MacPherson property of CSM classes

	12	13	14	23	24	34
$c^{sm}(\Omega_{12})$	$(z_3-z_1)(z_4-z_1)(z_3-z_2)(z_4-z_2)$	0	0	0	0	0
$c^{sm}(\Omega_{13})$	$\hbar \cdot \ldots$	$(z_2-z_1)(z_4-z_1)(z_4-z_3)(z_2-z_3+\hbar)$	0	0	0	0
$c^{sm}(\Omega_{14})$	$\hbar \cdot \ldots$	$\hbar \cdot (z_2-z_3+\hbar)\cdot \ldots$	$(z_2-z_1)(z_3-z_1)(z_2-z_4+\hbar)(z_3-z_4+\hbar)$	0	0	0
$c^{sm}(\Omega_{23})$	$\hbar \cdot \ldots$	$\hbar \cdot (z_2-z_3+\hbar)\cdot \ldots$	0	$(z_4-z_2)(z_4-z_3)(z_1-z_2+\hbar)(z_1-z_3+\hbar)$	0	0
$c^{sm}(\Omega_{24})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar)\cdot \ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)\cdot \ldots$	$\hbar(\ldots)(\ldots)\ldots$	$(z_3-z_2)(z_1-z_2+\hbar)(z_1-z_4+\hbar)(z_3-z_4+\hbar)$	0
$c^{sm}(\Omega_{34})$	$\hbar \cdot \ldots$	$\hbar(z_2-z_3+\hbar)\cdot \ldots$	$\hbar(z_2-z_4+\hbar)(z_3-z_4+\hbar)\cdot \ldots$	$\hbar(\ldots)(\ldots)\ldots$	$\hbar(\ldots)(\ldots)(\ldots)\ldots$	$(z_1-z_3+\hbar)(z_1-z_4+\hbar)(z_2-z_3+\hbar)(z_2-z_4+\hbar)$

Σ add together all

$$\sum_I c^{sm}(\Omega_I) = \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \\ \left(\begin{matrix} (z_3-z_1+\hbar)(z_3-z_2+\hbar) \\ (z_4-z_1+\hbar)(z_4-z_2+\hbar) \end{matrix} \right. & \begin{matrix} (z_2-z_1+\hbar)(z_4-z_1+\hbar) \\ (z_2-z_3+\hbar)(z_4-z_3+\hbar) \end{matrix} & \begin{matrix} (z_2-z_1+\hbar)(z_3-z_1+\hbar) \\ (z_2-z_4+\hbar)(z_3-z_4+\hbar) \end{matrix} & \begin{matrix} (z_1-z_2+\hbar)(z_4-z_2+\hbar) \\ (z_1-z_3+\hbar)(z_4-z_3+\hbar) \end{matrix} & \begin{matrix} (z_1-z_2+\hbar)(z_3-z_2+\hbar) \\ (z_1-z_4+\hbar)(z_3-z_4+\hbar) \end{matrix} & \left. \begin{matrix} (z_1-z_3+\hbar)(z_2-z_3+\hbar) \\ (z_1-z_4+\hbar)(z_2-z_4+\hbar) \end{matrix} \right) \end{array}$$

$\Rightarrow$

at each fixed point I it is $\displaystyle\prod_{\substack{i\in I \\ j\in \bar{I}}} (z_j - z_i + \hbar)$

but $\quad c(T\mathrm{Gr}_2\mathbb{C}^4)\big|_I = \displaystyle\prod_{\substack{i\in I \\ j\in \bar{I}}} (1 + z_j - z_i)$

$$\Rightarrow \quad \sum c^{sm}(\Omega_I) = c\Big(T\big(\overbrace{\textstyle\bigsqcup_I \Omega_I}^{\text{smooth}}\big)\Big) \qquad (\hbar = 1)$$

$$c^{sm}(\Omega_{23}) = \quad \dots \qquad \boxed{(z_4 - z_2)(z_4 - z_3)(z_1 - z_2 + \hbar)(z_1 - z_3 + \hbar)} \qquad 0 \qquad 0$$

$$c^{sm}(\Omega_{24}) = \quad \dots \qquad \hbar(\dots)(\dots)\dots \qquad \boxed{(z_3 - z_2)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar)} \qquad 0$$

$$c^{sm}(\Omega_{34}) = \quad \dots \qquad \hbar(\dots)(\dots)\dots \qquad \hbar(\cdot)(\cdot)(\cdot)\dots \qquad \boxed{(z_1 - z_3 + \hbar)(z_1 - z_4 + \hbar)(z_2 - z_3 + \hbar)(z_2 - z_4 + \hbar)}$$

$$\left.\begin{array}{l}(z_3 - z_2)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar) \\ \hbar(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar)\end{array}\right\} + \quad = (z_3 - z_2 + \hbar)(z_1 - z_2 + \hbar)(z_1 - z_4 + \hbar)(z_3 - z_4 + \hbar)$$

More generally

, if $\bigsqcup_J \Omega_J = M \overset{i}{\subseteq} Gr_k \mathbb{C}^n$,

smooth cpt

then

$$\boxed{\sum_J c^{sm}(\Omega_J) = i_*\Big(c(TM) \Big)}$$

$\hbar$-deformed Schubert Calculus

→ overdetermined notion

→ generalize to different settings

→ different conventions ($\hbar$ variable or $\hbar=1$)

→ different names :(

MacPherson

$c(TX) = ?$ if X is not smooth

Schubert Calc. not on X, but on T^*X

quantum integrable systems

quantum group representations

general recursions in Schubert Calculus

Hecke algebras

search for elliptic characteristic classes

KNOWN MATHEMATICS

More generally: $c^{sm}(f) \in H_T^*(Gr_k \mathbb{C}^n)$ is defined

constructible (T-invariant) function

meaning: can be written as

$$\sum c_V \cdot \underbrace{\mathbb{1}_V}$$

indicator function of variety V

- additive

$$c^{sm}(f+g) = c^{sm}(f) + c^{sm}(g)$$

$$c^{sm}(\lambda f) = \lambda \cdot c^{sm}(f)$$

- if $f = \mathbb{1}_M$ ← $\boxed{\text{compact smooth}}$ then $c^{sm}(f) = i_*(c(TM))$

$$c^{sm}(\Omega_I) = c^{sm}(\mathbb{1}_{\Omega_i})$$

Even more generally:

$$C_*^T : \underbrace{\mathcal{F}^T(-)}_{\text{functor of}} \longrightarrow \underbrace{H_*^T(-)}_{\text{T-equivariant}}$$

functor of T-invariant constructible functions on $\mathbb{C}$ algebraic varieties

[covariant, pushforward defined via χ]

T-equivariant homology functor

unique natural transformation of functors satisfying

- $\mathbb{1}_M \mapsto c(TM) \cap \mu_M^T$