COMPUTATION OF THE THOM POLYNOMIAL OF Σ^{1111} VIA SYMMETRIES OF SINGULARITIES

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ABSTRACT. In this paper we present the computation of the Thom polynomial of Σ^{1_4} with the aid of a new method based on the generalized Pontryagin-Thom construction [RSz] and announce some other results obtained with similar methods.

1. Introduction

The novelty presented in this paper is a new method to compute Thom polynomials. The basic tool of these new techniques is the "generalized Pontryagin-Thom construction" of [RSz], whose idea goes back as far as 1979 (see references therein). According to these results the properties of all maps with described singularities can be studied by considering only one such map, one which is universal in some sense. E.g. when we want to determine the coefficients of a particular Thom polynomial we only have to consider one map, the universal one. The disadvantage is that this map is between spaces that are not finite dimensional manifolds. Still, there are many similarities, which turn the theory manageable. Let us also remark that Szűcs conjectured in 1985 [Sz] that this 'universal map method' is capable to find Thom polynomials.

The new results, i.e. the new Thom polynomials will be presented in [R2]. However, to prove the strength of the method let us cite here the Thom polynomials of Σ^{1_n} $(n = 1, \dots, 8)$ between equal dimensional manifolds, since this was the setting where the most activity took place in the past:

$$
\begin{array}{ll}
n=1 & c_1 \\
n=2 & c_1^2 + c_2 \\
n=3 & c_1^3 + 3c_1c_2 + 2c_3 \\
n=4 & c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4 \\
n=5 & c_1^5 + 10c_1^3c_2 + 25c_1^2c_3 + 10c_1c_2^2 + 38c_1c_4 + 12c_2c_3 + 24c_5 \\
n=6 & c_1^6 + 15c_1^4c_2 + 55c_1^3c_3 + 30c_1^2c_2^2 + 141c_1^2c_4 + \\
 & +79c_1c_2c_3 + 5c_2^3 + 202c_1c_5 + 55c_2c_4 + 17c_3^2 + 120c_6 \\
n=7 & c_1^7 + 21c_1^5c_2 + 105c_1^4c_3 + 70c_1^3c_2^2 + 399c_1^3c_4 + 301c_1^2c_2c_3 + 35c_1c_2^3 + 960c_1^2c_5 + \\
 & +467c_1c_2c_4 + 139c_1c_3^2 + 58c_2^2c_3 + 1284c_1c_6 + 326c_2c_5 + 154c_3c_4 + 720c_7 \\
n=8 & c_1^8 + 28c_1^6c_2 + 182c_1^5c_3 + 140c_1^4c_2^2 + 952c_1^4c_4 + 868c_1^3c_2c_3 + 3383c_1^3c_5 + \\
 & +140c_1^2c_2^3 + 2229c_1^2c_2c_4 + 642c_1^2c_3^2 + 7552c_1^2c_6 + 501c_1c_2^2c_3 + 3455c_1c_2c_5 + \\
 & +1559c_1c_3c_4 + 9468c_1c_7 + 14c_2^4 + 364c_2^2c_4 + 202c_2c_3^2 + 2314c_2c_6 + \\
 & +954c_3c_5 + 332c_4^2 + 5040c_8.
\end{array}
$$

In this paper we give a detailed description of the calculation of the Thom polynomial Σ^{1_4}. This polynomial has been known since [G]. The reason for giving this as an example on one

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hand is that this is probably more easily achievable than e.g Σ^{1_8} but when dealing with this we have to face the typical problems ('competing singularities', see [G]). On the other hand this is the case which is probably the better exposed in the literature, so one might take the effort to understand the connections between this new method and the former one, which is usually called the desingularization method. The author believes that there is a close connection between the desingularization process and the *incidence class* defined here, see section 5.

In what follows A_i will be a shorthand notation for Σ^{1_i}. We will call a (multi)singularity η *more difficult* than another one ζ, if near an η-point in the target there is necessarily a ζ-point. For example, A_i is more difficult than A_j for $i > j$, or e.g. a singularity A_1 ($\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$) is more difficult than a double point. We will use Mather's notation $I_{2,2}$ for the simplest germ of type Σ^2 between equal dimensional manifolds (i.e. it is the germ with local algebra $\mathbb{C}[[x,y]]/(x^2,y^2)$).

2. On Thom polynomials

Recall that the Thom polynomial p_η of a singularity η is a polynomial in the Chern classes of the map f between complex analytic manifolds M, P, and this polynomial equals the Poincaré dual $[\eta(f)]$ of the cycle carried by the closure of

$$\eta(f) = \{x \in M \mid \text{the singularity of } f \text{ at } x \text{ is } \eta\ \}$$

for most maps. Here by Chern classes of a map $f : M \longrightarrow P$ we mean the Chern classes of the virtual bundle $f^*TP - TM$ over M.

The cohomology class $[\eta(f)]$ is most easily understood when $\eta(f)$ is a submanifold, which is often the case if f has no more complicated singularities than η. In this case $\eta(f)$ carries a fundamental homology class. We take the image of this class in the homology of M and apply Poincaré duality. The resulting class is $[\eta(f)] \in H^*(M;\mathbb{Z})$. Although the definition of $[\eta(f)]$ is not much more difficult when $\eta(f)$ is not a manifold (it has singularities along smaller dimensional strata), the interesting thing is that we will not need this. We will only use the definition of $[\eta(f)]$ in the mentioned case. Observe that this is a difference from the desingularization method, where the behaviour of $\eta(f)$ near the singular part is studied.

Let us make a few words about the history of Thom polynomials. The concept itself and the first computations go back to Thom [T]. A different approach is taken and new results are obtained by Porteous [P] ($\Sigma^i(f : M^n \longrightarrow P^{n+k})$ for any k) and Ronga [R] (an algorithm for Σ^{ij}). Their method in a more sophisticated form led Gaffney to [G] (Σ^{1111}, $k = 0$), whose method is used also in [Tu] ($\Sigma^{11111}, k = 0$). A different — geometrical — approach gives mod 2 results in [O] (e.g. $T.P.(A_7) \equiv c_1 \cdot T.P.(A_6) \mod 2$). Some more points about Thom polynomials can be found in the reviews [AVGL] or [SS].

3. Review on the generalized Pontryagin-Thom construction

Now we recall some notions and results from [RSz], with the notation of that paper. Please note that, overlines ($\overline{}$) are not meaning closures.

We will restrict ourselves to the case of maps between equal dimensional manifolds. Let τ be the following set of their multisingularities:

codim			
0	u_0A_0		$u_0 \leq 5$
1	$A_1 + u_1A_0$		$u_1 \leq 3$
2	$A_2 + u_2A_0,$	$2A_1 + u_4A_0$	$u_2 \leq 2, u_4 \leq 1$
3	$A_3 + u_3A_0,$	$A_1 + A_2$	$u_3 \leq 1$
4	$A_4,$	$I_{2,2}$	

Here, by e.g. $A_2 + 2A_0$ we mean a multisingularity $(\mathbb{C}^n, 0) \cup (\mathbb{C}^n, 0) \cup (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ whose restriction to one of the $\mathbb{C}^n$'s is A_2 and the restriction to the other two is A_0. [1] A map $f : M^n \longrightarrow P^n$ a called a τ-map if for any $y \in P$ the singularity of f at $f^{-1}(y)$ is from τ. Thus τ-maps are the maps between equal dimensional manifolds which have no more complicated multisingularities than A_4 and $I_{2,2}$.

Associated to this set τ a map $f\tau : Y\tau \longrightarrow X\tau$ is constructed in [RSz]. First let us describe the spaces $Y\tau$, $X\tau$. These topological spaces are stratified by submanifolds. By this we mean that $Y\tau$ is the union of its subspaces $\bar{K}_\eta$ for $\eta \in \tau$, and for each $\eta \in \tau$ a neighbourhood $\bar{U}_\eta$ of $\bar{K}_\eta$ in

$$Y\tau \setminus \{ \ \bar{K}_\zeta \mid \zeta \text{ is more difficult than } \eta \ \}$$

is fixed. Moreover $\bar{U}_\eta$ is homeomorphic to the total space of a vector bundle $\bar{\xi}_\eta$ over $\bar{K}_\eta$ (the 0-section being $\bar{K}_\eta$ itself). The space $X\tau$ is stratified in the same way by the 'submanifolds' K_η and their fixed neighbourhoods U_η are homeomorphic to the total space of a vector bundle ξ_η over K_η. We also have a concrete description of the bundles $\bar{\xi}_\eta$, ξ_η, which we present here only for $\eta = A_1, A_2, A_3, A_4, I_{2,2}$, since we will only need these cases and the other cases are slightly more difficult.

Let $\kappa : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^m, 0)$ be the "prototype" of η, i.e. let all germs $(\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ of singularity η be right-left equivalent to a suspension[2] of κ. Denote the maximal compact subgroup of the right-left automorphism group

$$Aut\ \kappa = \{ \ (\psi, \phi) \in Diff(\mathbb{C}^m, 0) \times Diff(\mathbb{C}^m, 0) \mid \phi \circ \kappa \circ \psi^{-1} = \kappa \ \}$$

by G_η. Well, $Aut\ \kappa$ is not a Lie group (at least not a finite dimensional Lie group), but — since it has many similarities with Lie groups — we can talk about its maximal compact subgroup [W], [R1]. Let G_η's representations on the source and target spaces be $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The fact is ([RSz]) that $\bar{K}_\eta$ and K_η are homeomorphic to BG_η (the base space of the universal principal G_η-bundle) and the vector bundles $\bar{\xi}_\eta$ and ξ_η are the vector bundles associated to the universal principal G_η-bundle using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$.

We have not said anything about the map $f\tau$ yet. The following knowledge — its behaviour near $\bar{K}_\eta$ — will be sufficient for us: restricted to $\bar{U}_\eta$ (= the total space of $\bar{\xi}_\eta$) the map $f\tau$ can

[1]In fact, the upper bounds for $u_0 \dots u_4$ are not very important. We might as well allow them to take any nonnegative values. The only reason to put the upper bounds is that e.g., near A_4 or $I_{2,2}$ there are only 5-tuple points and not 6-tuple ones — so τ is the smallest *ascending* set of multisingularities consisting A_4 and $I_{2,2}$.

[2]by suspension (= trivial unfolding) of $\kappa : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^m, 0)$ we mean a germ $S\kappa : (\mathbb{C}^{m+v}, 0) \longrightarrow (\mathbb{C}^{m+v}, 0)$, $(x, u) \mapsto (\kappa(x), u))$

be identified with a fibrewise map

$$\begin{array}{ccc} \bar{\xi}_\eta & \longrightarrow & \xi_\eta \\ \downarrow & & \downarrow \\ \bar{K}_\eta & \stackrel{\cong}{\longrightarrow} & K_\eta \end{array}$$

so, restricted to $\bar{K}_\eta$ it is a homeomorphism to K_η ($\eta = A_1, A_2, A_3, A_4, I_{2,2}$). In fact in each fibre above the map is right-left equivalent to the appropriate κ.

The main property of the map $f\tau$ is that it is a universal τ-map in the following sense. Whenever a τ-map $f : M^n \longrightarrow P^n$ is given, it can be induced from $f\tau$, i.e. there is a commutative diagram

$$\begin{array}{ccc} P & \stackrel{g}{\longrightarrow} & X\tau \\ \uparrow f & & \uparrow f\tau \\ M & \stackrel{h}{\longrightarrow} & Y\tau \end{array}$$

such that $\eta(f) = h^{-1}(\bar{K}_\eta)$, $f(\eta(f)) = g^{-1}(K_\eta)$ and a tubular neighbourhood of $\eta(f)$ in

$$M \setminus \{ \ \zeta(f) \mid \zeta \text{ is more difficult than } \eta \ \}$$

is diffeomorphic to the total space of a vector bundle induced from $\bar{\xi}_\eta$ by $h|_{\eta(f)}$. Also a tubular neighbourhood of $f(\eta(f))$ in

$$P \setminus \{ \ f(\zeta(f)) \mid \zeta \text{ is more difficult than } \eta \ \}$$

is diffeomorphic to the total space of a vector bundle induced from ξ_η by $g|_{f(\eta(f))}$.

In order to effectively study the topology of $f\tau$, of course, we need a better knowledge of the Lie group G_η ($\eta = A_1, A_2, A_3, A_4, I_{2,2}$) and its representations $\lambda_1(\eta)$, $\lambda_2(\eta)$. A general framework of their computation (for all stable singularities) is given in [R2]. However, the results for the occurring η's are easily checked, so we only give the groups and the representations with no proof here.

Theorem 3.1. *The prototype κ_i of A_i maps from $(\mathbb{C}^i, 0)$ to $(\mathbb{C}^i, 0)$; $G_{\kappa_i} = U(1)$ and the representations are*

$$\lambda_1(A_i) = \bigoplus_{j=1}^{i} \rho^j \qquad\qquad \lambda_2(A_i) = \bigoplus_{j=2}^{i+1} \rho^j.$$

(Here ρ is the standard 1-dimensional representation of $U(1)$ and its powers are meant tensor powers.)

Theorem 3.2. *The prototype κ of $I_{2,2}$ maps from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^4, 0)$; G_κ has an index 2 subgroup $G'_\kappa = U(1) \times U(1)$ and the representations restricted to G'_κ are*

$$\lambda_1(I_{2,2}) = \rho_1 \oplus \rho_2 \oplus (\rho_1^{-1} \otimes \rho_2^2) \oplus (\rho_1^2 \otimes \rho_2^{-1}) \qquad \lambda_2(I_{2,2}) = \rho_1^2 \oplus \rho_2^2 \oplus (\rho_1^{-1} \otimes \rho_2^2) \oplus (\rho_1^2 \otimes \rho_2^{-1}).$$

(Here ρ_1 and ρ_2 are the standard representations on the 1st and the 2nd factor.)

In fact one can easily write up κ_i and κ explicitly:

$$\begin{array}{lll} \kappa_i : & (x, u_1, \dots, u_{i-1}) & \mapsto \ (x^{i+1} + \sum_{j=1}^{i-1} u_j x^j, u_1, \dots, u_{i-1}) \\ \kappa : & (x, y, u, v) & \mapsto \ (x^2 + uy, y^2 + vx, u, v). \end{array}$$

Remark 3.3. Although it is not necessary to work with G'_κ instead of G_κ, it makes computations and (definitely) notations easier, and it will be enough for our purposes.

4. Computation of the Thom polynomial of A_4

Definition 4.1. *For each $\eta \in \tau$ let $c(\lambda_i(\eta))$, $e(\lambda_i(\eta))$ denote the total Chern class and the Euler class of the vector bundle associated to the universal principal G_η-bundle using the representation $\lambda_i(\eta)$ ($i = 1, 2$). Let us define*

$$c(\eta) = \frac{c(\lambda_2(\eta))}{c(\lambda_1(\eta))} \qquad \text{and} \qquad e(\eta) = e(\lambda_1(\eta)).$$

What allows us to compute the Thom polynomial of A_4 is the following result.

Theorem 4.2. *If $p(c) = p(c_1, c_2, c_3, c_4)$ is the (degree 4 weighted homogeneous) Thom polynomial of A_4 then*

- *$p(c(\eta)) = 0$ for $\eta = A_1, A_2, A_3, I_{2,2}$ and*
- *$p(c(A_4)) = e(A_4)$.*

This theorem can be justified in two ways. The first is more illuminating but making it precise needs more techniques. Here we are going to present this proof without the details. The second, which avoids the technical apparatus will be given in [R2].

Proof. The map $f\tau : Y\tau \longrightarrow X\tau$, although not a map between finite dimensional manifolds — can be regarded a τ-map, too. So, if $p(c)$ is the Thom polynomial of A_4 then $[\bar{K}_{A_4}] = [A_4(f\tau)]$ can be expressed as

$$[\bar{K}_{A_4}] = p(c(f\tau)) \quad \in H^4(Y\tau).$$

We restrict this cohomological identity to $\bar{K}_\eta$ for $\eta = A_1, A_2, A_3, A_4, I_{2,2}$. For this we must know the restriction of $[\bar{K}_{A_4}]$ and $c(f\tau)$ to the $\bar{K}_\eta$'s.

Lemma 4.3.

$$\begin{aligned} [\bar{K}_{A_4}]|_{\bar{K}_\eta} &= \begin{cases} e(A_4) & \text{if } \eta = A_4 \\ 0 & \text{if } \eta = A_1, A_2, A_3, I_{2,2} \end{cases} \\ c(f\tau)|_{\bar{K}_\eta} &= c(\eta). \end{aligned}$$

Proof. The first statement is a standard fact from differential topology, i.e. that the Poincaré dual of a submanifold restricted to the same submanifold is equal to the Euler class of its normal bundle. The second — $[\bar{K}_{A_4}]|_{\bar{K}_\eta} = 0$ for $\eta = A_1, A_2, A_3, I_{2,2}$ — comes from the construction of $Y\tau$. Indeed, the total spaces of the bundles $\bar{\xi}_\eta$ do not contain points of $\bar{K}_{A_4}$, so a 4-cycle in $\bar{K}_\eta$ (perturbed) does not intersect $\bar{K}_{A_4}$ at all. Let us remark, that this fact is based on the property that the η's are not *more difficult* singularities than A_4.

The third statement is proved as follows.

$$c(f\tau)|_{\bar{K}_\eta} = \frac{f\tau^* c(X\tau)}{c(Y\tau)}\Big|_{\bar{K}_\eta} = \frac{c(X\tau)|_{K_\eta}}{c(Y\tau)|_{\bar{K}_\eta}} = \frac{c(\xi_\eta \oplus T(BG_\eta))}{c(\bar{\xi}_\eta \oplus T(BG_\eta))} = \frac{c(\xi_\eta)}{c(\bar{\xi}_\eta)} = \frac{c(\lambda_2(\eta))}{c(\lambda_1(\eta))} = c(\eta).$$

□

The proof of the theorem is now complete. The only problem with this proof was that we worked with $Y\tau$, $X\tau$ like manifolds, we used their tangent bundles, characteristic classes, Poincaré duality etc. As mentioned before the theorem, these computation can be made precise by a careful definition of $[\bar{K}_{A_4}]$ (easy) and $c(f\tau)$ (a bit more delicate). This — and another way which avoids these difficulties — will appear in [R2]. $\square$

Theorems 3.1, 3.2 yield

Corollary 4.4.

$$\begin{array}{rcl} c(A_i) & = & \frac{1+(i+1)a}{1+a} = 1 + ia - ia^2 + ia^3 - \ldots \in \mathbb{Z}[[a]] \\ e(A_4) & = & 1 \cdot 2 \cdot 3 \cdot 4 \cdot a^4 \ \in \mathbb{Z}[a]. \end{array}$$

The image $c'(I_{2,2})$ of $c(I_{2,2})$ at the homomorphism $H^*(BG_\kappa) \longrightarrow H^*(BG'_\kappa)$ (see theorem 3.2) is

$$\begin{array}{rcl} c'(I_{2,2}) & = & \frac{(1+2a)(1+2b)}{(1+a)(1+b)} = \\ & = & 1 + (a+b) + (-a^2 + ab - b^2) + (a^3 - a^2b - ab^2 + b^3) + \ldots \in \mathbb{Z}[[a,b]] \end{array}$$

Now we are ready to determine the coefficients in $p(c) = Ac_1^4 + Bc_1^2c_2 + Cc_1c_3 + Dc_2^2 + Ec_4$. The substitutions $\eta = A_1, A_2, A_3, A_4, I_{2,2}$ into the formulas of theorem 4.2 give the following equations on A, B, C, D, E (in fact, we substitute $c'(I_{2,2})$ instead of $c(I_{2,2})$ for which the first formula of theorem 4.2 clearly holds, too):

$$\begin{array}{ccccccccccc} A(a)^4 & + & B(a)^2(-a^2) & + & C(a)(a^3) & + & D(-a^2)^2 & + & E(-a^4) & = & 0 \\ A(2a)^4 & + & B(2a)^2(-2a^2) & + & C(2a)(2a^3) & + & D(-2a^2)^2 & + & E(-2a^4) & = & 0 \\ A(3a)^4 & + & B(3a)^2(-3a^2) & + & C(3a)(3a^3) & + & D(-3a^2)^2 & + & E(-3a^4) & = & 0 \\ A(4a)^4 & + & B(4a)^2(-4a^2) & + & C(4a)(4a^3) & + & D(-4a^2)^2 & + & E(-4a^4) & = & 24 \end{array}$$

$$A(a+b)^4 + B(a+b)^2(-a^2+ab-b^2) + C(a+b)(a^3 - a^2b - ab^2 + b^3) + D(-a^2+ab-b^2)^2 +$$
$$E(-a^4 + a^3b + a^2b^2 + ab^3 - b^4) = 0$$

The first 4 of these equations give that

$$\begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 16 & -8 & 4 & 4 & -2 \\ 81 & -27 & 9 & 9 & -3 \\ 256 & -64 & 16 & 16 & -4 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$$

whose solution is $A = 1$, $B = 6$, $C + D = 11$, $E = 6$. We can substitute these values into our last equation and get the result: $D = 2$, $C = 9$. So the computation of the Thom polynomial of A_4 between equal dimensional manifolds is complete:

$$c_1^4 + 6c_1^2c_2 + 9c_1c_3 + 2c_2^2 + 6c_4.$$

5. Remarks

Remark 5.1. It turned out from the computation that considering $I_{2,2}$ was necessary. Without $I_{2,2}$ we could have only found that

$$p(c) = c_1^4 + 6c_1^2c_2 + Cc_1c_3 + Dc_2^2 + 6c_4 \qquad\qquad C + D = 11,$$

which is really the most general Thom polynomial of A_4 for maps *without* $I_{2,2}$ singularities. (This can be seen by the fact that the Thom polynomial of $I_{2,2}$ is $c_2^2 - c_1c_3$, so for a map without $I_{2,2}$ the class $c_2^2 - c_1c_3$ is 0.

Remark 5.2. As we saw the computation of other Thom polynomials with the same method basically depends on two things: the knowledge of the hierarchy of singularities (i.e. determining which singularity is necessarily near another one) and the symmetries of them (i.e. an effective method to determine the maximal compact subgroup of the right-left symmetry group of a given singularity). However, if we have already computed the Thom polynomial of some singularity then we can deduce some information on the mentioned two notions. The symmetries do not seem to be interesting here, but one can define the *incidence class* of singularities as

$$I(\eta,\zeta) := [\bar{K}_\eta]|_{\bar{K}_\zeta},$$

and compute it as

$$I(\eta,\zeta) = \text{Thom polynomial of } \eta\ \big(c(\zeta)\big)\ \in H^*(BG_\zeta),$$

which — the author thinks — is a well computable and very fine invariant of the incidence of singularities. For some properties, see [R2].

Remark 5.3. The method used in this paper, just like in [RSz], works for maps $M^n \longrightarrow P^{n+k}$ where $k \geq 0$. So we can use the method to compute Thom polynomials for maps with positive k, too, see [R2]. On the other hand, for $k < 0$ the techniques presented here do not work.

Remark 5.4. One can use the same method to compute so called multiple point formulas, e.g. the classical Herbert-Ronga formulas for immersions [AVGL], which can be considered as Thom polynomials of multisingularities.

Remark 5.5. Apparently one might use the same method to obtain Thom polynomials in other settings. E.g. one might be able to compute the integer (rational, $\mathbb{Z}_p$) cohomology class of some $[\eta(f)]$ in terms of integer (rational, $\mathbb{Z}_p$) characteristic classes (Pontryagin classes).

The very results, i.e. the Thom polynomials associated to singularities Σ^{1_n} and some other contact classes were circulated in a manuscript at the 5th Workshop on Real and Complex Singularities at Sao Carlos 1998 and afterwards. The author is sorry for the numerical errors in that manuscript — e.g. one in the Thom polynomial of Σ^{1_8}.

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