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THE DEGREE OF THE DISCRIMINANT OF IRREDUCIBLE REPRESENTATIONS

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Abstract

We present a formula for the degree of the discriminant of irreducible representations of a Lie group, in terms of the roots of the group and the highest weight of the representation. The proof uses equivariant cohomology techniques, namely, the theory of Thom polynomials, and a new method for their computation. We study the combinatorics of our formulas in various special cases.

1. Introduction

Let G be a complex connected reductive algebraic group, and let $\rho : G \to GL(V)$ be an irreducible algebraic representation. Then ρ induces an action of G on the projective space $\mathbb{P}(V)$. This action has a single closed orbit, the orbit of the weight vector of the highest weight λ. E.g., for $GL(n)$ acting on $\Lambda^k\mathbb{C}^n$, we get the Grassmannian $Gr_k(\mathbb{C}^n)$. The dual $\mathbb{P}D_\lambda$ of this orbit (or the affine cone D_λ over it) is called the *discriminant* of ρ since it generalizes the classical discriminant. The goal of the present paper is to give a formula for the degree of the discriminant in terms of the highest weight of the representation ρ and the roots of G (Theorems 5.2 and 5.6).

The classical approach to find the degree of dual varieties is due to Kleiman [16] and Katz [14]. Their method, however, does not produce a formula in the general setting. Special cases were worked out by Holme [13], Lascoux [18], Boole, Tevelev [21], Gelfand-Kapranov-Zelevinsky [12, Ch.13,14], see a summary in [22, Ch.7]. De Concini and Weyman [7] showed that, if G is fixed, then for regular highest weights the formula for the degree of the discriminant is a polynomial with positive coefficients, and they calculated the constant term of this polynomial. A corollary of our result is an explicit form for this polynomial (Cor. 5.8) with the additional fact that the same polynomial

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calculates the corresponding degrees for non-regular highest weights as well (modulo an explicit factor).

In special cases our formula can be expressed in terms of some basic concepts in the combinatorics of polynomials [20, 19], such as the Jacobi symmetrizer, divided difference operators, or the scalar product on the space of polynomials (Section 6).

For the group $GL(n)$ we further simplify the formula in many special cases in Section 7.

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2. Degree and Thom polynomials

In this paper we will use cohomology with rational coefficients. The Lie groups we consider are complex connected reductive algebraic groups. All varieties are over the complex numbers $\mathbb{C}$ and $GL(n) = GL(n;\mathbb{C})$ denotes the general linear group of $\mathbb{C}^n$.

2.1. Degree and cohomology. Suppose that Y is a smooth complex algebraic variety and $X \subset Y$ is a closed subvariety of complex codimension d. Then X represents a cohomology class $[X]$ in the cohomology group $H^{2d}(Y)$. This class is called the *Poincaré dual* of X. The existence of this class and its basic properties are explained e.g. in [10]. If Y is the projective space $\mathbb{P}^n$ then $H^*(Y) \cong \mathbb{Q}[x]/(x^{n+1})$ and $[X] = \deg(X)x^d$, where x is the class represented by a hyperplane. By definition, the cone $CX \subset \mathbb{C}^{n+1}$ of X has the same degree.

2.2. Degree and equivariant cohomology. We would also like to express the degree in terms of *equivariant cohomology.* Let G be a complex connected reductive algebraic group (though some definitions and claims hold for more general groups as well). Let G act on a topological space Y. Then the equivariant cohomology ring $H^*_G(Y)$ is defined as the ordinary cohomology of the Borel construction $EG \times_G Y$. Here EG denotes the universal principal G-bundle over the classifying space BG. If Y is a smooth complex algebraic variety, G is a complex Lie group acting on Y and $X \subset Y$ is a G-invariant subvariety of complex codimension d then X represents an *equivariant* cohomology class (sometimes called equivariant Poincaré dual or, if Y is contractible, Thom polynomial) $[X] \in H^{2d}_G(Y)$. This class has the following universal property.

Let $P \to M$ be an algebraic principal G-bundle. Then P is classified by a map $k : M \to BG$, in other words we have the diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{k}} & EG \\ \downarrow & & \downarrow \\ M & \xrightarrow{k} & BG. \end{array}$$

We also have an associated bundle $P \times_G Y \to M$ and an induced map $\hat{k} : P \times_G Y \to EG \times_G Y$. The universal property of $[X] \in H^{2d}_G(Y)$ is that for *every* $P \to M$ the ordinary cohomology class $[P \times_G X]$ represented by the subvariety $P \times_G X$ of $P \times_G Y$ satisfies

$$\hat{k}^*[X] = [P \times_G X]. \tag{2.1}$$

It is easy to see that (2.1) characterizes the equivariant cohomology class $[X] \in H^*(BG \times_G Y) = H^*_G(Y)$ uniquely. The existence of such a class is not so obvious, for details see [15] or [9].

Let $Y = V$ be a complex vector space and let $G = GL(1)$ act as scalars. If $X \subset Y$ is a G-invariant subvariety of complex codimension d then X is the cone of a projective variety $\mathbb{P}X \subset \mathbb{P}V$. Then $H^*_G(V) \cong \mathbb{Q}[x]$ and $[X] = \deg(\mathbb{P}X)x^d$, where x is the class of a hyperplane. For more general complex Lie groups we can expect to extract the degree if G 'contains the scalars'. As we will see, even this condition is not necessary: we can simply replace G with $G \times GL(1)$.

Suppose that $\rho : G \to GL(V)$ 'contains the scalars' i.e. there is a homomorphism $h : GL(1) \to G$, such that $\rho \circ h$ is the scalar representation on V: for $v \in V$ and $z \in GL(1)$ we have $\rho \circ h(z)v = zv$. By the basic properties of the equivariant Poincaré dual we have a 'change of action' formula:

$$[X]_{\rho \circ h} = h^*[X]_\rho, \tag{2.2}$$

where

$$h^* : H^*_G(V) \cong H^*(BG) \to H^*_{GL(1)}(V) \cong H^*(BGL(1))$$

is induced by the map $Bh : BGL(1) \to BG$ classifying the principal G-bundle $EGL(1) \times_h G$. The ρ and $\rho \circ h$ in the lower index indicates whether we take the G- or $GL(1)$-equivariant Poincaré dual of X. The 'change of action' formula implies that

$$\deg(X)x^d = h^*[X]_\rho.$$

It frequently happens that we can only find a homomorphism $h : GL(1) \to G$, such that $\rho \circ h(z)v = z^k v$ for some non zero integer k. Then, by the same way we obtain

$$k^d \deg(X) x^d = [X]_{\rho\circ h} = h^*[X]_\rho.$$

The calculation of h^* is fairly simple. Suppose that $m : GL(1)^r \to G$ is a (parametrized) maximal complex torus of G. Then by Borel's theorem [5, §27] $H^*(BG)$ is naturally isomorphic to the Weyl-invariant subring of $H^*(BT)$ (T is the image of m, a complex maximal torus of G), and $H^*(BT)$ can be identified with the symmetric algebra of the character group of T. Hence $H^*(BG) = \mathbb{Q}[\alpha_1, \dots, \alpha_r]^W$, where the α_i's generate the weight lattice of G (one can identify α_i with the i^{th} projection $\pi_i : GL(1)^r \to GL(1)$) and W is the Weyl group of G. Then $[X] = p(\alpha_1, \dots, \alpha_r)$ for some homogeneous polynomial p of degree d. We can assume that the homomorphism $h : GL(1) \to G$ factors through m, i.e. $h = m \circ \phi$, where $\phi(z) = (z^{k_1}, \dots, z^{k_r})$ for some integers $k_1, \dots, k_r$. Applying the 'change of action' formula (2.2) once more leads to

Proposition 2.1. *For the polynomial p, and integers $k, k_1, \dots, k_r$ defined above*

$$\deg(X) = p(k_1/k, \dots, k_r/k).$$

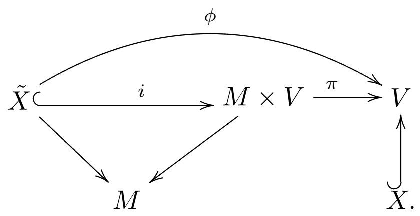
This innocent-looking statement provides a uniform approach for calculating the degree of degeneracy loci whenever a Chern–class formula is known. A similar but more involved argument (see [8]) calculates the equivariant Poincaré dual of $\mathbb{P}X$ in $\mathbb{P}V$.

3. Calculation of the equivariant Poincaré dual

Now we reduce the problem of computing an equivariant Poincaré dual to computing an integral. We will need G-equivariant characteristic classes: the equivariant Chern classes $c_i(E) \in H^{2i}_G(M)$ and the equivariant Euler class $e(E) \in H^{2n}_G(M)$ of a G-equivariant complex vector bundle $E \to M$ of rank n are defined via the Borel construction. We have $e(E) = c_n(E)$.

As in the case of ordinary cohomology, pushforward can be defined in the equivariant setting. An introduction can be found in [1]. Its properties are similar; for example if $\phi : \tilde{X} \to Y$ is a G-equivariant resolution of the d-codimensional invariant subvariety $X \subset Y$, then $[X] = \phi_* 1 \in H^{2d}_G(Y)$.

Let V be a vector space and $X \subset V$ a subvariety. Suppose that M is a compact manifold and let $\tilde{X} \to M$ be a sub-vector bundle S of $M \times V \to M$. Let $\pi : M \times V \to V$ be the projection, $i : \tilde{X} \subset M \times V$ the embedding, and let $\phi = \pi \circ i$, as in the diagram



Proposition 3.1. *Suppose that G acts on all spaces in the diagram above, and that all maps are G-invariant. Let Q be the quotient bundle $V \ominus S$ over M. We have*

$$\pi_* i_* 1 = \int_M e(Q) \qquad \in H^*_G(V) = H^*(BG). \tag{3.1}$$

If ϕ is a resolution of $X \subset V$ then

$$[X] = \int_M e(Q) \qquad \in H^*_G(V) = H^*(BG). \tag{3.2}$$

Proof. Observe that i_*1 is the Euler class of $Q = V \ominus S$ (we identify the cohomology of $M \times V$ with the cohomology of M); and π_* is integration along the fiber of π. $\square$

Remark 3.2. If $\phi(\tilde{X})$ has smaller dimension than $\tilde{X}$ then $\pi_* i_* 1 = \int_M e(Q)$ is zero since this cohomology class is supported on $\phi(\tilde{X})$ and its codimension is bigger than the rank of $\pi_* i_* 1$.

Below we will calculate the integral (3.2) using the Berline-Vergne-Atiyah-Bott integral formula [4], [1], that we recall now.

Proposition 3.3 (Berline-Vergne-Atiyah-Bott)**.** *Suppose that M is a compact oriented manifold, T is a torus acting smoothly on M and $C(M)$ is the set of components of the fixed point manifold. Then for any cohomology class $\alpha \in H^*_T(M)$*

$$\int_M \alpha = \sum_{F \in C(M)} \int_F \frac{i_F^* \alpha}{e(\nu_F)}. \tag{3.3}$$

Here $i_F : F \to M$ is the inclusion, $e(\nu_F)$ is the T-equivariant Euler class of the normal bundle ν_F of $F \subset M$. The right side is considered in the fraction field of the polynomial ring of $H^*_T(point) = H^*(BT)$ (see more on details in [1]*): part of the statement is that the denominators cancel when the sum is simplified.*

Remark 3.4. Here T denotes a *real* torus $T = U(1)^r$. Since $BU(1)^r$ is homotopy equivalent to $BGL(1)^r$ we can always restrict the complex torus action to the real one without losing any cohomological information.

Returning to the situation of Proposition 3.1, let us further assume that the group G is a torus, and that the fixed point set $F(M)$ on the compact manifold M is finite. For a fixed point $f \in F(M)$ the Euler class $e(V)$ restricted to f can be computed as the product of the Euler classes of the restrictions of S and Q to f, that is $e(V_f) = e(S_f)e(Q_f)$. Also, ν_f is just the tangent space $T_f M$. Hence, from Proposition 3.1 and 3.3 we obtain that

$$[X] = e(V) \sum_{f \in F(M)} \left(e(S_f) \cdot e(T_f M) \right)^{-1}. \tag{3.4}$$

This formula is a special case of [3, Prop. 3.2]. These type of formulas have their origin in [23].

4. The equivariant cohomology class of the dual of smooth varieties

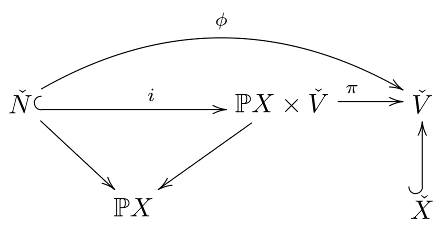
Let the torus T act on the complex vector space V and let $X \subset V$ be a T-invariant cone. Assume moreover that $\mathbb{P}X \subset \mathbb{P}V$ is smooth, and that the projective dual $\mathbb{P}\check{X}$ of $\mathbb{P}X$ is a hypersurface. As a general reference we refer the reader to the book of Tevelev [22] on Projectively Dual Varieties. For a smooth $\mathbb{P}X \subset \mathbb{P}V$ the projective dual of $\mathbb{P}X$ is simply the variety of projective hyperplanes tangent to $\mathbb{P}X$.

Our objective in this section is to find a formula for the T-equivariant rational cohomology class of the dual $\check{X}$ of $X \subset V$, under certain conditions ($\check{X}$ is the cone of $\mathbb{P}\check{X}$). The result will be a cohomology class in $H^*(BT)$ which we identify with the symmetric algebra of the character group of T. Since we use localization, our formulas will formally live in the fraction field of $H^*(BT)$, but part of the statements will be that in the outcome the denominators cancel.

Let $\check{V}$ denote the dual vector space of V and consider the conormal vector bundle

$$\check{N} = \{(x, \Lambda) \in \mathbb{P}X \times \check{V} \mid \mathbb{P}\Lambda \text{ is tangent to } \mathbb{P}X \text{ at } x\},$$

and the diagram



(cf. Proposition 3.1).

One can check that the assumption that $\mathbb{P}\check{X}$ is a hypersurface yields that the map ϕ is a resolution of $\check{X}$ (see [22, thm 1.10]). Hence from (3.4) we obtain the following statement.

Proposition 4.1. *Assume that the fixed point set $F(\mathbb{P}X)$ is finite. Then the equivariant cohomology class represented by the cone $\check{X}$ is*

$$[\check{X}] = \sum_{f\in F(\mathbb{P}X)} \frac{e(\check{V})}{e(T_f\mathbb{P}X)e(\check{N}_f\mathbb{P}X)}.$$

Let $\mathcal{W}(A)$ denote the set of weights of the T-module A counted with multiplicity. For a fixed point $f \in F(\mathbb{P}X)$ we have

$$\mathcal{W}(\check{V}) = \{-\omega(f)\} \cup \{-\omega(f) - \beta | \beta \in \mathcal{W}(T_f\mathbb{P}X)\} \cup \mathcal{W}(\check{N}_f\mathbb{P}X),$$

where $\omega(f)$ denotes the weight corresponding to the fixed point f. Notice the asymmetrical role of $\check{N}_f\mathbb{P}X$ and $T_f\mathbb{P}X$. The vector space $\check{N}_f\mathbb{P}X$ is a subspace of $\check{V}$ but $T_f\mathbb{P}X \subset T_f\mathbb{P}V \cong \hom(L_f, V \ominus L_f)$ where $L_f \leq V$ is the eigenline corresponding to the fixed point f. This isomorphism explains the shift by $-\omega(f)$ in the formula. Therefore we obtain

Theorem 4.2. *Suppose that the torus T acts on the complex vector space V linearly and $X \subset V$ is a T-invariant cone. Assume moreover that $\mathbb{P}X \subset \mathbb{P}V$ is smooth with finitely many T-fixed points and the projective dual $\mathbb{P}\check{X}$ is a hypersurface. Then the equivariant cohomology class represented by the cone $\check{X}$ of the dual $\mathbb{P}\check{X}$ is*

$$[\check{X}] = -\sum_{f\in F(\mathbb{P}X)} \omega(f) \prod_{\beta\in\mathcal{W}(T_f\mathbb{P}X)} \frac{\beta+\omega(f)}{-\beta}.$$

Notice that the fixed points f of the T-action on $\mathbb{P}X$ correspond to eigenlines $L_f \leq V$ and the weight $\omega(f)$ of this line is canonically identified with the cohomology class $e(L_f)$.

Remark 4.3. The difference of the dimensions of a hypersurface and the variety $\mathbb{P}\check{X}$ is called the *defect* of $\mathbb{P}X$. Hence Theorem 4.2 deals with the defect 0 case. It is customary to define the *cohomology class* of $\check{X}$ and the *degree* of $\mathbb{P}\check{X}$ to be 0 if $\mathbb{P}X$ has positive defect. Using this convention Theorem 4.2 remains valid without the condition on the defect. Indeed, by Remark 3.2 the right hand side, which is equal to the pushforward of 1, is automatically zero if the image of π_2 has smaller dimension.

Remark 4.4. Similar argument yields the cohomology class of $\mathbb{P}X$ itself: In Proposition 3.1 we choose S to be the tautological bundle over $\mathbb{P}X$. As a result we obtain a nontrivial special case of the **Duistermaat-Heckman**

formula: *If the set $F(\mathbb{P}X)$ of fixed points on $\mathbb{P}X$ is finite, then*

$$[X] = e(V) \sum_{f \in F(\mathbb{P}X)} (\omega(f) \cdot e(T_f \mathbb{P}X))^{-1}.$$

5. Cohomology and degree formulas for the discriminant

In this section we apply Theorem 4.2 to obtain formulas for the equivariant class and the degree of the discriminants of irreducible representations.

Let $\rho : G \to GL(V)$ be an irreducible representation of the complex connected reductive Lie group G on the complex vector space V, and let $\mathbb{P}X \subset \mathbb{P}V$ be the (closed, smooth) orbit of $[v]$ where $v \in V$ is a vector corresponding to the highest weight λ. Let $D_\lambda \subset \check{V}$ and $\mathbb{P}D_\lambda \subset \mathbb{P}\check{V}$ be the duals of X and $\mathbb{P}X$, they are called the "discriminants" of the representation of highest weight λ. The discriminant is "usually" a hypersurface, a complete list of representations of semisimple Lie groups for which the discriminant is not a hypersurfare (i.e. the defect is positive, c.f. Remark 4.3) can be found in [17], see also [22, Th.9.21].

Remark 5.1. For an irreducible representation of a reductive group G on V the action of the center of G is trivial on $\mathbb{P}V$, that is, only the semisimple part of G acts on $\mathbb{P}V$. Hence, seemingly it is enough to state the degree formula only for semisimple groups. However, we will calculate the degree of the discriminant from its equivariant Poicaré dual cohomology class, which is an element in $H^2_G(\text{point})$. For semisimple groups this cohomology group is 0. Therefore, if G is semisimple, we replace G with $G \times GL(1)$—and let $GL(1)$ act on V by multiplication. Consequently we are forced to state our theorems for reductive groups.

Let W be the Weyl group, let $R(G)$ be the set of roots of G, and $R^-(G)$ the set of negative roots. Let $\mathfrak{g}_\beta$ be the root space corresponding to the root β. Let H_β be the unique element in $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ with $\beta(H_\beta) = 2$.

Theorem 5.2. Main Formula—Short Version. *With G, W, V, λ, D_λ, $R^-(G)$ as above, the equivariant Poincaré dual of D_λ in $\check{V}$ is*

$$[D_\lambda] = -\sum_{\mu \in W\lambda} \mu \prod_{\beta \in T_\mu} \frac{\mu + \beta}{-\beta},$$

where

$$T_\lambda = \{\beta \in R^-(G) \mid \langle H_\beta, \lambda \rangle < 0\},$$

and $T_{w\lambda} = wT_\lambda$ for $w \in W$. Here we used the convention of Remark 4.3, i.e. the class of $[D_\lambda]$ is defined to be 0 if D_λ is not a hypersurface.

Let us remark that $\{\beta \in R^-(G) | B(\beta, \lambda) < 0\}$ is an equivalent description of T_λ (where B is the Killing form). The proof of Theorem 5.2 is based on the following two standard lemmas:

Lemma 5.3. *The fixed point set $F(\mathbb{P}X)$ of the maximal torus $T \subset G$ is equal to the orbit of $[v] \in \mathbb{P}V$ for the action of the Weyl group W.*

Notice that the Weyl group $W = N_G(T)/T$ indeed acts on $\mathbb{P}X$ since T fixes $[v] \in \mathbb{P}V$.

Proof. It is enough to show that if $[v] \in \mathbb{P}V$ and $g[v]$ are both fixed points of T and v is a maximal weight vector, then there exists a $\beta \in N_G(T)$ such that $g[v] = \beta[v]$.

Let $G_{[v]}$ be the stabilizer of $[v]$. Then, by the assumption, T and $g^{-1}Tg$ are contained in $G_{[v]}$. These are maximal tori in $G_{[v]}$, so there is a $p \in G_{[v]}$ such that $g^{-1}Tg = p^{-1}Tp$ (see e.g. [6, p. 263]). Then $\beta = gp^{-1} \in N_G(T)$ and $g[v] = \beta[v]$. □

Lemma 5.4. *The weights of the tangent space $T_f(\mathbb{P}X)$ as a T-space are*

$$T_f = \{\beta \in R^-(G) \mid \langle H_\beta, \omega(f)\rangle < 0\}$$

for any $f \in F(\mathbb{P}X)$.

For semisimple G the proof can be found in [11] and a more detailed version in [2, p.36]. The formula extends to the reductive case without change.

Proof of Theorem 5.2. It is enough to apply Theorem 4.2 to our situation. Lemma 5.3 determines the (finitely many) fixed points and Lemma 5.4 gives that

$$T_\lambda = \{\beta \in R^-(G) \mid \langle H_\beta, \lambda\rangle < 0\}.$$

The weights at other fixed points are obtained by applying the appropriate element of the Weyl group. □

For the Lie group $G = GL(n)$ the maximal torus can be identified with the subgroup of diagonal matrices $diag(z_1, \dots, z_n)$, $|z_i| = 1$, which is the product of n copies of S^1's. Let L_i be the character of this torus, which is the identity on the i'th S^1 factor, and constant 1 on the others. The Weyl group, the symmetric group on n letters, permutes the L_i's. The Weyl-invariant subring of the symmetric algebra $\mathbb{Q}[L_1, \dots, L_n]$ is identified with $H^*(BGL(n))$. Under this identification, the i'th elementary symmetric polynomial of the L_i's is the i'th Chern class ('splitting lemma'). Hence—as usual—we call the L_i's the *Chern roots* of $GL(n)$.

Example 5.5. Consider the dual of $Gr_3(\mathbb{C}^8)$ in its Plücker embedding. This is the discriminant of the representation of $GL(8)$ with highest weight $L_1 + L_2 + L_3$. Let $\binom{n}{k}$ stand for the set of k-element subsets of $\{1, \dots, n\}$. By

Theorem 5.2 the class of the discriminant is

$$-\sum_{S\in\binom{8}{3}}(L_{s_1}+L_{s_2}+L_{s_3})\cdot\prod_{i\in S}\prod_{j\notin S}\frac{L_{s_1}+L_{s_2}+L_{s_3}+L_j-L_i}{(L_i-L_j)}.$$

This is a 58-term sum. However, this class is in $H^2(BGL(8))$, hence we know that it must have the form of $v\cdot c_1=v\cdot\sum_{i=1}^{8}L_i$. Therefore we only need to determine the number v. A well chosen substitution will kill most of the terms, making the calculation of v easier. E.g. substitute $L_i=i-13/3$, then all terms—labeled by 3-element subsets S of $\{1,\dots,8\}$—are zero, except for the last one corresponding to $S=\{6,7,8\}$. For the last term we obtain -8. Hence, $-8=v\cdot(1-13/3+2-13/3+\ldots+8-13/3)$, which gives the value of $v=-6$. Thus we obtain the equivariant class of the dual of $Gr_3(\mathbb{C}^8)$ to be $-6(L_1+\ldots+L_8)$, and in turn its degree as $-6(-1/3-\ldots-1/3)=16$ (by Proposition 2.1).

Although similarly lucky substitutions cannot be expected in general, the $L_i=i$ substitution yields a formula for the degree of the dual variety of the Grassmannian $Gr_k(\mathbb{C}^n)$ in its Plücker embedding:

$$\deg(\check{G}r_k(\mathbb{C}^n))=\frac{2k}{n+1}\sum_{S\in\binom{n}{k}}l(S)\prod_{i\in S,j\notin S}\frac{l(S)+j-i}{i-j},\tag{5.1}$$

where $l(S)=\sum_{s\in S}s$.

To study how the degree depends on the highest weight λ for a fixed group G we introduce another expression for the degree where the sum is over all elements of the Weyl group instead of the orbit $W\lambda$.

For a dominant weight λ, let $O_\lambda=R^-(G)\setminus T_\lambda=\{\beta\in R^-(G)\mid\langle H_\beta,\lambda\rangle=0\}$, and let the sign $\varepsilon(\lambda)$ of λ be $(-1)^{|O_\lambda|}$. Let $W_\lambda\le W$ be the stabilizer subgroup of λ.

Theorem 5.6. Main Formula—Symmetric Version. *Under the conditions of Theorem 5.2 we have*

$$[D_\lambda]=-\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{w\in W}w\Big(\lambda\prod_{\beta\in R^-(G)}\frac{\lambda+\beta}{-\beta}\Big).$$

Proof. Let $w_1,\dots,w_m$ be left coset representatives of $W_\lambda\le W$, i.e. the disjoint union of the w_iW_λ's is W. Then

$$-\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{w\in W}w\left(\lambda\prod_{\beta\in R^-(G)}\frac{\lambda+\beta}{-\beta}\right)=$$

$$(5.2)\qquad = -\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{i=1}^{m} w_i\left(\sum_{w\in W_\lambda} w\left(\lambda \prod_{\beta\in T_\lambda}\frac{\lambda+\beta}{-\beta}\prod_{\beta\in O_\lambda}\frac{\lambda+\beta}{-\beta}\right)\right)$$

$$(5.3)\qquad = -\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{i=1}^{m} w_i\Big(\lambda \prod_{\beta\in T_\lambda}\frac{\lambda+\beta}{-\beta}\sum_{w\in W_\lambda} w\big(\prod_{\beta\in O_\lambda}\frac{\lambda+\beta}{-\beta}\big)\Big),$$

since $wT_\lambda = T_\lambda$ (but wO_λ is not necessarily equal to O_λ). Now we need the following lemma.

Lemma 5.7.

$$\sum_{w\in W_\lambda} w\big(\prod_{\beta\in O_\lambda}\frac{\lambda+\beta}{-\beta}\big) = \varepsilon(\lambda)|W_\lambda|.$$

Proof. The polynomial

$$P(x) = \sum_{w\in W_\lambda}\prod_{\beta\in O_\lambda}\frac{x+w(\beta)}{-w(\beta)} \in \mathbb{Z}(L_i)[x]$$

is W_λ-invariant, and has the form

$$\frac{q_k + q_{k-1}x + \ldots + q_0x^k}{\prod_{\beta\in O_\lambda}\beta}$$

where $k = |O_\lambda|$ and q_i is a degree i polynomial on the orthogonal complement of λ. This means that the numerator has to be anti-symmetric under W_λ (which is itself a Weyl group of a root system), hence it must have degree at least the number of positive roots. That is, all q_i, $i < k$ must vanish. This means that $P(x)$ is independent of x, i.e. $P(\lambda) = P(0) = \varepsilon(\lambda)|W_\lambda|$, as required. □

Using this Lemma, formula (5.3) is further equal to

$$-\sum_{i=1}^{m} w_i\Big(\lambda\prod_{\beta\in T_\lambda}\frac{\lambda+\beta}{-\beta}\Big),$$

which completes the proof of Theorem 5.6. □

The advantage of the Short Version (Theorem 5.2) is that for certain λ's (λ's on the boundary of the Weyl-chamber) the occurring products have only few factors, while the advantage of the Symmetric Version (Theorem 5.6) is that it gives a unified formula for all λ's of a fixed group. Now we will expand this latter observation.

Let G be semisimple and consider the representation with highest weight λ. Extend this action to an action of $G \times GL(1)$ with $GL(1)$ acting by scalar multiplication. Denoting the first Chern class of $GL(1)$ by u (that is $H^*(BU(1)) = \mathbb{Q}[u]$) we obtain that

$$[D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{w\in W} w\Big((\lambda+u)\prod_{\beta\in R^-(G)}\frac{\lambda+u+\beta}{-\beta}\Big). \tag{5.4}$$

Proposition 2.1 then turns this formula to a degree formula for the discriminant, by substituting $u=-1$:

$$\deg(D_\lambda) = -\frac{\varepsilon(\lambda)}{|W_\lambda|}\sum_{w\in W} w\Big((\lambda-1)\prod_{\beta\in R^-(G)}\frac{\lambda+\beta-1}{-\beta}\Big). \tag{5.5}$$

Part of the statement is that this formula is a constant, i.e. expression (5.4) is equal to a constant times u (although this can also be deduced from the fact that $H^2(BG)=0$ for semisimple groups).

Corollary 5.8. *Let G be a semisimple Lie group, and $\mathfrak{h}$ the corresponding Cartan subalgebra. There exists a polynomial $F_G : \mathfrak{h}^* \to \mathbb{Z}$, with degree equal the number of positive roots of G, such that*

$$\deg(D_\lambda) = \frac{\varepsilon(\lambda)}{|W_\lambda|}F_G(\lambda)$$

if λ is a dominant weight. The polynomial $F_G(\lambda)$ vanishes if and only if D_λ is not a hypersurface.

Proof. We have

$$F_G = -\sum_{w\in W} w\Big((\lambda-1)\prod_{\beta\in R^-(G)}\frac{\lambda+\beta-1}{-\beta}\Big), \tag{5.6}$$

and the last statement follows from Remark 4.3. □

Remark 5.9. The polynomial dependence of $\deg(D_\lambda)$ for *regular* weights λ (hence $\varepsilon(\lambda)=1$, $W_\lambda=\{1\}$), as well as a formula for a special value of the polynomial (the value at the sum of the fundamental weights) is given in [7]. Since $\deg(D_\lambda)$ is always non-negative the value of $\varepsilon(\lambda)$ is determined by the sign of $F_G(\lambda)$. The positive defect cases are known but this corollary provides an alternative and uniform way to find them.

Example 5.10. A choice of simple roots $\alpha_1,\alpha_2,\dots,\alpha_r$ of G determines the fundamental weights $\omega_1,\omega_2,\dots,\omega_r$ by $B(\omega_i,\frac{2\alpha_j}{B(\alpha_j,\alpha_j)})=\delta_{i,j}$ where $B(.,.)$ denotes the Killing form. We follow the convention of De Concini and Weyman [7] by writing F_G in the basis of fundamental weights (i.e. $y_1\omega_1+\ldots+y_r\omega_r \mapsto F_G(y_1,\dots,y_r)$) and substituting $y_i=x_i+1$. The advantage of this substitution is that in this way, according to [7], all the coefficients of the polynomial F_G become non-negative. Formula (5.6) gives the following polynomials for all semisimple Lie groups of rank at most 2 and for some of rank 3. [In these examples our convention for simple roots agrees with the one in the *coxeter/weyl* Maple package *www.math.lsa.umich.edu/˜jrs/maple.html* written by J. Stembridge.]

$\mathbf{A_1}$, $\alpha_1 = L_2 - L_1$:

$F = 2x_1$.

$\mathbf{A_1 + A_1}$, $\alpha = (L_2 - L_1, L'_2 - L'_1)$:

$F = 6x_1x_2 + 2x_2 + 2x_1 + 2$.

$\mathbf{A_2}$, $\alpha = (L_2 - L_1, L_3 - L_2)$:

$F = 6(x_1 + x_2 + 1)(2x_1x_2 + x_1 + x_2 + 1)$ [22, ex 7.18].

$\mathbf{B_2}$, $\alpha = (L_1, L_2 - L_1)$,

$F = 20(2x_2^3x_1 + 3x_2^2x_1^2 + x_2x_1^3) + 12(2x_2^3 + 12x_2^2x_1 + 11x_2x_1^2 + x_1^3) + 24(3x_2^2 + 7x_2x_1 + 2x_1^2) + 8(9x_2 + 8x_1) + 24$.

$\mathbf{G_2}$, $\alpha = (L_2 - L_1, L_1 - 2L_2 + L_3)$:

$F = 42(18x_2^5x_1 + 45x_2^4x_1^2 + 40x_2^3x_1^3 + 15x_2^2x_1^4 + 2x_2x_1^5) + 60(9x_2^5 + 90x_2^4x_1 + 150x_2^3x_1^2 + 90x_2^2x_1^3 + 20x_2x_1^4 + x_1^5) + 110(27x_2^4 + 132x_2^3x_1 + 144x_2^2x_1^2 + 52x_2x_1^3 + 5x_1^4) + 8(822x_2^3 + 2349x_2^2x_1 + 1527x_2x_1^2 + 248x_1^3) + 6(60x_2^2 + 1972x_2x_1 + 579x_1^2) + 4(1025x_2 + 727x_1) + 916$.

$\mathbf{A_1 + A_1 + A_1}$, $\alpha = (L_2 - L_1, L'_2 - L'_1, L''_2 - L''_1)$:

$F = 24x_2x_1x_3 + 12(x_2x_1 + x_2x_3 + x_1x_3) + 8(x_2 + x_1 + x_3) + 4$.

$\mathbf{A_1 + A_2}$, $\alpha = (L_2 - L_1, L'_2 - L'_1, L'_3 - L'_2)$:

$F = 60x_2^2x_1x_3 + 60x_2x_1x_3^2 + 36x_2^2x_1 + 36x_2^2x_3 + 144x_2x_1x_3 + 36x_2x_3^2 + 36x_1x_3^2 + 24x_2^2 + 72x_2x_1 + 96x_2x_3 + 72x_1x_3 + 24x_3^2 + 48x_2 + 36x_1 + 48x_3 + 24$.

$\mathbf{A_3}$, $\alpha = (L_2 - L_1, L_3 - L_2, L_4 - L_3)$:

$F = 420x_1x_3x_2(x_1 + x_2)(x_2 + x_3)(x_2 + x_1 + x_3) + 300(x_1^3x_2^2 + 4x_1^3x_2x_3 + x_1^3x_3^2 + 2x_1^2x_2^3 + 15x_1^2x_2^2x_3 + 12x_1^2x_2x_3^2 + x_1^2x_3^3 + x_1x_2^4 + 12x_1x_2^3x_3 + 15x_1x_2^2x_3^2 + 4x_1x_2x_3^3 + x_2^4x_3 + 2x_2^3x_3^2 + x_2^2x_3^3) + 220(3x_1^3x_2 + 3x_1^3x_3 + 12x_1^2x_2^2 + 33x_1^2x_2x_3 + 9x_1^2x_3^2 + 10x_1x_2^3 + 48x_1x_2^2x_3 + 33x_1x_2x_3^2 + 3x_1x_3^3 + x_2^4 + 10x_2^3x_3 + 12x_2^2x_3^2 + 3x_2x_3^3) + 8(42x_1^3 + 453x_1^2x_2 + 411x_1^2x_3 + 699x_1x_2^2 + 1566x_1x_2x_3 + 411x_1x_3^2 + 164x_2^3 + 699x_2^2x_3 + 453x_2x_3^2 + 42x_3^3) + 12(126x_1^2 + 491x_1x_2 + 407x_1x_3 + 239x_2^2 + 491x_2x_3 + 126x_3^2) + 16(133x_1 + 169x_2 + 133x_3) + 904$.

A Maple computer program computing F for any semisimple Lie group is available at www.unc.edu/˜rimanyi/progs/feherpolinom.mw.

Straightforward calculation gives that F_G for $G = A_1 + \ldots + A_1$ (n times) is

$$F_{nA_1}(y_1,\dots,y_n)=\sum_{k=0}^{n}(-2)^{n-k}(k+1)!\,\sigma_k(y_1,\dots,y_n),$$

where σ_i is the i'th elementary symmetric polynomial. This can also be derived from [12, Th.2.5,Ch.13].

6. Combinatorics of the degree formulas for $GL(n)$

Our main formulas, Theorems 5.2 and 5.6, can be encoded using standard notions from the combinatorics of symmetric functions. In Sections 6-7 we assume that $G = GL(n)$, and that the simple roots are $L_i - L_{i+1}$. Then $R^-(G) = \{L_i - L_j \,:\, i > j\}$. Irreducible representations correspond to the weights $\lambda = \sum_{i=1}^{n} a_i L_i$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. For such a weight one has $T_\lambda = \{L_i - L_j \,:\, i > j,\ a_i < a_j\}$.

6.1. Symmetrizer operators. Let

$$\lambda^+ = \lambda \prod_{\beta\in R^-(G)} (\lambda+\beta), \qquad \text{and} \qquad \Delta = \prod_{1\leq i<j\leq n} (L_i - L_j),$$

and recall the definition of the Jacobi-symmetrizer ([19]) of a polynomial $f(L_1,\dots,L_n)$:

$$J(f)(L_1,\dots,L_n) = \frac{1}{\Delta}\sum_{w\in S_n} \operatorname{sgn}(w) f(L_{w(1)},\dots,L_{w(n)}),$$

where $\operatorname{sgn}(w)$ is the sign of the permutation w, i.e. (-1) raised to the power of the number of transpositions in w. Then for $\lambda = \lambda_1 L_1 + \ldots + \lambda_n L_n$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, from Theorem 5.6 we obtain

$$[D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} J\Big(\lambda^+\Big), \tag{6.1}$$

and $\deg D_\lambda$ is obtained by multiplying this by the factor

$$\frac{-n}{|\lambda|\cdot\sigma_1}, \quad \text{where } |\lambda| := \sum \lambda_i \text{ and } \sigma_1 := \sum L_i. \tag{6.2}$$

The Jacobi symmetrizer is a special case of the divided difference operators ∂_w [19, Ch.7], corresponding to the maximal permutation $w_0 = [n, n-1,\dots,2,1]$ (i.e. $w_0(i) = n+1-i$). Shorter divided difference operators also turn up for certain representations. We will illustrate this with the case of the Plücker embedding of Grassmann varieties, i.e. $G = GL(n)$, $\lambda = L_1+\ldots+L_k$. Let $[k|n-k] \in S_n$ be the permutation $[n-k+1, n-k+2,\dots,n,1,2,\dots,n-k]$.

Theorem 5.2 gives

$$[D_{L_1+\ldots+L_k}] = -\partial_{[k|n-k]}\Big((L_1+\ldots+L_k)\prod_{i=1}^{k}\prod_{j=k+1}^{n}(L_1+\ldots+L_k+L_j-L_i)\Big),$$

and the degree of the discriminant of $Gr_k(\mathbb{C}^n)$ in its Plücker embedding is obtained by multiplying this with $-n/(k\sigma_1)$.

6.2. Scalar product. Now we show how to use the scalar product on function spaces to encode the formula of Theorem 5.6 in the case of $G = GL(n)$. Following [20], we define the scalar product of the polynomials f, g in n variables $L_1, \ldots, L_n$ as

$$\langle f, g\rangle = \frac{1}{n!}\Big[f\bar{g}\prod_{i\neq j}(1-\frac{L_i}{L_j})\Big]_1,$$

where $\bar{g}(L_1, \ldots, L_n) = g(1/L_1, \ldots, 1/L_n)$ and $[h]_1$ is the constant term (i.e. the coefficient of 1) of the Laurent polynomial $h \in \mathbb{Z}[L_1^{\pm}, \ldots, L_n^{\pm}]$. The Jacobi symmetrizer is basically a projection, thus for a degree $\binom{n}{2}+1$ polynomial f we have

$$\frac{1}{n!}\sum_w \operatorname{sgn}(w) f(L_{w(1)}, \ldots, L_{w(n)}) = \frac{\langle f, \sigma_1\Delta\rangle}{\langle \sigma_1\Delta, \sigma_1\Delta\rangle}\sigma_1\Delta.$$

Here $\langle\sigma_1\Delta, \sigma_1\Delta\rangle$ can be calculated to be $n(2n-3)!!$, where $(2k+1)!!$ denotes the semifactorial $1\cdot 3\cdots(2k-1)\cdot(2k+1)$, hence we obtain

$$[D_\lambda] = \frac{-\varepsilon(\lambda)n!}{|W_\lambda|\cdot n(2n-3)!!}\langle\lambda^+, \sigma_1\Delta\rangle\sigma_1,$$

and hence the following form of our Main Formula:

Theorem 6.1. *The degree of the discriminant of the irreducible representation of $GL(n)$ with highest weight λ is*

$$\deg D_\lambda = \frac{\varepsilon(\lambda)}{|\lambda|\cdot|W_\lambda|}\frac{n!}{(2n-3)!!}\langle\lambda^+, \sigma_1\Delta\rangle.$$

6.3. Permanent. For $(\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ and $w \in S_n$ let $w(\nu) = (\nu_{w(1)}, \ldots, \nu_{w(n)})$ and L^ν will denote the monomial $L_1^{\nu_1}\ldots L_n^{\nu_n}$.

Lemma 6.2. *Let $\mu = (n, n-2, n-3, \ldots, 2, 1, 0) \in \mathbb{N}^n$. If $\sum \nu_i = \binom{n}{2}+1$ then we have*

$$J(L^\nu) = \begin{cases}\operatorname{sgn}(w)\sigma_1 & \text{if } \nu_i = \mu_{w(i)}\\ 0 & \text{otherwise}\end{cases}$$

Proof. If $\nu_i = \nu_j$ then the terms of $J(L^\nu)$ turn up in cancelling pairs, hence $J(L^\nu) = 0$. This leaves only $\nu =$ permutations of μ for possible non-zero J-value. Direct check shows $J(L^\mu) = \sigma_1$ (c.f. the well known identity $J\left(L^{(n-1,n-2,\ldots,2,1,0)}\right) = 1$). □

The coefficient of a monomial L^ν in a polynomial f will be denoted by $c(f, L^\nu)$. Formula (6.1) and Lemma 6.2 yield that the class and degree of D_λ can be computed by counting coefficients.

Theorem 6.3. *The equivariant class of D_λ is*

$$[D_\lambda] = -\frac{\varepsilon(\lambda)}{|W_\lambda|} \sum_{w \in S_n} \operatorname{sgn}(w) c(\lambda^+, L^{w(\mu)}) \cdot \sigma_1,$$

and the degree of D_λ is obtained by multiplication by $-n/(|\lambda|\sigma_1)$.

Similar sums will appear later, hence we define the ν-permanent of a polynomial f as

$$\sum_{w \in S_n} \operatorname{sgn}(w) c(f, L^{w(\nu)}),$$

and denote it by $P(f, \nu)$.

The name *permanent* is justified by the following observation. Let $\nu = (\nu_1, \dots, \nu_n)$ be a partition. If the polynomial f is the product of $|\nu|$ linear factors $\sum_{j=1}^n a_j^{(i)} L_j$, $i = 1, \dots, |\nu|$, then $P(f, \nu)$ can be computed from the $n \times |\nu|$ matrix $(a_j^{(i)})$ as follows. For a permutation $w \in S_n$ we choose ν_1 entries from row $\nu(1)$ (i.e. $j = \nu(1)$), ν_2 entries from row $\nu(2)$, etc, such a way that no chosen entries are in the same column (hence they form a complete 'rook arrangement'). The product of the chosen entries with sign $\operatorname{sgn}(w)$ will be a term in $P(f, \nu)$, and we take the sum for all $w \in S_n$ and all choices. For example the $(2,1)$-permanent of the product $(aL_1 + a'L_2)(bL_1 + b'L_2)(cL_1 + c'L_2)$, considering the matrix $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$, is

$$abc' + ab'c + a'bc - a'b'c - a'bc' - ab'c'.$$

Theorem 6.4. *(Boole's formula,* [22, 7.1]*) If $\lambda = aL_1$, then the degree of D_λ is $n(a-1)^{n-1}$.*

Proof. We have

$$\lambda^+ = aL_1 \prod_{i=2}^n ((a-1)L_1 + L_i) \cdot \prod_{2 \le i < j \le n} (L_j - L_i + aL_1),$$

and let

$$\lambda^{++} = aL_1 \prod_{i=2}^n ((a-1)L_1 + L_i) \cdot \prod_{2 \le i < j \le n} (L_j - L_i).$$

Observe that $P(\lambda^+, \mu) = P(\lambda^{++}, \mu)$, since in the difference each term comes once with a positive, once with a negative sign. In $P(\lambda^{++}, \mu)$ only L_1 has degree $\geq n$ (namely n), hence we have

$$P(\lambda^{++},\mu) = a(a-1)^{n-1}P\left(\prod_{2\leq i<j\leq n}(L_i - L_j), (n-2, n-3, \ldots, 2, 1)\right)$$

which is further equal to $a(a-1)^{n-1}(-1)^{\binom{n-1}{2}}(n-1)!$ and Theorem 6.3 gives the result. □

6.4. Hyperdeterminants. The discriminant of the standard action of the product group $\prod_{u=1}^k GL(n_u)$ on $\bigotimes_{u=1}^k \mathbb{C}^{n_u}$ is called *hyperdeterminant* because it generalizes the case of the determinant for $k=2$. In other words, the hyperdeterminant is the dual of the Segre embedding $\prod_{u=1}^k \mathbb{P}(\mathbb{C}^{n_u}) \to \mathbb{P}(\bigotimes_{u=1}^k \mathbb{C}^{n_u})$. Gelfand, Kapranov and Zelevinsky [12] give a concrete description of the degree of the hyperdeterminants by giving a generator function. In the so-called boundary case when $n_k - 1 = \sum_{u=1}^{k-1}(n_u - 1)$ they get a closed formula. We show how to prove this using ν-permanents.

When considering representations of $\prod_{u=1}^k GL(n_u)$ $(n_1 \leq n_2 \leq \ldots \leq n_k)$ we will need k sets of variables (the k sets of Chern roots), we will call them $L_{(u),i}$, $u = 1,\ldots,k$, $i = 1,\ldots,n_u$. The hyperdeterminant is the discriminant corresponding to the representation with highest weight $\lambda = \sum_{u=1}^k L_{(u),1}$. If $\nu_{(u)} \in \mathbb{N}^{k_u}$, then a polynomial f in these variables has a $\nu = (\nu_{(1)},\ldots,\nu_{(k)})$-permanent, defined as

$$P(f,\nu) = \sum_{w^{(1)}\in S_{n_1}} \cdots \sum_{w^{(k)}\in S_{n_k}} \Big(\prod_{u=1}^k \operatorname{sgn}(w^{(u)})\Big) c\Big(f, \prod_u L_{(u)}^{w^{(u)}(\nu_{(u)})}\Big).$$

Theorem 6.5 ([12, 14.2.B]). *If $n_k - 1 > \sum_{u=1}^{k-1}(n_u - 1)$ then the discriminant is not a hypersurface. If $n_k - 1 = \sum_{u=1}^{k-1}(n_u - 1)$ then its degree is $n_k!/\prod_{u=1}^{k-1}(n_u-1)!$.*

Proof. Let $\mu^{(u)} = (\mu_{(1)},\ldots,\mu_{(k)})$, where $\mu_{(v)} = (n_v - 1, n_v - 2, \ldots, 1, 0)$ for $v \neq u$, and $\mu_{(u)} = (n_u, n_u - 2, n_u - 3, \ldots, 2, 1, 0)$. A straightforward generalization of the $k=1$ case gives

$$[D_\lambda] = -\frac{(-1)^{\sum\binom{n_u-1}{2}}}{\prod_{u=1}^k (n_u-1)!}\sum_{u=1}^k \Big(P(\lambda^+,\mu^{(u)})\sum_{i=1}^{n_u} L_{(u),i}\Big), \tag{6.3}$$

where

$$\lambda^+ = \lambda\left(\prod_{u=1}^k\prod_{i=2}^{n_u}(L_{(u),i} - L_{(u),1} + \lambda)\right)\left(\prod_{u=1}^k\prod_{2\leq i<j\leq n_u}(L_{(u),j} - L_{(u),i} + \lambda)\right).$$

Just like in the proof of the Boole formula above, we can change λ^+ to

$$\lambda^{++}=\lambda\left(\prod_{u=1}^{k}\prod_{i=2}^{n_u}(L_{(u),i}-L_{(u),1}+\lambda)\right)\left(\prod_{u=1}^{k}\prod_{2\leq i<j\leq n_u}(L_{(u),j}-L_{(u),i})\right)$$

without changing the value of the permanent. Observe that the highest power of an $L_{(k),i}$ variable in λ^{++} is $\sum_{u=1}^{k-1}(n_u-1)$, thus the last term in (6.3) is 0 if $n_k-1>\sum_{u=1}^{k-1}(n_u-1)$. This proves the first statement of the theorem.

If we have $n_k-1=\sum_{u=1}^{k-1}(n_u-1)$, then considering the power of $L_{(k),1}$ again, we get that

$$P(\lambda^{++},\mu_{(k)})=$$
$$P\Big(\big(\sum_{u=1}^{k-1}L_{(u),1}\big)^{n_k-1}\prod_{u=1}^{k}\prod_{2\leq i<j\leq n_u}(L_{(u),j}-L_{(u),i}),\big((n_1-1,\dots,1,0),\dots$$
$$\dots,(n_{k-1}-1,\dots,1,0),(n_k-2,\dots,1,0)\big)\Big)=$$
$$\frac{(n_k-1)!}{\prod_{u=1}^{k-1}(n_u-1)!}P\Big(\prod_{u=1}^{k}\prod_{2\leq i<j\leq n_u}(L_{(u),j}-L_{(u),i}),\big((n_1-2,\dots,1,0),\dots,(n_k-2,\dots,1,0)\big)\Big)=$$
$$(-1)^{\Sigma\binom{n_u-1}{2}}\frac{(n_k-1)!}{\prod_{u=1}^{k-1}(n_u-1)!}\prod_{u=1}^{k}(n_u-1)!=(-1)^{\Sigma\binom{n_u-1}{2}}(n_k-1)!^2.$$

Hence, in the expansion $-[D_\lambda]=c_1\sum L_{(1),i}+c_2\sum L_{(2),i}+\ldots+c_k\sum L_{(k),i}$ we have

$$c_k=(n_k-1)!^2/\prod_{u=1}^{k}(n_u-1)!=(n_k-1)!/\prod_{u=1}^{k-1}(n_u-1)!$$

Proposition 2.1 then gives $\deg(D_\lambda)$ by substituting (e.g.) $L_{(u),i}=0$ for $u<k$ and $L_{(k),i}=-1$ into $[D_\lambda]$, which proves the Theorem. □

7. Some more explicit formulae for $GL(n)$.

Notice that the expression of $[D_\lambda]$ from 5.2 is homogeneous in $L_1,\dots,L_n$ of degree 1. Moreover, $[D_\lambda]\cdot\Delta$ is anti-symmetric of degree $\deg\Delta+1$, hence it is the product of Δ and a symmetric polynomial of degree 1. Hence, by (6.2), for some constant $f_\lambda^{(n)}$ one has

$$[D_\lambda]=-f_\lambda^{(n)}\cdot\sigma_1\quad\text{and}\quad\deg(D_\lambda)=\frac{n}{|\lambda|}\cdot f_\lambda^{(n)},\tag{7.1}$$

where $\sigma_1 = \sum_{i=1}^n L_i$ as above. By similar argument one deduces that for a free variable t one has

$$R_\lambda^{(n)}(t) := \sum_{\mu\in W\lambda} (\mu+t) \prod_{\beta\in T_\mu} \frac{\mu+t+\beta}{-\beta} = A_\lambda^{(n)} t + f_\lambda^{(n)}\sigma_1$$

for another constant $A_\lambda^{(n)}$. Since the dual variety associated with the representation ρ_λ and the dual variety associated with the action $\rho_\lambda \times \mathrm{diag}$ of $GL(n)\times GL(1)$ are the same, their degrees are the same too. Applying our main result for these two groups and actions, one gets

$$R_\lambda^{(n)}(t) = f_\lambda^{(n)} \cdot \Big(\frac{n}{|\lambda|}\cdot t + \sigma_1\Big). \tag{7.2}$$

In fact, (7.2) is an entirely algebraic identity, and it is valid for *any* weight λ (i.e. not only for the dominant weights).

Using (7.2), we prove some identities connecting different weights.

Lemma 7.1. *For any $\lambda = \sum_i a_i L_i$ define $\overline{\lambda} := \sum_i a_i L_{n-i}$. Assume that either (i) $\lambda' = \lambda + a\sigma_1$, or (ii) $\overline{\lambda'} + \lambda = a\sigma_1$ for some $a \in \mathbb{Z}$. Then*

$$\deg(D_{\lambda'}) = \deg(D_\lambda).$$

Proof. In case (i), using (7.2), one gets $T_{\lambda+a\sigma_1} = T_\lambda$ and $f_{\lambda+a\sigma_1}^{(n)} = f_\lambda^{(n)}(\frac{na}{|\lambda|}+1)$, hence $f_{\lambda+a\sigma_1}^{(n)}/|\lambda+a\sigma_1| = f_\lambda^{(n)}/|\lambda|$. Then apply (7.1). For (ii) notice that $T_{\lambda'} = \{-\overline{\beta} : \beta \in T_\lambda\}$, hence $-[D_{\lambda'}]$ equals

$$\sum_{\mu\in W\lambda} \overline{a\sigma_1-\lambda} \prod_{\beta\in T_\mu} \frac{\overline{a\sigma_1-\lambda}+\overline{-\beta}}{\overline{\beta}} = -\sum_{\mu\in W\lambda} (\lambda-a\sigma_1) \prod_{\beta\in T_\mu} \frac{\lambda-a\sigma_1+\beta}{-\beta} =$$

$$-f_\lambda^{(n)}\Big(-\frac{na}{|\lambda|}+1\Big)\sigma_1.$$

Therefore, $f_{\lambda'}^{(n)} = f_\lambda^{(n)}(\frac{na}{|\lambda|}-1)$, or $f_{\lambda'}^{(n)}/|\lambda'| = f_\lambda^{(n)}/|\lambda|$.

Geometrically the lemma corresponds to the fact that tensoring with the one dimensional representation or taking the dual representation does not change the discriminant. □

It follows from the lemma that we can always assume that the last coefficient a_n of λ is zero.

Example 7.2. The case of $\lambda = (a+b)L_1 + b(L_2+\cdots+L_{n-1})$ with $a,b>0$ and $n\geq 3$.

Set $\lambda' := (a+b)L_1 + a(L_2+\cdots+L_{n-1})$. Then $\overline{\lambda'}+\lambda = (a+b)\sigma_1$, hence $\deg(D_\lambda) = \deg(D_{\lambda'})$ by 7.1. In particular, $\deg(D_\lambda)$ is a *symmetric* polynomial in variables (a,b). In the sequel we deduce its explicit form. For this notice that

$$T_\lambda = \{L_i - L_1,\ L_n - L_i : 2\leq i\leq n-1\} \cup \{L_n - L_1\}.$$

It is convenient to write λ as $aL_1-bL_n+b\sigma_1$. First, assume that $a\neq b$. Using (7.1) and Lemma 7.1 one gets

$$\deg(D_\lambda)=\frac{n}{|\lambda|}f^{(n)}_\lambda=\frac{n}{a+b(n-1)}f^{(n)}_{aL_1-bL_n}\cdot\Big(\frac{nb}{a-b}+1\Big)=\frac{n}{a-b}\cdot f^{(n)}_{aL_1-bL_n},$$

where $f^{(n)}_{aL_1-bL_n}\cdot\sigma_1$ equals

(7.3)

$$\sum_{i\neq j}(aL_i-bL_j)\frac{aL_i-bL_j+L_j-L_i}{L_i-L_j}\prod_{k\neq i,j}\frac{aL_i-bL_j+L_k-L_i}{L_i-L_k}\cdot\frac{aL_i-bL_j+L_j-L_k}{L_k-L_j}.$$

Clearly, $f^{(n)}_{aL_1-bL_n}=\lim_{L_n\to\infty}(f^{(n)}_{aL_1-bL_n}\cdot\sigma_1)/L_n$. (In the sequel we write simply lim for $\lim_{L_n\to\infty}$.) In order to determine this limit, we separate L_n in the expression E of (7.3). We get three types of contributions: $E=I+II+III$, where I contains those terms of the sum $\sum_{i\neq j}$ where $i,j\leq n-1$; II contains the terms with $i=n$; while III those terms with $j=n$.

One can see easily that $\lim I$ is finite (for any fixed $L_1,\dots,L_{n-1}$), hence $\lim I/L_n=0$.

The second term (after re-grouping) is

$$II=\frac{S}{\prod_{j\leq n-1}L_n-L_k},$$

where

$$S:=\sum_{j\leq n-1}\frac{\prod_{k\leq n-1}(aL_n-bL_j-L_n+L_k)(aL_n-bL_j+L_j-L_k)}{\prod_{n\neq k\neq j}L_k-L_j}.$$

By similar argument as above, S can be written as $S=\sum_{i=0}^n P_i\cdot L_n^{n-i}$, where P_i is a symmetric polynomial in $L_1,\dots,L_{n-1}$ of degree i (and clearly also depends on a,b and n). In particular, $\lim II/L_n$ is the 'constant' $P_0=P_0(a,b,n)$.

Set $t_1:=aL_n-bL_j-L_n$ and $t_2:=aL_n-bL_j+L_j$. Notice that if we modify the polynomial $P=\prod_{k\leq n-1}(t_1+L_k)(t_2-L_k)$ by any polynomial situated in the ideal generated by the symmetric polynomials $\sigma_1^{(n-1)}:=\sum_{i-1}^{n-1}L_i$, $\sigma_2^{(n-1)}$, ... (in $L_1,\dots,L_{n-1}$), then the modification has no effect in the 'constant' term P_0. In particular, since $P=(t_1^{n-1}+\sigma_1^{(n-1)}t_1^{n-2}+\cdots)(t_2^{n-1}-\sigma_1^{(n-1)}t_2^{n-2}+\cdots)$, in the expression of S, the polynomial P can be replaced by $t_1^{n-1}t_2^{n-1}$. Therefore, if $[Q(t)]_{t^i}$ denotes the coefficient of t^i in $Q(t)$, then one has:

$$P_0(a,b,n)=\left[\sum_{j\leq n-1}\frac{(aL_n-bL_j-L_n)^{n-1}(aL_n-bL_j+L_j)^{n-1}}{\prod_{n\neq k\neq j}L_k-L_j}\right]_{L_n^n}.$$

Set

$$P_{a,b}(t) = At^2 + Bt + C := (at - t - b)(at - b + 1).$$

Then

$$P_0(a,b,n) = \Big[P_{a,b}(t)^{n-1}\Big]_{t^n} \cdot \sum_{j\leq n-1} \frac{L_j^{n-2}}{\prod_{n\neq k\neq j} L_k - L_j}.$$

Notice that the last sum is exactly $(-1)^{n-2}$ by Lagrange interpolation formula.

A very similar computation provides $\lim III/L_n$, and using the symmetry of $P_{a,b}$ one finds that

$$\deg(D_\lambda) = \frac{(-1)^n n}{a-b}\left(\left[P_{a,b}(t)^{n-1}\right]_{t^n} - \left[P_{a,b}(t)^{n-1}\right]_{t^{n-2}}\right) =$$

$$\frac{(-1)^n n}{a-b}\left(\left[P_{a,b}(t)^{n-1}\right]_{t^n} - \left[P_{b,a}(t)^{n-1}\right]_{t^n}\right).$$

If ∂ denotes the 'divided difference' operator $\partial Q(a,b) := (Q(a,b) - Q(b,a))/(a-b)$, then the last expression reads as

$$\deg(D_\lambda) = (-1)^n n \cdot \partial\Big[P_{a,b}(t)^{n-1}\Big]_{t^n}.$$

By Corollary 5.8, this is valid for $a = b$ as well. By a computation (using the multinomial formula):

$$\deg(D_\lambda) = n!(a+b-1)\sum_{i=1}^{[n/2]} \frac{A^{i-1}C^{i-1}(-B)^{n-2i}}{i!(i-1)!(n-2i)!}. \tag{7.4}$$

This has the following factorization:

$$\deg(D_\lambda) = n(n-1)(a+b-1)(-B)^{n-2[n/2]} \cdot \prod_{i=1}^{[n/2]-1} (B^2 + \xi_i AC),$$

where

$$\prod_{i=1}^{[n/2]-1} (t+\xi_i) = t^{[n/2]-1} + \frac{(n-2)!}{2!1!(n-4)!} t^{[n/2]-2} + \frac{(n-2)!}{3!2!(n-6)!} t^{[n/2]-3} + \cdots.$$

E.g., for small values of n one has the following expressions for $\deg(D_\lambda)$:

$n = 3$ $\quad 6(a+b-1)(-B)$
$n = 4$ $\quad 12(a+b-1)(B^2 + AC)$
$n = 5$ $\quad 20(a+b-1)(-B)(B^2 + 3AC)$
$n = 6$ $\quad 30(a+b-1)(B^2 + \xi AC)(B^2 + \bar{\xi} AC)$, where $\xi^2 + 6\xi + 2 = 0$.

For $n = 3$, $\deg(D_\lambda) = 6(a+b-1)(2ab - a - b + 1)$ is in fact the universal polynomial $F_{GL(3)}$ (which already was determined in Example 5.10 case A_2, and from which one can determine the degrees of all the irreducible $GL(3)$-representations by Corollary 5.8), compare also with [22][7.18].

If we write $u := a-1$ and $v := b-1$, then all the coefficients of the polynomials $-B, A, C$, expressed in the new variables (u,v), are non-negative. Using (7.4), this remains true for $\deg(D_\lambda)$ as well; a fact compatible with [7].

For $a=1$ the formula simplifies drastically (since $A=0$): $\deg(D_\lambda) = n(n-1)b^{n-1}$. Symmetrically for $b=1$. For $a=b=1$ we recover the well known formula $\deg(D_\lambda) = n(n-1)$ for the degree of the discriminant of the adjoint representation. (A matrix is in the discriminant of the adjoint representation if it has multiple eigenvalues i.e. the equation of the discriminant is the discriminant of the characteristic polynomial.)

Example 7.3. The case of $\lambda = aL_1 + bL_2$ with $a > b \geq 1$ and $n \geq 3$. Since $T_\lambda = \{L_i - L_1,\ L_i - L_2\ :\ 3 \leq i \leq n\} \cup \{L_2 - L_1\}$, one has the following expression for $f_\lambda^{(n)} \cdot \sigma_1$:

(7.5)
$$\sum_{i\neq j}(aL_i+bL_j)\frac{aL_i+bL_j+L_j-L_i}{L_i-L_j}\prod_{k\neq i,j}\frac{aL_i+bL_j+L_k-L_i}{L_i-L_k}\cdot\frac{aL_i+bL_j+L_k-L_j}{L_j-L_k}.$$

We compute $f_\lambda^{(n)}$ as in 7.2. For this we write the expression E from (7.5) as $I + II + III$, where I, II and III are defined similarly as in 7.2. The computations of the first two contributions are similar as in 7.2: $\lim I/L_n = 0$ while

$$\lim II/L_n = \Big[(at-t+b)^{n-1}(at+b-1)^{n-1}\Big]_{t^n}.$$

On the other hand, III is slightly more complicated. For this write $t_1 := aL_i + bL_n - L_i$, $t_2 := aL_i + bL_n - L_n$, and $\Pi := \prod_{n\neq k\neq i} L_i - L_k$. Moreover, we define the relation $R_1 \equiv R_2$ whenever $\lim R_1/L_n = \lim R_2/L_n$. Then (for t_1 using the usual 'trick' as above by separating terms from the ideal $\mathcal{I}_{sym}$ generated by $\sigma_i^{(n-1)}$'s) we have that $III =$

$$\begin{aligned}
&= \prod_{k\leq n-1}\frac{1}{L_n-L_k}\cdot\sum_{i\leq n-1}\frac{\prod_{k\leq n-1}(t_1+L_k)\cdot(aL_i+bL_n+L_n-L_i)\cdot\prod_{n\neq k\neq i}(t_2+L_k)}{\Pi}\\
&\equiv \sum_{i\leq n-1}\frac{t_1^{n-1}}{\Pi\cdot L_n^{n-1}}\cdot(aL_i+bL_n+L_n-L_i)\cdot\prod_{n\neq k\neq i}(t_2+L_k)\\
&= \sum_{i\leq n-1}\frac{t_1^{n-1}}{\Pi\cdot L_n^{n-1}}\Big((b+1)L_n+(a-1)L_i\Big)\frac{t_2^{n-1}+\sigma_1^{(n-1)}t_2^{n-2}+\cdots}{t_2+L_i}\\
&= \sum_{i\leq n-1}\frac{t_1^{n-1}}{\Pi\cdot L_n^{n-1}}\Big((b+1)L_n+(a-1)L_i\Big)\cdot\\
&\qquad\cdot\Big(\frac{t_2^{n-1}-(-L_i)^{n-1}}{t_2+L_i}+\sigma_1^{(n-1)}\frac{t_2^{n-2}-(-L_i)^{n-2}}{t_2+L_i}+\cdots\Big)+
\end{aligned}$$

$$+\sum_{i\leq n-1}\frac{t_1^{n-1}}{\Pi\cdot L_n^{n-1}}\Big((b+1)L_n+(a-1)L_i\Big)\frac{(-L_i)^{n-1}+\sigma_1^{(n-1)}(-L_i)^{n-2}+\cdots}{t_2+L_i}$$

From the first sum we can again eliminate terms from $\mathcal{I}_{sym}$, while the second sum is a rational function in L_n, where the numerator and denominator both have degree n, hence this expression is $\equiv 0$. In particular,

$$III\equiv -\sum_{i\leq n-1}\frac{t_1^{n-1}}{\Pi\cdot L_n^{n-1}}\Big((b+1)L_n+(a-1)L_i\Big)\cdot\frac{t_2^{n-1}-(-L_i)^{n-1}}{t_2+L_i}.$$

Therefore, $\deg(D_\lambda)$ equals

$$\frac{n}{a+b}\left[(at-t+b)^{n-1}(at+b-1)^{n-1}-(bt+a-1)^{n-1}(bt+t+a-1)\cdot\frac{(bt-t+a)^{n-1}-(-1)^{n-1}}{bt-t+a+1}\right]_{t^n}.$$

For $n=3$ we recover the polynomial $F_{GL(3)}$ (cf. 5.8) – computed by 7.2 as well.

Also, for arbitrary n but for $b=1$ the formula becomes simpler:

$$\deg(D_\lambda)=\frac{n}{(a+1)^2}\Big((n-1)a^{n+1}-(n+1)a^{n-1}+2(-1)^{n-1}\Big).$$

This expression coincides with Tevelev's formula from [21], see also [22, 7.14]. (Notice that in [22, 7.2C], Tevelev provides a formula for arbitrary a and b, which is rather different from ours.)

Example 7.4. Specializations of $\lambda=aL_1+bL_2$.
Assume that λ_s is a specialization of λ (i.e. λ_s is either aL_1 – obtained by specialization $b=0$, or it is aL_1+aL_2 – by taking $b=a$). In this case, by Corollary 5.8 one has

$$\deg(D_{\lambda_s})=\frac{\varepsilon(\lambda)|W_\lambda|}{\varepsilon(\lambda_s)|W_{\lambda_s}|}\cdot\big(\deg(D_\lambda)_{|\text{specialized}}\big).$$

In the first case, $\deg(D_{aL_1+bL_2})|_{b=0}$ is $(-1)^n n(n-1)(a-1)^{n-1}$, and the correction factor is $(-1)^n(n-1)$, hence we recover Boole's formula

$$\deg(D_{aL_1})=n(a-1)^{n-1}.$$

(Evidently, this can be easily deduced by a direct limit computations – as above – as well.)

In the second case, $\deg(D_{aL_1+bL_2})|_{b=a}$ should be divided by -2. We invite the reader to verify that this gives

$$\deg(D_{aL_1+aL_2})=\frac{n}{2a}\left[(t-1)(at+a-1)^{n-1}\cdot\frac{(at-t+a)^{n-1}-(-1)^{n-1}}{at-t+a+1}\right]_{t^n}.$$

For $a=1$ we recover the well known formula of Holme [13] for the degree of the dual variety of the Grassmannian $Gr_2(\mathbb{C}^n)$:

$$\deg(D_{L_1+L_2}) = \frac{n}{2}\cdot\frac{1-(-1)^{n-1}}{2}.$$

Example 7.5. The case of $\lambda = L_1+L_2+L_3$, $n\geq 4$. This case was studied by Lascoux [18]. Using K-theory he gave an algorithm to calculate the degree of the dual of the Grassmannian $Gr_3(\mathbb{C}^n)$ and calculated many examples. In Proposition 7.6 we give a closed formula for the degree.

We write $\binom{n}{k}$ for the set of subsets of $\{1,2,\dots,n\}$ with k elements. For any $S\in\binom{n}{k}$ set $L_S := \sum_{i\in S} L_i$. Since $T_\lambda = \{L_i - L_j : i>3,\ j\leq 3\}$, one has

$$f_\lambda^{(n)}\cdot\sigma_1 = \sum_{S\in\binom{n}{3}} L_S \prod_{\substack{i\notin S\\ j\in S}} \frac{L_S+L_i-L_j}{L_j-L_i}.$$

Similarly as in the previous examples, we separate L_n, and we write the above expression as the sum $I+II$ of two terms, I corresponds to the subsets S with $n\notin S$, while II to the others. It is easy to see that $\lim_{L_n\to\infty} I/L_n = 0$. In order to analyze the second main contribution, it is convenient to introduce the following expression (in variables $L_1,\dots,L_n$ and a new free variable t):

$$R^{(n)}(t) :=$$

$$\sum_{\mathcal{J}=\{j_1,j_2\}\in\binom{n}{2}} (t+L_{\mathcal{J}})(t-L_{j_1})(t-L_{j_2}) \prod_{i\notin\mathcal{J}} \frac{(t+L_{j_1}+L_i)(t+L_{j_2}+L_i)(L_{\mathcal{J}}+L_i)}{(L_{j_1}-L_i)(L_{j_2}-L_i)}.$$

This is a homogeneous expression in variables (L,t) of degree $n+1$, hence it can be written as

$$R^{(n)}(t) = \sum_{k=0}^{n+1} P_k^{(n)} t^{n+1-k},$$

where $P_k^{(n)}$ is a symmetric polynomial in variables L. The point is that

$$II = \frac{R^{(n-1)}(L_n)}{\prod_{i\leq n-1} L_n - L_i}, \text{ hence } \lim II/L_n = P_0^{(n-1)}.$$

Therefore,

$$f_\lambda^{(n)} = P_0^{(n-1)} \text{ and } \deg(D_\lambda) = nP_0^{(n-1)}/3.$$

Next, we concentrate on the leading coefficient $P_0^{(n)}$ of $R^{(n)}(t)$. For two polynomials R_1 and R_2 of degree $n+1$, we write $R_1\equiv R_2$ if their leading coefficients are the same. Set $t_r := t+L_{j_r}$ for $r=1,2$; and for each k write (over the field $\mathbb{C}(L)$)

$$t_r^k = Q_{r,k}(t_r)(t_r+L_{j_1})(t_r+L_{j_2}) + A_{r,k}t_r + B_{r,k}$$

for some polynomial $Q_{r,k}$ of degree $k-2$ and constants $A_{r,k}$ and $B_{r,k}$. Then

$$\prod_{i\notin\mathcal{J}}(t+L_{j_r}+L_i)=\frac{t_r^n}{(t_r-L_{j_1})(t_r-L_{j_2})}+\sum_{k=1}^{n}\sigma_k Q_{r,n-k}+\sum_{k=1}^{n}\sigma_k\frac{A_{r,n-k}t_r+B_{r,n-k}}{(t_r-L_{j_1})(t_r-L_{j_2})}.$$

In $P_0^{(n)}$ the first sum has no contribution since it is in the ideal generated by the (non-constant) symmetric polynomials, the second sum has no contribution either, since its limit is zero when $t\to\infty$. Therefore, $R^{(n)}(t)$ is $\equiv$ with

$$\sum_{\mathcal{J}\in\binom{n}{2}}(t+L_{\mathcal{J}})(t-L_{j_1})(t-L_{j_2})\cdot\frac{(t+L_{j_1})^n}{(t+2L_{j_1})(t+L_{\mathcal{J}})}\cdot\frac{(t+L_{j_2})^n}{(t+2L_{j_2})(t+L_{\mathcal{J}})}\cdot\prod_{i\notin\mathcal{J}}\frac{L_{\mathcal{J}}+L_i}{(L_{j_1}-L_i)(L_{j_2}-L_i)},$$

which equals

$$\sum_{\mathcal{J}\in\binom{n}{2}}\frac{1}{t+L_{\mathcal{J}}}\cdot\frac{(t-L_{j_1})(t+L_{j_1})^n}{t+2L_{j_1}}\cdot\frac{(t-L_{j_2})(t+L_{j_2})^n}{t+2L_{j_2}}\cdot\prod_{i\notin\mathcal{J}}\frac{L_{\mathcal{J}}+L_i}{(L_{j_1}-L_i)(L_{j_2}-L_i)}.$$

Let us define the—binomial-like—coefficients $\left\{ n \atop k \right\}$ by the expansion (near $t=\infty$):

$$\frac{(t-1)(t+1)^n}{t+2}=\sum_{k\le n}\left\{ n \atop k \right\}t^k, \tag{7.6}$$

Using

$$\frac{1}{t+1}=\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t^3}-\cdots,$$

it provides the expansions

$$\frac{(t-L_{j_r})(t+L_{j_r})^n}{t+2L_{j_r}}=\sum_{k\le n}\left\{ n \atop k \right\}t^k L_{j_r}^{n-k},\quad\text{and}\quad\frac{1}{t+L_{\mathcal{J}}}=\frac{1}{t}-\frac{L_{\mathcal{J}}}{t^2}+\frac{L_{\mathcal{J}}^2}{t^3}-\cdots.$$

Therefore,

$$P_0^{(n)}=\sum_{\substack{k\le n,\,l\le n\\ k+l\ge n+2}}(-1)^{k+l+n}\left\{ n \atop k \right\}\left\{ n \atop l \right\}\mathcal{L}^{(n)}_{n-k,n-l}, \tag{7.7}$$

where

$$\mathcal{L}^{(n)}_{n-k,n-l}:=\frac{1}{2}\cdot\sum_{\mathcal{J}=\{j_1,j_2\}\in\binom{n}{2}}L_{\mathcal{J}}^{k+l-n-2}(L_{j_1}^{n-k}L_{j_2}^{n-l}+L_{j_1}^{n-l}L_{j_2}^{n-k})\cdot\prod_{i\notin\mathcal{J}}\frac{L_{\mathcal{J}}+L_i}{(L_{j_1}-L_i)(L_{j_2}-L_i)}.$$

(Here we already symmetrized $\mathcal{L}$ in order to be able to apply in its computation the machinery of symmetric polynomials. Using this index-notation for $\mathcal{L}^{(n)}_{n-k,n-l}$ has the advantage that in this way this expression is independent of n, as we will see later.) Next, we determine the constants $\left\{ n \atop k \right\}$ and $\mathcal{L}^{(n)}_{n-k,n-l}$. The index-set of the sum of (7.7) says that we only need these constants for $k\ge 2$ and $l\ge 2$.

The constants $\left\{{n \atop k}\right\}$. The identity

$$\frac{(t+1)^n(t-1)}{t+2} = (t+1)^n + 3\sum_{i\geq 1}(-1)^i(t+1)^{n-i}$$

provides

$$\left\{{n \atop k}\right\} = \binom{n}{k} + 3\sum_{i\geq 1}(-1)^i\binom{n-i}{k} \quad \text{for any } 0 \leq k \leq n. \tag{7.8}$$

Notice that these constants satisfy 'Pascal's triangle rule': $\left\{{n \atop k}\right\} = \left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$. This law together with the 'initial values' $\left\{{n \atop 0}\right\} = 1$ for n even and $= -2$ for n odd, and with $\left\{{n \atop n}\right\} = 1$ determines completely all $\left\{{n \atop k}\right\}$ for $0 \leq k \leq n$. E.g., the first values are:

5	-2	-1	-2	1	2	1
4	1	-2	0	1	1	
3	-2	0	0	1		
2	1	-1	1			
1	-2	1				
0	1					
n/k	0	1	2	3	4	5

The constants $\mathcal{L}^{(n)}_{n-k,n-l}$. For $a \geq 0$, $b \geq 0$, $n \geq 3$ and $a+b = n-2$ we set

$$X^{(n)}_{a,b} := \sum_{\mathcal{J}=\{j_1,j_2\}\in\binom{n}{2}} (L^a_{j_1}L^b_{j_2} + L^b_{j_1}L^a_{j_2}) \cdot \prod_{i\notin\mathcal{J}} \frac{L_{\mathcal{J}} + L_i}{(L_{j_1}-L_i)(L_{j_2}-L_i)}.$$

Then $X^{(n)}_{a,b} = X^{(n)}_{b,a}$. By the above homogeneity argument $X^{(n)}_{a,b}$ is a constant. For $a > 0$ and $b > 0$ we separate L_n and substitute $L_n = 0$, and we get

$$X^{(n)}_{a,b} = X^{(n-1)}_{a-1,b} + X^{(n-1)}_{a,b-1}. \tag{7.9}$$

Therefore, $X^{(n)}_{a,b}$ can be determined from this 'Pascal rule' and the values $X^{(n)}_{n-1,0}$ (for $n \geq 3$). Next we compute these numbers. We write $X^{(n)}_{n-2,0}$ as $I + II + III$, where I corresponds to $\mathcal{J} \in \binom{n-1}{2}$, and

$$II = \sum_{j\leq n-1} L^{n-2}_j \prod_{j\neq i\neq n} \frac{L_n + L_j + L_i}{(L_n - L_i)(L_j - L_i)},$$

$$III = \sum_{j\leq n-1} L^{n-2}_n \prod_{j\neq i\neq n} \frac{L_n + L_j + L_i}{(L_n - L_i)(L_j - L_i)}.$$

Clearly, $X^{(n)}_{n-2,0} = \lim_{L_n \to 0}(I + II + III)$. It is easy to see that $\lim I = 0$ and $\lim II = 1$. Moreover, using (7.2), we get

$$III = \frac{L_n^{n-2}}{\prod_{i \le n-1}(L_n - L_i)} \cdot \Big(-\frac{1}{2} R^{(n-1)}_{2L_1}(L_n) + \frac{3}{2} L_n \sum_{j \le n-1} \prod_{j \ne i \ne n} \frac{L_j + L_i}{L_j - L_i} \Big),$$

hence

$$X^{(n)}_{n-2,0} = 1 - \frac{n-1}{2} + \frac{3}{4}\Big(1 + (-1)^n\Big). \tag{7.10}$$

Now, we return back to $\mathcal{L}^{(n)}_{n-k,n-l}$. The binomial formula for $(L_{j_1} + L_{j_2})^{k+l-n-2}$ and (7.9) gives

$$2\mathcal{L}^{(n)}_{n-k,n-l} = \sum_{i=0}^{k+l-n-2} \binom{k+l-n-2}{i} X^{(n)}_{l-2-i,n-l+i} = X^{(k+l-2)}_{l-2,k-2}.$$

Let $\rangle^{(n)}_{a,b}$ be defined (for $a, b \ge 0$ and $a + b = n - 2$) by the Pascal rule and initial values $\rangle^{(n)}_{n-2,0} = X^{(n)}_{n-2,0}$ and $\rangle^{(n)}_{0,n-2} = 0$. Symmetrically, define $\langle^{(n)}_{a,b} = \rangle^{(n)}_{b,a}$, hence $\rangle^{(n)}_{a,b} + \langle^{(n)}_{a,b} = X^{(n)}_{a,b}$. It is really surprising that X is another incarnation of the constants $\left\{ {n \atop k} \right\}$ (for $k \ge 1$). Indeed, comparing the initial values of $\rangle^{(n)}_{a,b}$ and $\left\{ {n \atop k} \right\}$ we get that

$$\rangle^{(k+l-2)}_{l-2,k-2} = \left\{ {k+l-5 \atop k-1} \right\} \quad (l \ge 2,\ k \ge 2).$$

Therefore, we proved the following fact.

Proposition 7.6. *For any $n \ge 3$ and $2 \le k \le n$ consider the contants defined by (7.8), or by (7.6). Consider the weight $\lambda = L_1 + L_2 + L_3$ of $GL(3)$ which provides the Plücker embedding of $Gr_3(\mathbb{C}^n)$. Then the degree of the dual variety of $Gr_3(\mathbb{C}^n)$ is*

$$\deg(D_\lambda) = \frac{n}{3} \cdot \sum_{\substack{k \le n-1, \\ 4 \le l \le n-1, \\ k+l \ge n+1}} (-1)^{k+l+n-1} \left\{ {n-1 \atop k} \right\} \left\{ {n-1 \atop l} \right\} \left\{ {k+l-5 \atop k-1} \right\}.$$

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