Solution to Midterm-2, Problem 4

Since $f' : [a, b] \to \mathbb{R}$ is continuous on a closed interval, it attains its maximum $f'(x_1) = M$ and minimum $f'(x_2) = m$ (EVT). That is, for all $x \in [a, b]$ we have $m \le f'(x) \le M$. This implies that

$$|f'(x)| \le \max(|m|, |M|) \qquad \text{for all } x \in [a, b].$$

Let $c = \max(|m|, |M|)$. Since both m and M are values of f', we have $c < 1$. Let x and y be in $[a, b]$. By MVT (a.k.a. Lagrange's theorem) there exists a $u \in [a, b]$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(u).$$

We have

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(u)| \le c,$$

what is a rearrangement of what we wanted to prove.