{N'} : N \rightarrow Y\tau';
\end{aligned}$$

and the transversality and pull-back relations and the restriction equalities just mentioned ensure that  $g$  and  $\bar{g}$  satisfy the requirements of the theorem. The proof of Theorem 1 is complete.  $\square$

Now suppose that  $P = S^{m+k}$ . Then the operation in the homotopy group  $\pi_{m+k}(X\tau) = [S^{m+k}, X\tau]$  clearly corresponds to the (remote) disjoint union operation in  $Cob_m(S^{m+k}; \tau)$ . Therefore we have the group isomorphism:

$$\pi_{m+k}(X\tau) \cong Cob_m(S^{m+k}; \tau).$$

Since  $XEmb(k) = MO(k)$ , as a special case we obtained the theorem of Thom [22]:

$$\pi_{m+k}(MO(k)) \cong Cob_m(S^{m+k}; \{Emb(k)\}).$$

We can generalize the analogous statement of Thom dealing with *oriented* cobordisms of *oriented* embeddings:

$$\pi_{m+k}(MSO(k)) \cong Cob_m^{SO}(S^{m+k}; \{Emb(k)\}).$$

To perform this generalization for  $\tau$ -maps we need some definitions. Denote by  $Diff^+(\mathbb{R}^n)$  the subgroup (of index two) in  $Diff(\mathbb{R}^n)$  containing the elements whose differentials at 0 have positive determinant and let

$$Diff^-(\mathbb{R}^n) = Diff(\mathbb{R}^n) - Diff^+(\mathbb{R}^n).$$

*Definition.* If  $G \leq Diff(\mathbb{R}^n) \times Diff(\mathbb{R}^{n+k})$  then

$$G^{SO} := G \cap \left( Diff^+(\mathbb{R}^n) \times Diff^+(\mathbb{R}^{n+k}) \cup Diff^-(\mathbb{R}^n) \times Diff^-(\mathbb{R}^{n+k}) \right).$$

Now for every singularity type  $[\eta]$  with multiplicity 1 in  $\tau$  change the group  $G = MC\ Aut_{\mathcal{A}}\eta$  to  $G^{SO}$  in the definition of  $X\tau$  and  $Y\tau$  (and perform the analogous changes for singularities with higher multiplicities). Denote the resulting spaces by  $X^{SO}\tau$ ,  $Y^{SO}\tau$ . Now it is clear that the oriented cobordism set  $Cob_m^{SO}(P, \tau)$  is in a one-to-one correspondence with  $[P, X^{SO}\tau]$ . In case  $P = S^{m+k}$  this correspondence is also a group isomorphism.

## Symmetry of singularities

The main theorem asserts an isomorphism between a group defined in differential topological terms and another group defined in algebraic topological terms. Therefore — using this isomorphism — algebraic topological computations can lead to differential topological results. For this we must have some information about the space  $X\tau$ . We saw that  $X\tau$  is glued together from blocks, each of which is a disc bundle of a vector bundle  $\xi_\eta$ . This vector bundle  $\xi_\eta$  is associated to a universal principal bundle  $EG \rightarrow BG$  with a representation  $\lambda_2$ . (Recall that  $G$  is the maximal compact subgroup of  $Aut_{\mathcal{A}}\eta$  and  $\lambda_2$  is its representation on the target space of  $\eta$ .) Therefore it is clear that the effective usage of this “bridge” between differential and algebraic topology requires a *method* to compute the maximal compact

subgroup  $G$  of  $\text{Aut}_{\mathcal{A}}\eta$  and its representation  $\lambda_2$ . By *method* here we mean that this problem (“infinite dimensional” in nature) of determining  $G$  should be reduced to a “finite dimensional” one.

Many analogues of this reduction have already been studied, see e.g. [9], [10], [11]. The cases occurring in our theorem (right-left equivalence of real, smooth, stable singularities of codimension  $k \geq 0$ ) are different from the ones in the literature, but the main idea and the results are shown to be very similar. Therefore here we give only the statement which reduces the computation of  $G$  and  $\lambda_1, \lambda_2$  to classical mathematics (a detailed proof can be found in [13]).

**THEOREM 5.** *Let  $Q_\eta$  be the local algebra of  $\eta$  and let  $d$  denote its defect, i.e. the minimal value of  $b - a$  when  $Q_\eta$  can be presented as*

$$\mathbb{R}[[x_1, \dots, x_a]] / (r_1, \dots, r_b).$$

*Then  $MC \text{Aut}_{\mathcal{A}}\eta \leq MC \text{Aut } Q_\eta \times O(k - d)$ .*

The representations  $\lambda_1, \lambda_2$  can be determined as follows. Since  $G$  acts on  $Q_\eta$  (the action of  $O(k - d)$  is trivial)  $G$  also acts on

$$h : (x_1, \dots, x_a) \mapsto (r_1, \dots, r_b)$$

as an  $\mathcal{A}$ -equivalence group, where  $a$  and  $b$  are minimal and  $Q_\eta$  is presented as above. This action induces an  $\mathcal{A}$ -action on

$$f : (x_1, \dots, x_a) \mapsto (r_1, \dots, r_b, 0, \dots, 0)$$

( $k - d$  0's at the end), with the standard  $O(k - d)$ -action on  $\mathbb{R}^{k-d}$  “at the end”. Since  $\eta$  is the miniversal unfolding of  $f$ , the  $G$ -action on  $f$  induces a  $G$ -action on  $\eta$  (see [25]).

*Examples.*

*Notation.* In what follows  $\rho_l$  will always mean the usual representation of  $O(l)$  on  $\mathbb{R}^l$ . If  $\rho_l$  is written as a representation of a direct product  $O(l) \times H$  then  $\rho_l$  is really meant to be  $\rho_l \circ pr_{O(l)}$ .

Let  $\eta_{r,k}$  be the root of  $\Sigma^{1r}(k)$  (the “isolated Morin singularity of type  $\Sigma^{1r}$  in codimension  $k$ ”). It has local algebra  $Q_{\eta_{r,k}} = \mathbb{R}[[x]]/(x^{r+1})$  (defect=0).

**THEOREM 6.**

$$MC \text{Aut}_{\mathcal{A}}\eta_{r,k} \cong \mathbb{Z}_2 \times O(k) = O(1) \times O(k).$$

*Its representations  $\lambda_1$  and  $\lambda_2$  on the source and target spaces are*

$$\lambda_1 = \mu_1 \oplus \mu_V \quad \lambda_2 = \mu_2 \oplus \mu_V,$$

*where*

$$\mu_1 := \rho_1,$$

$$\mu_2 := \rho_1^{r+1} \oplus \rho_k$$

and

$$\begin{aligned} \mu_V &:= \left( \sum_{l=r+2}^{2r} \rho_1^{\otimes l} \right) \oplus \left( \sum_{i=1}^r \rho_k \otimes \rho_1^{\otimes i} \right) = \\ &= \left[ \frac{r-1}{2} \right] 1 \oplus \left[ \frac{r-1}{2} \right] \rho_1 \oplus \left[ \frac{r}{2} \right] \rho_k \oplus \left[ \frac{r}{2} \right] \rho_1 \otimes \rho_k. \end{aligned}$$

□

Now let  $\eta$  be the simplest singularity type of Thom-Boardman symbol  $\Sigma^{2,0}$  (corresponding to the algebra  $\mathbb{R}[[x, y]]/(x^2, y^2, xy)$ ).

Denote by  $\rho_2^l$  the map  $O(2) \rightarrow O(2)$  which sends

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \cos l\alpha & -\sin l\alpha \\ \sin l\alpha & \cos l\alpha \end{pmatrix}$$

and  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  to itself.

**THEOREM 7.** *The group  $MC \text{ Aut}_{\mathcal{A}} \eta \cong O(2) \times O(k-1)$ , and its representation on the source and target spaces are*

$$\lambda_1 = \mu_1 \oplus \mu_V \qquad \lambda_2 = \mu_2 \oplus \mu_V,$$

where

$$\mu_1 := \rho_2 \qquad \mu_2 := 1 \oplus \rho_2^2 \oplus \rho_{k-1},$$

and

$$\mu_V := \rho_2 \oplus \rho_2^3 \oplus (\rho_2 \otimes \rho_{k-1}).$$

□

## Applications to differential topology

To illustrate how our Main Theorem works, here we present four groups of differential topological applications. Some of them are already present in the literature (since special cases of the Main Theorem have already been proved by the second author) some others are detailed in the Ph. D. Thesis of the first author or in preprints.

### • Orientability questions

If  $f : N^m \rightarrow P^{m+k}$  is a stable smooth map then the submanifold consisting of the points  $y \in P$  for which the germ of  $f$  at  $f^{-1}(y)$  (has multiplicity 1 and) is of Thom-Boardman class  $\Sigma^i$  will be denoted by  $\Sigma^i(f)$ .

THEOREM 8. Let  $k, i > 1$ . The following two statements are equivalent:

- for every stable smooth map  $f : N^* \rightarrow P^{*+k}$  where  $P$  is orientable, the manifold  $\Sigma^i(f)$  is orientable;
- $k$  is even and  $i$  is odd.

THEOREM 9. Let  $k, i > 1$ . The following two statements are equivalent:

- for every stable smooth map  $f : N^* \rightarrow P^{*+k}$  where  $N$  is orientable, the manifold  $\Sigma^i(f)$  is orientable;
- $k$  is even and  $i$  is even.

THEOREM 10. Let  $k, i > 1$ . The following two statements are equivalent:

- for every stable smooth map  $f : N^* \rightarrow P^{*+k}$  where both  $N$  and  $P$  are orientable, the manifold  $\Sigma^i(f)$  is orientable;
- $k$  is even.

*Sketch of the proof.* The orientability of a submanifold in an orientable manifold is equivalent to the orientability of its normal bundle. We proved that the normal bundle of the simplest  $\Sigma^i$  points (we can forget about the others, since if  $k > 1$  then the others form a subspace of codimension  $> 1$  in this, which has no effect on orientability) is a pull-back bundle of certain universal bundle whose structure group is  $\lambda_1(G)$  or  $\lambda_2(G)$  (according to whether we are working in the source or the target space). The study of these representations give the results of the theorems.  $\square$

### • Cobordism group of embeddings and immersions

Let us have an embedding or an immersion  $f : N^m \rightarrow \mathbb{R}^{m+k}$ . Compose it with the standard projection of  $\mathbb{R}^{m+k}$  to a hyperplane. Then we have a (special)  $\Sigma^1$ -map. The cobordism group of embeddings or immersions in codimension  $k$  is therefore isomorphic to the cobordism group of those special  $\Sigma^1$ -maps in codimension  $k - 1$ . The careful analysis of the latter gave results for those dimension pairs  $(m, k)$  for which the classical Thom construction could not. The detailed proofs for the following theorems are given in [20], [16].

THEOREM 11. If  $k$  is even and  $m \leq 3k$  then the group  $Cob_m(S^{m+k}, Emb(k))$  is isomorphic modulo 2-primary torsion to  $\Omega_{m-k}$  (the abstract cobordism group of oriented  $m - k$ -manifolds).  $\square$

*Remark.* The same group is known to be finite 2-primary if  $k$  is odd.

THEOREM 12. If  $m < 2k$  then the cobordism group  $Cob_m^{SO}(S^{m+k}, Imm(k))$  is isomorphic modulo 2-primary torsion to

$$\begin{cases} \Omega_m \oplus \Omega_{m-k} & \text{if } k \text{ is even,} \\ \Omega_m & \text{if } k \text{ is odd.} \end{cases} \quad \square$$

An analogous result holds for  $m \leq 3k$ , too, see [21].

THEOREM 13. If  $m \leq 3k$  then the cobordism group  $Cob_m(S^{m+k}, Imm(k))$  is isomorphic modulo 2-torsion to  $Cob_m(S^{m+k}, Emb(k))$ .  $\square$

## • Removing singularities

Using our generalized Thom construction we can answer questions of the following type: when can a map (a  $\tau$ -map) be approximated with another one, which is less singular ( $\tau'$ -map) than the original? The proof of this kind of theorems leads to the comparison of the homotopic or homologic properties of  $X\tau$  and  $X\tau'$ . The detailed proofs of the following theorems are in [18], [16] and [13].

**THEOREM 14.** *a) Let  $M$  be a smooth oriented manifold and  $x \in \Omega_i(M^m)$  an element of its oriented  $i$ -dimensional bordism group,  $i < m$ . If  $x$  contains a map having only  $\Sigma^{1,0}$  singular points, then a non-zero multiple of  $x$  contains an immersion.*

*b) In particular, if  $i < \frac{2}{3}m$ , then a non-zero multiple of any class  $x \in \Omega_i(M^m)$  contains an immersion. This non-zero multiple can be chosen to be a power of 2.  $\square$*

**THEOREM 15.** *If  $m \leq 3k$  and  $f : N^m \rightarrow \mathbb{R}^{m+k}$  is the composition of an immersion  $N \rightarrow \mathbb{R}^{m+k+2}$  and the standard projection, then the only obstacle to the existence of a  $g$  cobordant (among the maps that are compositions of immersions and the projections just mentioned) to  $f$  is the abstract cobordism class of  $\Sigma^2(f)$ .  $\square$*

## • CW models for various loop spaces

If  $X\tau$  is the classifying space (in the sense used in the Main Theorem) of those  $\Sigma^1$ -maps  $M^m \rightarrow \mathbb{R}^{m+k}$  that can be lifted to an embedding  $M^m \rightarrow \mathbb{R}^{m+k+1}$  then clearly  $X\tau$  is weakly homotopic equivalent to  $\Omega MO(k+1)$ . Indeed,

$$\pi_{m+k}(X\tau) \cong Cob_m(S^{m+k}, \bar{\Sigma}^1(k)) \cong$$

$$\cong Cob_m(S^{m+k+1}, Emb(k+1)) \cong \pi_{m+k+1}(MO(k+1)) \cong \pi_{m+k}(\Omega MO(k+1)).$$

In fact our main theorem — as stated — does not deal with this  $X\tau$ , but some modification of the statement can cover this case, too. Some more detail on this kind of CW-representations of loop spaces can be found in [17].

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