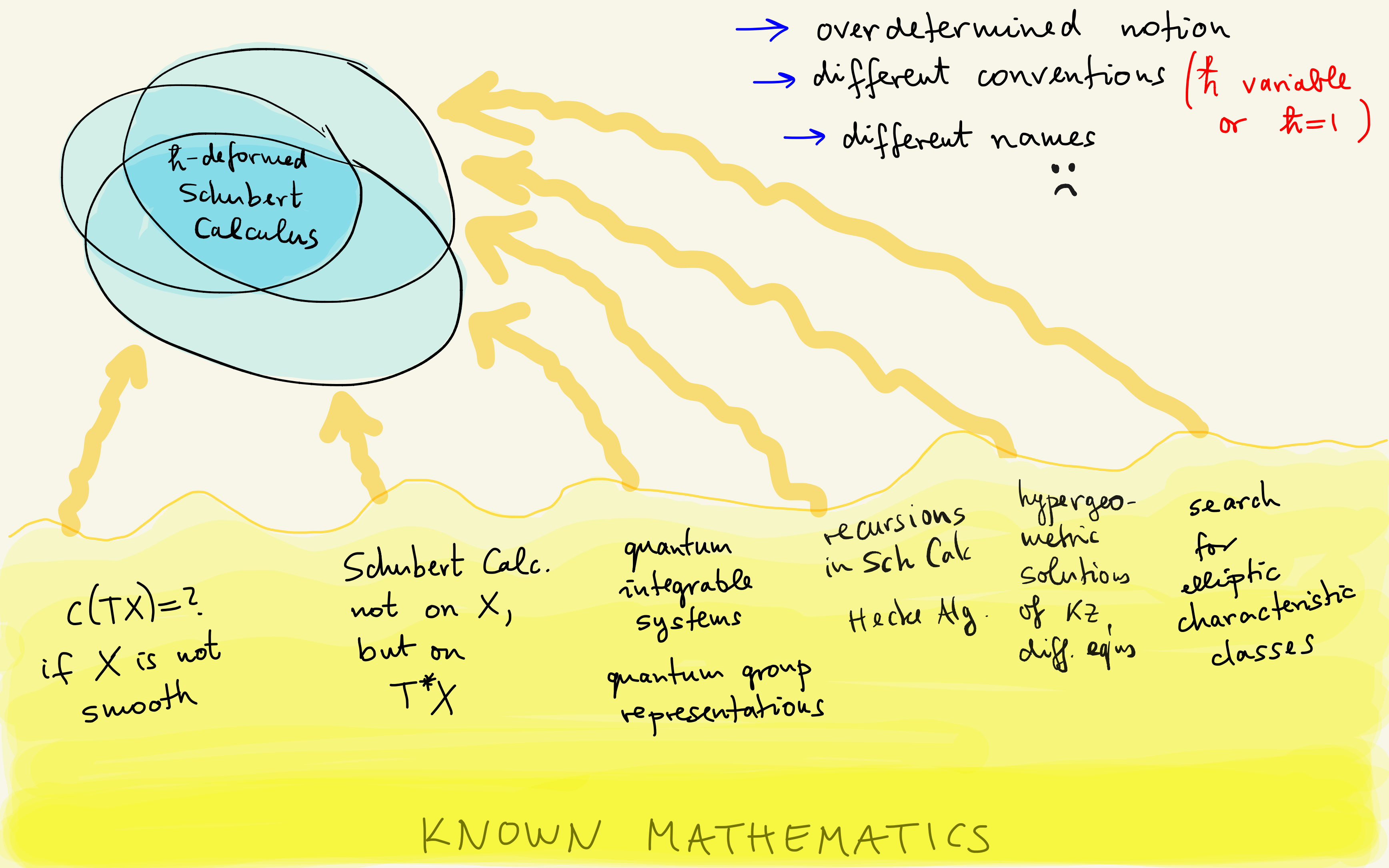
Introduction

+ $H^*_T(Gr)$


→ overdetermined notion
→ different conventions ($\hbar$ variable or $\hbar=1$)
→ different names
:(
$\hbar$-deformed Schubert Calculus
$c(TX)=?$ if X is not smooth
Schubert Calc. not on X, but on T^*X
quantum integrable systems
quantum group representations
recursions in Sch Calc
Hecke Alg.
hypergeometric solutions of KZ, diff. eq'ns
search for elliptic characteristic classes
KNOWN MATHEMATICS

GEOMETRY

$$T \circlearrowright \left(\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathbb{C}^n \arrow[r] \arrow[d] & Q \arrow[dl] \\ & Gr_k\mathbb{C}^n & \end{tikzcd} \right)$$

bundles

$Gr_k\mathbb{C}^n \supset \Omega_I$

Schubert cells

ALGEBRA-COMBIN.

$H^*_T(Gr_k\mathbb{C}^n)$

$\ni [\overline{\Omega}_I]$ fundamental class of closure of Ω_I (aka as Schubert class)

$\ni c^{sm}(\Omega_I)$ $\hbar$-deformed Sch class (aka Chern-Schwartz-MacPherson class)

Ex $k=1$ $n=2$

$$H_T^*(Gr_1\mathbb{C}^2) =$$

$$\left\{ \left(f_1(z_1,z_2),\, f_2(z_1,z_2)\right) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \;:\; \overbrace{f_1(u,u) = f_2(u,u)}^{\text{consistency condition}} \right\}$$

e.g.

$$(z_2 - z_1 \,,\; 0)$$

$$(1 \,,\; z_1 - z_2 + 1)$$

$$(2z_1^2 \,,\; z_1z_2 + z_2^2)$$

multiplication defined componentwise

Ex $\quad k=1 \quad n=2$

$$H^*_T(Gr_1 \mathbb{C}^2) =$$

$$\left\{ \left(f_1(z_1,z_2), f_2(z_1,z_2)\right) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \; : \; \overbrace{f_1(u,u) = f_2(u,u)}^{\text{consistency condition}} \right\}$$

equivalently: $\quad z_1 - z_2 \mid f_1(z_1,z_2) - f_2(z_1,z_2)$

$(z_2 - z_1, \quad 0) \qquad f_1 - f_2 = z_2 - z_1$

$(1, \quad z_1 - z_2 + 1) \qquad z_2 - z_1$

$(2z_1^2, \quad z_1 z_2 + z_2^2) \qquad 2z_1 - z_1 z_2 - z_2^2 = (z_1 - z_2)(2z_1 + z_2)$

<u>Ex</u> $k=1 \quad n=2$

$$H_T^*(Gr_1 \mathbb{C}^2) =$$

$$\left\{ \left(f_1(z_1,z_2), f_2(z_1,z_2)\right) \in \mathbb{Z}[z_1,z_2] \times \mathbb{Z}[z_1,z_2] \;:\; \overbrace{f_1(u,u) = f_2(u,u)}^{\text{consistency condition}} \right\}$$

$$z_1 - z_2 \mid f_1 - f_2$$

another rephrasing:

$\exists f(t, z_1, z_2)$ such that
$$f_1(z_1,z_2) = f(z_1, z_1, z_2)$$
$$f_2(z_1,z_2) = f(z_2, z_1, z_2)$$

$(z_2 - z_1, \quad 0) \qquad f = z_2 - t$

$(1, \quad z_1 - z_2 + 1) \qquad f = z_1 - t + 1$

$(2z_1^2, \quad z_1 z_2 + z_2^2) \qquad f = t^2 + t z_1$

Ex $k=1$ $n=2$

$$H^*_T(Gr_1\mathbb{C}^2) =$$

$$\{(f_1(z_1,z_2), f_2(z_1,z_2)) \in \mathbb{Z}[z_1,z_2]\times\mathbb{Z}[z_1,z_2] \;:\; f_1(u,u) = f_2(u,u)\}$$

$z_1 - z_2 \mid f_1 - f_2$

$\exists f(t, z_1, z_2)$ s.t.

$f_1 = f(z_1, z_1, z_2)$

$f_2 = f(z_2, z_1, z_2)$

three equivalent ways of phrasing the consistency condition

Ex $k=2$ $n=4$

$$H_T^*(Gr_2\mathbb{C}^4) = \left\{ (f_{12}, f_{13}, \dots, f_{34}) \in \mathbb{Z}[z_1,z_2,z_3,z_4]^6 : \text{consistency condition} \right\}$$

2-element subsets of $\{1,2,3,4\}$

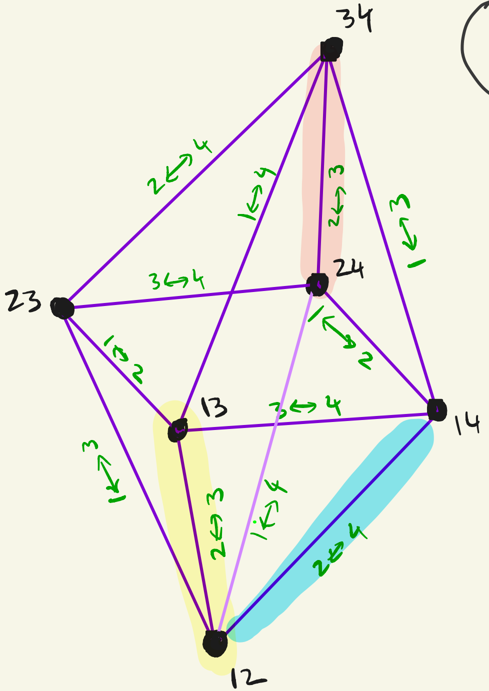
...

three equivalent ways of phrasing

⋮

Ex $k=2$ $n=4$

$$H_T^*(Gr_2\mathbb{C}^4) = \Big\{ \underbrace{(f_{12}, f_{13}, \dots, f_{34})}_{\text{six vertices}} \in \mathbb{Z}[z_1, z_2, z_3, z_4]^6 : \underbrace{\begin{array}{l} z_2 - z_3 \mid f_{12} - f_{13} \\ z_2 - z_4 \mid f_{12} - f_{14} \\ \quad\vdots \\ z_2 - z_3 \mid f_{24} - f_{34} \end{array}}_{\text{twelve edges}} \Big\}$$

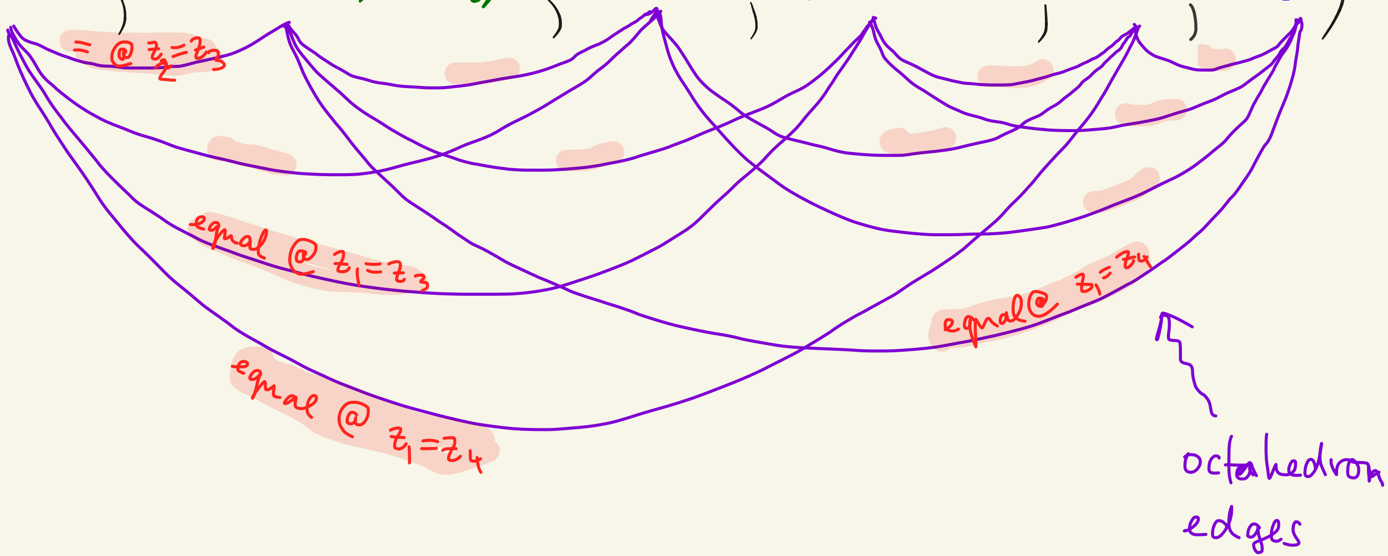


rem say, $z_2 - z_3 \mid f_{12} - f_{13}$ is equivalent

$$f_{12}(z_1, u, u, z_4) = f_{13}(z_1, u, u, z_4)$$

$$
\begin{array}{cccccc}
f_{12} & f_{13} & f_{14} & f_{23} & f_{24} & f_{34} \\
\end{array}
$$

$$
\Big((z_4-z_1)(z_4-z_2),\ (z_4-z_1)(z_4-z_3),\ 0,\ (z_4-z_2)(z_4-z_3),\ 0,\ 0 \Big)
$$



Third equivalent rephrasing of consistency condition

$$H_T^*(\mathrm{Gr}_2\mathbb{C}^4) = \Big\{ (f_{12}, \dots, f_{34}) \in \mathbb{Z}[z_1, z_2, z_3, z_4]^6 :$$

$$\exists\, f(t_1, t_2, z_1, z_2, z_3, z_4) \in \mathbb{Z}[\overbrace{t_1, t_2}^{S_2}, z_1, z_2, z_3, z_4] \text{ such that}$$

$$f_{12} = f(z_1, z_2, z_1, z_2, z_3, z_4)$$

$$f_{13} = f(z_1, z_3, z_1, z_2, z_3, z_4)$$

$$\vdots$$

$$f_{34} = f(z_3, z_4, z_1, z_2, z_3, z_4) \Big\}$$

General $k \leq n$.

$$H^*_T(Gr_k \mathbb{C}^n) = \left\{ (f_I) \in \mathbb{Z}[z_1, \dots, z_n]^{\binom{n}{k}} \ : \ \text{consistency} \right\}$$

$\uparrow$ k-element subset of $\{1 \dots n\}$

General $k \leq n$.

$$H_T^*(Gr_k \mathbb{C}^n) = \left\{ (f_I) \in \mathbb{Z}[z_1, \dots, z_n]^{\binom{n}{k}} \ : \ \text{consistency} \right\}$$

$\forall$ I, J satisfying $I = K \cup \{i\}$
$J = K \cup \{j\}$

$$z_i - z_j \mid f_I - f_J$$

$$\exists f \in \mathbb{Z}[\overbrace{t_1, \dots, t_k}, z_1, \dots, z_n]^{S_k}$$

$$f_I = f(z_I, z_1, \dots, z_n)$$

So far:

$$\boxed{H^*_T(Gr_k\mathbb{C}^n) = \text{explicit description}}$$

enough for most of this lecture