Recursions

(c.f. Lascoux-Schützenberger
recursion in Schubert Calculus
[lecture yesterday])

$\mathcal{F}(3)$	123	132	213	231	312	321
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	0	0
$c^{sm}(\Omega_{132})$						
$c^{sm}(\Omega_{213})$						
$c^{sm}(\Omega_{231})$						
$c^{sm}(\Omega_{312})$						
$c^{sm}(\Omega_{321})$						

on full flag variety $= \{ \ell^1 \subseteq V^2 \subseteq \mathbb{C}^3 \}$

T-fixpoints $\leftrightarrow$ Schubert cells

$\leftrightarrow$ permutations

X = full flag variety $\qquad s_k = (k\ k+1) \in S_n$

Bott-Samelson recursion

$$c^{sm}(\Omega_{ws_k})\Big|_{\sigma} = \frac{\hbar}{z_{\sigma(k+1)} - z_{\sigma(k)}}\, c^{sm}(\Omega_w)\Big|_{\sigma} - \frac{z_{\sigma(k+1)} - z_{\sigma(k)} + \hbar}{z_{\sigma(k+1)} - z_{\sigma(k)}}\, c^{sm}(\Omega_w)\Big|_{\sigma s_k}$$

$\forall w, \sigma$

only for full flag varieties

R-matrix recursion

$$c^{sm}(\Omega_{s_k w})\Big|_{\sigma} = \frac{\hbar}{z_{k+1} - z_k}\, c^{sm}(\Omega_w)\Big|_{\sigma} + \frac{z_k - z_{k+1} + \hbar}{z_k - z_{k+1}} \left[c^{sm}(\Omega_w)\Big|_{s_k \sigma} \right]_{z_k \leftrightarrow z_{k+1}}$$

$\forall w, \sigma$

also for partial flag varieties

$\mathfrak{S}(3)$	123	132	213	231	312	321
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	0	0
$c^{sm}(\Omega_{132})$						

$$c^{sm}(\Omega_{s_k w})\Big|_{\sigma} = \frac{\hbar}{z_{k+1}-z_k}\, c^{sm}(\Omega_w)\Big|_{\sigma} + \frac{z_k - z_{k+1} + \hbar}{z_k - z_{k+1}} \Big[c^{sm}(\Omega_w)\Big|_{s_k \sigma} \Big]_{z_k \leftrightarrow z_{k+1}}$$

$w = 123 \quad k = 2$

$\sigma = 123$

$$c^{sm}(\Omega_{132})\Big|_{123} = \frac{\hbar}{z_3 - z_2}(z_2-z_1)(z_3-z_1)(z_3-z_2) + \frac{z_2 - z_3 + \hbar}{z_2 - z_3}\cdot 0 = \hbar(z_2-z_1)(z_3-z_1)$$

$w = 123 \quad k = 2$

$\sigma = 132$

$$c^{sm}(\Omega_{132})\Big|_{132} = \frac{\hbar}{z_3 - z_2}\cdot 0 + \frac{z_2 - z_3 + \hbar}{z_2 - z_3}\Big[(z_2-z_1)(z_3-z_1)(z_3-z_2)\Big]_{z_2 \leftrightarrow z_3} =$$

$$= (z_2 - z_3 + \hbar)(z_3 - z_1)(z_2 - z_1)$$

$\mathcal{F}(3)$	123	132	213	231	312 321 $\rightarrow$ .....
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	...
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0	...
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0	...
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	...
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0	...
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	$\boxed{\text{Homework}}$	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	...

$$= \hbar\left(\hbar^2 + z_2 z_1 - z_1 z_3 - z_2^2 + z_2 z_3\right)$$

$F(3)$	123	132	213	231	312 321 $\cdots\rightarrow$
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0	
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0	
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0	
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	Homework	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	

Bott-Samelson $w=312$ $k=2$
$\sigma=123$

$$\frac{\hbar}{z_3-z_2}\,\hbar^2(z_3-z_1)-\frac{z_3-z_2+\hbar}{z_3-z_2}\,\hbar(z_2-z_1)(z_2-z_3+\hbar)$$

Bott-Samelson $w=231$ $k=1$
$\sigma=123$

$$\frac{\hbar}{z_2-z_1}\,\hbar^2(z_3-z_1)-\frac{(z_2-z_1+\hbar)}{z_2-z_1}\,(z_1-z_2+\hbar)(z_3-z_2)\hbar$$

$\mathcal{F}(3)$	123	132	213	231	312 321 ...... $\rightarrow$
$c^{sm}(\Omega_{123})$	$(z_2-z_1)(z_3-z_1)(z_3-z_2)$	0	0	0	
$c^{sm}(\Omega_{132})$	$\hbar(z_2-z_1)(z_3-z_1)$	$(z_3-z_1)(z_2-z_1)(z_2-z_3+\hbar)$	0	0	
$c^{sm}(\Omega_{213})$	$\hbar(z_3-z_1)(z_3-z_2)$	0	$(z_1-z_2+\hbar)(z_3-z_1)(z_3-z_2)$	0	
$c^{sm}(\Omega_{231})$	$\hbar^2(z_3-z_1)$	$\hbar(z_3-z_1)(z_2-z_3+\hbar)$	$(z_1-z_2+\hbar)(z_3-z_2)\hbar$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	
$c^{sm}(\Omega_{312})$	$\hbar^2(z_3-z_1)$	$\hbar(z_2-z_1)(z_2-z_3+\hbar)$	$(z_3-z_1)(z_1-z_2+\hbar)\hbar$	0	
$c^{sm}(\Omega_{321})$		$\hbar^2(z_2-z_3+\hbar)$	Homework	$\hbar(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	

R-matrix recursion
$w = 312 \quad k=1$
$\sigma = 123$

$$\frac{\hbar}{z_2-z_1}\,\hbar^2(z_3-z_1) + \frac{z_1-z_2+\hbar}{z_1-z_2}\Big[(z_3-z_1)(z_1-z_2+\hbar)\hbar\Big]_{z_1\leftrightarrow z_2}$$

R-matrix recursion
$w = 231 \quad k=2$
$\sigma = 123$

$$\frac{\hbar}{z_3-z_2}\,\hbar^2(z_3-z_1) + \frac{z_2-z_3+\hbar}{z_2-z_3}\Big[(z_3-z_1)\hbar(z_2-z_3+\hbar)\Big]_{z_2\leftrightarrow z_3}$$

Remarks

- Bott-Samelson recursion only for full flag varieties
- R-matrix recursion for G/P (partial flag var's) too
- both recursions can be phrased "globally" as well, on $c^{sm}(\Omega_w)$'s not on "local" classes $c^{sm}(\Omega_w)|_\sigma$
- either one can serve as a definition of $c^{sm}(\Omega_I)$ (together with the obvious $c^{sm}(\Omega_{id}) = c^{sm}(\text{point})$)
- $c^{sm}(\Omega_w)|_\sigma$ overdetermined $\to$ identities for $\boxed{\text{rational}}$ functions

"K-theoretic version"

"elliptic cohomology version"

$\downarrow$

$\boxed{\text{trigonometric}}$

$\downarrow$

$\boxed{\text{elliptic}}$

- full flag variety having 2 "dual" recursions is an incarnation of the fact that

$$T^*\mathcal{F}(n) \xleftrightarrow[\text{symmetry}]{\text{3d mirror}} T^*\mathcal{F}(n)$$

in general

$$X \xleftrightarrow[\text{symmetry}]{\text{3d mirror}} X'$$

$\Vert$

their (elliptic) Schubert calculus "match" (in a complicated sense)