BASIC NOTIONS IN COTANGENT SCHUBERT CALCULUS
INTRO WORKSHOP ON COMBINATORIAL ALGEBARIC GEOMETRY
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PROBLEMS

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These problems use the notations, and refer to notions from my lecture *"Basic notions in cotangent Schubert Calculus"* at CIRM 2021. Lecture notes are available upon request. The meaning and relevance of the statements made in these problems are also explained in that lecture.

Problem 1 (*on* $H_T^*(\mathbb{P}^1)$)
Consider the ring homomorphism

$$\begin{aligned}\mathrm{Loc} : \mathbb{Z}[t, z_1, z_2] &\to \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2]\\ f(t, z_1, z_2) &\mapsto (f(z_1, z_1, z_2), f(z_2, z_1, z_2)) .\end{aligned}$$

Prove the following two characterizations of the image (range) of Loc:

$$Im(\mathrm{Loc}) = \{(f_1(z_1, z_2), f_2(z_1, z_2)) \in \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2] \ : \ f_1(u, u) = f_2(u, u)\},$$
$$Im(\mathrm{Loc}) = \{(f_1(z_1, z_2), f_2(z_1, z_2)) \in \mathbb{Z}[z_1, z_2] \oplus \mathbb{Z}[z_1, z_2] \ : \ (z_1 - z_2)|(f_1 - f_2)\}.$$

Problem 2 (*on equivariant Schubert classes in* $H_T^*(\mathbb{P}^{n-1})$)
Let $j \leqslant n$ be non-negative integers. Invent (that is, give a formula for) a polynomial $f(t, z_1, z_2, \ldots, z_n)$ such that

- f is of homogeneous degree $n - j$ (where $\deg t = \deg z_i = 1$);
- $f(z_j, z_1, z_2, \ldots, z_n) = \prod_{i=j+1}^{n}(z_i - z_j)$;
- $f(z_i, z_1, z_2, \ldots, z_n) = 0$ if $j < i \leqslant n$.

Problem 3 (*on equivariant CSM classes in* $H_T^*(\mathbb{P}^{n-1})$)
Let $j \leqslant n$ be non-negative integers. Invent a polynomial $f(t, z_1, z_2, \ldots, z_n, \hbar)$ such that

- f is of homogeneous degree $n - 1$ (where $\deg t = \deg z_i = \deg \hbar = 1$);
- $f(z_j, z_1, z_2, \ldots, z_n) = \prod_{i=1}^{j-1}(z_i - z_j + \hbar) \prod_{i=j+1}^{n}(z_i - z_j)$;
- $f(z_i, z_1, z_2, \ldots, z_n) = 0$ if $j < i \leqslant n$;
- $f(z_i, z_1, z_2, \ldots, z_n)$ is divisible by $\hbar$ for $i < j$;
- $f(z_i, z_1, z_2, \ldots, z_n)$ is divisible by $\prod_{s=1}^{i-1}(z_s - z_i + \hbar)$.

Problem 4 (*on the MacPherson property of CSM classes*)
Consider the polynomial f you defined in Problem 3, and let us call it $f_{j,n}$. Define $F_n = \sum_{j=1}^n f_{j,n}$. Show that $F_n|_{t=z_i}$ is a product of linear factors, for all n and i.

Problem 5 (*on equivariant Littlewood-Richardson coefficients on* $\mathbb{P}^1$)
In the lecture we saw that in $H_T^*(\mathbb{P}^1)$ we have

$$\begin{aligned}[\overline{\Omega}_1] =& (z_2 - z_1, 0),\\ [\overline{\Omega}_2] =& (\quad 1 \quad, 1).\end{aligned}$$

Calculate the products $[\overline{\Omega}_i] \cdot [\overline{\Omega}_j]$ as $\mathbb{Z}[z_1, z_2]$-linear combinations of $[\overline{\Omega}_1]$ and $[\overline{\Omega}_2]$.

Problem 6 (*on CSM versions of equivariant Littlewood-Richardson coefficients on* $\mathbb{P}^1$)
In the lecture we saw that in $H_T^*(\mathbb{P}^1)$ we have

$$\begin{aligned}\mathrm{c}^{\mathrm{sm}}(\Omega_1) =& (z_2 - z_1, \qquad 0 \qquad),\\ \mathrm{c}^{\mathrm{sm}}(\Omega_2) =& (\quad \hbar \quad, z_1 - z_2 + \hbar).\end{aligned}$$

Calculate the products $\mathrm{c}^{\mathrm{sm}}(\Omega_i) \cdot \mathrm{c}^{\mathrm{sm}}(\Omega_j)$ as $\mathbb{Z}[z_1, z_2, \hbar]$-linear combinations of $\mathrm{c}^{\mathrm{sm}}(\Omega_1)$ and $\mathrm{c}^{\mathrm{sm}}(\Omega_2)$.

Problem 7 (*on the R-matrix property on CSM classes*)
In the lecture we claimed that

$$\begin{pmatrix} \mathrm{c}^{\mathrm{sm}}(\Omega_{\varnothing}^{\mathrm{opposite}}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{1}^{\mathrm{opposite}}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{2}^{\mathrm{opposite}}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{12}^{\mathrm{opposite}}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & \frac{\hbar}{z_1 - z_2 + \hbar} & 0 \\ 0 & \frac{\hbar}{z_1 - z_2 + \hbar} & \frac{z_1 - z_2}{z_1 - z_2 + \hbar} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{c}^{\mathrm{sm}}(\Omega_{\varnothing}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{1}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{2}) \\ \mathrm{c}^{\mathrm{sm}}(\Omega_{12}) \end{pmatrix} \tag{1}$$

(and that the occurring matrix satisfies the parameterized Yang-Baxter equation). Verify (1). Find the analogous matrix if we replace $\mathrm{c}^{\mathrm{sm}}(\Omega_I)$'s with $[\overline{\Omega}_I]$'s.

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