# COHOMOLOGY OF A FLAG VARIETY AS A YANGIAN BETHE ALGEBRA

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Abstract. We interpret the equivariant cohomology algebra $H^*_{GL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})$ of the cotangent bundle of a partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing chains of subspaces $0=F_0\subset F_1\subset\cdots\subset F_N=\mathbb{C}^n$, $\dim F_i/F_{i-1}=\lambda_i$, as the Yangian Bethe algebra $\mathcal{B}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ of the $\mathfrak{gl}_N$-weight subspace $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$ of a $Y(\mathfrak{gl}_N)$-module $\frac{1}{D}\mathcal{V}^-$. Under this identification the dynamical connection of [TV1] turns into the quantum connection of [BMO] if $\mathcal{F}_{\boldsymbol\lambda}$ is the full flag variety. For an arbitrary $\boldsymbol\lambda$, we conjecture a description of the small quantum equivariant cohomology algebra of the cotangent bundle $T^*\mathcal{F}_{\boldsymbol\lambda}$ as the Yangian Bethe algebra $\mathcal{B}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$.

## 1. Introduction

A Bethe algebra of a quantum integrable model is a commutative algebra of linear operators (Hamiltonians) acting on the space of states of the model. An interesting problem is to describe the Bethe algebra as the algebra of functions on a suitable scheme. Such a description can be considered as an instance of the geometric Langlands correspondence, see [MTV2, MTV3]. The $\mathfrak{gl}_N$ $XXX$ model is an example of a quantum integrable model. The Bethe algebra $\mathcal{B}^q$ of the $XXX$ model is a commutative subalgebra of the Yangian $Y(\mathfrak{gl}_N)$. The algebra $\mathcal{B}^q$ depends on the parameters $q=(q_1,\dots,q_N)\in\mathbb{C}^N$. Having a $Y(\mathfrak{gl}_N)$-module $M$, one obtains the commutative subalgebra $\mathcal{B}^q(M)\subset\mathrm{End}(M)$ as the image of $\mathcal{B}^q$. The geometric interpretation of the algebra $\mathcal{B}^q(M)$ as the algebra of functions on a scheme leads to interesting objects, see for example, [MTV4].

In this paper we consider an extension of the Yangian $Y(\mathfrak{gl}_N)$, denoted $\widetilde{Y}(\mathfrak{gl}_N)$, which is a subalgebra of $Y(\mathfrak{gl}_N)\otimes\mathbb{C}[h]$, and we work with the corresponding Bethe subalgebra $\widetilde{\mathcal{B}}^q\subset\widetilde{Y}(\mathfrak{gl}_N)$.

One of the most interesting Yangian modules is the vector space $\mathcal{V}=V\otimes\mathbb{C}[z_1,\dots,z_n,h]$ of $V$-valued polynomials in $z_1,\dots,z_n$, where $V=(\mathbb{C}^N)^{\otimes n}$ is the tensor power of the standard vector representation of $\mathfrak{gl}_N$. We introduce on $\mathcal{V}$ two Yangian actions, called

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$\phi^{\pm}$, and two actions of the symmetric group $S_n$, called $S_n^{\pm}$-actions. The Yangian action $\phi^+$ commutes with the $S_n^+$-action. Hence, the subspace $\mathcal{V}^+ \subset \mathcal{V}$ of $S_n^+$-invariants is a Yangian module. Similarly, the Yangian action $\phi^-$ commutes with the $S_n^-$-action. Hence, the subspace $\mathcal{V}^- \subset \mathcal{V}$ of $S_n^-$-skew-invariants is a Yangian module. The Yangian module structure on $\mathcal{V}^-$ induces a Yangian module structure on $\frac{1}{D}\mathcal{V}^- = \{\frac{1}{D}f \mid f \in \mathcal{V}^-\}$, where $D = \prod_{1\leqslant i<j\leqslant n}(z_i - z_j + h)$. The Bethe algebra $\widetilde{\mathcal{B}}^q$ preserves the $\mathfrak{gl}_N$-weight decompositions $\mathcal{V}^+ = \bigoplus_{\boldsymbol{\lambda}} \mathcal{V}^+_{\boldsymbol{\lambda}}$ and $\frac{1}{D}\mathcal{V}^- = \bigoplus_{\boldsymbol{\lambda}} \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$.

In this paper we study the limit of the algebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ as $q$ tends to infinity so that $q_{i+1}/q_i \to 0$ for all $i = 1, \dots, N-1$, and $q_N = 1$. We show that in this limit each of the Bethe algebras $\widetilde{\mathcal{B}}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ can be identified with the algebra of the equivariant cohomology $H_{\boldsymbol{\lambda}} := H^*_{GL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$ of the cotangent bundle of the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \ldots \subset F_N = \mathbb{C}^n, \qquad \dim F_i/F_{i-1} = \lambda_i\,.$$

More precisely, we construct an algebra isomorphism $\mu^+_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \widetilde{\mathcal{B}}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$ and a vector space isomorphism $\nu^+_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \mathcal{V}^+_{\boldsymbol{\lambda}}$, which identify the $\widetilde{\mathcal{B}}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$-module $\mathcal{V}^+_{\boldsymbol{\lambda}}$ with the action of the algebra $H_{\boldsymbol{\lambda}}$ on itself by multiplication operators, see Theorem 5.4. Similarly, we construct an algebra isomorphism $\mu^-_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ and a vector space isomorphism $\nu^+_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, which identify the $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ with the action of the algebra $H_{\boldsymbol{\lambda}}$ on itself by multiplication operators, see Theorem 5.6.

In Section 6, using the discrete Wronskian determinant we introduce an algebra $\mathcal{H}^q_{\boldsymbol{\lambda}}$ and construct vector space isomorphisms $\mu^{q+}_{\boldsymbol{\lambda}} : \mathcal{H}^q_{\boldsymbol{\lambda}} \to \mathcal{V}^+_{\boldsymbol{\lambda}}$, $\mu^{q-}_{\boldsymbol{\lambda}} : \mathcal{H}^q_{\boldsymbol{\lambda}} \to \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ and algebra isomorphisms $\nu^{q+}_{\boldsymbol{\lambda}} : \mathcal{H}^q_{\boldsymbol{\lambda}} \to \widetilde{\mathcal{B}}(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\nu^{q-}_{\boldsymbol{\lambda}} : \mathcal{H}^q_{\boldsymbol{\lambda}} \to \widetilde{\mathcal{B}}(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$. The isomorphisms $\mu^{q+}_{\boldsymbol{\lambda}}$ and $\nu^{q+}_{\boldsymbol{\lambda}}$ identify the $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$-module $\mathcal{V}^+_{\boldsymbol{\lambda}}$ and the regular representation of $\mathcal{H}^q_{\boldsymbol{\lambda}}$, see Theorem 6.3. Similarly, the isomorphisms $\mu^{q-}_{\boldsymbol{\lambda}}$ and $\nu^{q-}_{\boldsymbol{\lambda}}$ identify the $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ and the regular representation of $\mathcal{H}^q_{\boldsymbol{\lambda}}$, see Theorem 6.5. In particular, the algebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$ and $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ are isomorphic, and the $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$-module $\mathcal{V}^+_{\boldsymbol{\lambda}}$ is isomorphic to the $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$.

Under the isomorphisms $\nu^+_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \mathcal{V}^+$ and $\nu^-_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \frac{1}{D}\mathcal{V}^-$ the subalgebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$ and $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ induce respectively commutative subalgebras $\widetilde{\mathcal{B}}^{q+}_{\boldsymbol{\lambda}}$ and $\widetilde{\mathcal{B}}^{q-}_{\boldsymbol{\lambda}}$ of $\mathrm{End}(H_{\boldsymbol{\lambda}})$. In Section 8 we describe the subalgebra $\widetilde{\mathcal{B}}^{q+}_{\boldsymbol{\lambda}}$ for the special case $N = n$ and $\boldsymbol{\lambda} = (1, \dots, 1)$. Namely, we describe the preimages under $\nu^{\pm}_{\boldsymbol{\lambda}}$ of the dynamical Hamiltonians $\phi^{\pm}(X^q_i)$, $i = 1, \dots, n$, which are generating elements of the Bethe algebras $\widetilde{\mathcal{B}}(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$. We give a formula for the corresponding elements of $\widetilde{\mathcal{B}}^{q\pm}_{\boldsymbol{\lambda}}$ in terms of appropriate actions of the degenerate affine Hecke algebra on $H_{\boldsymbol{\lambda}}$, see Theorem 8.5.

One may expect that for arbitrary $N, n$ and $\boldsymbol{\lambda}$ the subalgebras $\widetilde{\mathcal{B}}^{q+}$ and $\widetilde{\mathcal{B}}^{q-}$ are related to the algebra of quantum multiplication on $H_{\boldsymbol{\lambda}}$. In [BMO], the quantum multiplication on $H_{\boldsymbol{\lambda}}$ was calculated for $N = n$ and $\boldsymbol{\lambda} = (1, \dots, 1)$. Our Theorem 8.5 and Section 8.4 show that for $N = n$ and $\boldsymbol{\lambda} = (1, \dots, 1)$ the algebra $\widetilde{\mathcal{B}}^{q-}$ is the algebra of quantum multiplication. We conjecture a similar statement for arbitrary $\boldsymbol{\lambda}$ in Section 7. Namely, we introduce a deformed multiplication on $H_{\boldsymbol{\lambda}}$ for any $\boldsymbol{\lambda}$ in terms of representation theory and conjecture that the deformed multiplication is the quantum multiplication. The quantum multiplication on $H_{\boldsymbol\lambda}$ for arbitrary $\boldsymbol\lambda$ is not known to us yet.

In more general terms, our identification of the quantum multiplication on $H_{\boldsymbol\lambda}$ for $N = n$ and $\boldsymbol\lambda = (1, \dots , 1)$ and the Yangian Bethe algebra $\widetilde{\mathcal B}^{q-}$ indicates a relation between quantum integrable chain models and quantum cohomology algebras. That relation was discussed in physics literature in recent works by Nekrasov and Shatashvili, see for example [NS].

The identification of the quantum multiplication on $H_{\boldsymbol\lambda}$ and the Yangian Bethe algebra $\widetilde{\mathcal B}^{q-}$ gives a relation between interesting objects in both theories. For example, the eigenvectors of the Bethe algebra is the main object of the $XXX$ model and finding eigenvectors is the subject of the highly developed Bethe ansatz theory. Under the above identification the eigenvectors correspond to idempotents of the quantum multiplication and the idempotents are an important object in the theory of Frobenius manifolds, see [D]. According to the above identification we can find the idempotents of the quantum cohomology algebra by the $XXX$ Bethe ansatz, see an example in Section 8.4.

The above identification brings another interesting relation. Namely, Theorem 8.5 shows that the isomorphism $\nu^-_{\boldsymbol\lambda}$ identifies the trigonometric dynamical connection of [TV1] with the quantum connection of [BMO]. It is known that the flat sections of the trigonometric dynamical connection are given by multidimensional hypergeometric integrals, see [TV1, MV, SV], cf. [TV2]. These hypergeometric integrals provide flat sections of the quantum connection of [BMO] by Theorem 8.5. This presentation of flat sections of the quantum connection as hypergeometric integrals is in the spirit of mirror symmetry, see Candelas et al. [COGP], Givental [G1, G2], and [BCK, BCKS, GKLO, I, JK], see also an example in Section 8.4.

The results of this paper are parallel to the results of paper [RSTV], where we consider the Bethe subalgebra of $U(\mathfrak{gl}_N[t])$ instead of the Bethe algebra of $\widetilde Y(\mathfrak{gl}_N)$.

In Section 2 we discuss the spaces $\mathcal V^+$ and $\frac1D\mathcal V^-$. In Section 3 we define Bethe algebras. In Section 4 we introduce and study Yangian actions on $\mathcal V^+$ and $\frac1D\mathcal V^-$. In Section 5 we describe the relations of the Bethe algebras $\widetilde{\mathcal B}^\infty(\mathcal V^+_{\boldsymbol\lambda})$, $\widetilde{\mathcal B}^\infty(\frac1D\mathcal V^-_{\boldsymbol\lambda})$ and the equivariant cohomology of partial flag varieties. In particular we identify the Shapovalov pairing on $\mathcal V$ with the Poincare pairing on $H_{\boldsymbol\lambda}$, see Propositions 5.2 and 5.3. In Section 6 we describe the Bethe algebras $\widetilde{\mathcal B}^q(\mathcal V^+_{\boldsymbol\lambda})$, $\widetilde{\mathcal B}^q(\frac1D\mathcal V^-_{\boldsymbol\lambda})$ and the relevant deformation $\mathcal H^q_{\boldsymbol\lambda}$ of $H_{\boldsymbol\lambda}$. In Section 7 we introduce the deformed multiplication on $H_{\boldsymbol\lambda}$ and formulate the conjecture that the deformed multiplication on $H_{\boldsymbol\lambda}$ is the quantum multiplication. In Section 8 we prove the conjecture for $N = n$ and $\boldsymbol\lambda = (1, \dots , 1)$.

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## 2. Spaces $\mathcal V^+$ and $\frac1D\mathcal V^-$

2.1. **Lie algebra $\mathfrak{gl}_N$.** Let $e_{i,j}$, $i, j = 1, \dots , N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - \delta_{i,l}e_{k,j}$. We denote by $\mathfrak h \subset \mathfrak{gl}_N$ the subalgebra generated by $e_{i,i}$, $i = 1, \dots , N$. For a Lie algebra $\mathfrak g$, we denote by $U(\mathfrak g)$ the universal enveloping algebra of $\mathfrak g$.

A vector $v$ of a $\mathfrak{gl}_N$-module $M$ has weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{i,i} v = \lambda_i v$ for $i = 1, \dots, N$. We denote by $M_{\boldsymbol{\lambda}} \subset M$ the weight subspace of weight $\boldsymbol{\lambda}$.

Let $\mathbb{C}^N$ be the standard vector representation of $\mathfrak{gl}_N$ with basis $v_1, \dots, v_N$ such that $e_{i,j} v_k = \delta_{j,k} v_i$ for all $i, j, k$. A tensor power $V = (\mathbb{C}^N)^{\otimes n}$ of the vector representation has a basis given by the vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$, where $i_j \in \{1, \dots, N\}$. Every such sequence $(i_1, \dots, i_n)$ defines a decomposition $I = (I_1, \dots, I_N)$ of $\{1, \dots, n\}$ into disjoint subsets $I_1, \dots, I_N$, where $I_j = \{k \mid i_k = j\}$. We denote the basis vector $v_{i_1} \otimes \dots \otimes v_{i_n}$ by $v_I$.

Let
$$V = \bigoplus_{\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0},\, |\boldsymbol{\lambda}| = n} V_{\boldsymbol{\lambda}}$$
be the weight decomposition. Denote $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all indices $I$ with $|I_j| = \lambda_j$, $j = 1, \dots N$. The vectors $\{v_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}\}$ form a basis of $V_{\boldsymbol{\lambda}}$. The dimension of $V_{\boldsymbol{\lambda}}$ equals the multinomial coefficient $d_{\boldsymbol{\lambda}} := \frac{n!}{\lambda_1! \dots \lambda_N!}$.

Let $\mathcal{S}$ be the bilinear form on $V$ such that the basis $\{v_I\}$ is orthonormal. We call $\mathcal{S}$ the Shapovalov form.

2.2. **$S_n$-actions.** Define an action of the symmetric group $S_n$ on $\mathcal{I}_{\boldsymbol{\lambda}}$. Let $I = (I_1, \dots, I_N) \in \mathcal{I}_{\boldsymbol{\lambda}}$, where $I_j = \{i_1, \dots, i_{\lambda_j}\} \subset \{1, \dots, n\}$. For $\sigma \in S_n$, define $\sigma(I) = (\sigma(I_1), \dots, \sigma(I_N))$, where $\sigma(I_j) = \{\sigma(i_1), \dots, \sigma(i_{\lambda_j})\}$.

Let $P^{(i,j)}$ be the permutation of the $i$-th and $j$-th factors of $V = (\mathbb{C}^N)^{\otimes n}$. Define two $S_n$-actions on $V$-valued functions of $z_1, \dots, z_n, h$, called $S_n^{\pm}$-actions. For the $S_n^+$-action, the $i$-th elementary transposition $s_i \in S_n$ acts by the formula:

$$s_i : f(z_1, \dots, z_n, h) \mapsto \frac{(z_i - z_{i+1})\, P^{(i,i+1)} - h}{z_i - z_{i+1}}\, f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h) + {} + \frac{h}{z_i - z_{i+1}}\, f(z_1, \dots, z_i, z_{i+1}, \dots, z_n, h)\,. \tag{2.1}$$

For the $S_n^-$-action, the $i$-th elementary transposition $s_i \in S_n$ acts by the formula:

$$s_i : f(z_1, \dots, z_n, h) \mapsto \frac{(z_i - z_{i+1})\, P^{(i,i+1)} + h}{z_i - z_{i+1}}\, f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h) - {} - \frac{h}{z_i - z_{i+1}}\, f(z_1, \dots, z_i, z_{i+1}, \dots, z_n, h)\,. \tag{2.2}$$

**Lemma 2.1.** *The $S_n^{\pm}$-actions* (2.1) *and* (2.2) *are well-defined.* □

Define operators $\tilde{s}_1^{\pm}, \dots, \tilde{s}_{n-1}^{\pm}$ acting on $V$-valued functions of $z_1, \dots, z_n, h$ by

$$\tilde{s}_i^{\pm} f(z_1, \dots, z_n, h) = \frac{(z_i - z_{i+1})\, P^{(i,i+1)} \mp h}{z_i - z_{i+1} \mp h}\, f(z_1, \dots, z_{i+1}, z_i, \dots, z_n, h)\,. \tag{2.3}$$

**Lemma 2.2.** *The assignment $s_i \mapsto \tilde{s}_i^+$, $i = 1, \dots, n-1$, defines an action of $S_n$.* □

**Lemma 2.3.** *The assignment $s_i \mapsto \tilde{s}_i^-$, $i = 1, \dots, n-1$, defines an action of $S_n$.* □

**Lemma 2.4.** *The function $f(z_1, \dots, z_n, h)$ is invariant with respect to the $S_n^+$-action (resp. $S_n^-$-action) if and only if $f = \tilde{s}_i^+ f$ (resp. $f = \tilde{s}_i^- f$ ) for every $i = 1, \dots, n-1$.* □

Define operators $\hat s_1,\dots,\hat s_{n-1}$ acting on functions of $z_1,\dots,z_n,h$ by

$$\hat s_i^{\pm} f(z_1,\dots,z_n,h) \,=\, \frac{z_i-z_{i+1}\pm h}{z_i-z_{i+1}}\, f(z_1,\dots,z_{i+1},z_i,\dots,z_n,h)\,\mp \mp\, \frac{h}{z_i-z_{i+1}}\, f(z_1,\dots,z_i,z_{i+1},\dots,z_n,h) \tag{2.4}$$

**Lemma 2.5.** *The assignment* $s_i\mapsto \hat s_i^+$, $i=1,\dots,n-1$, *defines an action of* $S_n$. □

**Lemma 2.6.** *The assignment* $s_i\mapsto \hat s_i^-$, $i=1,\dots,n-1$, *defines an action of* $S_n$. □

Let $f(z_1,\dots,z_n,h)$ be a $V$-valued function with coordinates $\{f_I(z_1,\dots,z_n,h)\}$:

$$f(z_1,\dots,z_n,h)\,=\,\sum_I f_I(z_1,\dots,z_n,h)\,v_I\,.$$

**Lemma 2.7.** *The function* $f(z_1,\dots,z_n,h)$ *is invariant with respect to the* $S_n^+$*-action (resp.* $S_n^-$*-action) if and only if* $f_{\sigma(I)}=\hat\sigma^- f_I$ *(resp.* $f_{\sigma(I)}=\hat\sigma^+ f_I$*) for any* $I\in\mathcal I_{\boldsymbol\lambda}$ *and any* $\sigma\in S_n$. □

**Lemma 2.8.** *The function* $f(z_1,\dots,z_n,h)$ *is skew-invariant with respect to the* $S_n^+$*-action (resp.* $S_n^-$*-action) if and only if* $f_{\sigma(I)}=(-1)^\sigma\hat\sigma^+ f_I$ *(resp.* $f_{\sigma(I)}=(-1)^\sigma\hat\sigma^- f_I$*) for any* $I\in\mathcal I_{\boldsymbol\lambda}$ *and any* $\sigma\in S_n$. □

Let
$$D\,=\prod_{1\leqslant i<j\leqslant n}(z_i-z_j+h)\,,\qquad \check D\,=\prod_{1\leqslant i<j\leqslant n}(z_j-z_i+h)\,.$$

**Lemma 2.9.** *The function* $f$ *is skew-invariant with respect to the* $S_n^+$*-action (resp.* $S_n^-$*-action) if and only if the function* $\frac1{\check D}f$ *(resp.* $\frac1D f$*) is invariant with respect to the* $S_n^+$*-action (resp.* $S_n^-$*-action).* □

2.3. **Spaces** $\mathcal V^+$**,** $\frac1{\check D}\mathcal V^=$ **and** $\frac1D\mathcal V^-$**.** Let $\mathcal V^+=(V\otimes\mathbb C[z_1,\dots,z_n,h])^{S_n^+}$ be the space of $S_n^+$-invariant $V$-valued polynomials. Let $\frac1{\check D}\mathcal V^=$ be the space of $S_n^+$-invariant $V$-valued functions of the form $\frac1{\check D}f$, $f\in V\otimes\mathbb C[z_1,\dots,z_n,h]$. Let $\frac1D\mathcal V^-$ be the space of $S_n^-$-invariant $V$-valued functions of the form $\frac1D f$, $f\in V\otimes\mathbb C[z_1,\dots,z_n,h]$. All $\mathcal V^+$, $\frac1{\check D}\mathcal V^=$ and $\frac1D\mathcal V^-$ are $\mathbb C[z_1,\dots,z_n]^{S_n}\otimes\mathbb C[h]$-modules.

The $S^\pm$-actions on $V\otimes\mathbb C[z_1,\dots,z_n,h]$ commute with the $\mathfrak{gl}_N$-action. Hence $\mathcal V^+$, $\frac1{\check D}\mathcal V^=$ and $\frac1D\mathcal V^-$ are $\mathfrak{gl}_N$-modules. Consider the $\mathfrak{gl}_N$-weight decompositions

$$\mathcal V^+=\bigoplus_{\substack{\boldsymbol\lambda\in\mathbb Z^N_{\geqslant0}\\|\boldsymbol\lambda|=n}}\mathcal V^+_{\boldsymbol\lambda},\qquad \tfrac1{\check D}\mathcal V^==\bigoplus_{\substack{\boldsymbol\lambda\in\mathbb Z^N_{\geqslant0}\\|\boldsymbol\lambda|=n}}\tfrac1{\check D}\mathcal V^=_{\boldsymbol\lambda},\qquad \tfrac1D\mathcal V^-=\bigoplus_{\substack{\boldsymbol\lambda\in\mathbb Z^N_{\geqslant0}\\|\boldsymbol\lambda|=n}}\tfrac1D\mathcal V^-_{\boldsymbol\lambda}.$$

Define a partial ordering on subsets of $\{1,\dots,n\}$ of the same cardinality. Say $A\leqslant B$ if $A=\{a_1<\dots<a_i\}$, $B=\{b_1<\dots<b_i\}$, and $a_j\leqslant b_j$ for all $j=1,\dots,i$.

Fix $\boldsymbol\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb Z_{\geqslant0}$, $|\boldsymbol\lambda|=n$. For $I,J\in\mathcal I_{\boldsymbol\lambda}$, say $I<J$ if $I_k=J_k$ for $k=1,\dots,l-1$ with some $l$, and $I_l<J_l$. Define $I^{\min},I^{\max}\in\mathcal I_{\boldsymbol\lambda}$ by

$$I^{\min}\,=\,\bigl(\{1,\dots,\lambda_1\},\{\lambda_1+1,\dots,\lambda_1+\lambda_2\},\,\dots\,,\{n-\lambda_N+1,\dots,n\}\bigr)\,, \tag{2.5}$$
$$I^{\max}\,=\,\bigl(\{n-\lambda_1+1,\dots,n\},\{n-\lambda_1-\lambda_2+1,\dots,n-\lambda_1\},\,\dots\,,\{1,\dots,\lambda_N\}\bigr)\,.$$

Clearly, $I^{\min} \leqslant I \leqslant I^{\max}$ for any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$.

Given $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, denote

$$Q(\boldsymbol{z}_I) \,=\, \prod_{1\leqslant a<b\leqslant N}\,\prod_{i\in I_a}\prod_{j\in I_b}(z_i-z_j+h)\,. \tag{2.6}$$

For any function $f(z_1,\dots,z_n,h)$, set $\check f(z_1,\dots,z_n,h)=f(z_n,\dots,z_1,h)$. Let

$$Q_{\boldsymbol{\lambda}}(z_1,\dots,z_n,h) \,=\, Q(\boldsymbol{z}_{I^{\min}})\,, \tag{2.7}$$

so that $\check Q_{\boldsymbol{\lambda}}(z_1,\dots,z_n,h)=Q(\boldsymbol{z}_{I^{\max}})$.

Define

$$\vartheta^+_{\boldsymbol{\lambda}} : \mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h] \,\to\, \mathcal{V}^+_{\boldsymbol{\lambda}}\,, \tag{2.8}$$

$$\vartheta^+_{\boldsymbol{\lambda}}(f) \,=\, \frac{1}{\lambda_1!\dots\lambda_N!}\sum_{\sigma\in S_n}\hat\sigma^+\check f\; v_{\sigma(I^{\max})}\,,$$

$$\vartheta^=_{\boldsymbol{\lambda}} : \mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h] \,\to\, \tfrac{1}{\check D}\mathcal{V}^=_{\boldsymbol{\lambda}}\,, \tag{2.9}$$

$$\vartheta^=_{\boldsymbol{\lambda}}(f) \,=\, \frac{1}{\lambda_1!\dots\lambda_N!}\sum_{\sigma\in S_n}\hat\sigma^+\bigl(\tfrac{1}{\check Q}\check f\bigr)\; v_{\sigma(I^{\min})}\,,$$

and

$$\vartheta^-_{\boldsymbol{\lambda}} : \mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h] \,\to\, \tfrac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}\,, \tag{2.10}$$

$$\vartheta^-_{\boldsymbol{\lambda}}(f) \,=\, \frac{1}{\lambda_1!\dots\lambda_N!}\sum_{\sigma\in S_n}\hat\sigma^-\bigl(\tfrac{1}{Q}f\bigr)\; v_{\sigma(I^{\min})}\,.$$

**Lemma 2.10.** *The maps $\vartheta^+_{\boldsymbol{\lambda}}$, $\vartheta^=_{\boldsymbol{\lambda}}$, and $\vartheta^-_{\boldsymbol{\lambda}}$ are isomorphisms of the $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$-module $\mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]$ with the $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$-modules $\mathcal{V}^+_{\boldsymbol{\lambda}}$, $\frac{1}{\check D}\mathcal{V}^=_{\boldsymbol{\lambda}}$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, respectively.*

*Proof.* The operators $\hat\sigma^+$, $\sigma\in S_n$, preserve $\mathbb{C}[z_1,\dots,z_n,h]$. Thus $\vartheta^+_{\boldsymbol{\lambda}}(f)=\sum_I f_I v_I$, and $f_I\in\mathbb{C}[z_1,\dots,z_n,h]$ for any $I$. Moreover, $f_{\sigma(I)}=\hat\sigma^+ f_I$ for any $\sigma\in S_n$, and $f_{I^{\max}}=\check f$. Hence, the map $\vartheta^+_{\boldsymbol{\lambda}}$ is well-defined by Lemma 2.7. If $g\in\mathcal{V}^+_{\boldsymbol{\lambda}}$, $g=\sum_I g_I v_I$, then $g=\vartheta^+_{\boldsymbol{\lambda}}(\check g_{I^{\max}})$ by Lemma 2.7, so $\vartheta^+_{\boldsymbol{\lambda}}$ is an isomorphism.

The proof for the cases of $\vartheta^=_{\boldsymbol{\lambda}}$ and $\vartheta^-_{\boldsymbol{\lambda}}$ are similar because the operators $\hat\sigma^+$, $\sigma\in S_n$, preserve the localized algebra $\mathbb{C}[z_1,\dots,z_n,h,\check D^{-1}]$, and the operators $\hat\sigma^-$, $\sigma\in S_n$, preserve the localized algebra $\mathbb{C}[z_1,\dots,z_n,h,D^{-1}]$. □

It is known in Schubert calculus that $\mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}$ is a free $\mathbb{C}[z_1,\dots,z_n]$-module of rank $d_{\boldsymbol{\lambda}}$. This yields that for any $\boldsymbol{\lambda}$, the subspaces $\mathcal{V}^+_{\boldsymbol{\lambda}}$, $\frac{1}{\check D}\mathcal{V}^=_{\boldsymbol{\lambda}}$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ are free $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$-modules of rank $d_{\boldsymbol{\lambda}}$.

The Shapovalov form $\mathcal{S}$ on $V$ induces a pairing

$$(V\otimes\mathbb{C}[z_1,\dots,z_n,h])\otimes(V\otimes\mathbb{C}[z_1,\dots,z_n,h])\to\mathbb{C}[z_1,\dots,z_n,h] \tag{2.11}$$

denoted by the same letter.

**Lemma 2.11.** *The pairing* (2.11) *induces a pairing*

$$\mathcal{V}^+ \otimes \tfrac{1}{D}\mathcal{V}^- \to \mathbb{C}[z_1,\dots,z_n]^{S_n} \otimes \mathbb{C}[h]\,. \tag{2.12}$$

*Proof.* For any $f \in \mathcal{V}^+$ and $g \in \frac{1}{D}\mathcal{V}^-$, the scalar function $\mathcal{S}(f,g)$ is a symmetric function of $z_1,\dots,z_n$ with possible poles only at the hyperplanes $z_i - z_j + h = 0$ for $1 \leqslant i < j \leqslant n$. Since this arrangement of hyperplanes is not invariant under the permutations of $z_1,\dots,z_n$, the poles are absent. □

**Lemma 2.12.** *For any* $\boldsymbol{\lambda}$, *the pairing* $\mathcal{V}^+_{\boldsymbol{\lambda}} \otimes \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} \to \mathbb{C}[z_1,\dots,z_n]^{S_n} \otimes \mathbb{C}[h]$, *induced by pairing* (2.12), *is surjective.*

Lemma 2.12 is proved in Section 4.1

The pairing (2.11) also induces a pairing

$$\tfrac{1}{\check{D}}\mathcal{V}^= \otimes \tfrac{1}{D}\mathcal{V}^- \to Z^{-1}\,\mathbb{C}[z_1,\dots,z_n]^{S_n} \otimes \mathbb{C}[h]\,, \tag{2.13}$$

where

$$Z = D\check{D} = \prod_{\substack{i,j=1\\ i\neq j}}^{n} (z_i - z_j + h)\,. \tag{2.14}$$

The pairing (2.12) and (2.13) will also be denoted by $\mathcal{S}$.

**Lemma 2.13.** *The pairings* (2.12) *and* (2.13) *are nondegenerate. That is, if* $\mathcal{S}(f,g) = 0$ *for a given* $f \in \mathcal{V}^+$ *and every* $g \in \frac{1}{D}\mathcal{V}^-$, *then* $f = 0$. *And similarly, for all other possible cases.* □

2.4. **Vectors** $\xi^{\pm}_I$**.** Fix $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_N) \in \mathbb{Z}_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$. Given $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, denote

$$R(\boldsymbol{z}_I) = \prod_{1\leqslant a<b\leqslant N} \prod_{i\in I_a} \prod_{j\in I_b} (z_i - z_j)\,. \tag{2.15}$$

**Proposition 2.14.** *There exist unique elements* $\{\xi^+_I \in V \otimes \mathbb{C}[z_1,\dots,z_n,h,\check{D}^{-1}] \mid I \in \mathcal{I}_{\boldsymbol{\lambda}}\}$ *such that* $\xi^+_{I^{\min}} = v_{I^{\min}}$ *and*

$$\xi^+_{s_i(I)} = \tilde{s}^+_i\, \xi^+_I \tag{2.16}$$

*for every* $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ *and* $i = 1,\dots,n-1$. *Moreover,*

$$\xi^+_I = \sum_{J\leqslant I} X^+_{I,J}\, v_J\,, \tag{2.17}$$

*where* $X^+_{I,J} \in \mathbb{C}[z_1,\dots,z_n,h,\check{D}^{-1}]$, $X^+_{I,I} \neq 0$, *and*

$$X^+_{I^{\max},I^{\max}} = \frac{R(\boldsymbol{z}_{I^{\max}})}{Q(\boldsymbol{z}_{I^{\max}})}\,. \tag{2.18}$$

For example, if $N = n = 2$ and $\boldsymbol{\lambda} = (1,1)$, then $\xi^+_{(1,2)}(z_1,z_2,h) = v_{(1,2)}$ and

$$\xi^+_{(2,1)}(z_1,z_2,h) = \frac{z_2 - z_1}{z_2 - z_1 + h}\, v_{(2,1)} + \frac{h}{z_2 - z_1 + h}\, v_{(1,2)}\,.$$

**Proposition 2.15.** *There exist unique elements* $\{\xi^-_I \in V\otimes\mathbb{C}[z_1,\dots,z_n,h,D^{-1}] \mid I\in\mathcal{I}_{\boldsymbol\lambda}\}$ *such that* $\xi^-_{I^{\max}} = v_{I^{\max}}$ *and*

$$\xi^-_{s_i(I)} \,=\, \tilde s_i^{\,-}\,\xi^-_I \tag{2.19}$$

*for every* $I\in\mathcal{I}_{\boldsymbol\lambda}$ *and* $i=1,\dots,n-1$. *Moreover,*

$$\xi^-_I \,=\, \sum_{J\geqslant I} X^-_{I,J}\, v_J\,, \tag{2.20}$$

*where* $X^-_{I,J}\in\mathbb{C}[z_1,\dots,z_n,h,D^{-1}]$, $X^-_{I,I}\neq 0$, *and*

$$X^-_{I^{\min},I^{\min}} \,=\, \frac{R(\boldsymbol z_{I^{\min}})}{Q(\boldsymbol z_{I^{\min}})}\,. \tag{2.21}$$

*Proof of Propositions 2.14, 2.15.* Proposition 2.14 follows from formula (2.3) and Lemma 2.2. Notice that $\tilde s_i^{\,+} v_{I^{\min}} = v_{I^{\min}}$ if and only if $s_i(I^{\min}) = I^{\min}$. Similarly, Proposition 2.15 follows from formula (2.3) and Lemma 2.3. □

For example, if $N=n=2$ and $\boldsymbol\lambda=(1,1)$, then $\xi^-_{(2,1)}(z_1,z_2,h) = v_{(2,1)}$ and

$$\xi^-_{(1,2)}(z_1,z_2,h) \,=\, \frac{z_1-z_2}{z_1-z_2+h}\, v_{(1,2)} + \frac{h}{z_1-z_2+h}\, v_{(2,1)}\,.$$

Let $f_I(z_1,\dots,z_n,h)$, $I\in\mathcal{I}_{\boldsymbol\lambda}$, be a collection of scalar functions.

**Lemma 2.16.** *The V-valued function* $\sum_I f_I(z_1,\dots,z_n,h)\,\xi^+_I$ $\Big($*resp.* $\sum_I f_I(z_1,\dots,z_n,h)\,\xi^-_I\Big)$, *is invariant with respect to the* $S^+_n$*-action (resp.* $S^-_n$*-action) if and only if*

$$f_{\sigma(I)}(z_1,\dots,z_n,h) \,=\, f_I(z_{\sigma_1},\dots,z_{\sigma_n},h)$$

*for any* $I\in\mathcal{I}_{\boldsymbol\lambda}$ *and any* $\sigma\in S_n$. □

**Proposition 2.17.** *For any* $f\in\mathbb{C}[z_1,\dots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]$, *we have*

$$\vartheta^+_{\boldsymbol\lambda}(f) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol\lambda}} \frac{f(\boldsymbol z_I,h)\,Q(\boldsymbol z_I)}{R(\boldsymbol z_I)}\;\xi^+_I\,, \tag{2.22}$$

$$\vartheta^=_{\boldsymbol\lambda}(f) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol\lambda}} \frac{f(\boldsymbol z_I,h)}{R(\boldsymbol z_I)}\;\xi^+_I\,, \tag{2.23}$$

$$\vartheta^-_{\boldsymbol\lambda}(f) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol\lambda}} \frac{f(\boldsymbol z_I,h)}{R(\boldsymbol z_I)}\;\xi^-_I\,. \tag{2.24}$$

*Proof.* By formulae (2.17), (2.18), we have $\vartheta^+_{\boldsymbol\lambda}(f) = \sum_I c_I\,\xi^+_I$ for some coefficients $c_I$, and $c_{I^{\max}} = f(\boldsymbol z_{I^{\max}},h)\,Q(\boldsymbol z_{I^{\max}})/R(\boldsymbol z_{I^{\max}})$. Now formulae (2.22), (2.23) follow from Lemma 2.16.

Formula (2.24) follows similarly from formulae (2.20), (2.21), and Lemma 2.16. □

**Theorem 2.18.** *For* $I,J\in\mathcal{I}_{\boldsymbol\lambda}$, *we have*

$$\mathcal{S}(\xi^+_I,\xi^-_J) \,=\, \delta_{I,J}\,\frac{R(\boldsymbol z_I)}{Q(\boldsymbol z_I)}\,. \tag{2.25}$$

*Proof.* If $I = I^{\min}$ or $J = I^{\max}$, the claim follows from formulae (2.17) – (2.21). Then formula (2.25) extends to arbitrary $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$ by properties (2.16), (2.19). □

2.5. **Distinguished elements** $v^+_{\boldsymbol{\lambda}}$**,** $v^=_{\boldsymbol{\lambda}}$**, and** $v^-_{\boldsymbol{\lambda}}$**.** Let $v^+_{\boldsymbol{\lambda}} = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I$. Clearly, $v^+_{\boldsymbol{\lambda}} \in \mathcal{V}^+ \cap \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$. Moreover, $v^+_{\boldsymbol{\lambda}} = \vartheta^+_{\boldsymbol{\lambda}}(1)$, $v^+_{\boldsymbol{\lambda}} = \vartheta^=_{\boldsymbol{\lambda}}(Q_{\boldsymbol{\lambda}})$ and $v^+_{\boldsymbol{\lambda}} = \vartheta^-_{\boldsymbol{\lambda}}(Q_{\boldsymbol{\lambda}})$, where $Q_{\boldsymbol{\lambda}}$ is given by (2.7).

Denote $v^=_{\boldsymbol{\lambda}} = \vartheta^=_{\boldsymbol{\lambda}}(1) \in \frac{1}{D}\mathcal{V}^=_{\boldsymbol{\lambda}}$ and $v^-_{\boldsymbol{\lambda}} = \vartheta^-_{\boldsymbol{\lambda}}(1) \in \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$. By Proposition 2.17 we have

$$v^+_{\boldsymbol{\lambda}} = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{Q(\boldsymbol{z}_I)}{R(\boldsymbol{z}_I)}\,\xi^+_I = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{Q(\boldsymbol{z}_I)}{R(\boldsymbol{z}_I)}\,\xi^-_I\,, \tag{2.26}$$

$$v^=_{\boldsymbol{\lambda}} = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{1}{R(\boldsymbol{z}_I)}\,\xi^+_I\,, \qquad v^-_{\boldsymbol{\lambda}} = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{1}{R(\boldsymbol{z}_I)}\,\xi^-_I\,. \tag{2.27}$$

**Lemma 2.19.** *The vector* $v^+_{\boldsymbol{\lambda}}$ *belongs to the irreducible* $\mathfrak{gl}_N$*-submodule of* $V$ *of highest weight* $(n,0,\dots,0)$*, generated by the vector* $v_1\otimes\dots\otimes v_1$*.*

**Lemma 2.20** ([RTVZ])**.** *The functions* $v^-_{\boldsymbol{\lambda}}$ *and* $v^=_{\boldsymbol{\lambda}}$ *are singular vectors with respect to the pointwise* $\mathfrak{gl}_N$*-action on* $V$*.*

Notice that the functions $v^-_{\boldsymbol{\lambda}}$ and $v^=_{\boldsymbol{\lambda}}$ under certain conditions become quantized conformal blocks and satisfy *qKZ* equations with respect to $z_1,\dots,z_n$, see [RTVZ], cf. [V, RV, RSV].

## 3. Bethe subalgebras

3.1. **Yangian** $Y(\mathfrak{gl}_N)$**.** The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T^{\{s\}}_{i,j}$ for $i,j=1,\dots,N$, $s\in\mathbb{Z}_{>0}$, subject to relations

$$(u-v)\bigl[T_{i,j}(u),T_{k,l}(v)\bigr] = T_{k,j}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v)\,, \qquad i,j,k,l=1,\dots,N\,, \tag{3.1}$$

where
$$T_{i,j}(u) = \delta_{i,j} + \sum_{s=1}^{\infty} T^{\{s\}}_{i,j}\,u^{-s}\,.$$

Let $T(u) = \sum_{i,j=1}^N E_{i,j}\otimes T_{i,j}(u)$, where $E_{i,j}\in\mathrm{End}(\mathbb{C}^N)$ is the image of $e_{i,j}\in\mathfrak{gl}_N$. Relations (3.1) can be written as the equality of series with coefficients in $\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)\otimes Y(\mathfrak{gl}_N)$:

$$(u-v+P^{(1,2)})\,T^{(1)}(u)T^{(2)}(v) = T^{(2)}(v)T^{(1)}(u)\,(u-v+P^{(1,2)})\,, \tag{3.2}$$

where $P^{(1,2)}$ is the permutation of the $\mathbb{C}^N$ factors, $T^{(1)}(u) = \sum_{i,j=1}^N E_{i,j}\otimes 1\otimes T_{i,j}(u)$ and $T^{(2)}(u) = 1\otimes T(u)$.

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct $\Delta : Y(\mathfrak{gl}_N)\to Y(\mathfrak{gl}_N)\otimes Y(\mathfrak{gl}_N)$ given by $\Delta\bigl(T_{i,j}(u)\bigr) = \sum_{k=1}^N T_{k,j}(u)\otimes T_{i,k}(u)$ for $i,j=1,\dots,N$. The Yangian $Y(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a Hopf subalgebra, the embedding given by $e_{i,j}\mapsto T^{\{1\}}_{j,i}$.

Notice that $\bigl[T^{\{1\}}_{i,j}, T^{\{s\}}_{k,l}\bigr] = \delta_{i,l} T^{\{s\}}_{k,j} - \delta_{j,k} T^{\{s\}}_{i,l}$ for $i,j,k,l = 1,\dots,N$, $s \in \mathbb{Z}_{>0}$, which implies that the Yangian $Y(\mathfrak{gl}_N)$ is generated by the elements $T^{\{1\}}_{i,i+1}, T^{\{1\}}_{i+1,i}$, $i = 1,\dots,N-1$, and $T^{\{s\}}_{1,1}$, $s > 0$.

The assignment

$$\varpi : T^{\{s\}}_{i,j} \mapsto T^{\{s\}}_{j,i}, \qquad i,j = 1,\dots,N, \quad s > 0, \tag{3.3}$$

defines a Hopf anti-automorphism of the Yangian $Y(\mathfrak{gl}_N)$.

More information on the Yangian $Y(\mathfrak{gl}_N)$ can be found in [M]. Notice that the series $T_{i,j}(u)$ here corresponds to the series $T_{j,i}(u)$ in [M].

3.2. **Bethe algebra.** For $p = 1,\dots,N$, $\boldsymbol{i} = \{1 \leqslant i_1 < \dots < i_p \leqslant N\}$, $\boldsymbol{j} = \{1 \leqslant j_1 < \dots < j_p \leqslant N\}$, define

$$M_{\boldsymbol{i},\boldsymbol{j}}(u) = \sum_{\sigma \in S_p} (-1)^\sigma T_{i_1, j_{\sigma(1)}}(u) \dots T_{i_p, j_{\sigma(p)}}(u-p+1).$$

For $\boldsymbol{i} = \{1,\dots,N\}$, the series $M_{\boldsymbol{i},\boldsymbol{i}}(u)$ is called the quantum determinant and denoted by $\mathrm{qdet}\, T(u)$. Its coefficients generate the center of the Yangian $Y(\mathfrak{gl}_N)$.

We have

$$\Delta\bigl(M_{\boldsymbol{i},\boldsymbol{j}}(u)\bigr) = \sum_{\boldsymbol{k} = \{1 \leqslant k_1 < \dots < k_p \leqslant N\}} M_{\boldsymbol{k},\boldsymbol{j}}(u) \otimes M_{\boldsymbol{i},\boldsymbol{k}}(u), \tag{3.4}$$

see, for example, [NT, Proposition 1.11] or [MTV1, Lemma 4.3]. Notice that the Yangian coproduct used here is opposite to that used in [NT]. We also have

$$\varpi\bigl(M_{\boldsymbol{i},\boldsymbol{j}}(u)\bigr) = M_{\boldsymbol{j},\boldsymbol{i}}(u), \tag{3.5}$$

see, for example, [NT, Lemma 1.5].

For $q = (q_1,\dots,q_N) \in (\mathbb{C}^*)^N$ and $p = 1,\dots,N$, define

$$B^q_p(u) = \sum_{\boldsymbol{i} = \{1 \leqslant i_1 < \dots < i_p \leqslant N\}} q_{i_1} \dots q_{i_p} M_{\boldsymbol{i},\boldsymbol{i}}(u) = \sigma_p(q_1,\dots,q_N) + \sum_{s=1}^\infty B^q_{p,s} u^{-s}, \tag{3.6}$$

where $\sigma_p$ is the $p$-th elementary symmetric function and $B^q_{p,s} \in Y(\mathfrak{gl}_N)$. Let $\mathcal{B}^q \subset Y(\mathfrak{gl}_N)$ be the unital subalgebra generated by the elements $B^q_{p,s}$, $p = 1,\dots,N$, $s \geqslant 0$. It is easy to see that the subalgebra $\mathcal{B}^q$ does not change if all $q_1,\dots,q_N$ are multiplied simultaneously by the same number. The algebra $\mathcal{B}^q$ is called a *Bethe subalgebra* of $Y(\mathfrak{gl}_N)$.

**Theorem 3.1** ([KS])**.** *The subalgebra $\mathcal{B}^q$ is commutative and commutes with the subalgebra $U(\mathfrak{h}) \subset Y(\mathfrak{gl}_N)$.* □

The series $B^q_p(u)$ depends polynomially on $q_1,\dots,q_N$. Let $q$ tend to infinity so that $q_{i+1}/q_i \to 0$ for all $i = 1,\dots,N-1$, and $q_N = 1$. In this limit,

$$B^q_p(u) = q_1 \dots q_p \bigl(M_{\boldsymbol{i},\boldsymbol{i}}(u) + o(1)\bigr), \qquad \boldsymbol{i} = \{1,\dots,p\}. \tag{3.7}$$

Introduce the elements $B^{\infty}_{p,s}\in Y(\mathfrak{gl}_N)\,,\ p=1,\dots,N,\ s\in\mathbb{Z}_{>0}$, by the formula

$$M_{\boldsymbol{i},\boldsymbol{i}}(u)\,=\,1+\sum_{s=1}^{\infty}B^{\infty}_{p,s}\,u^{-s}\,,\qquad \boldsymbol{i}=\{1,\dots,p\}\,. \tag{3.8}$$

Let $\mathcal{B}^{\infty}\subset Y(\mathfrak{gl}_N)$ be the unital subalgebra generated by the elements $B^{\infty}_{p,s}\,,\ p=1,\dots,N$, $s>0$. The algebra $\mathcal{B}^{\infty}$ is called a *Gelfand-Zetlin subalgebra* of $Y(\mathfrak{gl}_N)$. It has been studied in [NT].

The subalgebra $\mathcal{B}^{\infty}$ is commutative by Theorem 3.1. Since $B^{\infty}_{p,1}=T^{\{1\}}_{1,1}+\ldots+T^{\{1\}}_{p,p}$ for $p=1,\dots,N$, the subalgebra $\mathcal{B}^{\infty}$ contains the subalgebra $U(\mathfrak{h})\subset Y(\mathfrak{gl}_N)$.

Assume that $q_1,\dots,q_N$ are distinct numbers. Define the elements $S^q_{i,s}\in\mathcal{B}^q$ for $i=1,\dots,N,\ s\in\mathbb{Z}_{>0}\,$, by the rule

$$S^q_{i,s}\,=\,\sum_{p=1}^{N}(-1)^{p-1}\,B^q_{p,s}\,q_i^{N-p-1}\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{q_i-q_j}\,, \tag{3.9}$$

so that

$$\prod_{i=1}^{N}\frac{1}{x-q_i}\,\sum_{p=1}^{N}\sum_{s=1}^{\infty}(-1)^p\,B^q_p(u)\,x^{N-p}\,=\,1-\sum_{i=1}^{N}\sum_{s=1}^{\infty}\frac{q_i}{x-q_i}\,S^q_{i,s}\,u^{-s}\,.$$

In particular,

$$S^q_{i,1}\,=\,T^{\{1\}}_{i,i}\,,\qquad\quad S^q_{i,2}\,=\,T^{\{2\}}_{i,i}-\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{q_i-q_j}\,\bigl(T^{\{1\}}_{i,i}T^{\{1\}}_{j,j}-T^{\{1\}}_{i,j}T^{\{1\}}_{j,i}+T^{\{1\}}_{j,j}\bigr)\,. \tag{3.10}$$

**Lemma 3.2.** *For distinct numbers $q_1,\dots,q_N$, the subalgebra $\mathcal{B}^q$ contains the subalgebra $U(\mathfrak{h})\subset Y(\mathfrak{gl}_N)$.* □

**Lemma 3.3.** *Let $q_{i+1}/q_i\to 0$ for all $i=1,\dots,N-1$. Then $S^q_{i,s}\to B^{\infty}_{i,s}-B^{\infty}_{i-1,s}$ for $i=1,\dots,N$, where $B^{\infty}_{0,s}=0$.*

*Proof.* Write formula (3.9) as

$$S^q_{i,s}\,=\,\sum_{p=1}^{N}(-1)^{p-i}\,h^{s-1}\,B^q_{p,s}\,\frac{q_i^{i-p-1}}{q_1\ldots q_{i-1}}\,\prod_{j=1}^{i-1}\frac{1}{1-q_i/q_j}\,\prod_{j=i+1}^{N}\frac{1}{1-q_j/q_i}\,.$$

Now the claim follows from formulae (3.7), (3.8). □

**Lemma 3.4.** *For any $X\in\mathcal{B}^q$, we have $\varpi(X)=X$, and for any $X\in\mathcal{B}^{\infty}$, we have $\varpi(X)=X$.*

*Proof.* By formulae (3.5)–(3.8), see also [MTV1, Proposition 4.11], for any $p=1,\dots,N$ and $s>0$, we have $\varpi(B^q_{p,s})=B^q_{p,s}$ and $\varpi(B^{\infty}_{p,s})=B^{\infty}_{p,s}$. Since the algebras $\mathcal{B}^q$ and $\mathcal{B}^{\infty}$ are commutative, the statement follows. □

As a subalgebra of $Y(\mathfrak{gl}_N)$, the Bethe algebra $\mathcal{B}^q$ acts on any $Y(\mathfrak{gl}_N)$-module $M$. Since $\mathcal{B}^q$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces $M_{\boldsymbol{\lambda}}$. If $L\subset M$ is a $\mathcal{B}^q$-invariant subspace, then the image of $\mathcal{B}^q$ in $\mathrm{End}(L)$ will be called the Bethe algebra of $L$ and denoted by $\mathcal{B}^q(L)$.

3.3. **Algebras $\widetilde{Y}(\mathfrak{gl}_N)$, $\widetilde{\mathcal{B}}^q$, $\widetilde{\mathcal{B}}^\infty$.** Let $\widetilde{Y}(\mathfrak{gl}_N)$ be the subalgebra of $Y(\mathfrak{gl}_N)\otimes\mathbb{C}[h]$ generated over $\mathbb{C}$ by $\mathbb{C}[h]$ and the elements $h^{s-1}T^{\{s\}}_{i,j}$ for $i,j=1,\dots,N$, $s>0$. Equivalently, the subalgebra $\widetilde{Y}(\mathfrak{gl}_N)$ is generated over $\mathbb{C}$ by $\mathbb{C}[h]$ and the elements $T^{\{1\}}_{i,i+1}, T^{\{1\}}_{i+1,i}$, $i=1,\dots,N-1$, and $h^{s-1}T^{\{s\}}_{1,1}$, $s>0$.

Let $\widetilde{\mathcal{B}}^q$ (resp. $\widetilde{\mathcal{B}}^\infty$) be the subalgebra of $Y(\mathfrak{gl}_N)\otimes\mathbb{C}[h]$ generated over $\mathbb{C}$ by $\mathbb{C}[h]$ and the elements $h^{s-1}B^q_{p,s}$ (resp. $h^{s-1}B^\infty_{p,s}$) for $p=1,\dots,N$, $s>0$. It is easy to see that $\widetilde{\mathcal{B}}^q$ and $\widetilde{\mathcal{B}}^\infty$ are subalgebras of $\widetilde{Y}(\mathfrak{gl}_N)$.

Introduce the series $A_1(u),\dots,A_N(u)$, $E_1(u),\dots,E_{N-1}(u)$, $F_1(u),\dots,F_{N-1}(u)$:

$$A_p(u) \,=\, M_{\boldsymbol{i},\boldsymbol{i}}(u/h) \,=\, 1+\sum_{s=1}^\infty h^s B^\infty_{p,s}\,u^{-s}\,, \tag{3.11}$$

$$E_p(u) \,=\, h^{-1}M_{\boldsymbol{j},\boldsymbol{i}}(u/h)\bigl(M_{\boldsymbol{i},\boldsymbol{i}}(u/h)\bigr)^{-1} \,=\, \sum_{s=1}^\infty h^{s-1}E_{p,s}\,u^{-s}\,, \tag{3.12}$$

$$F_p(u) \,=\, h^{-1}\bigl(M_{\boldsymbol{i},\boldsymbol{i}}(u/h)\bigr)^{-1}M_{\boldsymbol{i},\boldsymbol{j}}(u/h) \,=\, \sum_{s=1}^\infty h^{s-1}F_{p,s}\,u^{-s}\,,$$

where $\boldsymbol{i}=\{1,\dots,p\}$, $\boldsymbol{j}=\{1,\dots,p-1,p+1\}$. Observe that $E_{p,1}=T^{\{1\}}_{p+1,p}$, $F_{p,1}=T^{\{1\}}_{p,p+1}$ and $B^\infty_{1,s}=T^{\{s\}}_{1,1}$, so the coefficients of the series $E_p(u)$, $F_p(u)$ and $h^{-1}\bigl(A_p(u)-1\bigr)$ together with $\mathbb{C}[h]$ generate $\widetilde{Y}(\mathfrak{gl}_N)$.

In what follows we will describe actions of the algebra $\widetilde{Y}(\mathfrak{gl}_N)$ by using series (3.11), (3.12).

The quotient algebra $\widetilde{Y}(\mathfrak{gl}_N)/(h-1)\widetilde{Y}(\mathfrak{gl}_N)$ is canonically isomorphic to $Y(\mathfrak{gl}_N)$. Also, the quotient algebra $\widetilde{Y}(\mathfrak{gl}_N)\big/h\widetilde{Y}(\mathfrak{gl}_N)$ is naturally isomorphic to the universal enveloping algebra $U(\mathfrak{gl}_N[t])$, the element $h^{s-1}T^{\{s\}}_{i,j}$ projecting to $e_{j,i}\otimes t^{s-1}$.

3.4. **Dynamical Hamiltonians and dynamical connection.** Assume that $q_1,\dots,q_N$ are distinct numbers. Define the elements $X^q_1,\dots,X^q_N\in\widetilde{\mathcal{B}}^q$ by the rule

$$\begin{aligned} X^q_i \,&=\, h\,S^q_{i,2}-\frac{h}{2}\,T^{\{1\}}_{i,i}\bigl(T^{\{1\}}_{i,i}-1\bigr)+h\sum_{\substack{j=1\\ j\neq i}}^N \frac{q_j}{q_i-q_j}\,T^{\{1\}}_{i,i}\,T^{\{1\}}_{j,j} \\ &=\, hT^{\{2\}}_{i,i}-\frac{h}{2}\,T^{\{1\}}_{i,i}\bigl(T^{\{1\}}_{i,i}-1\bigr)+h\sum_{\substack{j=1\\ j\neq i}}^N \frac{q_j}{q_i-q_j}\,G_{i,j}\,, \end{aligned} \tag{3.13}$$

where $S^q_{i,2}$ is given by (3.10) and

$$G_{i,j} \,=\, T^{\{1\}}_{i,j}\,T^{\{1\}}_{j,i}-T^{\{1\}}_{j,j} \,=\, T^{\{1\}}_{j,i}\,T^{\{1\}}_{i,j}-T^{\{1\}}_{i,i}\,.$$

Here the second formula is obtained from the first one by using commutation relations of the elements $T^{\{1\}}_{k,l}$. Taking the limit $q_{i+1}/q_i\to 0$ for all $i=1,\dots,N-1$, we define the elements

$X_1^\infty,\ldots,X_N^\infty\in\widetilde{\mathcal{B}}^\infty$,

$$X_i^\infty = h B_{i,2}^\infty - h B_{i-1,2}^\infty - \frac{h}{2} T_{i,i}^{\{1\}}\bigl(T_{i,i}^{\{1\}}-1\bigr) - h T_{i,i}^{\{1\}}\bigl(T_{1,1}^{\{1\}}+\ldots+T_{i-1,i-1}^{\{1\}}\bigr) \tag{3.14}$$

$$= h T_{i,i}^{\{2\}} - \frac{h}{2} T_{i,i}^{\{1\}}\bigl(T_{i,i}^{\{1\}}-1\bigr) - h\,(G_{i,1}+\ldots+G_{i,i-1})\,.$$

cf. Lemma 3.3. We will call the elements $X_i^q$, $X_i^\infty$, $i=1,\ldots,N$, the *dynamical Hamiltonians*. Observe that

$$X_i^q = X_i^\infty + h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,G_{i,j} + h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,G_{i,j}\,. \tag{3.15}$$

Notice that the elements $T_{i,i}^{\{1\}}$ and $G_{i,j}$ belong to the subalgebra $U(\mathfrak{gl}_N)$ embedded in $\widetilde{Y}(\mathfrak{gl}_N)$,

$$T_{i,i}^{\{1\}} = e_{i,i}\,,\qquad G_{i,j} = e_{i,j}\,e_{j,i} - e_{i,i} = e_{j,i}\,e_{i,j} - e_{j,j}\,.$$

In addition, we define the elements $X_1^{q\pm},\ldots,X_N^{q\pm}\in\widetilde{\mathcal{B}}^q$,

$$X_i^{q+} = X_i^q - h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,e_{i,i}\,e_{j,j} - h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,e_{i,i}\,e_{j,j}\,, \tag{3.16}$$

$$X_i^{q-} = X_i^q + h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,e_{i,i} + h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,e_{j,j} \tag{3.17}$$

$$= X_i^\infty + h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,e_{i,j}\,e_{j,i} + h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,e_{j,i}\,e_{i,j}\,.$$

It is straightforward to verify that for any nonzero complex number $\kappa$, the formal differential operators

$$\nabla_i = \kappa\,q_i\frac{\partial}{\partial q_i} - X_i^q,\qquad i=1,\ldots,N, \tag{3.18}$$

pairwise commute and, hence, define a flat connection for any $\widetilde{Y}(\mathfrak{gl}_N)$-module.

**Lemma 3.5.** *The connections $\nabla^\pm$ defined by*

$$\nabla_i^+ = \kappa\,q_i\frac{\partial}{\partial q_i} - X_i^{q+},\qquad \nabla_i^- = \kappa\,q_i\frac{\partial}{\partial q_i} - X_i^{q-}, \tag{3.19}$$

*$i=1,\ldots,N$, are flat for any $\kappa$.*

*Proof.* Connections $\nabla^\pm$ are gauge equivalent to connection (3.18),

$$\nabla_i^+ = (\Theta^+)^{-1}\,\nabla_i\,\Theta^+,\qquad \Theta^+ = \prod_{1\leqslant i<j\leqslant n}(1-q_j/q_i)^{-h\,e_{i,i}e_{j,j}/\kappa}\,,$$

$$\nabla_i^- = (\Theta^-)^{-1}\,\nabla_i\,\Theta^-,\qquad \Theta^- = \prod_{1\leqslant i<j\leqslant n}(1-q_j/q_i)^{h\,e_{j,j}/\kappa}\,, \tag{3.20}$$

which yields the statement. □

Connection (3.18) was introduced in [TV1], see also [MTV1], and is called the *trigonometric dynamical connection*. The connection is defined over $\mathbb{C}^n$ with coordinates $q_1, \dots, q_N$, and has singularities at the union of the diagonals $q_i = q_j$. In the case of a tensor product of evaluation $\widetilde{Y}(\mathfrak{gl}_N)$-modules, the trigonometric dynamical connection commutes with the associated $qKZ$ difference connection, see [TV1]. Under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$ duality, the trigonometric dynamical connection and the associated $qKZ$ difference connection are respectively identified with the trigonometric $KZ$ connection and the dynamical difference connection, see [TV1].

Notice that the trigonometric dynamical connection in [TV1] differs from the connection here by a gauge transformation.

## 4. Yangian actions

4.1. $\widetilde{Y}(\mathfrak{gl}_N)$**-actions.** Let $\mathbb{C}[z_1, \dots, z_n, h]$ act on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ by multiplication on the second factor. Set

$$L^+(u) \,=\, (u - z_n + hP^{(0,n)}) \dots (u - z_1 + hP^{(0,1)}) \,, \tag{4.1}$$

$$L^-(u) \,=\, (u - z_1 + hP^{(0,1)}) \dots (u - z_n + hP^{(0,n)}) \,,$$

where the factors of $\mathbb{C}^N \otimes V$ are labeled by $0, 1, \dots, n$. Both $L^+(u)$ and $L^-(u)$ are polynomials in $u, z_1, \dots, z_n, h$ with values in $\mathrm{End}(\mathbb{C}^N \otimes V)$. We consider $L^\pm(u)$ as $N \times N$ matrices with $\mathrm{End}(V) \otimes \mathbb{C}[u, z_1, \dots, z_n, h]$-valued entries $L^\pm_{i,j}(u)$.

**Proposition 4.1.** *The assignments*

$$\phi^\pm\bigl(T_{i,j}(u/h)\bigr) \,=\, L^\pm_{i,j}(u) \prod_{a=1}^{n} (u - z_a)^{-1} \tag{4.2}$$

*define the actions $\phi^+$ and $\phi^-$ of the algebra $\widetilde{Y}(\mathfrak{gl}_N)$ on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$. Here the right-hand side of* (4.2) *is a series in $u^{-1}$ with coefficients in* $\mathrm{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n, h]$.

*Proof.* The claim follows from relations (3.2) and the Yang-Baxter equation

$$(u - v + hP^{(1,2)})\,(u + hP^{(1,3)})\,(v + hP^{(2,3)}) \,= \tag{4.3}$$

$$= \,(v + hP^{(2,3)})\,(u + hP^{(1,3)})\,(u - v + hP^{(1,2)})\,.$$ □

For both actions $\phi^\pm$, the subalgebra $U(\mathfrak{gl}_N) \subset \widetilde{Y}(\mathfrak{gl}_N)$ acts on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ in the standard way: any element $x \in \mathfrak{gl}_N$ acts as $x^{(1)} + \ldots + x^{(n)}$. The actions $\phi^\pm$ clearly commute with the $\mathbb{C}[z_1, \dots, z_n, h]$-action.

**Lemma 4.2.** *The Yangian actions $\phi^\pm$ are contravariantly related through the Shapovalov pairing* (2.11)*:*

$$\mathcal{S}\bigl(\phi^+(X)\, f, g\bigr) \,=\, \mathcal{S}\bigl(f, \phi^-(\varpi(X))\, g\bigr)\,,$$

*for any $X \in \widetilde{Y}(\mathfrak{gl}_N)$ and any $V$-valued functions $f, g$ of $z_1, \dots, z_n, h$.*

*Proof.* The claim follows from formulae (3.3), (4.1) and (4.2). □

**Corollary 4.3.** *For any* $X \in \widetilde{\mathcal{B}}^q$ *or* $X \in \widetilde{\mathcal{B}}^\infty$, *and any* $V$*-valued functions* $f, g$ *of* $z_1, \dots, z_n, h$, *we have*

$$\mathcal{S}\big(\phi^+(X)\, f, g\big) \,=\, \mathcal{S}\big(f, \phi^-(X)\, g\big)$$

*Proof.* The claim follows from Lemma 3.4. □

**Lemma 4.4.** *The Yangian action* $\phi^+$ *(resp.* $\phi^-$*) on* $V$*-valued functions of* $z_1, \dots, z_n, h$ *commutes with the* $S_n^+$*-action* (2.1) *(resp. the* $S_n^-$*-action* (2.2)*).*

*Proof.* The claim follows from the Yang-Baxter equation (4.3) and the fact that the actions $\phi^\pm$ commute with multiplication by functions of $z_1, \dots, z_n, h$. □

By Lemma 4.4, the action $\phi^+$ makes the spaces $\mathcal{V}^+$ and $\frac{1}{D}\mathcal{V}^=$ into a $\widetilde{Y}(\mathfrak{gl}_N)$-modules, and the action $\phi^-$ on $V \otimes \mathbb{C}[z_1, \dots, z_n, h]$ makes the spaces $\frac{1}{D}\mathcal{V}^-$ into a $\widetilde{Y}(\mathfrak{gl}_N)$-modules.

Let $\Pi \in \mathrm{End}(V)$ be the following linear map, $\Pi\,(g_1 \otimes \dots \otimes g_n) = g_n \otimes \dots \otimes g_1$ for any $g_1, \dots, g_n \in \mathbb{C}^N$. For any $V$-valued function $f(z_1, \dots, z_n, h)$ set

$$\widetilde{\Pi}\, f(z_1, \dots, z_n, h) \,=\, \Pi\big(f(z_n, \dots, z_1, h)\big)\,. \tag{4.4}$$

**Lemma 4.5.** *The map* (4.4) *induces an isomorphism* $\frac{1}{D}\mathcal{V}^= \to \frac{1}{D}\mathcal{V}^-$ *of* $\widetilde{Y}(\mathfrak{gl}_N)$*-modules. Moreover,* $\widetilde{\Pi}\, v^=_{\boldsymbol{\lambda}} = v^-_{\boldsymbol{\lambda}}$ *for any* $\boldsymbol{\lambda}$.

*Proof.* We have $\widetilde{\Pi}\, \tilde{s}^+_i = \tilde{s}^-_{n-i}\, \widetilde{\Pi}$ for every $i = 1, \dots, n-1$, where the operators $\tilde{s}^\pm_i$ are given by (2.3). Thus by Lemma 2.4, $\widetilde{\Pi}$ induces a vector space isomorphism $\frac{1}{D}\mathcal{V}^= \to \frac{1}{D}\mathcal{V}^-$. Since

$$\widetilde{\Pi}\, \phi^+(X) \,=\, \phi^-(X)\, \widetilde{\Pi} \tag{4.5}$$

for every $X \in \widetilde{Y}(\mathfrak{gl}_N)$, see (4.1), (4.2), $\widetilde{\Pi}$ induces an isomorphism of $\widetilde{Y}(\mathfrak{gl}_N)$-modules. The equality $\widetilde{\Pi}\, v^=_{\boldsymbol{\lambda}} = v^-_{\boldsymbol{\lambda}}$ follows from formulae (2.9), (2.10). □

**Proposition 4.6.** *The* $\widetilde{Y}(\mathfrak{gl}_N)$*-module* $\mathcal{V}^+$ *is generated by the vector* $v_1 \otimes \dots \otimes v_1$.

*Proof.* For every $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, the subspace $\mathcal{V}^+_{\boldsymbol{\lambda}}$ is generated by the $\widetilde{\mathcal{B}}^\infty$-action on $v^+_{\boldsymbol{\lambda}}$, see Theorem 5.4 in Section 5.3. Since

$$v^+_{\boldsymbol{\lambda}} \,=\, \phi^+\Big(\big(T^{\{1\}}_{2,1}\big)^{\lambda_1-\lambda_2} \dots \big(T^{\{1\}}_{N-1,1}\big)^{\lambda_{N-1}-\lambda_N} \big(T^{\{1\}}_{N,1}\big)^{\lambda_N}\Big)\, \frac{v_1 \otimes \dots \otimes v_1}{(\lambda_1-\lambda_2)! \dots (\lambda_{N-1}-\lambda_N)!\, \lambda_N!}\,,$$

the $\widetilde{Y}(\mathfrak{gl}_N)$-module $\mathcal{V}^+$ is generated by $v_1 \otimes \dots \otimes v_1$. □

Notice that the $\widetilde{Y}(\mathfrak{gl}_N)$-modules $\mathcal{V}^+$ is not isomorphic to $\frac{1}{D}\mathcal{V}^= \simeq \frac{1}{D}\mathcal{V}^-$ because for any homomorphism $\mathcal{V}^+ \to \frac{1}{D}\mathcal{V}^=$ of $\widetilde{Y}(\mathfrak{gl}_N)$-modules, the image of the vector $v_1 \otimes \dots \otimes v_1$ belongs to $(v_1 \otimes \dots \otimes v_1) \otimes \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ by the weight reasoning and generates a proper $\widetilde{Y}(\mathfrak{gl}_N)$-submodule of $\frac{1}{D}\mathcal{V}^=$.

Write $n = kN + l$ for $k, l \in \mathbb{Z}_{\geqslant 0}$, $l < N$. Set $\boldsymbol{\mu} = (k+1, \dots, k+1, k, \dots, k)$ with $l$ parts equal to $k+1$ and $N - l$ parts equal to $k$.

**Proposition 4.7.** *The* $\widetilde{Y}(\mathfrak{gl}_N)$*-module* $\frac{1}{D}\mathcal{V}^-$ *(resp.* $\frac{1}{D}\mathcal{V}^=$*) is generated by the function* $v^-_{\boldsymbol{\mu}}$ *(resp.* $v^=_{\boldsymbol{\mu}}$*).*

*Proof.* For every $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, the subspace $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ is generated by the $\widetilde{\mathcal{B}}^\infty$-action on $v^-_{\boldsymbol{\lambda}}$, see Theorem 5.6 in Section 5.3. By formula (4.11) below, we have

$$\phi^-(h^{\lambda_p-\lambda_{p+1}+1}E_{p,\lambda_p-\lambda_{p+1}+2})\,v^-_{\boldsymbol{\lambda}} \,=\, (-1)^{\lambda_p}\,v^-_{\boldsymbol{\lambda}-\alpha_p}\,, \tag{4.6}$$

if $\lambda_p - \lambda_{p+1} \geqslant -1$. Here $\alpha_p = (0, \dots, 0, 1, -1, 0, \dots, 0)$, with $p-1$ first zeros. Similarly, by formula (4.12) below, we have

$$\phi^-(h^{\lambda_{p+1}-\lambda_p+1}F_{p,\lambda_{p+1}-\lambda_p+2})\,v^-_{\boldsymbol{\lambda}} \,=\, (-1)^{\lambda_p-1}\,v^-_{\boldsymbol{\lambda}+\alpha_p}\,,$$

if $\lambda_{p+1} - \lambda_p \geqslant -1$. Thus every $v^-_{\boldsymbol{\lambda}}$ can be obtained from $v^-_{\boldsymbol{\mu}}$ by the $\widetilde{Y}(\mathfrak{gl}_N)$-action. This proves the statement for $\frac{1}{D}\mathcal{V}^-$. The proof for $\frac{1}{D}\mathcal{V}^=$ is similar. □

*Proof of Lemma 2.12.* Clearly, if the statement holds for $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, then it holds for $\boldsymbol{\lambda}^\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)})$ for any $\sigma \in S_N$. So it suffices to prove the statement assuming that $\lambda_1 \geqslant \dots \geqslant \lambda_N$. Then by (4.6), there is $X \in \widetilde{Y}(\mathfrak{gl}_N)$ such that $\phi^-(X)\,v^-_{\boldsymbol{\lambda}} = v_1 \otimes \dots \otimes v_1$. Thus by Lemma 4.2,

$$\mathcal{S}\bigl(\phi^+(X)\,v_1 \otimes \dots \otimes v_1\,, v^-_{\boldsymbol{\lambda}}\bigr) \,=\, \mathcal{S}\bigl(v_1 \otimes \dots \otimes v_1\,, \phi^-(X)\,v^-_{\boldsymbol{\lambda}}\bigr) \,=\, 1\,. \qquad □$$

**Proposition 4.8.** *The Bethe algebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+)$, $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=)$, and $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-)$ are isomorphic. Similarly, the Bethe algebras $\widetilde{\mathcal{B}}^\infty(\mathcal{V}^+)$, $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^=)$, and $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-)$ are isomorphic.*

*Proof.* The statement follows from Corollary 4.3 and Lemma 2.13. For the pairs $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=)$, $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-)$ and $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^=)$, $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-)$, the algebras are isomorphic by Lemma 4.5 as well. □

4.2. $Y(\mathfrak{gl}_N)$**-actions.** Let $\phi_0^\pm : Y(\mathfrak{gl}_N) \to \mathrm{End}(V)$ be the homomorphisms obtained from $\phi^\pm$ by taking $z_1 = \dots = z_n = 0$ and $h = 1$. They make the space $V$ into $Y(\mathfrak{gl}_N)$-modules, respectively denoted by $V^{\langle\pm\rangle}$. The following statement is well known.

**Proposition 4.9.** *The $Y(\mathfrak{gl}_N)$-modules $V^{\langle\pm\rangle}$ are irreducible and isomorphic. The isomorphism $V^{\langle+\rangle} \to V^{\langle-\rangle}$ is given by the map $x_1 \otimes \dots \otimes x_n \mapsto x_n \otimes \dots \otimes x_1$ for any $x_1, \dots, x_n \in \mathbb{C}^N$. The modules $V^{\langle\pm\rangle}$ are contravariantly dual,*

$$\mathcal{S}\bigl(\phi_0^+(X)\,f, g\bigr) \,=\, \mathcal{S}\bigl(f, \phi_0^-(\varpi(X))\,g\bigr)\,,$$

*for any $X \in Y(\mathfrak{gl}_N)$ and $f, g \in V$.* □

Let $\mathcal{J} \subset \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[h]$ be the ideal generated by the relations $h = 1$ and $\sigma_i(z_1, \dots, z_n) = 0$ for all $i = 1, \dots, n$, where $\sigma_i$ is the $i$-th elementary symmetric function. The quotient $\widehat{V} = V \otimes \mathbb{C}[z_1, \dots, z_n, h]\,/\,\mathcal{J}(V \otimes \mathbb{C}[z_1, \dots, z_n, h]) \simeq V \otimes (\mathbb{C}[z_1, \dots, z_n, h]\,/\,\mathcal{J})$ is a complex vector space of dimension $n!\,N^n$. The actions $\phi^\pm$ make it into $Y(\mathfrak{gl}_N)$-modules respectively denoted by $\widehat{V}^{\langle\pm\rangle}$.

**Lemma 4.10.** *The evaluation assignment*

$$f(z_1, \dots, z_n, h) \,\mapsto\, f(0, \dots, 0, 1) \tag{4.7}$$

*defines homomorphisms $\widehat{V}^{\langle+\rangle} \to V^{\langle+\rangle}$ and $\widehat{V}^{\langle-\rangle} \to V^{\langle-\rangle}$ of $Y(\mathfrak{gl}_N)$-modules.* □

Notice that the $Y(\mathfrak{gl}_N)$-modules $\widehat{V}^{\langle\pm\rangle}$ are respectively isomorphic to the $Y(\mathfrak{gl}_N)$-modules $V^{\langle\pm\rangle}\otimes(\mathbb{C}[z_1,\dots,z_n,h]/\mathcal{J})$ where the second factor $\mathbb{C}[z_1,\dots,z_n,h]/\mathcal{J}$ is the trivial $Y(\mathfrak{gl}_N)$-module with all the generators $T_{i,j}^{\{s\}}$ acting by zero. However, the isomorphisms are not given by the identity operator on $V\otimes(\mathbb{C}[z_1,\dots,z_n,h]/\mathcal{J})$.

The quotients $\mathcal{V}^+/\mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ are complex vector spaces of dimension $N^n$. The $\widetilde{Y}(\mathfrak{gl}_N)$-module structure on the respective spaces $\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^=$ $\frac{1}{D}\mathcal{V}^-$ make $\mathcal{V}^+/\mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ into $Y(\mathfrak{gl}_N)$-modules.

**Theorem 4.11.** *The assignment* (4.7) *defines isomorphisms*

$$\eta^+ : \mathcal{V}^+/\mathcal{J}\mathcal{V}^+ \to V^{\langle+\rangle}, \qquad \eta^= : \tfrac{1}{D}\mathcal{V}^=/\mathcal{J}\tfrac{1}{D}\mathcal{V}^= \to V^{\langle+\rangle}, \qquad \eta^- : \tfrac{1}{D}\mathcal{V}^-/\mathcal{J}\tfrac{1}{D}\mathcal{V}^- \to V^{\langle-\rangle}$$

*of $Y(\mathfrak{gl}_N)$-modules.*

*Proof.* Clearly, the maps $\eta^+$, $\eta^=$ and $\eta^-$ are nonzero homomorphisms of $Y(\mathfrak{gl}_N)$-modules, see Lemma 4.10. Since the $Y(\mathfrak{gl}_N)$-modules $V^{\langle\pm\rangle}$ are irreducible and all the spaces $\mathcal{V}^+/\mathcal{J}\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$, $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ and $V^{\langle\pm\rangle}$ have the same dimension, the maps $\eta^+$, $\eta^=$ and $\eta^-$ are isomorphisms. □

**Corollary 4.12.** *The Shapovalov pairing*

$$\mathcal{S} : \mathcal{V}^+/\mathcal{J}\mathcal{V}^+ \otimes \tfrac{1}{D}\mathcal{V}^-/\mathcal{J}\tfrac{1}{D}\mathcal{V}^- \to (\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h])/\mathcal{J} \simeq \mathbb{C} \tag{4.8}$$

*induced by pairing* (2.12) *is nondegenerate. The $Y(\mathfrak{gl}_N)$-modules $\mathcal{V}^+/\mathcal{J}\mathcal{V}^+$ and $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ are contravariantly dual:*

$$\mathcal{S}\bigl(\phi^+(X)\,f,g\bigr) \,=\, \mathcal{S}\bigl(f,\phi^-(\varpi(X))\,g\bigr)\,,$$

*for any $X\in Y(\mathfrak{gl}_N)$ and $f\in\mathcal{V}^+/\mathcal{J}\mathcal{V}^+$, $g\in\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$.* □

Alternatively, the nondegeneracy of pairing (4.8) follows from Lemma 2.11, Proposition 5.2 in Section 5 and the nondegeneracy of the Poincare pairing on $\mathcal{F}_{\boldsymbol{\lambda}}$.

**Corollary 4.13.** *The Shapovalov pairing*

$$\mathcal{S} : \tfrac{1}{D}\mathcal{V}^=/\mathcal{J}\tfrac{1}{D}\mathcal{V}^= \otimes \tfrac{1}{D}\mathcal{V}^-/\mathcal{J}\tfrac{1}{D}\mathcal{V}^- \to (Z^{-1}\,\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h])/\mathcal{J} \simeq \mathbb{C}$$

*induced by pairing* (2.13) *is nondegenerate. The $Y(\mathfrak{gl}_N)$-modules $\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$ and $\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$ are contravariantly dual:*

$$\mathcal{S}\bigl(\phi^+(X)\,f,g\bigr) \,=\, \mathcal{S}\bigl(f,\phi^-(\varpi(X))\,g\bigr)\,,$$

*for any $X\in Y(\mathfrak{gl}_N)$ and $f\in\frac{1}{D}\mathcal{V}^=/\mathcal{J}\frac{1}{D}\mathcal{V}^=$, $g\in\frac{1}{D}\mathcal{V}^-/\mathcal{J}\frac{1}{D}\mathcal{V}^-$.* □

4.3. **Yangian actions on $\xi^{\pm}_I$.** Denote

$$A^{\pm}_p(u) = \phi^{\pm}\bigl(A_p(u)\bigr)\,, \qquad E^{\pm}_p(u) = \phi^{\pm}\bigl(E_p(u)\bigr)\,, \qquad F^{\pm}_p(u) = \phi^{\pm}\bigl(F_p(u)\bigr)\,. \tag{4.9}$$

**Lemma 4.14.** *The series $A^{\pm}_N(u)$ act on $V\otimes\mathbb{C}[z_1,\dots,z_n,h]$ as the operator of multiplication by $\prod_{j=1}^n \bigl(1+h\,(u-z_j)^{-1}\bigr)$.*

*Proof.* For $n=1$, the proof is straightforward. For $n>1$, the claim follows from the coproduct formula (3.4) and the $n=1$ case. □

**Corollary 4.15.** *Each of the images $\phi^{\pm}(\widetilde{\mathcal{B}}^{\infty})$ contains the subalgebra of multiplication operators by symmetric polynomials in variables $z_1,\dots,z_n$.*

*Proof.* The elements $\phi^{\pm}(h^{s-1}B^{\infty}_{N,s}) \in \mathrm{End}(V)\otimes\mathbb{C}[z_1,\dots,z_n,h]$ have the form $\sum_{a=1}^n z_a^{s-1} + h\,g^{\pm}_{p,s}$, where $g^{\pm}_{p,s}\in\mathrm{End}(V)\otimes\mathbb{C}[z_1,\dots,z_n,h]$ are symmetric in $z_1,\dots,z_n$ and have homogeneous degree $s-2$. The claim follows. □

**Theorem 4.16.** *For each choice of the sign, + or −, we have*

$$A^{\pm}_p(u)\,\xi^{\pm}_I \,=\, \xi^{\pm}_I\,\prod_{a=1}^{p}\prod_{i\in I_a}\Bigl(1+\frac{h}{u-z_i}\Bigr)\,, \tag{4.10}$$

$$E^{\pm}_p(u)\,\xi^{\pm}_I \,=\, \sum_{i\in I_{p+1}}\frac{\xi^{\pm}_{I^{i\prime}}}{u-z_i}\prod_{\substack{k\in I_{p+1}\\ k\neq i}}\frac{z_i-z_k+h}{z_i-z_k}\,, \tag{4.11}$$

$$F^{\pm}_p(u)\,\xi^{\pm}_I \,=\sum_{j\in I_p}\frac{\xi^{\pm}_{I^{\prime j}}}{u-z_j}\prod_{\substack{k\in I_p\\ k\neq j}}\frac{z_j-z_k-h}{z_j-z_k}\,, \tag{4.12}$$

*where the sequences $I^{i\prime}$ and $I^{\prime j}$ are defined as follows: $I^{i\prime}_a = I^{\prime j}_a = I_a$ for $a\neq p,p+1$, and $I^{i\prime}_p = I_p\cup\{i\}$, $I^{i\prime}_{p+1} = I_{p+1}-\{i\}$, $I^{\prime j}_p = I_p-\{j\}$, $I^{\prime j}_{p+1} = I_{p+1}\cup\{j\}$.*

*Proof.* Formulae (4.10) – (4.12) can be obtained from the results of [NT] as a particular case. Here we outline a partially alternative proof. We consider the case of the plus sign. The other case can be done similarly.

First observe that by formula (2.16) and Lemma 4.4, it suffices to prove formulae (4.10) – (4.12) only for $I=I^{\min}$. In this case, formula (4.10) for $n>1$ follows from the coproduct formula (3.4) and the $n=1$ case of (4.10). The proof of (4.10) for $n=1$ is straightforward.

To get formula (4.11) for $I=I^{\min}$, observe that by formulae (2.17), (3.4), (4.10), we have $E^{+}_p(u)\,\xi^{+}_{I^{\min}} = \sum_{i\in I_{p+1}} c_i\,\xi^{+}_{I^{\min,i\prime}}$. The largest element of $I^{\min}_{p+1}$ equals $i_{\max} = \lambda_1+\ldots+\lambda_{p+1}$, the coefficient $c_{i_{\max}}$ can be calculated due to the triangularity property (2.17), and $c_{i_{\max}}$ has the required form. The coefficient $c_i$ for other $i\in I_{p+1}$ can be obtained from $c_{i_{\max}}$ by permuting $z_i$ and $z_{i_{\max}}$ because $I^{\min}$ is invariant under the transposition of $i$ and $i_{\max}$. Thus all the coefficients $c_i$ are as required, which proves formula (4.11).

The proof of formula (4.12) is similar. □

4.4. **Actions of the dynamical Hamiltonians.**

**Lemma 4.17.** *For actions $\phi^{\pm}$, see* (4.2)*, of the dynamical Hamiltonians $X_1^{\infty},\dots,X_N^{\infty}\in\widetilde{\mathcal{B}}^{\infty}$ on $V$-valued functions of $z_1,\dots,z_n,h$, we have*

$$\phi^{+}(X_i^{\infty}) \,=\, \sum_{a=1}^{n} z_a\, e_{i,i}^{(a)} + \frac{h}{2}\,(e_{i,i}-e_{i,i}^2) + h\sum_{j=1}^{N}\,\sum_{1\leqslant a<b\leqslant n} e_{i,j}^{(a)}\, e_{j,i}^{(b)} - h\sum_{j=1}^{i-1} G_{i,j}\,, \tag{4.13}$$

$$\phi^{-}(X_i^{\infty}) \,=\, \sum_{a=1}^{n} z_a\, e_{i,i}^{(a)} + \frac{h}{2}\,(e_{i,i}-e_{i,i}^2) + h\sum_{j=1}^{N}\,\sum_{1\leqslant b<a\leqslant n} e_{i,j}^{(a)}\, e_{j,i}^{(b)} - h\sum_{j=1}^{i-1} G_{i,j}\,,$$

*where $G_{i,j}=e_{i,j}\,e_{j,i}-e_{i,i}=e_{j,i}\,e_{i,j}-e_{j,j}$ and $e_{k,l}=e_{k,l}^{(1)}+\ldots+e_{k,l}^{(n)}$ for every $k,l$.*

*Proof.* The statement follow from formulae (4.1) and (3.13). □

The operators $\phi^{\pm}(X_i^{q})$, $\phi^{\pm}(X_i^{q+})$, $\phi^{\pm}(X_i^{q-})$, $i=1,\dots,N$, can be found from formulae (3.15) – (3.17).

**Lemma 4.18.** *For any $\boldsymbol{\lambda}$ and any $i=1,\dots,N$, we have $\phi^{+}(X_i^{q+})\,v_{\boldsymbol{\lambda}}^{+}\,=\,\phi^{+}(X_i^{\infty})\,v_{\boldsymbol{\lambda}}^{+}$,*

$$\phi^{+}(X_i^{q-})\,v_{\boldsymbol{\lambda}}^{=}\,=\,\phi^{+}(X_i^{\infty})\,v_{\boldsymbol{\lambda}}^{=}\,,\qquad \phi^{-}(X_i^{q-})\,v_{\boldsymbol{\lambda}}^{-}\,=\,\phi^{-}(X_i^{\infty})\,v_{\boldsymbol{\lambda}}^{-}\,.$$

*Proof.* By formulae (3.15) – (3.17), the statement is equivalent to

$$(e_{i,i}\,e_{j,j}-e_{i,j}\,e_{j,i}+e_{i,i})\,v_{\boldsymbol{\lambda}}^{+}\,=\,0$$

for any $i\neq j$, and

$$e_{j,i}\,e_{i,j}\,v_{\boldsymbol{\lambda}}^{=}\,=\,0\,,\qquad e_{j,i}\,e_{i,j}\,v_{\boldsymbol{\lambda}}^{-}\,=\,0$$

for $i<j$. These equalities hold by Lemmas 2.19, 2.20. □

4.5. ***qKZ* difference connection.** Let

$$R^{(i,j)}(u)\,=\,\frac{u+h\,P^{(i,j)}}{u+h}\,,\qquad i,j=1,\dots,n\,,\quad i\neq j\,.$$

Define operators $K_1^{\pm},\dots,K_n^{\pm}\in\mathrm{End}(V)\otimes\mathbb{C}[z_1,\dots,z_n,h]$,

$$K_i^{+}\,=\,R^{(i,i-1)}(z_i-z_{i-1})\,\ldots\,R^{(i,1)}(z_i-z_1)\,\times$$
$$\times\;q_1^{e_{1,1}^{(i)}}\ldots\,q_N^{e_{N,N}^{(i)}}\,R^{(i,n)}(z_i-z_n-\kappa)\,\ldots\,R^{(i,i+1)}(z_i-z_{i+1}-\kappa)\,,$$

$$K_i^{-}\,=\,R^{(i,i+1)}(z_i-z_{i+1})\,\ldots\,R^{(i,n)}(z_i-z_n)\,\times$$
$$\times\;q_1^{e_{1,1}^{(i)}}\ldots\,q_N^{e_{N,N}^{(i)}}\,R^{(i,1)}(z_i-z_1-\kappa)\,\ldots\,R^{(i,i-1)}(z_i-z_{i-1}-\kappa)\,.$$

Consider the difference operators $\widehat{K}_1^{\pm},\dots,\widehat{K}_n^{\pm}$ acting on $V$-valued functions of $z_1,\dots,z_n,h$,

$$\widehat{K}_i^{\pm}\,f(z_1,\dots,z_n,h)\,=\,K_i^{\pm}(z_1,\dots,z_n,h)\,f(z_1,\dots,z_{i-1},z_i-\kappa,z_{i+1},\dots,z_n)\,.$$

**Theorem 4.19** ([FR]). *The operators* $\widehat{K}_1^+, \dots, \widehat{K}_n^+$ *pairwise commute. Similarly, the operators* $\widehat{K}_1^-, \dots, \widehat{K}_n^-$ *pairwise commute.* □

**Theorem 4.20** ([TV1]). *The operators* $\widehat{K}_1^+, \dots, \widehat{K}_n^+$, $\phi^+(\nabla_1^+), \dots, \phi^+(\nabla_N^+)$ *pairwise commute. Similarly, the operators* $\widehat{K}_1^-, \dots, \widehat{K}_n^-$, $\phi^-(\nabla_1^-), \dots, \phi^-(\nabla_N^-)$ *pairwise commute.* □

**Corollary 4.21.** *The operators* $K_1^+|_{\kappa=0}, \dots, K_n^+|_{\kappa=0}$, $\phi^+(X_1^{q+}), \dots, \phi^+(X_N^{q+})$ *pairwise commute. Similarly, the operators* $K_1^-|_{\kappa=0}, \dots, K_n^-|_{\kappa=0}$, $\phi^-(X_1^{q-}), \dots, \phi^-(X_N^{q-})$ *pairwise commute.*

The commuting difference operators $\widehat{K}_1^+, \dots, \widehat{K}_n^+$ define the *qKZ difference connection.* Similarly, the commuting difference operators $\widehat{K}_1^-, \dots, \widehat{K}_n^-$ define another *qKZ* difference connection. Theorem 4.20 says that the *qKZ* difference connections commute with the corresponding dynamical connections

**Proposition 4.22.** *For every* $X \in \widetilde{\mathcal{B}}^q$, *the operators* $K_1^+|_{\kappa=0}, \dots, K_n^+|_{\kappa=0}$ *commute with* $\phi^+(X)$, *and the operators* $K_1^-|_{\kappa=0}, \dots, K_n^-|_{\kappa=0}$ *commute with* $\phi^-(X)$.

*Proof.* By formulae (4.1), (4.2), (3.6), we have

$$K_i^{\pm}|_{\kappa=0} \,=\, \Bigl(\phi^{\pm}\bigl(B_1^q(u)\bigr) \prod_{j=1}^{n} \frac{u-z_j}{u-z_j+h}\Bigr)\Big|_{u=z_i}, \qquad i=1,\dots,n\,,$$

so the statement follows from Theorem 3.1. Alternatively, the proposition follows from the Yang-Baxter equation (4.3). □

## 5. Equivariant cohomology of the cotangent bundles of partial flag varieties

5.1. **Partial flag varieties.** For $\boldsymbol{\lambda} \in \mathbb{Z}_{\geqslant 0}^N$, $|\boldsymbol{\lambda}| = n$, consider the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing chains of subspaces

$$0 \,=\, F_0 \subset F_1 \subset \ldots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i=1,\dots,N$. Denote by $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ the cotangent bundle of $\mathcal{F}_{\boldsymbol{\lambda}}$.

Let $T^n \subset GL_n = GL_n(\mathbb{C})$ be the torus of diagonal matrices. The groups $T^n \subset GL_n$ act on $\mathbb{C}^n$ and hence on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$. Let the group $\mathbb{C}^*$ act on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ by multiplication in each fiber.

The set of fixed points $(T^*\mathcal{F}_{\boldsymbol{\lambda}})^{T^n\times\mathbb{C}^*}$ of the torus action lies in the zero section of the cotangent bundle and consists of the coordinate flags $F_I = (F_0 \subset \ldots \subset F_N)$, $I = (I_1, \dots, I_N) \in \mathcal{I}_{\boldsymbol{\lambda}}$, where $F_i$ is the span of the standard basis vectors $u_j \in \mathbb{C}^n$ with $j \in I_1 \cup \ldots \cup I_i$. The fixed points are in a one-to-one correspondence with the elements of $\mathcal{I}_{\boldsymbol{\lambda}}$ and hence with the basis vectors $v_I$ of $V_{\boldsymbol{\lambda}}$, see Section 2.1.

We consider the $GL_n(\mathbb{C})\times\mathbb{C}^*$-equivariant cohomology algebra

$$H_{\boldsymbol{\lambda}} \,=\, H^*_{GL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\,.$$

Denote by $\Gamma_i = \{\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}\}$ the set of the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber $F_i/F_{i-1}$. Let $\Gamma = (\Gamma_1; \dots; \Gamma_N)$. Denote by $\boldsymbol{z} = \{z_1, \dots, z_n\}$ the Chern roots corresponding to the factors of the torus $T^n$. Denote by $h$ the Chern root corresponding to the factor $\mathbb{C}^*$ acting on the fibers of $T^*\mathcal{F}_{\boldsymbol\lambda}$ by multiplication. Then

$$H_{\boldsymbol\lambda} \,=\, \mathbb{C}[\boldsymbol z;\Gamma]^{S_n\times S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]\Big/\Big\langle\prod_{i=1}^N\prod_{j=1}^{\lambda_i}(u-\gamma_{i,j}) \,=\, \prod_{a=1}^n(u-z_a)\Big\rangle\,. \tag{5.1}$$

The cohomology $H_{\boldsymbol\lambda}$ is a module over $H^*_{GL_n\times\mathbb{C}^*}(pt;\mathbb{C})=\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$.

For $A\subset\{1,\ldots,n\}$ denote $\boldsymbol z_A=\{z_a,\,a\in A\}$, and for $I=(I_1,\ldots,I_N)\in\mathcal{I}_{\boldsymbol\lambda}$ denote $\boldsymbol z_I=(\boldsymbol z_{I_1};\ldots;\boldsymbol z_{I_N})$. Set

$$\zeta_{\boldsymbol\lambda}: H_{\boldsymbol\lambda}\to\,\mathbb{C}[z_1,\ldots,z_n]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]\,,\qquad [f(\boldsymbol z;\Gamma;h)]\,\mapsto\, f(\boldsymbol z;\boldsymbol z_{I^{\min}};h)\,. \tag{5.2}$$

Clearly, $\zeta_{\boldsymbol\lambda}$ is an isomorphism of $\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$-algebras.

Let $\mathcal{J}\subset H_{\boldsymbol\lambda}$ be the ideal generated by the relations $h=1$ and $\sigma_i(z_1,\ldots,z_n)=0,\ i=1,\ldots,n$. Then $H_{\boldsymbol\lambda}/\mathcal{J}=H^*(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})$.

5.2. **Equivariant integration.** We will need the *integration on $\mathcal{F}_{\boldsymbol\lambda}$ map*

$$\int :\, H_{\boldsymbol\lambda}\to\, H^*_{GL_n\times\mathbb{C}^*}(pt,\mathbb{C})\,,$$

that is, the composition

$$H_{\boldsymbol\lambda}\,=\,H^*_{GL_n}(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})\otimes\mathbb{C}[h]\xrightarrow{\int_{\mathcal{F}_{\boldsymbol\lambda}}\otimes\,\mathrm{id}}H^*_{GL_n}(pt;\mathbb{C})\otimes\mathbb{C}[h]\,=\,H^*_{GL_n\times\mathbb{C}^*}(pt;\mathbb{C})\,.$$

The Atiyah-Bott equivariant localization theorem [AB] gives the integration on $\mathcal{F}_{\boldsymbol\lambda}$ map in terms of the fixed point set $(T^*\mathcal{F}_{\boldsymbol\lambda})^{T^n\times\mathbb{C}^*}$: for any $[f(\boldsymbol z;\Gamma;h)]\in H_{\boldsymbol\lambda}$,

$$\int[f]\,=\,(-1)^{\sum_{i<j}\lambda_i\lambda_j}\sum_{I\in\mathcal{I}_{\boldsymbol\lambda}}\frac{f(\boldsymbol z;\boldsymbol z_I;h)}{R(\boldsymbol z_I)}\,, \tag{5.3}$$

where $R(\boldsymbol z_I)$ is given by (2.15). Clearly, the right-hand side in (5.3) lies in $\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$. The integration on $\mathcal{F}_{\boldsymbol\lambda}$ map induces the *Poincare pairing on $\mathcal{F}_{\boldsymbol\lambda}$*,

$$H_{\boldsymbol\lambda}\otimes H_{\boldsymbol\lambda}\to\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]\,,\qquad [f]\otimes[g]\mapsto\int[fg]\,. \tag{5.4}$$

After factorization by the ideal $\mathcal{J}$ we obtain the nondegenerate Poincare pairing

$$H^*(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})\otimes H^*(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})\,\to\,\mathbb{C}\,. \tag{5.5}$$

We will also use the *integration on $T^*\mathcal{F}_{\boldsymbol\lambda}$ map*,

$$⨍ :\, H^*(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})\,\to\,Z^{-1}\,\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]\,,$$

$$⨍\,[f]\,=\,(-1)^{\sum_{i<j}\lambda_i\lambda_j}\sum_{I\in\mathcal{I}_{\boldsymbol\lambda}}\frac{f(\boldsymbol z;\boldsymbol z_I;h)}{Q(\boldsymbol z_I)R(\boldsymbol z_I)}\,. \tag{5.6}$$

Here $Z=\prod_{i\neq j}(z_i-z_j+h)$, cf. (2.14). Notice that $Q(\boldsymbol{z}_I)\,R(\boldsymbol{z}_I)$ is the Euler class of the tangent space at the point $F_I\in T^*\mathcal{F}_{\boldsymbol{\lambda}}$. This integration map was used in [BMO], see also [HP]. The integration on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ map induces the *Poincare paring on* $T^*\mathcal{F}_{\boldsymbol{\lambda}}$,

$$H_{\boldsymbol{\lambda}}\otimes H_{\boldsymbol{\lambda}}\to Z^{-1}\,\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]\,,\qquad [f]\otimes[g]\mapsto\fint[fg]\,,\tag{5.7}$$

which will be called the *Poincare paring on* $T^*\mathcal{F}_{\boldsymbol{\lambda}}$.

**Example.** Let $N=n=2$ and $\boldsymbol{\lambda}=(1,1)$. Then

$$\int[1]\,=\,0\,,\qquad\quad \int[\gamma_{1,1}]\,=\,-1\,,$$

$$\fint[1]\,=\,\frac{2}{(z_1-z_2+h)\,(z_2-z_1+h)}\,,\qquad\quad \fint[\gamma_{1,1}]\,=\,\frac{z_1+z_2-h}{(z_1-z_2+h)\,(z_2-z_1+h)}\,.$$

5.3. **$H_{\boldsymbol{\lambda}}$ and $\mathcal{V}^+_{\boldsymbol{\lambda}}$, $\frac{1}{D}\mathcal{V}^=_{\boldsymbol{\lambda}}$, $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$.** Consider the maps $\nu^+_{\boldsymbol{\lambda}}:H_{\boldsymbol{\lambda}}\to\mathcal{V}^+_{\boldsymbol{\lambda}}$, $\nu^=_{\boldsymbol{\lambda}}:H_{\boldsymbol{\lambda}}\to\frac{1}{D}\mathcal{V}^=_{\boldsymbol{\lambda}}$, and $\nu^-_{\boldsymbol{\lambda}}:H_{\boldsymbol{\lambda}}\to\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$,

$$\nu^+_{\boldsymbol{\lambda}}\,:\,[f]\,\mapsto\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z};\boldsymbol{z}_I;h)\,Q(\boldsymbol{z}_I)}{R(\boldsymbol{z}_I)}\;\xi^+_I\,,\tag{5.8}$$

$$\nu^=_{\boldsymbol{\lambda}}\,:\,[f]\,\mapsto\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z};\boldsymbol{z}_I;h)}{R(\boldsymbol{z}_I)}\;\xi^+_I\,.\tag{5.9}$$

$$\nu^-_{\boldsymbol{\lambda}}\,:\,[f]\,\mapsto\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z};\boldsymbol{z}_I;h)}{R(\boldsymbol{z}_I)}\;\xi^-_I\,.\tag{5.10}$$

In particular, $\nu^+_{\boldsymbol{\lambda}}\,:\,[1]\,\mapsto\,v^+_{\boldsymbol{\lambda}}$, $\nu^=_{\boldsymbol{\lambda}}\,:\,[1]\,\mapsto\,v^=_{\boldsymbol{\lambda}}$, $\nu^-_{\boldsymbol{\lambda}}\,:\,[1]\,\mapsto\,v^-_{\boldsymbol{\lambda}}$, see (2.26). We have $\nu^+_{\boldsymbol{\lambda}}=\vartheta^+_{\boldsymbol{\lambda}}\,\zeta_{\boldsymbol{\lambda}}$, $\nu^=_{\boldsymbol{\lambda}}=\vartheta^=_{\boldsymbol{\lambda}}\,\zeta_{\boldsymbol{\lambda}}$, $\nu^-_{\boldsymbol{\lambda}}=\vartheta^-_{\boldsymbol{\lambda}}\,\zeta_{\boldsymbol{\lambda}}$, where the maps $\vartheta^+_{\boldsymbol{\lambda}}$, $\vartheta^=_{\boldsymbol{\lambda}}$, $\vartheta^-_{\boldsymbol{\lambda}}$ and $\zeta_{\boldsymbol{\lambda}}$ are given by formulae (2.8) – (2.10) and (5.2). Observe that

$$\nu^-_{\boldsymbol{\lambda}}\,=\,\widetilde{\Pi}\,\nu^=_{\boldsymbol{\lambda}}\,,\tag{5.11}$$

where $\widetilde{\Pi}$ is given by formula (4.4).

**Lemma 5.1.** *The maps $\nu^+_{\boldsymbol{\lambda}}$, $\nu^=_{\boldsymbol{\lambda}}$, $\nu^-_{\boldsymbol{\lambda}}$ are isomorphisms of $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$-modules.*

*Proof.* The claim follows from Lemma 2.10. □

**Proposition 5.2.** *The Shapovalov form and Poincare pairing on $\mathcal{F}_{\boldsymbol{\lambda}}$ are related by the formula*

$$\mathcal{S}\big(\nu^+_{\boldsymbol{\lambda}}[f],\nu^-_{\boldsymbol{\lambda}}[g]\big)\,=\,(-1)^{\sum_{i<j}\lambda_i\lambda_j}\int[f][g]\,.$$

*Proof.* The statement follows from formulae (2.25), (5.3), (5.8), (5.10), and Lemma 5.1. □

**Proposition 5.3.** *The Shapovalov form and Poincare pairing on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ are related by the formula*

$$\mathcal{S}\big(\nu^=_{\boldsymbol{\lambda}}[f],\nu^-_{\boldsymbol{\lambda}}[g]\big)\,=\,(-1)^{\sum_{i<j}\lambda_i\lambda_j}\fint[f][g]\,.$$

*Proof.* The statement follows from formulae (2.25), (5.6), (5.9), (5.10), and Lemma 5.1. □

Introduce the elements $f_{p,s} \in \mathbb{C}[\boldsymbol{z};\Gamma]^{S_n\times S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]$, $p=1,\ldots,N$, $s\in\mathbb{Z}_{>0}$, by

$$\prod_{i=1}^{\lambda_p}\Bigl(1+\frac{h}{u-\gamma_{p,i}}\Bigr) \,=\, 1+h\sum_{s=1}^{\infty} f_{p,s}\,u^{-s}\,. \tag{5.12}$$

Since $f_{p,s}=\sum_{i=1}^{\lambda_p}\gamma_{p,i}^{s-1}+h\,g_{p,s}$, where $g_{p,s}\in\mathbb{C}[\boldsymbol{z};\Gamma]^{S_n\times S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]$ are of homogeneous degree $s-2$, the elements $[f_{p,s}]$ $p=1,\ldots,N$, $s>0$, generate $H_{\boldsymbol{\lambda}}$ over $\mathbb{C}[h]$.

Define the elements $C^{\pm}_{p,s}\in\mathrm{End}(V)\otimes\mathbb{C}[z_1,\ldots,z_n,h]$, $p=1,\ldots,N$, $s\in\mathbb{Z}_{>0}$, by

$$\bigl(A^{\pm}_{p-1}(u)\bigr)^{-1}A^{\pm}_p(u) \,=\, 1+h\sum_{s=1}^{\infty} C^{\pm}_{p,s}\,u^{-s}\,, \tag{5.13}$$

where $A^{\pm}_0(u)=1$ and $A^{\pm}_1(u),\ldots,A^{\pm}_N(u)$ are given by (4.9).

Let $\mathcal{A}$ be a commutative algebra. The algebra $\mathcal{A}$ considered as a module over itself is called the regular representation of $\mathcal{A}$.

**Theorem 5.4.** *The assignment* $\mu^{+}_{\boldsymbol{\lambda}}:[f_{p,s}]\mapsto C^{+}_{p,s}$, $p=1,\ldots,N$, $s>0$, *extends uniquely to an isomorphism* $H_{\boldsymbol{\lambda}}\to\widetilde{\mathcal{B}}^{\infty}(\mathcal{V}^{+}_{\boldsymbol{\lambda}})$ *of* $\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras. The maps* $\mu^{+}_{\boldsymbol{\lambda}}$ *and* $\nu^{+}_{\boldsymbol{\lambda}}$ *identify the* $\widetilde{\mathcal{B}}^{\infty}(\mathcal{V}^{+}_{\boldsymbol{\lambda}})$*-module* $\mathcal{V}^{+}_{\boldsymbol{\lambda}}$ *with the regular representation of the algebra* $H_{\boldsymbol{\lambda}}$.

**Theorem 5.5.** *The assignment* $\mu^{=}_{\boldsymbol{\lambda}}:[f_{p,s}]\mapsto C^{+}_{p,s}$, $p=1,\ldots,N$, $s>0$, *extends uniquely to an isomorphism* $H_{\boldsymbol{\lambda}}\to\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}})$ *of* $\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras. The maps* $\mu^{=}_{\boldsymbol{\lambda}}$ *and* $\nu^{=}_{\boldsymbol{\lambda}}$ *identify the* $\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}})$*-module* $\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}}$ *with the regular representation of the algebra* $H_{\boldsymbol{\lambda}}$.

**Theorem 5.6.** *The assignment* $\mu^{-}_{\boldsymbol{\lambda}}:[f_{p,s}]\mapsto C^{-}_{p,s}$, $p=1,\ldots,N$, $s>0$, *extends uniquely to an isomorphism* $H_{\boldsymbol{\lambda}}\to\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}})$ *of* $\mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras. The maps* $\mu^{-}_{\boldsymbol{\lambda}}$ *and* $\nu^{-}_{\boldsymbol{\lambda}}$ *identify the* $\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}})$*-module* $\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}}$ *with the regular representation of the algebra* $H_{\boldsymbol{\lambda}}$.

*Proofs of Theorems 5.4 – 5.6.* The operator $\nu^{+}_{\boldsymbol{\lambda}}\,C^{+}_{p,s}\,(\nu^{+}_{\boldsymbol{\lambda}})^{-1}$ acts on $H_{\boldsymbol{\lambda}}$ as the multiplication by $[f_{p,s}]$, see Lemma 5.1 and formula (4.10). This yields Theorem 5.4. The proofs of Theorems 5.5 and 5.6 are similar. □

Theorems 5.5 and 5.6 are equivalent by Lemma 4.5 and relation (5.11). In particular,

$$\widetilde{\Pi}\;\mu^{=}_{\boldsymbol{\lambda}}(f) \,=\, \mu^{-}_{\boldsymbol{\lambda}}(f)\;\widetilde{\Pi} \tag{5.14}$$

for any $f\in H_{\boldsymbol{\lambda}}$, see (4.5).

**Corollary 5.7.** *The* $\widetilde{\mathcal{B}}^{\infty}$*-modules* $\mathcal{V}^{+}$, $\frac{1}{D}\mathcal{V}^{=}$, *and* $\frac{1}{D}\mathcal{V}^{-}$ *are isomorphic.*

*Proof.* The isomorphisms restricted to weight subspaces are $\nu^{=}_{\boldsymbol{\lambda}}\,(\nu^{+}_{\boldsymbol{\lambda}})^{-1}:\mathcal{V}^{+}_{\boldsymbol{\lambda}}\to\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}}$ and $\nu^{-}_{\boldsymbol{\lambda}}\,(\nu^{+}_{\boldsymbol{\lambda}})^{-1}:\mathcal{V}^{+}_{\boldsymbol{\lambda}}\to\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}}$. □

**Corollary 5.8.** *The subalgebras* $\widetilde{\mathcal{B}}^{\infty}(\mathcal{V}^{+}_{\boldsymbol{\lambda}})\subset\mathrm{End}(\mathcal{V}^{+}_{\boldsymbol{\lambda}})$, $\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}})\subset\mathrm{End}(\frac{1}{D}\mathcal{V}^{=}_{\boldsymbol{\lambda}})$ *and* $\widetilde{\mathcal{B}}^{\infty}(\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}})\subset\mathrm{End}(\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}})$ *are maximal commutative subalgebras.* □

**Lemma 5.9.** *For any* $f \in H_{\boldsymbol\lambda}$, *and any* $g_1 \in \mathcal V^{+}_{\boldsymbol\lambda}$, $g_2 \in \frac{1}{\check D}\mathcal V^{=}_{\boldsymbol\lambda}$, $g_3 \in \frac{1}{\check D}\mathcal V^{=}_{\boldsymbol\lambda}$, *we have*

$$\mathcal S\bigl(\mu^{+}_{\boldsymbol\lambda}(f)\,g_1, g_3\bigr) \,=\, \mathcal S\bigl(g_1, \mu^{-}_{\boldsymbol\lambda}(f)\,g_3\bigr)\,, \qquad \mathcal S\bigl(\mu^{=}_{\boldsymbol\lambda}(f)\,g_2, g_3\bigr) \,=\, \mathcal S\bigl(g_2, \mu^{-}_{\boldsymbol\lambda}(f)\,g_3\bigr)\,.$$

*Proof.* The statement follows from the definition of the maps $\mu^{+}_{\boldsymbol\lambda}$, $\mu^{=}_{\boldsymbol\lambda}$, $\mu^{-}_{\boldsymbol\lambda}$ and Corollary 4.3. □

**Example.** Let $N = n = 2$ and $\boldsymbol\lambda = (1,1)$. Then the weight subspace $V_{\boldsymbol\lambda}$ is two-dimensional, $V_{\boldsymbol\lambda} = \mathbb C\, v_{(1,2)} \oplus \mathbb C\, v_{(2,1)}$, $v_{(1,2)} = v_1 \otimes v_2$, $v_{(2,1)} = v_2 \otimes v_1$. The vectors $\xi^{\pm}_I$ are

$$\xi^{+}_{(1,2)} = v_{(1,2)}\,, \qquad \xi^{+}_{(2,1)} = \frac{z_2 - z_1}{z_2 - z_1 + h}\, v_{(2,1)} + \frac{h}{z_2 - z_1 + h}\, v_{(1,2)}\,,$$

$$\xi^{-}_{(2,1)} = v_{(2,1)}\,, \qquad \xi^{-}_{(1,2)} = \frac{z_1 - z_2}{z_1 - z_2 + h}\, v_{(1,2)} + \frac{h}{z_1 - z_2 + h}\, v_{(2,1)}\,.$$

Elements of $\mathcal V^{+}_{\boldsymbol\lambda}$ have the form

$$f(z_1, z_2, h)\,\frac{z_1 - z_2 + h}{z_1 - z_2}\,\xi^{+}_{(1,2)} + f(z_2, z_1, h)\,\frac{z_2 - z_1 + h}{z_2 - z_1}\,\xi^{+}_{(2,1)}\,,$$

elements of $\frac{1}{\check D}\mathcal V^{=}_{\boldsymbol\lambda}$ have the form

$$\frac{f(z_1, z_2, h)}{z_1 - z_2}\,\xi^{+}_{(1,2)} + \frac{f(z_2, z_1, h)}{z_2 - z_1}\,\xi^{+}_{(2,1)}\,,$$

and elements of $\frac{1}{D}\mathcal V^{-}_{\boldsymbol\lambda}$ have the form

$$\frac{f(z_1, z_2, h)}{z_1 - z_2}\,\xi^{-}_{(1,2)} + \frac{f(z_2, z_1, h)}{z_2 - z_1}\,\xi^{-}_{(2,1)}\,.$$

The series (4.9), (5.13) for generators of the Bethe algebras $\widetilde{\mathcal B}^{\infty}(\mathcal V^{+}_{\boldsymbol\lambda})$ and $\widetilde{\mathcal B}^{\infty}(\frac{1}{D}\mathcal V^{-}_{\boldsymbol\lambda})$ are

$$A^{\pm}_2(u) \,=\, \Bigl(1 + \frac{h}{u - z_1}\Bigr)\Bigl(1 + \frac{h}{u - z_2}\Bigr),$$

$$A^{+}_1(u) \,=\, \begin{pmatrix} 1 + \dfrac{h}{u - z_1} & h^2 \\ 0 & 1 + \dfrac{h}{u - z_2} \end{pmatrix}, \qquad \bigl(A^{+}_1(u)\bigr)^{-1} A^{+}_2(u) \,=\, \begin{pmatrix} 1 + \dfrac{h}{u - z_2} & -h^2 \\ 0 & 1 + \dfrac{h}{u - z_1} \end{pmatrix},$$

$$A^{-}_1(u) \,=\, \begin{pmatrix} 1 + \dfrac{h}{u - z_1} & 0 \\ h^2 & 1 + \dfrac{h}{u - z_2} \end{pmatrix}, \qquad \bigl(A^{-}_1(u)\bigr)^{-1} A^{-}_2(u) \,=\, \begin{pmatrix} 1 + \dfrac{h}{u - z_2} & 0 \\ -h^2 & 1 + \dfrac{h}{u - z_1} \end{pmatrix},$$

where we are using the basis $v_{(1,2)}, v_{(2,1)}$ of $V_{\boldsymbol\lambda}$. For the maps $\nu^{+}_{\boldsymbol\lambda}$, $\nu^{=}_{\boldsymbol\lambda}$, $\nu^{-}_{\boldsymbol\lambda}$, and $\nu^{+}_{\boldsymbol\lambda}$, $\nu^{=}_{\boldsymbol\lambda}$, $\nu^{-}_{\boldsymbol\lambda}$, we have

$$\mu^{+}_{\boldsymbol\lambda} : [1] \,\mapsto\, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mu^{+}_{\boldsymbol\lambda} : [\gamma_{1,1}] \,\mapsto\, \begin{pmatrix} z_1 & h \\ 0 & z_2 \end{pmatrix},$$

$$\nu^+_{\boldsymbol\lambda} : [1] \mapsto \frac{z_1-z_2+h}{z_1-z_2}\,\xi^+_{(1,2)} + \frac{z_2-z_1+h}{z_2-z_1}\,\xi^+_{(2,1)} = v_{(1,2)} + v_{(2,1)} = v^+_{\boldsymbol\lambda},$$

$$\nu^+_{\boldsymbol\lambda} : [\gamma_{1,1}] \mapsto z_1\frac{z_1-z_2+h}{z_1-z_2}\,\xi^+_{(1,2)} + z_2\frac{z_2-z_1+h}{z_2-z_1}\,\xi^+_{(2,1)} = (z_1+h)\,v_{(1,2)} + z_2\,v_{(2,1)}\,,$$

$$\mu^=_{\boldsymbol\lambda} : [1] \mapsto \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \mu^=_{\boldsymbol\lambda} : [\gamma_{1,1}] \mapsto \begin{pmatrix}z_1&h\\0&z_2\end{pmatrix},$$

$$\nu^=_{\boldsymbol\lambda} : [1] \mapsto \frac{1}{z_1-z_2}\,\xi^+_{(1,2)} + \frac{1}{z_2-z_1}\,\xi^+_{(2,1)} = \frac{v_{(1,2)} - v_{(2,1)}}{z_1-z_2-h} = v^=_{\boldsymbol\lambda},$$

$$\nu^=_{\boldsymbol\lambda} : [\gamma_{1,1}] \mapsto \frac{z_1}{z_1-z_2}\,\xi^+_{(1,2)} + \frac{z_2}{z_2-z_1}\,\xi^+_{(2,1)} = \frac{(z_1-h)\,v_{(1,2)} - z_2\,v_{(2,1)}}{z_1-z_2-h}\,,$$

$$\mu^-_{\boldsymbol\lambda} : [1] \mapsto \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \mu^-_{\boldsymbol\lambda} : [\gamma_{1,1}] \mapsto \begin{pmatrix}z_1&0\\h&z_2\end{pmatrix}.$$

$$\nu^-_{\boldsymbol\lambda} : [1] \mapsto \frac{1}{z_1-z_2}\,\xi^-_{(1,2)} + \frac{1}{z_2-z_1}\,\xi^-_{(2,1)} = \frac{v_{(1,2)} - v_{(2,1)}}{z_1-z_2+h} = v^-_{\boldsymbol\lambda},$$

$$\nu^-_{\boldsymbol\lambda} : [\gamma_{1,1}] \mapsto \frac{z_1}{z_1-z_2}\,\xi^-_{(1,2)} + \frac{z_2}{z_2-z_1}\,\xi^-_{(2,1)} = \frac{z_1\,v_{(1,2)} - (z_2-h)\,v_{(2,1)}}{z_1-z_2+h}\,,$$

5.4. **Cohomology as $\widetilde Y(\mathfrak{gl}_N)$-modules.** The isomorphisms

$$\nu^+ = \bigoplus_{\boldsymbol\lambda}\nu^+_{\boldsymbol\lambda} : \bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda} \to \mathcal V^+, \qquad \nu^- = \bigoplus_{\boldsymbol\lambda}\nu^-_{\boldsymbol\lambda} : \bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda} \to \tfrac{1}{D}\mathcal V^-$$

induce two $\widetilde Y(\mathfrak{gl}_N)$-module structures on $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ denoted respectively by $\rho^+$ and $\rho^-$,

$$\rho^\pm(X) = (\nu^\pm)^{-1}\,\phi^\pm(X)\,\nu^\pm$$

for any $X\in\widetilde Y(\mathfrak{gl}_N)$. The $\widetilde Y(\mathfrak{gl}_N)$-module structure on $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ induced by the isomorphism

$$\nu^= = \bigoplus_{\boldsymbol\lambda}\nu^=_{\boldsymbol\lambda} : \bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda} \to \tfrac{1}{\tilde D}\mathcal V^=$$

coincides with the $\rho^-$-structure

$$(\nu^=)^{-1}\,\phi^+(X)\,\nu^= = \rho^-(X)\,, \tag{5.15}$$

since $\nu^-_{\boldsymbol\lambda} = \widetilde\Pi\,\nu^=_{\boldsymbol\lambda}$ and $\widetilde\Pi\,\phi^+(X) = \phi^-(X)\,\widetilde\Pi$, see (5.11), (4.5).

By formulae (5.8), (5.10) and (4.10), we have $\rho^\pm\bigl(A_p(u)\bigr) : H_{\boldsymbol\lambda} \to H_{\boldsymbol\lambda}$,

$$\rho^\pm\bigl(A_p(u)\bigr) : [f] \mapsto \Bigl[f(\boldsymbol z;\Gamma;h)\prod_{a=1}^{p}\prod_{i=1}^{\lambda_p}\Bigl(1+\frac{h}{u-\gamma_{p,i}}\Bigr)\Bigr], \tag{5.16}$$

for $p=1,\dots,N$. In particular, by (3.11), (3.14),

$$\rho^\pm(X^\infty_i) : [f] \mapsto [(\gamma_{i,1}+\ldots+\gamma_{i,\lambda_i})\,f(\boldsymbol z;\Gamma;h)]\,, \qquad i=1,\dots,N\,. \tag{5.17}$$

Let $\alpha_1, \dots, \alpha_{N-1}$ be simple roots, $\alpha_p = (0, \dots, 0, 1, -1, 0, \dots, 0)$, with $p-1$ first zeros.

**Theorem 5.10.** *We have*

$$\rho^+\big(E_p(u)\big) \,:\, H_{\boldsymbol{\lambda}-\alpha_p} \,\mapsto\, H_{\boldsymbol{\lambda}}\,,$$

$$\rho^+\big(E_p(u)\big) \,:\, [f] \,\mapsto\, \Bigg[\, \sum_{i=1}^{\lambda_p} \frac{f(\boldsymbol{z};\Gamma'^i;h)}{u-\gamma_{p,i}} \prod_{\substack{j=1\\ j\neq i}}^{\lambda_p} \frac{\gamma_{p,i}-\gamma_{p,j}-h}{\gamma_{p,i}-\gamma_{p,j}} \,\Bigg],$$

$$\rho^+\big(F_p(u)\big) \,:\, H_{\boldsymbol{\lambda}+\alpha_p} \,\mapsto\, H_{\boldsymbol{\lambda}}\,,$$

$$\rho^+\big(F_p(u)\big) \,:\, [f] \,\mapsto\, \Bigg[\, \sum_{i=1}^{\lambda_{p+1}} \frac{f(\boldsymbol{z};\Gamma^{i\prime};h)}{u-\gamma_{p+1,i}} \prod_{\substack{j=1\\ j\neq i}}^{\lambda_{p+1}} \frac{\gamma_{p+1,i}-\gamma_{p+1,j}+h}{\gamma_{p+1,i}-\gamma_{p+1,j}} \,\Bigg],$$

*where*

$$\Gamma'^i = \,(\Gamma_1;\dots;\Gamma_{p-1};\Gamma_p - \{\gamma_{p,i}\};\Gamma_{p+1}\cup\{\gamma_{p,i}\};\Gamma_{p+2};\dots;\Gamma_N)\,,$$

$$\Gamma^{i\prime} = \,(\Gamma_1;\dots;\Gamma_{p-1};\Gamma_p \cup \{\gamma_{p+1,i}\};\Gamma_{p+1}-\{\gamma_{p+1,i}\};\Gamma_{p+2};\dots;\Gamma_N)\,.$$

*Similarly,*

$$\rho^-\big(E_p(u)\big) \,:\, H_{\boldsymbol{\lambda}-\alpha_p} \,\mapsto\, H_{\boldsymbol{\lambda}}\,,$$

$$\rho^-\big(E_p(u)\big) \,:\, [f] \,\mapsto\, \Bigg[\, \sum_{i=1}^{\lambda_p} \frac{f(\boldsymbol{z};\Gamma'^i;h)}{u-\gamma_{p,i}} \prod_{\substack{j=1\\ j\neq i}}^{\lambda_p} \frac{1}{\gamma_{p,j}-\gamma_{p,i}} \prod_{k=1}^{\lambda_{p+1}} (\gamma_{p,i}-\gamma_{p+1,k}+h) \Bigg],$$

$$\rho^-\big(F_p(u)\big) \,:\, H_{\boldsymbol{\lambda}+\alpha_p} \,\mapsto\, H_{\boldsymbol{\lambda}}\,,$$

$$\rho^-\big(F_p(u)\big) \,:\, [f] \,\mapsto\, \Bigg[\, \sum_{i=1}^{\lambda_{p+1}} \frac{f(\boldsymbol{z};\Gamma^{i\prime};h)}{u-\gamma_{p+1,i}} \prod_{\substack{j=1\\ j\neq i}}^{\lambda_{p+1}} \frac{1}{\gamma_{p+1,i}-\gamma_{p+1,j}} \prod_{k=1}^{\lambda_p} (\gamma_{p,k}-\gamma_{p+1,i}+h) \Bigg].$$

*Proof.* The statement follows from formulae (5.8) – (5.10) and (4.11), (4.12). □

The $\widetilde{Y}(\mathfrak{gl}_N)$-module structures $\rho^\pm$ on $\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ are the Yangian versions of representations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}_N})$ considered in [Vas1, Vas2].

The $\rho^-$-structure appears to be more preferable, it was used in [RTVZ] to construct quantized conformal blocks in the tensor power $V^{\otimes n}$, see also Sections 6 – 8.

**Corollary 5.11.** *The $\widetilde{Y}(\mathfrak{gl}_N)$-module structures $\rho^\pm$ on $\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ are contravariantly related through the Poincare pairing on $\mathcal{F}_{\boldsymbol{\lambda}}$,*

$$\int [f]\; \rho^+(X)\,[g] \;=\; \int [g]\; \rho^-\big(\varpi(X)\big)\,[f]$$

*for any $X \in \widetilde{Y}(\mathfrak{gl}_N)$ and any $f, g \in \bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$.*

*Proof.* The statement follows from Proposition 5.2 and Lemma 4.2. ☐

**Corollary 5.12.** *The $\widetilde{Y}(\mathfrak{gl}_N)$-module structure $\rho^-$ on $\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ is contravariantly related to itself through the Poincare pairing on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$,*

$$\fint [f]\,\rho^-(X)\,[g] \;=\; \fint [g]\,\rho^-\bigl(\varpi(X)\bigr)\,[f]$$

*for any $X\in\widetilde{Y}(\mathfrak{gl}_N)$ and any $f,g\in\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$.*

*Proof.* The statement follows from relation (5.15), Proposition 5.3 and Lemma 4.2. ☐

**Corollary 5.13.** *For any $X\in\widetilde{\mathcal{B}}^q$ or $X\in\widetilde{\mathcal{B}}^\infty$, and any $f,g\in\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$, we have*

$$\int [f]\,\rho^+(X)\,[g] \;=\int [g]\,\rho^-(X)\,[f]\,,\qquad \fint [f]\,\rho^-(X)\,[g] \;=\fint [g]\,\rho^-(X)\,[f]\,.$$

*Proof.* The statement follows from Corollaries 5.11, 5.12 and 4.3. ☐

The $\widetilde{Y}(\mathfrak{gl}_N)$-module structures on $\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ descend to two $Y(\mathfrak{gl}_N)$-module structures on the cohomology with constant coefficients

$$H(\mathbb{C})\,:=\bigoplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},\,|\boldsymbol{\lambda}|=n} H^*(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\,,$$

denoted by the same letters $\rho^+$ and $\rho^-$. The obtained $Y(\mathfrak{gl}_N)$-modules are isomorphic to the irreducible $Y(\mathfrak{gl}_N)$-modules $V^{\langle\pm\rangle}$, and hence isomorphic among themselves, see Proposition 4.9.

**Corollary 5.14.** *The $Y(\mathfrak{gl}_N)$-module $H(\mathbb{C})$ with the $\rho^+$-structure is contravariantly dual to the $Y(\mathfrak{gl}_N)$-module $H(\mathbb{C})$ with the $\rho^-$-structure with respect to the Poincare pairing on $\mathcal{F}_{\boldsymbol{\lambda}}$,*

$$\int [f]\,\rho^+(X)\,[g] \;=\; \int [g]\,\rho^-\bigl(\varpi(X)\bigr)\,[f]$$

*for any $X\in\widetilde{Y}(\mathfrak{gl}_N)$ and any $f,g\in\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$.*

*Proof.* The statement follows from Corollary 5.11. ☐

**Corollary 5.15.** *The $Y(\mathfrak{gl}_N)$-module $H(\mathbb{C})$ with the $\rho^-$-structure is self-dual with respect to the Poincare pairing on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$,*

$$\fint [f]\,\rho^-(X)\,[g] \;=\; \fint [g]\,\rho^-\bigl(\varpi(X)\bigr)\,[f]$$

*for any $X\in\widetilde{Y}(\mathfrak{gl}_N)$ and any $f,g\in\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$.*

*Proof.* The statement follows from Corollary 5.12. ☐

5.5. **Topological interpretation of the Yangian actions on cohomology.** The $\widetilde{Y}(\mathfrak{gl}_N)$-actions $\rho^{\pm}$ on $\bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ have topological interpretations. Consider the partitions $\boldsymbol{\mu}' = (\lambda_1, \dots, \lambda_p - 1, 1, \lambda_{p+1}, \dots, , \lambda_N)$ and $\boldsymbol{\mu}'' = (\lambda_1, \dots, \lambda_p, 1, \lambda_{p+1} - 1, \dots, \lambda_N)$. There are natural forgetful maps

$$\mathcal{F}_{\boldsymbol{\lambda}-\alpha_p} \xleftarrow{\pi_1'} \mathcal{F}_{\boldsymbol{\mu}'} \xrightarrow{\pi_2'} \mathcal{F}_{\boldsymbol{\lambda}}\,, \qquad \mathcal{F}_{\boldsymbol{\lambda}+\alpha_p} \xleftarrow{\pi_1''} \mathcal{F}_{\boldsymbol{\mu}''} \xrightarrow{\pi_2''} \mathcal{F}_{\boldsymbol{\lambda}}\,. \tag{5.18}$$

The rank $\lambda_p - 1$, $1$, $\lambda_{p+1}$ bundles over $\mathcal{F}_{\boldsymbol{\mu}'}$ with fibers $F_p/F_{p-1}$, $F_{p+1}/F_p$, $F_{p+2}/F_{p+1}$ will be respectively denoted by $A'$, $B'$, $C'$. The rank $\lambda_p$, $1$, $\lambda_{p+1} - 1$ bundles over $\mathcal{F}_{\boldsymbol{\mu}''}$ with fibers $F_p/F_{p-1}$, $F_{p+1}/F_p$, $F_{p+2}/F_{p+1}$ will be respectively denoted by $A''$, $B''$, $C''$.

For a $T^n$-equivariant bundle $\xi$, let $e(\xi)$ be its equivariant Euler class. We can make the extra $\mathbb{C}^*$ (whose Chern root is $h$) act on any bundle by fiberwise action with weight $k \cdot h$ ($k \in \mathbb{Z}$). The equivariant Euler class with this extra action will be denoted by $e_{k\cdot h}(\xi)$. Note that $B'$ and $B''$ are line bundles, hence their Euler class is their first Chern class.

Recall that a proper map $f : X \to Y$ induces the pullback $f^* : H^*(Y) \to H^*(X)$ and push-forward (a.k.a. Gysin) $f_* : H^*(X) \to H^*(Y)$ maps on cohomology.

**Theorem 5.16.** *The operators $\rho^{\pm}\big(E_p(u)\big)$, $\rho^{\pm}\big(F_p(u)\big)$, are equal to the following topological operations*

$$\rho^{+}\big(E_p(u)\big) \,:\, x \,\mapsto \pi_{2*}'\left(\pi_1'^{*}(x) \cdot \frac{e_{-h}\big(\mathrm{Hom}(A', B')\big)}{u - e(B')}\right),$$

$$\rho^{+}\big(F_p(u)\big) \,:\, x \,\mapsto \pi_{2*}''\left(\pi_1''^{*}(x) \cdot \frac{e_{-h}\big(\mathrm{Hom}(B'', C'')\big)}{u - e(B'')}\right),$$

$$\rho^{-}\big(E_p(u)\big) \,:\, x \,\mapsto\, (-1)^{\lambda_p - \lambda_{p+1} + 1}\, \pi_{2*}'\left(\pi_1'^{*}(x) \cdot \frac{e_{-h}\big(\mathrm{Hom}(B', C')\big)}{u - e(B')}\right),$$

$$\rho^{-}\big(F_p(u)\big) \,:\, x \,\mapsto\, (-1)^{\lambda_p - \lambda_{p+1} + 1}\, \pi_{2*}''\left(\pi_1''^{*}(x) \cdot \frac{e_{-h}\big(\mathrm{Hom}(A'', B'')\big)}{u - e(B'')}\right).$$

*Proof.* The proof is a straightforward application of the equivariant localization formulation of push-forward maps. We omit the details, because they are completely analogous to the arguments in the Appendix of [RSTV]. Notice that the sign $(-1)^{\lambda_p - \lambda_{p+1} + 1}$ in the formulae for $\rho^-$ comes from the fact that the functions $R(\boldsymbol{z}_I)$ here and $R(\boldsymbol{z}_{I_1}| \dots |\boldsymbol{z}_{I_N})$ in [RSTV] differ by the sign $(-1)^{\sum_{i<j} \lambda_i \lambda_j}$. □

**Remark.** Observe that the Euler classes in the four expressions above are Euler classes of fiberwise tangent or cotangent bundles of some of the forgetful maps (5.18), following the convention of Section 5.1 that $\mathbb{C}^*$ acts on these bundles by fiberwise multiplication.

## 6. Bethe algebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$, and discrete Wronskian

The algebra $H_{\boldsymbol{\lambda}}$, see (5.1), is a free polynomial algebra over $\mathbb{C}$ with generators $h$ and elementary symmetric functions $\sigma_i(\gamma_{p,1}, \dots, \gamma_{p,\lambda_p})$ for $p = 1, \dots, N$, $i = 1, \dots, \lambda_p$. Hence, $H_{\boldsymbol{\lambda}}$ is a free polynomial algebra over $\mathbb{C}$ with generators $h$ and the elements $f_{p,s}$, see (5.12), for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$. Then Theorems 5.4, 5.6 show that the Bethe algebras

$\widetilde{\mathcal{B}}^\infty(\mathcal{V}^+_{\boldsymbol\lambda})$ and $\widetilde{\mathcal{B}}^\infty(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ are free polynomial algebras over $\mathbb{C}$ with generators $h$ and the respective elements $C^\pm_{p,s}$, see (5.13), for $p=1,\dots,N$, $s=1,\dots,\lambda_p$.

In this section we give similar statements for the Bethe algebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$ and $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$. We also formulate counterparts of Theorems 5.4, 5.6, see Theorems 6.3, 6.5 below.

In the rest of the paper we assume that $q_1,\dots,q_N$ are distinct numbers. Set

$$W^q(u) \,=\, \det\biggl(q_i^{N-j}\prod_{k=1}^{\lambda_i}\bigl(u-\gamma_{i,k}+h\,(i-j)\bigr)\biggr)_{i,j=1}^N . \tag{6.1}$$

The function $W^q(u)$ is essentially a Casorati determinant (discrete Wronskian) of functions $g_i(u)=q_i^{u/h}\prod_{j=1}^{\lambda_i}\bigl(u-\gamma_{i,j}+h\,(i-1)\bigr)$,

$$W^q(u) \,=\, \det\Bigl(g_i\bigl(u-h\,(j-1)\bigr)\Bigr)_{i,j=1}^N\,\prod_{i=1}^N q_i^{N-1-u/h} .$$

Define the algebra

$$\mathcal{H}^q_{\boldsymbol\lambda} \,=\, \mathbb{C}[\boldsymbol z;\Gamma]^{S_n\times S_{\lambda_1}\times\ldots\times S_{\lambda_N}}\otimes\mathbb{C}[h]\Big/\Big\langle W^q(u) \,=\prod_{1\leqslant i<j\leqslant N}(q_i-q_j)\,\prod_{a=1}^n(u-z_a)\Big\rangle . \tag{6.2}$$

For example, if $N=n=2$ and $\boldsymbol\lambda=(1,1)$, then the relations are

$$\gamma_{1,1}+\gamma_{2,1} \,=\, z_1+z_2\,, \qquad \gamma_{1,1}\,\gamma_{2,1}+\frac{q_2}{q_1-q_2}\,h\,(\gamma_{1,1}-\gamma_{2,1}+h) \,=\, z_1\,z_2\,.$$

It is easy to see that the subalgebra $\mathcal{H}^q_{\boldsymbol\lambda}$ does not change if all $q_1,\dots,q_N$ are multiplied simultaneously by the same number. Notice that in the limit $q_{i+1}/q_i\to 0$ for all $i=1,\dots,N-1$, the relations in $\mathcal{H}^q_{\boldsymbol\lambda}$ turn into the relations in $H_{\boldsymbol\lambda}$, see (5.1).

Below we describe isomorphisms of the regular representation of $\mathcal{H}^q_{\boldsymbol\lambda}$ with the $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$-module $\mathcal{V}^+_{\boldsymbol\lambda}$, as well as with the $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$.

Let $x$ be a complex variable. Set

$$\widehat{W}^q(u,x) \,=\, \det\biggl(q_i^{N-j}\prod_{k=1}^{\lambda_i}\bigl(u-\gamma_{i,k}+h\,(i-j)\bigr)\biggr)_{i,j=0}^N , \tag{6.3}$$

where $q_0=x$ and $\lambda_0=0$. Clearly, $\widehat{W}^q(u,x)=x^N\,W^q(u)+\ldots+(-1)^N\,W^q(u+h)$. Define the elements $W^q_{p,s}\in\mathcal{H}^q_{\boldsymbol\lambda}$ for $p=1,\dots,N$, $s\in\mathbb{Z}_{>0}$, by the rule

$$\frac{\widehat{W}^q(u,x)}{W^q(u)} \,=\, \prod_{i=1}^N(x-q_i)\,+\,h\sum_{p=1}^N\sum_{s=1}^\infty(-1)^p\,W^q_{p,s}\,x^{N-p}\,u^{-s} . \tag{6.4}$$

Define also the elements $U^q_{i,s}\in\mathcal{H}^q_{\boldsymbol\lambda}$ for $i=1,\dots,N$, $s\in\mathbb{Z}_{>0}$ by the rule

$$U^q_{i,s} \,=\, \sum_{p=1}^N(-1)^{p-1}\,W^q_{p,s}\,q_i^{N-p-1}\prod_{\substack{j=1\\ j\neq i}}^N\frac{1}{q_i-q_j}\,, \tag{6.5}$$

so that

$$\frac{\widehat{W}^q(u,x)}{W^q(u)} \prod_{i=1}^{N} \frac{1}{x-q_i} \,=\, 1 \,-\, h \sum_{i=1}^{N} \sum_{s=1}^{\infty} \frac{q_i}{x-q_i}\, U^q_{i,s}\, u^{-s}\,.$$

**Proposition 6.1.** *The algebra* $\mathcal{H}^q_{\boldsymbol\lambda}$ *is a free polynomial algebra over* $\mathbb{C}$ *with generators* $h$ *and* $U^q_{i,s}$ *for* $i=1,\dots,N,\ s=2,\dots,\lambda_i+1$.

*Proof.* Formulae (6.1) and (6.3) – (6.5) imply that $U^q_{i,1}=\lambda_i$ for all $i=1,\dots,N,$ and

$$U^q_{i,s} \,=\, (s-1)\,\sigma_{s-1}(\gamma_{i,1},\dots,\gamma_{i,\lambda_i}) \,+\, U'^q_{i,s}\,, \qquad s\geqslant 1\,, \tag{6.6}$$

where $\sigma_{s-1}$ is the $(s-1)$-th elementary symmetric function and $U'^q_{i,s}$ is a polynomial in $h$ and $\sigma_r(\gamma_{j,1},\dots,\gamma_{j,\lambda_j})$ for $r<s-1$ and $j=1,\dots,N$. For instance,

$$U^q_{i,2} \,=\, \sigma_1(\gamma_{i,1},\dots,\gamma_{i,\lambda_i}) \,-\, h\,i\,\lambda_i + \frac{h}{2}\,\lambda_i\,(\lambda_i+1) \,-\, h\,\lambda_i \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{q_j\,(\lambda_j+1)}{q_i-q_j}\,. \tag{6.7}$$

Since $\mathcal{H}^q_{\boldsymbol\lambda}$ is clearly a free polynomial algebra over $\mathbb{C}$ with generators $h$ and $\sigma_s(\gamma_{i,1},\dots,\gamma_{i,\lambda_i})$ for $i=1,\dots,N,\ s=1,\dots,\lambda_i$, the statement follows. □

**Corollary 6.2.** *The elements* $W^q_{p,s}$ *for* $p=1,\dots,N,\ s\geqslant 2$, *together with* $\mathbb{C}[h]$ *generate the algebra* $\mathcal{H}^q_{\boldsymbol\lambda}$. □

**Theorem 6.3** ([MTV5])**.** *The assignment* $\mu^{q+}_{\boldsymbol\lambda} : W^q_{p,s} \mapsto \phi^+(h^{s-1}B^q_{p,s}),\ p=1,\dots,N,\ s>0,$ *extends uniquely to an isomorphism of* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras* $\mu^{q+}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$. *The map*

$$\nu^{q+}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \mathcal{V}^+_{\boldsymbol\lambda}\,, \qquad \nu^{q+}_{\boldsymbol\lambda} : [f] \,\mapsto\, \mu^{q+}_{\boldsymbol\lambda}([f])\,v^+_{\boldsymbol\lambda}\,, \tag{6.8}$$

*is an isomorphism of vector spaces identifying the* $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$*-module* $\mathcal{V}^+_{\boldsymbol\lambda}$ *and the regular representation of* $\mathcal{H}^q_{\boldsymbol\lambda}$.

**Theorem 6.4** ([MTV5])**.** *The assignment* $\mu^{q=}_{\boldsymbol\lambda} : W^q_{p,s} \mapsto \phi^+(h^{s-1}B^q_{p,s}),\ p=1,\dots,N,\ s>0,$ *extends uniquely to an isomorphism of* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras* $\mu^{q=}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda})$. *The map*

$$\nu^{q=}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \tfrac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}\,, \qquad \nu^{q=}_{\boldsymbol\lambda} : [f] \,\mapsto\, \mu^{q=}_{\boldsymbol\lambda}([f])\,v^=_{\boldsymbol\lambda}\,, \tag{6.9}$$

*is an isomorphism of vector spaces identifying the* $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda})$*-module* $\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}$ *and the regular representation of* $\mathcal{H}^q_{\boldsymbol\lambda}$.

**Theorem 6.5** ([MTV5])**.** *The assignment* $\mu^{q-}_{\boldsymbol\lambda} : W^q_{p,s} \mapsto \phi^-(h^{s-1}B^q_{p,s}),\ p=1,\dots,N,\ s>0,$ *extends uniquely to an isomorphism of* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$*-algebras* $\mu^{q-}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$. *The map*

$$\nu^{q-}_{\boldsymbol\lambda} : \mathcal{H}^q_{\boldsymbol\lambda} \to \tfrac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}\,, \qquad \nu^{q-}_{\boldsymbol\lambda} : [f] \,\mapsto\, \mu^{q-}_{\boldsymbol\lambda}([f])\,v^-_{\boldsymbol\lambda}\,, \tag{6.10}$$

*is an isomorphism of vector spaces identifying the* $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$*-module* $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$ *and the regular representation of* $\mathcal{H}^q_{\boldsymbol\lambda}$.

The proofs of Theorems 6.3 – 6.5 are similar to the proof of Theorem 6.7 in [MTV3] and will be published elsewhere.

Theorems 6.3 – 6.5 are instances of the discrete geometric Langlands correspondence, see the corresponding versions of the geometric Langlands correspondence in [MTV2, MTV3], cf. also [RSTV].

Theorems 6.4 and 6.5 are equivalent by Lemma 4.5. In particular,

$$\widetilde{\Pi}\, \mu^{q=}_{\boldsymbol\lambda}(f) \,=\, \mu^{q-}_{\boldsymbol\lambda}(f)\, \widetilde{\Pi}\,, \qquad \widetilde{\Pi}\, \nu^{q=}_{\boldsymbol\lambda} \,=\, \nu^{q-}_{\boldsymbol\lambda} \tag{6.11}$$

where $\widetilde{\Pi}$ is given by (4.4) and $f \in \mathcal{H}^q_{\boldsymbol\lambda}$.

**Corollary 6.6.** *The $\widetilde{\mathcal{B}}^q$-modules $\mathcal{V}^+$, $\frac{1}{D}\mathcal{V}^=$, and $\frac{1}{D}\mathcal{V}^-$ are isomorphic.*

*Proof.* The isomorphisms restricted to weight subspaces are $\nu^{q=}_{\boldsymbol\lambda}(\nu^{q+}_{\boldsymbol\lambda})^{-1} : \mathcal{V}^+_{\boldsymbol\lambda} \to \frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}$ and $\nu^{q-}_{\boldsymbol\lambda}(\nu^{q+}_{\boldsymbol\lambda})^{-1} : \mathcal{V}^+_{\boldsymbol\lambda} \to \frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$. □

**Corollary 6.7.** *The subalgebras $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda}) \subset \mathrm{End}(\mathcal{V}^+_{\boldsymbol\lambda})$, $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}) \subset \mathrm{End}(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda})$ and $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}) \subset \mathrm{End}(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ are maximal commutative subalgebras.* □

**Corollary 6.8.** *The Bethe algebra $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$ is a free polynomial algebra over $\mathbb{C}$ with generators $h$ and $\phi^+(S^q_{p,s})$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$, where the elements $S^q_{p,s} \in \widetilde{\mathcal{B}}^q$ are defined by* (3.9)*. The Bethe algebra $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda})$ is a free polynomial algebra over $\mathbb{C}$ with generators $h$ and $\phi^+(S^q_{p,s})$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$. The Bethe algebra $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$ is a free polynomial algebra over $\mathbb{C}$ with generators $h$ and $\phi^-(S^q_{p,s})$ for $p = 1, \dots, N$, $s = 1, \dots, \lambda_p$.*

*Proof.* Since $\mu^{q+}_{\boldsymbol\lambda}(U_{i,s}) = \mu^{q=}_{\boldsymbol\lambda}(U_{i,s}) = \phi^+(S^q_{i,s})$ and $\mu^{q-}_{\boldsymbol\lambda}(U_{i,s}) = \phi^-(S^q_{i,s})$, see (6.5), (3.9), the statement follows from Theorems 6.3 – 6.5, and Proposition 6.1. □

**Lemma 6.9.** *For any $f \in H_{\boldsymbol\lambda}$, and any $g_1 \in \mathcal{V}^+_{\boldsymbol\lambda}$, $g_2 \in \frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}$, $g_3 \in \frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda}$, we have*

$$\mathcal{S}\big(\mu^{q+}_{\boldsymbol\lambda}(f)\, g_1, g_3\big) \,=\, \mathcal{S}\big(g_1, \mu^{q-}_{\boldsymbol\lambda}(f)\, g_3\big)\,, \qquad \mathcal{S}\big(\mu^{q=}_{\boldsymbol\lambda}(f)\, g_2, g_3\big) \,=\, \mathcal{S}\big(g_2, \mu^{q-}_{\boldsymbol\lambda}(f)\, g_3\big)\,.$$

*Proof.* The statement follows from the definition of the maps $\mu^{\pm}_{\boldsymbol\lambda}$ and Corollary 4.3. □

Recall that $\widetilde{Y}(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a subalgebra. Denote by $\widetilde{Y}(\mathfrak{gl}_N)^{\mathfrak h}$ the subalgebra of $\widetilde{Y}(\mathfrak{gl}_N)$ commuting with $U(\mathfrak h)$, where $\mathfrak h \subset \mathfrak{gl}_N$ in the Lie subalgebra generated by $e_{i,i}$, $i = 1, \dots, N$. The isomorphisms $\nu^{q\pm}_{\boldsymbol\lambda}$ induce homomorphisms

$$\rho^{q\pm}_{\boldsymbol\lambda} : \widetilde{Y}(\mathfrak{gl}_N)^{\mathfrak h} \to \mathrm{End}(\mathcal{H}^q_{\boldsymbol\lambda})\,, \qquad \rho^{q\pm}_{\boldsymbol\lambda} : X \,\mapsto\, (\nu^{q\pm}_{\boldsymbol\lambda})^{-1}\, \phi^{\pm}(X)\, \nu^{q\pm}_{\boldsymbol\lambda}\,. \tag{6.12}$$

By relations (6.11) and (4.5), we also have $\rho^{q-}(X) \,=\, (\nu^{q=}_{\boldsymbol\lambda})^{-1}\, \phi^+(X)\, \nu^{q=}_{\boldsymbol\lambda}$.

**Lemma 6.10.** *For $i = 1, \dots, N$, we have*

$$\rho^{q\pm}_{\boldsymbol\lambda}(X^{q-}_i) \,:\, [f] \;\mapsto\; \bigg[ f(\boldsymbol z; \Gamma; h) \bigg( \sum_{k=1}^{\lambda_i} \gamma_{i,k} - h \sum_{j=i+1}^{N} \frac{q_j}{q_i - q_j}\, (\lambda_i - \lambda_j) \bigg) \bigg]\,.$$

*Proof.* The statement follow from formulae (6.7), (3.13), (3.17) because $\rho^{q\pm}_{\boldsymbol\lambda}(S^q_{i,2})$ acts as multiplication by the element $U^q_{i,2}$. ☐

Define pairings

$$(\,,\,):\,\mathcal{H}^q_{\boldsymbol\lambda}\otimes\mathcal{H}^q_{\boldsymbol\lambda}\,\to\,\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]\,,\qquad (f,g)\,=\,\mathcal{S}(\nu^{q+}_{\boldsymbol\lambda}f\,,\,\nu^{q-}_{\boldsymbol\lambda}g)\,. \tag{6.13}$$

and

$$\langle\,,\,\rangle:\,\mathcal{H}^q_{\boldsymbol\lambda}\otimes\mathcal{H}^q_{\boldsymbol\lambda}\,\to\,Z^{-1}\,\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]\,,\qquad \langle f,g\rangle\,=\,\mathcal{S}(\nu^{q=}_{\boldsymbol\lambda}f\,,\,\nu^{q-}_{\boldsymbol\lambda}g)\,. \tag{6.14}$$

Here $Z=\prod_{i\neq j}(z_i-z_j+h)$, cf. (2.14).

**Lemma 6.11.** *Pairings* (6.13) *and* (6.14) *are symmetric and invariant,*

$$(f_1\,,f_2)\,=\,(f_2\,,f_1)\,,\qquad (f_1\,f_2\,,f_3)\,=\,(f_2\,,f_1\,f_3)\,,$$
$$\langle f_1\,,f_2\rangle\,=\,\langle f_2\,,f_1\rangle\,,\qquad \langle f_1\,f_2\,,f_3\rangle\,=\,\langle f_2\,,f_1\,f_3\rangle\,,$$

*for any* $f_1\,,f_2\,,f_3\in\mathcal{H}^q_{\boldsymbol\lambda}$.

*Proof.* Theorems 6.3 – 6.5 and Corollary 4.3 yield $(f_1\,,f_2)\,=\,(1,f_1\,f_2)$ and $\langle f_1\,,f_2\rangle\,=\,\langle 1,f_1\,f_2\rangle$, which proves the statement. ☐

**Example.** Let $N=n=2$, $\boldsymbol\lambda=(1,1)$, so that $V_{\boldsymbol\lambda}=\mathbb{C}\,v_{(1,2)}\oplus\mathbb{C}\,v_{(2,1)}$. Then

$$\mu^{q+}_{\boldsymbol\lambda}:[1]\,\mapsto\,\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \mu^{q+}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\begin{pmatrix}z_1&h\\0&z_2\end{pmatrix}+\frac{q_2}{q_1-q_2}\begin{pmatrix}h&h\\h&h\end{pmatrix},$$

$$\nu^{q+}_{\boldsymbol\lambda}:[1]\,\mapsto\,v_{(1,2)}+v_{(2,1)}\,,$$

$$\nu^{q+}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\Big(z_1+h\,\frac{q_1+q_2}{q_1-q_2}\Big)\,v_{(1,2)}+\Big(z_2+\frac{2\,h\,q_2}{q_1-q_2}\Big)\,v_{(2,1)}\,,$$

$$\mu^{q=}_{\boldsymbol\lambda}:[1]\,\mapsto\,\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \mu^{q=}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\begin{pmatrix}z_1&h\\0&z_2\end{pmatrix}+\frac{q_2}{q_1-q_2}\begin{pmatrix}h&h\\h&h\end{pmatrix},$$

$$\nu^{q=}_{\boldsymbol\lambda}:[1]\,\mapsto\,\frac{v_{(1,2)}-v_{(2,1)}}{z_1-z_2-h}\,,\qquad \nu^{q=}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\frac{(z_1-h)\,v_{(1,2)}-z_2\,v_{(2,1)}}{z_1-z_2-h}\,,$$

$$\mu^{q-}_{\boldsymbol\lambda}:[1]\,\mapsto\,\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \mu^{q-}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\begin{pmatrix}z_1&0\\h&z_2\end{pmatrix}+\frac{q_2}{q_1-q_2}\begin{pmatrix}h&h\\h&h\end{pmatrix},$$

$$\nu^{q-}_{\boldsymbol\lambda}:[1]\,\mapsto\,\frac{v_{(1,2)}-v_{(2,1)}}{z_1-z_2+h}\,,\qquad \nu^{q-}_{\boldsymbol\lambda}:[\gamma_{1,1}]\,\mapsto\,\frac{z_1\,v_{(1,2)}-(z_2-h)\,v_{(2,1)}}{z_1-z_2+h}\,,$$

$$(1,1)\,=\,0\,,\qquad (1,\gamma_{1,1})\,=\,1\,,\qquad (\gamma_{1,1},\gamma_{1,1})\,=\,z_1+z_2+\frac{2\,h\,q_2}{q_1-q_2}\,.$$

$$\langle 1,1\rangle \,=\, \frac{2}{(z_1-z_2+h)\,(z_1-z_2-h)}\,, \qquad \langle 1,\gamma_{1,1}\rangle \,=\, \frac{z_1+z_2-h}{(z_1-z_2+h)\,(z_1-z_2-h)}\,,$$

$$\langle \gamma_{1,1}\,,\gamma_{1,1}\rangle \,=\, \frac{z_1^2+z_2^2-h\,(z_1+z_2)}{(z_1-z_2+h)\,(z_1-z_2-h)}\,.$$

Notice that up to a simple change of basis, the form $\langle\,,\rangle$ in this example coincides with the bilinear form $\langle\,,\rangle$ in [BGr, Section 3.2], namely, with the form associated with the equivariant quantum cohomology of the cotangent bundle $T^*\mathbb{P}^1$ of the projective line $\mathbb{P}^1$. That form was used in [BMO], see also [BGh].

## 7. Quantum multiplication on $H_{\boldsymbol\lambda}$

Recall that $q_1,\dots,q_N$ are distinct numbers. Let $\beta^\pm_{\boldsymbol\lambda}: H_{\boldsymbol\lambda}\to\mathcal{H}^q_{\boldsymbol\lambda}$ be the following isomorphisms of vector spaces

$$\beta^+_{\boldsymbol\lambda} = (\nu^{q+}_{\boldsymbol\lambda})^{-1}\,\nu^+_{\boldsymbol\lambda}\,, \qquad \beta^-_{\boldsymbol\lambda} = (\nu^{q-}_{\boldsymbol\lambda})^{-1}\,\nu^-_{\boldsymbol\lambda}\,.$$

We also have $\beta^-_{\boldsymbol\lambda} = (\nu^{q=}_{\boldsymbol\lambda})^{-1}\nu^=_{\boldsymbol\lambda}$, because $\nu^-_{\boldsymbol\lambda} = \widetilde\Pi\,\nu^=_{\boldsymbol\lambda}$ and $\nu^{q-}_{\boldsymbol\lambda} = \widetilde\Pi\,\nu^{q=}_{\boldsymbol\lambda}$, where $\widetilde\Pi$ is given by (4.4). Notice that $\beta^\pm_{\boldsymbol\lambda}(1) = 1\in\mathcal{H}^q_{\boldsymbol\lambda}$.

Define new multiplications $\star$ and $\bullet$ on $H_{\boldsymbol\lambda}$ by the rule

$$\beta^+_{\boldsymbol\lambda}(f\star g) \,=\, \beta^+_{\boldsymbol\lambda}(f)\,\beta^+_{\boldsymbol\lambda}(g)\,, \qquad \beta^-_{\boldsymbol\lambda}(f\bullet g) \,=\, \beta^-_{\boldsymbol\lambda}(f)\,\beta^-_{\boldsymbol\lambda}(g) \tag{7.1}$$

for any $f,g\in H_{\boldsymbol\lambda}$. The defined multiplications are associative and commutative. The products $f\star g$ and $f\bullet g$ tend to the ordinary product $fg$ as $q_{i+1}/q_i\to 0$ for every $i=1,\dots,N-1$,

Denote $f\star$ and $f\bullet$ the corresponding operators on $H_{\boldsymbol\lambda}$, $f\star: g\mapsto f\star g$, $f\bullet: g\mapsto f\bullet g$.

Recall the $\widetilde Y(\mathfrak{gl}_N)$-actions $\rho^\pm$ on $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$, defined in Section 5.4. Given $X\in\widetilde Y(\mathfrak{gl}_N)^{\mathfrak h}$, denote by $\rho^\pm_{\boldsymbol\lambda}(X)\in\mathrm{End}(H_{\boldsymbol\lambda})$ the restriction of $\rho^\pm(X)$ to $H_{\boldsymbol\lambda}$.

**Lemma 7.1.** *Let $f\in H_{\boldsymbol\lambda}$ and $X\in\widetilde{\mathcal B}^q$. If $\nu^+_{\boldsymbol\lambda}(f) = \phi^+(X)\,v^+_{\boldsymbol\lambda}$, then $f\star = \rho^+_{\boldsymbol\lambda}(X)$. Similarly, if $\nu^=_{\boldsymbol\lambda}(f) = \phi^+(X)\,v^=_{\boldsymbol\lambda}$ or $\nu^-_{\boldsymbol\lambda}(f) = \phi^-(X)\,v^-_{\boldsymbol\lambda}$, then $f\bullet = \rho^-_{\boldsymbol\lambda}(X)$.*

*Proof.* We will prove the statement for the map $\nu^-_{\boldsymbol\lambda}$. Other statements are proved similarly. Since $\nu^-_{\boldsymbol\lambda}(g) = \nu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-_{\boldsymbol\lambda}(g)\bigr)$ for any $g\in H_{\boldsymbol\lambda}$, we have

$$\nu^-_{\boldsymbol\lambda}(f\bullet g) \,=\, \nu^-_{\boldsymbol\lambda}(g\bullet f) \,=\, \mu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-_{\boldsymbol\lambda}(g)\bigr)\,\nu^-_{\boldsymbol\lambda}(f) \,=\, \mu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-_{\boldsymbol\lambda}(g)\bigr)\,\phi^-(X)\,v^-_{\boldsymbol\lambda} \,=$$

$$=\, \phi^-(X)\,\mu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-_{\boldsymbol\lambda}(g)\bigr)\,v^-_{\boldsymbol\lambda} \,=\, \phi^-(X)\,\nu^-_{\boldsymbol\lambda}(g) \,=\, \nu^-_{\boldsymbol\lambda}\bigl(\rho^-(X)\,g\bigr)\,,$$

which proves the claim. □

Denote by $\widetilde{\mathcal B}^{q\pm}_{\boldsymbol\lambda}$ the images in $\mathrm{End}(H_{\boldsymbol\lambda})$ of the Bethe subalgebra $\widetilde{\mathcal B}^q\subset\widetilde Y(\mathfrak{gl}_N)$ under the homomorphisms $\rho^\pm$, respectively. The algebra $\widetilde{\mathcal B}^{q+}_{\boldsymbol\lambda}$ is isomorphic to $\widetilde{\mathcal B}^q(\mathcal V^+_{\boldsymbol\lambda})$, and the algebra $\widetilde{\mathcal B}^{q-}_{\boldsymbol\lambda}$ is isomorphic to $\widetilde{\mathcal B}^q(\frac1D\mathcal V^=_{\boldsymbol\lambda})$ and to $\widetilde{\mathcal B}^q(\frac1D\mathcal V^-_{\boldsymbol\lambda})$.

**Corollary 7.2.** *For any $f \in H_{\boldsymbol\lambda}$, the operator $f\star$ is the unique element of $\widetilde{\mathcal B}^{q+}_{\boldsymbol\lambda}$ which sends $1 \in H_{\boldsymbol\lambda}$ to $f$, and the operator $f\bullet$ is the unique element of $\widetilde{\mathcal B}^{q-}_{\boldsymbol\lambda}$ which sends $1 \in H_{\boldsymbol\lambda}$ to $f$.* □

**Corollary 7.3.** *For every $i = 1, \dots, N$, we have*

$$(\gamma_{i,1} + \ldots + \gamma_{i,\lambda_i})\star = \rho^+_{\boldsymbol\lambda}(X^{q+}_i), \qquad (\gamma_{i,1} + \ldots + \gamma_{i,\lambda_i})\bullet = \rho^-_{\boldsymbol\lambda}(X^{q-}_i), \tag{7.2}$$

*where $\gamma_{i,j}$ are the Chern roots, see* (5.1), *and $X^{q\pm}_i \in \widetilde{\mathcal B}^q$ are given by* (3.16), (3.17).

*Proof.* The statement follows from Lemmas 7.1 and 4.18. □

**Proposition 7.4.** *The map $\chi^+_{\boldsymbol\lambda} : f \mapsto f\star$ is an isomorphism $H_{\boldsymbol\lambda} \to \widetilde{\mathcal B}^{q+}_{\boldsymbol\lambda}$ of vector spaces. The map $\beta^+_{\boldsymbol\lambda}(\chi^+_{\boldsymbol\lambda})^{-1} : \widetilde{\mathcal B}^{q+}_{\boldsymbol\lambda} \to \mathcal H^q_{\boldsymbol\lambda}$ is an isomorphism of $\mathbb C[z_1, \dots, z_n]^{S_n} \otimes \mathbb C[h]$-algebras. The maps $\beta^+_{\boldsymbol\lambda}(\chi^+_{\boldsymbol\lambda})^{-1}$ and $\beta^+_{\boldsymbol\lambda}$ identify the $\widetilde{\mathcal B}^{q+}_{\boldsymbol\lambda}$-module $H_{\boldsymbol\lambda}$ with the regular representation of the algebra $\mathcal H^q_{\boldsymbol\lambda}$.* □

**Proposition 7.5.** *The map $\chi^-_{\boldsymbol\lambda} : f \mapsto f\bullet$ is an isomorphism $H_{\boldsymbol\lambda} \to \widetilde{\mathcal B}^{q-}_{\boldsymbol\lambda}$ of vector spaces. The map $\beta^-_{\boldsymbol\lambda}(\chi^-_{\boldsymbol\lambda})^{-1} : \widetilde{\mathcal B}^{q-}_{\boldsymbol\lambda} \to \mathcal H^q_{\boldsymbol\lambda}$ is an isomorphism of $\mathbb C[z_1, \dots, z_n]^{S_n} \otimes \mathbb C[h]$-algebras. The maps $\beta^-_{\boldsymbol\lambda}(\chi^-_{\boldsymbol\lambda})^{-1}$ and $\beta^-_{\boldsymbol\lambda}$ identify the $\widetilde{\mathcal B}^{q-}_{\boldsymbol\lambda}$-module $H_{\boldsymbol\lambda}$ with the regular representation of the algebra $\mathcal H^q_{\boldsymbol\lambda}$.* □

Consider $q_1, \dots, q_n$ as variables. For a nonzero complex number $\kappa$, define the connections $\nabla^\star$ and $\nabla^\bullet$

$$\nabla^\star_i = \kappa\, q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \ldots + \gamma_{i,\lambda_i})\star, \qquad \nabla^\bullet_i = \kappa\, q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \ldots + \gamma_{i,\lambda_i})\bullet, \tag{7.3}$$

$i = 1, \dots, n$.

**Proposition 7.6.** *The connections $\nabla^\star$ and $\nabla^\bullet$ are flat for any $\kappa$.*

*Proof.* The statement follows from Corollary 7.3 and Lemma 3.5. □

**Lemma 7.7.** *For any $f, g \in H_{\boldsymbol\lambda}$, we have*

$$⨍ f g = ⨍ f \bullet g = \langle \beta^-_{\boldsymbol\lambda}(f), \beta^-_{\boldsymbol\lambda}(g)\rangle,$$

*where the pairing $\langle\,,\rangle$ on $\mathcal H^q_{\boldsymbol\lambda}$ is defined by* (6.14).

*Proof.* Propositions 5.2, 5.3 and Lemma 6.11 imply

$$⨍ f g = \mathcal S\bigl(\nu^=_{\boldsymbol\lambda}(f), \nu^-_{\boldsymbol\lambda}(g)\bigr) = \mathcal S\bigl(\nu^{q=}_{\boldsymbol\lambda}\bigl(\beta^-(f)\bigr), \nu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-(g)\bigr)\bigr) = \langle \beta^-_{\boldsymbol\lambda}(f), \beta^-_{\boldsymbol\lambda}(g)\rangle =$$

$$= \langle 1, \beta^-_{\boldsymbol\lambda}(f)\beta^-_{\boldsymbol\lambda}(g)\rangle = \mathcal S\bigl(v^=_{\boldsymbol\lambda}, \nu^{q-}_{\boldsymbol\lambda}\bigl(\beta^-_{\boldsymbol\lambda}(f)\beta^-_{\boldsymbol\lambda}(g)\bigr)\bigr) = \mathcal S\bigl(v^=_{\boldsymbol\lambda}, \nu^-_{\boldsymbol\lambda}(f\bullet g)\bigr) = ⨍ f\bullet g.$$ □

**Corollary 7.8.** *For any $f_1, f_2, f_3 \in H_{\boldsymbol\lambda}$, we have*

$$⨍ (f_1 \bullet f_2)\, f_3 = ⨍ f_1 \bullet f_2 \bullet f_3 = ⨍ f_2\,(f_1 \bullet f_3).$$ □

**Corollary 7.9.** *The operators* $f\bullet$ *are symmetric with respect to pairing* (5.7).

**Conjecture 7.10.** *The space* $H_{\boldsymbol\lambda}$ *with the multiplication* $\bullet$ *and pairing* (5.7) *is the small equivariant quantum cohomology algebra* $QH_{GL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})$ *of the cotangent bundle* $T^*\mathcal{F}_{\boldsymbol\lambda}$ *of a partial flag variety* $\mathcal{F}_{\boldsymbol\lambda}$.

We will verify this conjecture for $N=n$ and $\boldsymbol\lambda=(1,\dots,1)$ in Section 8.4.

Observe that under Conjecture 7.10, the connection $\nabla^\bullet$ is the quantum connection on $H_{\boldsymbol\lambda}$.

**Corollary 7.11.** *The small equivariant quantum cohomology algebra* $QH_{GL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol\lambda};\mathbb{C})$ *of the cotangent bundle* $T^*\mathcal{F}_{\boldsymbol\lambda}$ *of a partial flag variety* $\mathcal{F}_{\boldsymbol\lambda}$ *is isomorphic to the algebra* $\mathcal{H}^q_{\boldsymbol\lambda}$.

This corollary gives a description of the small equivariant quantum cohomology algebra by generators and relations, see formula (6.2).

## 8. Special case $N=n$, $\boldsymbol\lambda=(1,\dots,1)$, and Hecke algebra

In this section we consider the special case when $\mathcal{F}_{\boldsymbol\lambda}$ is the variety of full flags in $\mathbb{C}^n$.

8.1. **Hecke algebra.** Consider the algebra $\mathfrak{H}_n$ generated by the central element $c$, pairwise commuting elements $y_1,\dots,y_n$, and elements of the symmetric group $S_n$, subject to relations

$$s_i\,y_i-y_{i+1}\,s_i\,=\,c\,,\qquad i=1,\dots,n-1\,,\tag{8.1}$$
$$s_i\,y_j-y_j\,s_i\,=\,0\,,\qquad j\neq i,i+1\,,$$

where $s_i$ is the transposition of $i$ and $i+1$. For every nonzero complex number $t$, the quotient algebra $\mathfrak{H}_n/(c=t)$ is the degenerate affine Hecke algebra corresponding to the parameter $t$.

For $i=1,\dots,n$, set

$$Y^q_i\,=\,y_i+c\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,s_{i,j}\,+\,c\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,s_{i,j}\,.\tag{8.2}$$

where $s_{i,j}\in\mathfrak{H}_n$ is the transposition of $i$ and $j$. It is known that for any nonzero complex number $p$, the formal differential operators

$$\nabla^{\mathfrak{H}}_i\,=\,\kappa\,q_i\frac{\partial}{\partial q_i}\,-\,Y^q_i,\qquad i=1,\dots,N,\tag{8.3}$$

pairwise commute and, hence, define a flat connection for any $\mathfrak{H}_n$-module. This connection is called the *affine KZ connection*, see [C2, formula (1.1.41)].

8.2. **The case** $N=n$ **and** $\boldsymbol\lambda=(1,\dots,1)$**.** Recall that $q_1,\dots,q_N$ are distinct numbers.

**Theorem 8.1.** *Let* $N=n$ *and* $\boldsymbol\lambda=(1,\dots,1)$. *Then the operators* $\phi^+(X^q_i)$, $i=1,\dots,N$, *and* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$ *generate the Bethe algebra* $\widetilde{\mathcal{B}}^q(\mathcal{V}^+_{\boldsymbol\lambda})$. *Similarly, the operators* $\phi^+(X^q_i)$, $i=1,\dots,N$, *and* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$ *generate the Bethe algebra* $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^=_{\boldsymbol\lambda})$, *and the operators* $\phi^-(X^q_i)$, $i=1,\dots,N$, *and* $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$ *generate the Bethe algebra* $\widetilde{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$.

*Proof.* The statement follows from formula (3.13) and Corollary 6.8. □

For any $\sigma \in S_n$ define $w_\sigma \in \mathrm{End}(V)$ by the rule

$$w_\sigma\,(v_{i_1}\otimes\ldots\otimes v_{i_n}) \,=\, v_{\sigma(i_1)}\otimes\ldots\otimes v_{\sigma(i_n)}$$

for any sequence $i_1,\ldots,i_n$. The operators $w_\sigma$ define the action of $S_n$ on $V$. We will write $w_{(i,j)} = w_\sigma$ for $\sigma$ being the transposition of $i$ and $j$.

Recall the elements $G_{i,j} \,=\, e_{i,j}\,e_{j,i} - e_{i,i}$.

**Lemma 8.2.** *Let $N = n$ and $\boldsymbol{\lambda} = (1,\ldots,1)$. For any $i \neq j$, the element $G_{i,j}$ acts on $V_{\boldsymbol{\lambda}}$ as $w_{(i,j)}$.* □

**Proposition 8.3.** *Let $N = n$ and $\boldsymbol{\lambda} = (1,\ldots,1)$. The assignments $y_i \mapsto \phi^+(X_i^\infty)$, $c \mapsto h$, $s_j \mapsto G_{(j,j+1)}$, $i = 1,\ldots,n$, $j = 1,\ldots,n-1$, define a representation of $\mathfrak{H}_n$ on $\mathcal{V}^+_{\boldsymbol{\lambda}}$. Similarly, the assignments $y_i \mapsto \phi^-(X_i^\infty)$, $c \mapsto -h$, $s_j \mapsto -G_{(j,j+1)}$, $i = 1,\ldots,n$, $j = 1,\ldots,n-1$, define a representation of $\mathfrak{H}_n$ on $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$.*

*Proof.* The verification is straightforward using formulae (4.13), (8.1), and Lemma 8.2. □

Denote the obtained representations of $\mathfrak{H}_n$ on $\mathcal{V}^+_{\boldsymbol{\lambda}}$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ respectively by $\psi^+$ and $\psi^-$. By Proposition 8.3 and formulae (3.15), (8.2), we have

$$\phi^\pm(X_i^q) \,=\, \psi^\pm(Y_i^q)\,, \qquad i = 1,\ldots,n\,. \tag{8.4}$$

So for $N = n$ and $\boldsymbol{\lambda} = (1,\ldots,1)$, the trigonometric dynamical connection (3.18) defined on $\mathcal{V}^+_{\boldsymbol{\lambda}}$ through $\phi^+$ coincides with the affine *KZ* connection (8.3) defined on $\mathcal{V}^+_{\boldsymbol{\lambda}}$ through $\psi^+$, and similarly, the trigonometric dynamical connection (3.18) defined on $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ through $\phi^-$ coincides with the affine *KZ* connection (8.3) defined on $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ through $\psi^-$.

**Lemma 8.4.** *Let $N = n$ and $\boldsymbol{\lambda} = (1,\ldots,1)$. For the vectors $v^+_{\boldsymbol{\lambda}} \in \mathcal{V}^+_{\boldsymbol{\lambda}}$ and $v^-_{\boldsymbol{\lambda}} \in \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, we have*

$$v^+_{\boldsymbol{\lambda}} = \sum_{\sigma\in S_n} v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}\,, \qquad v^-_{\boldsymbol{\lambda}} \,=\, \frac{1}{D}\sum_{\sigma\in S_n}(-1)^\sigma\, v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}\,. \tag{8.5}$$

*Thus $w_{(i,j)}\,v^\pm_{\boldsymbol{\lambda}} \,=\, \pm\, v^\pm_{\boldsymbol{\lambda}}$ for any $i \neq j$.* □

For $N = n$ and $\boldsymbol{\lambda} = (1,\ldots,1)$, denote the Chern roots $\gamma_{1,1},\ldots,\gamma_{n,1}$ of the corresponding line bundles over $\mathcal{F}_{\boldsymbol{\lambda}}$ by $x_1,\ldots,x_n$, respectively. Then

$$H_{\boldsymbol{\lambda}} = \mathbb{C}[z_1,\ldots,z_n]^{S_n}\otimes\mathbb{C}[x_1,\ldots,x_n]\otimes\mathbb{C}[h]\Big/\Big\langle\prod_{i=1}^n(u-x_i) \,=\, \prod_{i=1}^n(u-z_i)\Big\rangle\,, \tag{8.6}$$

see (5.1). The assignment $f(z_1,\ldots,z_n;x_1,\ldots,x_n;h) \mapsto f(x_1,\ldots,x_n;x_1,\ldots,x_n;h)$ defines an isomorphism $H_{\boldsymbol{\lambda}}$ with $\mathbb{C}[x_1,\ldots,x_n]\otimes\mathbb{C}[h]$.

Consider the following two $\mathfrak{H}_n$-actions $\tau^\pm$ on $\mathbb{C}[x_1,\ldots,x_n]\otimes\mathbb{C}[h] \simeq H_{\boldsymbol{\lambda}}$. For any $i = 1$, $\ldots, N$, $\tau^\pm(y_i)$ is the multiplication by $x_i$, and $\tau^\pm(c)$ is the multiplication by $\pm h$. The elements $s_1,\dots,s_{n-1}$ act by the rule

$$\begin{aligned}\tau^{\pm}(s_i) : f(x_1,\dots,x_n,h) \mapsto \frac{x_i-x_{i+1}\mp h}{x_i-x_{i+1}}\, f(x_1,\dots,x_{i+1},x_i,\dots,x_n,h) \pm \\ \pm\, \frac{h}{x_i-x_{i+1}}\, f(x_1,\dots,x_i,x_{i+1},\dots,x_n,h)\,,\end{aligned}$$

cf. (2.4). The actions of $s_1,\dots,s_{n-1}$ are uniquely determined by relations in $\mathfrak{H}_n$, see (8.1), and the property $\tau^{\pm}(s_i) : 1 \mapsto 1$ for all $i=1,\dots,n-1$.

**Theorem 8.5.** *For $i=1,\dots,n$, we have*

$$\rho^{\pm}(X_i^q) \,=\, \tau^{\pm}(Y_i^q) \,=\, x_i \pm h \sum_{j=1}^{i-1} \frac{q_i}{q_i-q_j}\,\tau^{\pm}(s_{i,j}) \pm h \sum_{j=i+1}^{n} \frac{q_j}{q_i-q_j}\,\tau^{\pm}(s_{i,j})\,.$$

*Proof.* Since $\rho^{\pm}(X) = (\nu_{\boldsymbol\lambda}^{\pm})^{-1}\,\phi^{\pm}(X)\,\nu_{\boldsymbol\lambda}^{\pm}$ for any $X\in \widetilde{Y}(\mathfrak{gl}_N)$, the statement follows from formula (8.4), Lemma 8.4, and the definition of $\tau^{\pm}$. □

In particular, by Theorem 8.5 and formula (3.17), for any $i=1,\dots,n$, we get

$$\rho^{-}(X_i^{q-}) \,=\, x_i - h \sum_{j=1}^{i-1} \frac{q_i}{q_i-q_j}\,\bigl(\tau^{-}(s_{i,j})-1\bigr) - h \sum_{j=i+1}^{n} \frac{q_j}{q_i-q_j}\,\bigl(\tau^{-}(s_{i,j})-1\bigr)\,. \tag{8.7}$$

8.3. **Relations in $\mathcal{H}_{\boldsymbol\lambda}^q$ and trigonometric Calogero-Moser model.** For $N=n$ and $\boldsymbol\lambda=(1,\dots,1)$, recall the algebra $\mathcal{H}_{\boldsymbol\lambda}^q$ defined in (6.2),

$$\mathcal{H}_{\boldsymbol\lambda}^q \,=\, \mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[x_1,\dots,x_n]\otimes\mathbb{C}[h]\Big/\Big\langle W^q(u) \,=\, \prod_{1\leqslant i<j\leqslant n}(q_i-q_j)\prod_{a=1}^{n}(u-z_a)\Big\rangle\,.$$

where $x_1,\dots,x_n$ denote the Chern roots $\gamma_{1,1},\dots,\gamma_{n,1}$ and

$$W^q(u) \,=\, \det\Bigl(q_i^{n-j}\,\bigl(u-x_i+h\,(i-j)\bigr)\Bigr)_{i,j=1}^{n}\,.$$

**Lemma 8.6.** *The isomorphism $\beta_{\boldsymbol\lambda}^- : H_{\boldsymbol\lambda}\to\mathcal{H}_{\boldsymbol\lambda}^q$ is such that $\beta_{\boldsymbol\lambda}^-(x_i)=x_i$, $i=1,\dots,n$.* □

Proposition 7.5 for the case $N=n$, $\boldsymbol\lambda=(1,\dots,1)$ reads as follows.

**Proposition 8.7.** *The assignment $\chi^{\bullet} : x_i \mapsto x_i\,\bullet$, $i=1,\dots,n$, extends uniquely to an isomorphism $\mathcal{H}_{\boldsymbol\lambda}^q\to\widetilde{\mathcal{B}}_{\boldsymbol\lambda}^{q-}$ of $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$-algebras. The maps $\chi_{\boldsymbol\lambda}^{\bullet}$ and $\beta_{\boldsymbol\lambda}^-$ identify the $\widetilde{\mathcal{B}}_{\boldsymbol\lambda}^{q-}$-module $H_{\boldsymbol\lambda}$ with the regular representation of the algebra $\mathcal{H}_{\boldsymbol\lambda}^q$.* □

Define the $n\times n$ matrix $C$ with entries

$$C_{i,i} \,=\, x_i - h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j} - h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,, \qquad i=1,\dots,n\,,$$

$$C_{i,j} \,=\, \frac{h\,q_i}{q_i-q_j}\,, \qquad i,j=1,\dots,n\,, \quad i\neq j\,.$$

Notice that

$$C_{i,i} = \rho_{\boldsymbol{\lambda}}^{q-}(X_i^q), \qquad i = 1, \dots, n,$$

where $X_1^q, \dots, X_n^q$ are the dynamical Hamiltonians (3.15), see (3.17) and Lemma 6.10.

**Lemma 8.8.** $W^q(u) = \det(u - C) \prod_{1 \leqslant i < j \leqslant n} (q_i - q_j)$.

*Proof.* The verification is an exercise in linear algebra, see [MTV6, Section 6.1]. □

Recall the function $\widehat{W}^q(u, x)$, see (6.3). Define the diagonal matrix $Q = \operatorname{diag}(q_1, \dots, q_n)$.

**Lemma 8.9.** $\widehat{W}^q(u) = \det\big((u - C)(x - Q) - hQ\big) \prod_{1 \leqslant i < j \leqslant n} (q_i - q_j)$.

*Proof.* The statement follows from Lemma 8.8 for $(n+1) \times (n+1)$ matrices by taking an appropriate limit, see similar calculation in [MTV7, Lemma 4.1] for the ordinary Wronskian. □

**Corollary 8.10.** *For $N = n$, $\boldsymbol{\lambda} = (1, \dots, 1)$, we have*

$$\mathcal{H}_{\boldsymbol{\lambda}}^q = \mathbb{C}[z_1, \dots, z_n]^{S_n} \otimes \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[h] \Big/ \Big\langle \det\big((u - C)(x - Q) - hQ\big) = \prod_{a=1}^{n} (u - z_a) \Big\rangle.$$

Cf. this statement with part (4) of Theorem 3.2 in [BMO].

The coefficients of the determinant $\det(u - C)$ in Lemma 8.8 form a complete set of integrals in involution of the trigonometric Calogero-Moser model, see for example [E]. There $q_1$, $\dots, q_n$ are exponentials of coordinates, and the diagonal entries $C_{1,1}, \dots, C_{n,n}$ are momenta. The Hamiltonian of the trigonometric Calogero-Moser model equals

$$\frac{1}{2} \operatorname{tr} C^2 = \frac{1}{2} \sum_{i=1}^{n} C_{i,i}^2 + \frac{h^2}{2} \sum_{1 \leqslant i < j \leqslant n} \frac{q_i q_j}{(q_i - q_j)^2}.$$

The matrices $Q$ and $C$ satisfy the relation $\operatorname{rank}(CQ - QC - hQ) = 1$. Thus if $x_1, \dots,$ $x_n, h$ are numbers, the matrices $Q$ and $C/h$ define a point of the trigonometric Calogero-Moser space. Notice that the matrices $Q$ and $\widetilde{C} = Q^{-1}C/h$ satisfy the relation $\operatorname{rank}(\widetilde{C}Q - Q\widetilde{C} - 1) = 1$. So the correspondence $(Q, C/h) \mapsto (Q, \widetilde{C})$ describes an embedding of the trigonometric Calogero-Moser space into the rational Calogero-Moser space as a submanifold of points with invertible matrix $Q$.

8.4. **Bethe algebra and quantum equivariant cohomology.** In lectures [O], Okounkov, in particular, considers the equivariant quantum cohomology algebra $QH_{SL_n \times \mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$ of the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of a partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$. Namely, he considers the standard equivariant cohomology $H^*_{SL_n \times \mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$ as a module over the algebra of quantum multiplication. The operators of quantum multiplication depend on the equivariant parameters $z_1, \dots, z_n, h$ and additional $N - 1$ parameters $q_2/q_1, \dots, q_N/q_{N-1}$ corresponding to quantum deformation. Okounkov describes this module as the Yangian Bethe algebra of an $XXX$-type model associated with $V^{\otimes n}$. The absence of published notes of the lectures does not allow us to be more precise here.

In [BMO], Braverman, Maulik and Okounkov consider the case $N = n$, $\boldsymbol{\lambda} = (1,\dots,1)$, of the equivariant cohomology $H^*_{SL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ of the full flag variety. The authors describe the cohomology as a module over the algebra of quantum multiplication. More precisely, they describe the associated quantum connection on the trivial bundle with fiber $H^*_{SL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ over the space $(\mathbb{C}^*)^{n-1}$, which is the maximal torus of the group $SL_n$.

Let $N = n$ and $\boldsymbol{\lambda} = (1,\dots,1)$. Consider the quotient algebras

$$\bar H_{\boldsymbol{\lambda}} = H_{\boldsymbol{\lambda}} \big/ \langle\, z_1 + \ldots + z_n = 0\,\rangle\,,$$

$$\bar{\mathcal{B}}^q(\tfrac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}) = \widetilde{\mathcal{B}}^q(\tfrac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}) \big/ \langle\, z_1 + \ldots + z_n = 0\,\rangle\,, \qquad \bar{\mathcal{B}}^{q-}_{\boldsymbol{\lambda}} = \widetilde{\mathcal{B}}^{q-}_{\boldsymbol{\lambda}} \big/ \langle\, z_1 + \ldots + z_n = 0\,\rangle\,.$$

We identify the notation of [BMO] with our notation as follows: $H^*_{SL_n\times\mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C}) = \bar H_{\boldsymbol{\lambda}}$, $\kappa = -1$, $t = -h$, $q^{\alpha_i^\vee} = q_{i+1}/q_i$, $i = 1,\dots,n-1$, where $\alpha_1,\dots,\alpha_{n-1}$ are simple roots of the Lie algebra $\mathfrak{sl}_n$, and all functions of $q_1,\dots,q_n$ are of homogeneous degree zero. Then the quantum connection $\nabla^{\rm BMO}$ of [BMO] takes the form

$$\nabla^{\rm BMO}_{\alpha_i} = q_i\frac{\partial}{\partial q_i} - q_{i+1}\frac{\partial}{\partial q_{i+1}} - (x_i - x_{i+1})\,{\bullet}\,, \qquad i = 1,\dots,n\,, \tag{8.8}$$

see (7.2), (8.7).

**Lemma 8.11.** *The algebra $\bar{\mathcal{B}}^{q-}_{\boldsymbol{\lambda}}$ is generated by the elements $\rho^-(X^{q-}_i - X^{q-}_{i+1})$, $i = 1,\dots,n-1$, and $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$.*

*Proof.* By Theorem 8.1, the algebra $\bar{\mathcal{B}}^q(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ is generated by the elements $\phi^-(X^{q-}_i)$, $i = 1,\dots,n$, and $\mathbb{C}[z_1,\dots,z_n]^{S_n}\otimes\mathbb{C}[h]$. The lemma follows from the identity $\phi^-(X^{q-}_1 + \ldots + X^{q-}_n) = z_1 + \ldots + z_n$, see formula (3.17). □

The algebra of quantum multiplication on $\bar H_{\boldsymbol{\lambda}}$ determined in [BMO] is generated by the elements $(x_i - x_{i+1})\,{\bullet}$, $i = 1,\dots,n-1$, see [BMO]. Thus by Corollary 7.3 and Lemma 8.11, we have the following result verifying Conjecture 7.10 for $N = n$ and $\boldsymbol{\lambda} = (1,\dots,1)$.

**Theorem 8.12.** *For $N = n$, $\boldsymbol{\lambda} = (1,\dots,1)$, the quantum multiplication on $\bar H_{\boldsymbol{\lambda}}$ determined in* [BMO] *coincides with the multiplication $\bullet$ on $\bar H_{\boldsymbol{\lambda}}$ introduced in* (7.1).

Consider the quotient algebra $\bar{\mathcal{H}}^q_{\boldsymbol{\lambda}} = \mathcal{H}^q_{\boldsymbol{\lambda}} / \langle\, z_1 + \ldots + z_n = 0\,\rangle$. Since $x_1 + \ldots + x_n = z_1 + \ldots + z_n$, the algebra $\bar{\mathcal{H}}^q_{\boldsymbol{\lambda}}$ is generated by the elements $x_i - x_{i+1}$, $i = 1,\dots,n-1$. By Proposition 8.7 we have the following result.

**Corollary 8.13.** *For $N = n$, $\boldsymbol{\lambda} = (1,\dots,1)$, the space $H_{\boldsymbol{\lambda}}$ as the module over the algebra of quantum multiplication on $H_{\boldsymbol{\lambda}}$ defined in* [BMO] *is isomorphic to the regular representation of the algebra $\bar{\mathcal{H}}^q_{\boldsymbol{\lambda}}$ by the isomorphisms $x_i - x_{i+1} \mapsto x_i - x_{i+1}$, $(x_i - x_{i+1})\,{\bullet} \mapsto x_i - x_{i+1}$, $i = 1,\dots,n-1$.*

It is known that the flat sections of the trigonometric dynamical connection are given by multidimensional hypergeometric integrals, see [MV], cf. [TV2]. These hypergeometric integrals provide flat sections of the quantum connection defined by (3.20). This presentation of flat sections of the quantum connection as hypergeometric integrals is in the spirit of mirror symmetry, see Candelas et al. [COGP], Givental [G1, G2], and [BCK, BCKS, GKLO, I, JK].

**Example.** Let $N = n = 2$ and $\boldsymbol{\lambda} = (1,1)$. Set $x = x_1 - x_2$, $z = z_1 - z_2$, $q = q_2/q_1$. Then

$$\bar{H}_{\boldsymbol{\lambda}} = \mathbb{C}[x, z^2, h] / \langle x^2 = z^2 \rangle , \qquad \bar{\mathcal{H}}^q_{\boldsymbol{\lambda}} = \mathbb{C}[x, z^2, h] \Big/ \Big\langle x^2 - \frac{4hq}{1-q}(x+h) = z^2 \Big\rangle ,$$

and

$$x \bullet \,=\, x - \frac{2hq}{1-q}\big(\tau^-(s) - 1\big)$$

is the operator of quantum multiplication by $x$ acting on $\bar{H}_{\boldsymbol{\lambda}}$. Here the operator $\tau^-(s)$ is

$$\tau^-(s) : f(x, z^2, h) \,\mapsto\, \frac{x+h}{x} f(-x, z^2, h) - \frac{h}{x} f(x, z^2, h) .$$

The isomorphism of the algebra of quantum multiplication on $\bar{H}_{\boldsymbol{\lambda}}$ and $\bar{\mathcal{H}}^q_{\boldsymbol{\lambda}}$ is $x \bullet \mapsto x$, that is, the operator $x\bullet$ acting on $\bar{H}_{\boldsymbol{\lambda}}$ satisfies the equation

$$(x \bullet)^2 - \frac{4hq}{1-q}(h + x \bullet) \,=\, z^2 .$$

The quantum differential equation is

$$-2\kappa q \frac{\partial}{\partial q} I \,=\, x \bullet I . \tag{8.9}$$

The $q$-hypergeometric solutions of this equation are given by the formula

$$I \,=\, q^{z/2\kappa}(x+h) \int_C q^{u/\kappa}\, \Gamma\Big(\frac{u-h}{\kappa}\Big)\, \Gamma\Big(\frac{u+z-h}{\kappa}\Big)\, \Gamma\Big(-\frac{u}{\kappa}\Big)\, \Gamma\Big(-\frac{u+z}{\kappa}\Big)\,(2u + z - x)\, du ,$$

where the integration contour $C$ is a deformation of the imaginary line separating the poles of $\Gamma\big((u-h)/\kappa\big)\,\Gamma\big((u+z-h)/\kappa\big)$ from those of $\Gamma(-u/\kappa)\,\Gamma\big(-(u+z)/\kappa\big)$. Linearly independent solutions can be obtained by choosing various branches of $q^{u/\kappa}$.

Alternatively, a basis of solutions can be obtained by taking the sum of residues of the integrand at $u \in \kappa\mathbb{Z}_{\geqslant 0}$ or at $u \in -z + \kappa\mathbb{Z}_{\geqslant 0}$. The sum of residues at $u \in \kappa\mathbb{Z}_{\geqslant 0}$ gives

$$J_1 \,=\, (x+h)\, \Gamma\Big(-\frac{h}{\kappa}\Big)\, \Gamma\Big(-\frac{z}{\kappa}\Big)\, \Gamma\Big(\frac{z-h}{\kappa}\Big) \,\times$$
$$\times \sum_{d=0}^{\infty} q^{d + z/2\kappa}\, \frac{z - x + 2\kappa d}{\kappa^{d-1}\, d!} \prod_{i=0}^{d-1} \frac{(h-\kappa i)(h - z - \kappa i)}{z + \kappa(i+1)} ,$$

and the sum of residues at $u \in -z + \kappa\mathbb{Z}_{\geqslant 0}$ gives

$$J_2 \,=\, (x+h)\, \Gamma\Big(-\frac{h}{\kappa}\Big)\, \Gamma\Big(\frac{z}{\kappa}\Big)\, \Gamma\Big(-\frac{z+h}{\kappa}\Big) \,\times$$
$$\times \sum_{d=0}^{\infty} q^{d - z/2\kappa}\, \frac{(2\kappa d - z - x)}{\kappa^{d-1}\, d!} \prod_{i=0}^{d-1} \frac{(h-\kappa i)(h + z - \kappa i)}{\kappa(i+1) - z} .$$

The hypergeometric solutions of equation (8.9) are given by the formula

$$I = q^{z/2\kappa}(1-q)^{2h/\kappa}\int_C u^{(h-z)/\kappa}(u-1)^{-h/\kappa}(u-q)^{-h/\kappa}\Big(\frac{x-z+2h}{u-1}+\frac{x+z}{u-q}\Big)\,du\,,$$

where $C$ is an appropriate integration contour. We plan to discuss $q$-hypergeometric and hypergeometric formulae for flat sections of the quantum connection in a separate paper.

The dynamical connection commutes with the difference *qKZ* connection as explained in Section 4.5. For $N = n$, $\boldsymbol{\lambda} = (1,\dots,1)$, the dynamical connection is identified with the quantum connection by Theorem 8.12. Then the *qKZ* connection of Section 4.5 induce on $H_{\boldsymbol{\lambda}}$ a difference connection that commutes with the quantum differential connection in the sense of Section 4.5. This discrete structure was discussed in [BMO] under the name of shift operators. We will discuss the *qKZ* discrete connection in this situation in more details in another paper.

The isomorphism of the Bethe algebra and quantum multiplication establishes a correspondence between the eigenvectors of the operators of the Bethe algebra and idempotents of the quantum algebra. Hence we may find the idempotents of the quantum algebra by the *XXX* Bethe ansatz.

**Example.** Let $N=n=2$, $\boldsymbol{\lambda}=(1,1)$. If $u_1,u_2$ are solutions of the Bethe ansatz equation

$$(u-z)(u+z) \,=\, q(u-z+2h)(u+z+2h)\,,$$

then the elements

$$w_1 \,=\, \frac{x-u_2}{u_1-u_2}\,,\qquad w_2 \,=\, \frac{x-u_1}{u_2-u_1}\,,$$

of $\bar H_{\boldsymbol{\lambda}}$ satisfy the equations $w_i \bullet w_j = \delta_{i,j}\,w_i$.

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