Grassmannians

$0 \leq k \leq n$ integers

$$\mathrm{Gr}_k \mathbb{C}^n := \{ V^k \leq \mathbb{C}^n \} \qquad \mathrm{Gr}_1 \mathbb{C}^n =: \mathbb{P}^{n-1}$$

Geometry

- torus action, fixpts
- bundles over $\mathrm{Gr}_k \mathbb{C}^n$
- Schubert decomposition

torus action on $Gr_k \mathbb{C}^n$

$\underbrace{(\mathbb{C}^*)^n}_{\text{torus} = T^n = T} \circlearrowright \mathbb{C}^n$ by $(\zeta_1, \dots, \zeta_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \zeta_1 x_1 \\ \zeta_2 x_2 \\ \vdots \\ \zeta_n x_n \end{pmatrix}$

<u>induces</u> $(\mathbb{C}^*)^n \circlearrowright Gr_k \mathbb{C}^n$ by $\zeta \cdot V^k = \{ \zeta x : x \in V^k \}$

<u>fixed points</u>: coordinate k-planes $\overset{1:1}{\longleftrightarrow}$ k-element subsets of $\{1 \dots n\}$

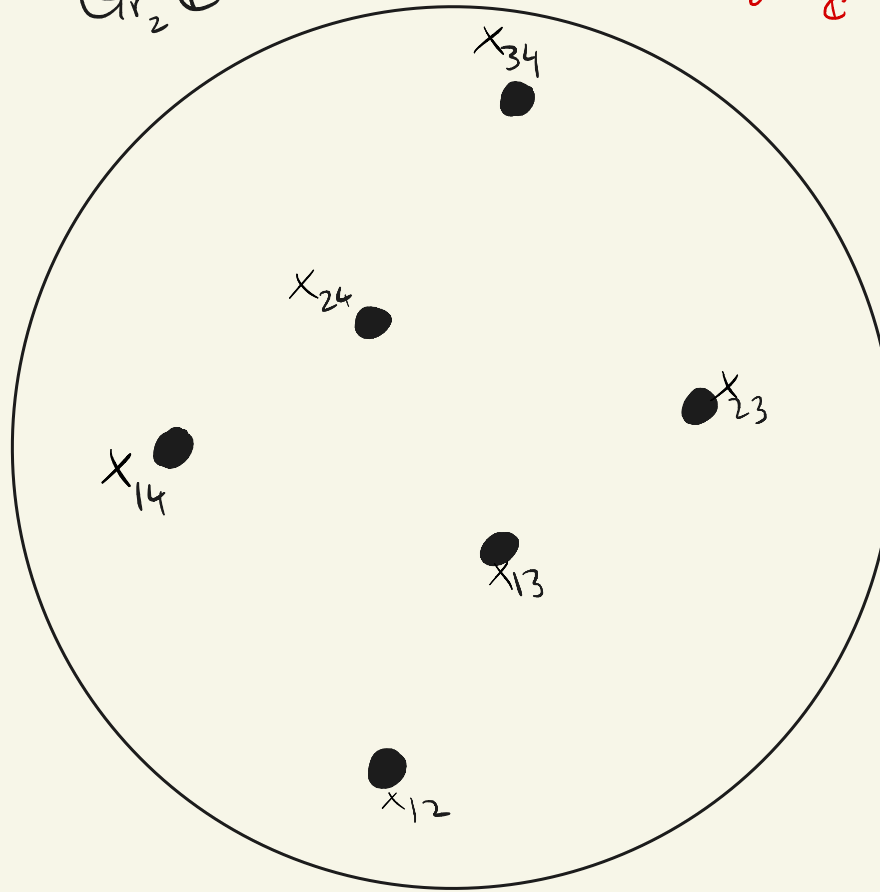
$x_I \longleftrightarrow I$

$\mathrm{Gr}_1 \mathbb{C}^2 = \mathbb{P}^1$

$\dim_{\mathbb{C}} = 1$

$\mathrm{Gr}_2 \mathbb{C}^4$

$\dim_{\mathbb{C}} = 4$



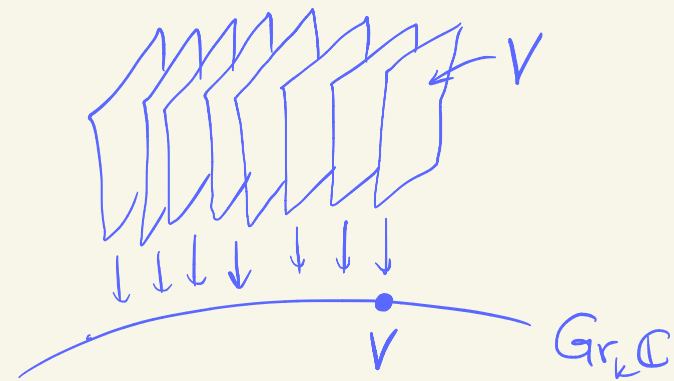
Tautological bundle over $Gr_k\mathbb{C}^n$

$$S = \{ (V, x) \in Gr_k\mathbb{C}^n \times \mathbb{C}^n \ : \ x \in V \}$$

$T^n C$

$$\begin{tikzcd} S \arrow[d, "\mathbb{C}^k"] \\ Gr_k\mathbb{C}^n \end{tikzcd}$$

" preimage of V is V "



Role: bundles determine elements in the cohomology of the base

" Chern class (bundle) "

Facts

$$\mathbb{Z}[\overbrace{t_1,\dots,t_k}, z_1\dots z_n]^{S_k} \overset{q}{\twoheadrightarrow} H^*_T(Gr_k\mathbb{C}^n) \overset{Loc}{\rightarrowtail} \bigoplus_{\substack{|I|=k \\ I\subseteq\{1\dots n\}}} \underbrace{H^*_T(X_I)}_{\mathbb{Z}[z_1,\dots,z_n]}$$

- Loc = restrictions to fixed pts

  Loc injective

  and $\mathrm{im}(Loc) = \{ (f_I) : z_i - z_j \mid f_I - f_J \}$

  $I = K \cup \{i\}$

  $J = K \cup \{j\}$

- $t_1,\dots,t_k$ are Chern roots of S

  q surjective

- $$(Loc \circ q)_I = \begin{cases} t_1 \longmapsto z_{i_1} \\ t_2 \longmapsto z_{i_2} \\ \vdots \\ t_k \longmapsto z_{i_k} \end{cases} \qquad I = \{i_1,\dots,i_k\}$$

Where are we so far

$Gr_k\,\mathbb{C}^n$

torus action

bundle

$\to H^*_T(Gr_k\,\mathbb{C}^n)$

explicit description

Schubert decomposition

coming up