Motivic characteristic classes of coincident root loci

The generating function approach

(work in progress)

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Introduction

Brasselet, Schürmann and Yokura (2005) introduced the so-called *motivic Chern class transformation* and its "homology shadow"[1] which could be called 'Hirzebruch transformation', and which unifies:

- the Chern-Schwartz-MacPherson transformation,
- the Baum-Fulton-MacPherson-Todd transformation, and
- the Cappell-Shaneson L-class transformation.

These can be used to define characteristic classes of singular varieties embedded into (for simplicity) smooth compact varieties.

My goal today is to compute these classes for a concrete example, which (in my opinion) falls into the sweet spot between easy and hopeless.

[1] my wording

Motivic characteristic classes

By a "motivic characteristic class" $m(f)$ I mean (loosely after BSY), a functor $m : \mathrm{Var}/X \to A(X)$, from the category of quasi-projective varieties over X to some (co)homology-like theory $A(X)$, which satisfies the following 3+1 properties:

- additivity: for $f_i : Y_i \to X$ we have
$$m(f_1 \coprod f_2) = m(f_1) + m(f_2)$$
- exterior product: for $f_i : Y_i \to X_i$ we have
$$m(f_1 \times f_2) = m(f_1) \times m(f_2)$$
- functoriality: for $f : Y \to X$ and $g : X \to Z$ <u>proper</u>, we have
$$m(g \circ f) = g_!(m(f))$$
- normalization: there is some natural expression for $m(X) := m(\mathrm{id}_X)$ when X is smooth (and compact?).

Examples of motivic characteristic classes

Shorthand / abuse of notation: $m(X) = m(\mathrm{id}_X)$, which is a characteristic class living on X.

Examples of such motivic classes:

- in constant rings:
  - the Euler characteristic $\chi(X) \in \mathbb{Z}$
  - the Hirzebruch χ_y genus $\chi_y(X) \in \mathbb{Z}[t]$
  - the Hodge polynomial (or E-polynomial) $E(X) \in \mathbb{Z}[u, v]$ (?)
- in homology $H_*(X)$:
  - the Chern(-Schwartz-MacPherson) class $c_{\mathrm{SM}}(X)$
  - the (Baum-Fulton-MacPherson-)Todd class $\mathrm{td}(X)$
  - the (Cappell-Shaneson-)Thom-Milnor L-class $L(X)$
  - the (BSY-)Hirzebruch class $T_y(X)$
- in K-theory $K_0(X)$:
  - the Brasselet-Schürmann-Yokura motivic Chern class $\mathrm{mc}(X)$
- equivariant versions of these

What are these classes exactly?

The 3 properties plus normalization uniquely[2] determine these classes, so it's enough to specify the normalization, that is, their values for X smooth:

- the CSM class is $c_{\mathrm{SM}}(X) = c(TX) = \prod_i (1+\alpha_i)$
- the Todd class is $\mathrm{td}(X) = \prod_i \frac{\alpha_i}{1-e^{-\alpha_i}}$
- the L class is $L(X) = \prod_i \frac{\alpha_i}{\tanh(\alpha_i)}$
- the (normalized) Hirzebruch class is

$$T_y(X) = \prod_i \left\{ \frac{\alpha_i(1+t)}{1-e^{-\alpha_i(1+t)}} - \alpha_i t \right\} \in H^*(X)[t]$$

where α_i are the Chern roots of the tangent bundle TX;

And for the motivic Chern class it is

$$\mathrm{mc}(X) = \lambda_t(T^*X) = \sum \left[\wedge^i (T^*X) \right] \cdot t^i \in K^0(X)[t]$$

[2] I hope

The case of smoooth ambient space

It often makes sense to assume that the ambient space X is smooth and compact (and the map $f : Y \to X$ is an embedding); especially since we are dealing with stratifications of smooth spaces.

In that case we can use Poincaré duality of X and switch to the <u>co</u>homology version $m(Y \subset X) \in H^*(X)$. When the ambient space and the embedding is clear from the context, it also makes sense to simply write $m(Y)$ instead of $m(Y \subset X)$.

In this talk the ambient space will be always smooth and compact, in fact it will be either a projective space or a product of projective spaces. Because of this, instead of $m(Y \subset \mathbb{P}^{\underline{n}})$ I will simply write $m(Y)$.

Coincident root loci

The space $\mathbb{P}^n = \mathbb{P}\mathrm{Sym}^n\mathbb{C}^2$ is the configuration space of unordered n-tuples of points in $\mathbb{P}^1 = \mathbb{P}\mathbb{C}^2$ (roots of degree n binary forms).

This space is naturally stratified by specifying the multiplicities of the points (roots): Given a partition $\mu = (\mu_1, \ldots, \mu_k)$ of n, define $X_\mu \subset \mathbb{P}^n$ be the set of configurations (or forms) which have k distinct roots, with multiplicities $\mu_1, \mu_2, \ldots, \mu_k$.

$$\mathbb{P}^n = \coprod_{\mu \vdash n} X_\mu$$

For example the partition $\mu = (3, 2, 2, 1)$ of 8 means that we have a triple point, two disjoint double points and a singleton point.

We call the loci X_μ *coincident root loci*.[3]

[3] also called: multiple root loci, pejorative manifolds, discriminant strata, factorization manifolds, λ-Chow varieties, etc.

What about the singularities?

The loci X_μ are *smooth* (but not compact!). What we would normally be interested is their closures $\overline{X}_\mu$, which are singular.

It is easy to see that

$$\overline{X}_\mu = \coprod_{\nu \preceq \mu} X_\nu$$

where $\prec$ is the refinement partial order. Since we assume additivity, this means

$$m(\overline{X}_\mu) = \sum_{\nu \preceq \mu} m(X_\nu),$$

thus modulo understanding this partial ordering, it's enough to consider the 'open' case. It's not clear if we can expect nice results for the closures (this partial order is known to be "not so nice").

In any case, here I will concentrate on the open strata.

The goal: Compute $m(X_\mu \subset \mathbb{P}^n)$ for different m-s

Last year I talked about computing the equivariant CSM classes of the loci X_μ. It is then a natural question to ask about the other motivic classes?

(Un)fortunately, the method I presented last year works *only* for the CSM class, as it uses, in a crucial way, the following property of the CSM classes: For $f : Y \subset X$ and a proper map $\pi : X \to X'$, we have

$$c_{\mathrm{SM}}(\pi \circ f) = \pi_* \, c_{\mathrm{SM}}(Y \subset X) = \sum \chi(\pi^{-1}(z_i) \cap Y) \cdot c_{\mathrm{SM}}(Z_i)$$

with $\coprod_i Z_i = \pi(Y)$ such that $\chi(\pi^{-1}(z) \cap Y)$ is constant on each Z_i.

The analogue of this does *not* hold for the other motivic classes[4]. So I figured out a different method, which in principle works for *all* motivic classes[5]. And it turns out to have some nice and surprising consequences!

[4] it only holds if π is a *Zariski locally trivial* fibration (?)

[5] in the sense explained before

Motivation

I like this problem because:

- it is simple enough that we are actually able to compute these classes;
- it is complicated enough to have a very rich structure;
- the equivariant versions should have applications in enumerative geometry;
- it is basically the simplest configuration / moduli problem one can think of.

So this is a really nice and “simple” example to work with.

In this talk I'm concentrating on the non-equivariant case, but of course what we really want are the equivariant versions. I don't see any serious obstacles to do that, but it's more (heavy) work, which is not yet done. And the non-equivariant case is already rather interesting!

Some previous results

The loci $\overline{X}_\mu$ were studied by many people:

- Cayley (1857): defining equations (for $n \leq 5$)
- Schubert (1886): enumerative consequences for some particular μ-s
- Hilbert (1887): the degree of $\overline{X}_\mu$
- Kirwan (1992): the fundamental class for some specific μ-s (in the context of GIT)
- Aluffi (1998): the non-equivariant CSM class
- Chipalkatti (2001): the defining ideal (for some cases at least?)
- Fehér, Némethi, Rimányi ($\sim$2003; published in 2006): the equivariant fundamental class via localization
- Kőműves (2003): the same equivariant fundamental class via restriction equations
- Kőműves (2016, unpublished): the equivariant CSM class

The plan

My plan for today (and tomorrow):

- explain the geometry, and introduce another family of interesting loci $D(\underline{n})$, closely related to X_μ
- present an *algorithm* to compute $m(D(\underline{n}))$, which is generic over all motivic classes m (modulo some "plugins")
- figure out the necessary "plugins" for cohomology (and K-theory)
- convert the algorithm to equations about the generating functions of the classes $m(D(\underline{n}))$
- solve these equations in some particular cases, resulting in very explicit formulas for the generating functions
- recover the generating functions for $m(X_\mu)$

The basic maps

We have two basic maps between spaces of points we will use.

The diagonal map:

$$\Delta^k \ : \ \mathbb{P}^n \ \longrightarrow \ \underbrace{\mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{k \text{ times}}$$

And the multiplication map:

$$\Psi \ : \ \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r} \ \longrightarrow \ \mathbb{P}^{n_1+n_2+\cdots+n_r}$$

merging different sets of points.

Composing these together, we get the "power map":

$$\Psi \circ \Delta^k \ =: \ \Omega^k \ : \ \mathbb{P}^n \ \longrightarrow \ \mathbb{P}^{nk}$$

which replicates (duplicates, triplicates, etc) the points.

The space of distinct points

Let $D(n) = X_{(1^n)} \subset \mathbb{P}^n$ be the configuration of n distinct points, and define

$$\underbrace{D(n_1, n_2, \ldots, n_r)}_{D(\underline{n})} \subset D(n_1) \times \cdots \times D(n_r) \subset \underbrace{\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}}_{\mathbb{P}^{\underline{n}}}$$

to be the configuration of $N = n_1 + n_2 + \cdots + n_r$ points which are all distinct.

Observation: For a partition $\mu = (1^{e_1}, 2^{e_2}, 3^{e_3}, \ldots)$ we have

$$\Omega^\mu : D(e_1, e_2, e_3, \ldots) \xrightarrow{\sim} X_\mu$$

$$\begin{array}{ccccccccccc}
D(\underline{e}) & \subset & \mathbb{P}^{e_1} & \times & \mathbb{P}^{e_2} & \times \cdots \times & \mathbb{P}^{e_r} & & & & \\
 & & \downarrow \Omega^1 & & \downarrow \Omega^2 & & \downarrow \Omega^r & & & & \\
 & & \mathbb{P}^{1e_1} & \times & \mathbb{P}^{2e_2} & \times \cdots \times & \mathbb{P}^{re_r} & \xrightarrow{\Psi} & \mathbb{P}^N & \supset & X_\mu
\end{array}$$

The motivic recursion, page I.

The core idea: products of $D(\underline{n})$-s stratify naturally, where all the strata are isomorphic images of other $D(\underline{m})$ loci.

The simplest example is take just two: $D(p) \times D(q) \subset \mathbb{P}^p \times \mathbb{P}^q$. In this case we have

$$D(p) \times D(q) = \coprod_{k \geq 0} D(p-k, q-k, k)$$

where the stratification is based on how many of the p and q points in $D(p)$ and $D(q)$ coincide. Since they are already indistinguishable from each other, nothing else can happen.

We have a natural map

$$\begin{array}{ccc}
\mathbb{P}^{p-k} \times \mathbb{P}^{q-k} \times \mathbb{P}^k & \xrightarrow{\mathrm{id}\times\mathrm{id}\times\Delta^2} & \mathbb{P}^{p-k} \times \mathbb{P}^{q-k} \times (\mathbb{P}^k \times \mathbb{P}^k) \\
\downarrow & & \downarrow \mathrm{id}\times\mathrm{swap}\times\mathrm{id} \\
\mathbb{P}^p \times \mathbb{P}^q & \xleftarrow{\Psi\times\Psi} & (\mathbb{P}^{p-k} \times \mathbb{P}^k) \times (\mathbb{P}^{q-k} \times \mathbb{P}^k)
\end{array}$$

The motivic recursion, page II.

More generally, we have

$$D(p) \times D(\underline{n}) = \coprod_{\substack{\underline{n}=\underline{d}+\underline{e} \\ p=k+\sum e}} D(k, \underline{d}, \underline{e})$$

using a similar diagram:

$$\begin{tikzcd}
\mathbb{P}^k \times \mathbb{P}^{\underline{d}} \times \mathbb{P}^{\underline{e}} \arrow[r, "\mathrm{id}\times\mathrm{id}\times\Delta^2"] \arrow[d] & \mathbb{P}^k \times \mathbb{P}^{\underline{d}} \times (\mathbb{P}^{\underline{e}} \times \mathbb{P}^{\underline{e}}) \arrow[d] \\
\mathbb{P}^p \times \mathbb{P}^{\underline{n}} & (\mathbb{P}^k \times \mathbb{P}^{\underline{e}}) \times (\mathbb{P}^{\underline{d}} \times \mathbb{P}^{\underline{e}}) \arrow[l, "\Psi_{k,e}\times\Psi_{d,e}"']
\end{tikzcd}$$

Observation: The open stratum on the RHS (corresponding to $\underline{e}=0$) is just $D(p, \underline{n})$, and all the other strata has smaller "total dimension" (meaning $\sum \underline{n}$). Therefore, if we knew how to compute the pushforwards, and knew $m(D(k)) = m(X_{1^k})$, then we could recursively compute all $m(D(\underline{n}))$, and thus $m(X_\mu)$.

The double recursion

Alas, we don't know $m(D(k)) = m(X_{1^k})$. This is in some sense the hardest of all X_μ.

However, we noted before that $X_\mu = \Omega^\mu(D(e_1, e_2, \dots))$ and that

$$\mathbb{P}^n = \coprod_{\mu \vdash n} X_\mu = \coprod_{\underline{e}} \Omega^{\underline{e}}(D(\underline{e}))$$

where $\mu = (1^{e_1}, 2^{e_2}, 3^{e_3}, \dots)$, and $\underline{e}$ runs over sequences such that $n = \sum_i i e_i$. Applying the m class, we now have two mutually recursive equations:

$$m(D(p)) \times m(D(\underline{n})) = \sum_{k,\underline{d},\underline{e}} \Psi_! \Delta_! \, m(D(k, \underline{d}, \underline{e})) \qquad (*)$$

$$m(\mathbb{P}^n) = \sum_{\underline{e}} \Omega_! \, m(D(\underline{e})) \qquad (**)$$

which together determine all $m(D(\underline{n}))$ (since they give an explicit recursive algorithm to compute them).

The "plugins"

So to compute $m(D(n))$ and $m(X_\mu)$, the only things we need to know are:

- $m(\mathbb{P}^n)$
- how to compute the pushforward $\Delta_!$
- and how to compute the pushforward $\Psi_!$

(recall the Ω is just a composition of these).

The rest of the algorithm is completely generic over different m-s! That's why I call these "plugins".

I worked out these for all the non-equivariant classes mentioned above[6], and wrote a compute program to compute these classes. This works in practice for X_μ with $|\mu| \leq 10$, or for example $D(p,q)$ for $p+q \leq 16$, but the program could be made more efficient.

[6]strictly speaking, I don't yet have a proof for the K-theory pushforward formulas

Generating functions

OK, we now have an *algorithm*, which is great, but not very insightful.

I mean, knowning that the Hirzebruch class of $X_{(5,2,2,1)}$ is

$$\begin{aligned} T_y(X_{(5,2,2,1)}) = & (240u^6 - 294u^7 + 66u^8) + \\ & + t \cdot (-162u^7 + 116u^8 - 9u^9 - u^{10}) + \\ & + t^2 \cdot (50u^8 - 18u^9 - u^{10}) + t^3 \cdot (-9u^9 + u^{10}) + t^4 \cdot u^{10} \end{aligned}$$

does not help me much, does it?

In the remaining part I will work out the *generating functions* of all these classes, which I hope will convince you that generating functions are both a really powerful tool to have, and a very compact way to encode a large amount of information.

The χ_y genus

Let's start with the χ_y genus because it's the simplest interesting example[7]. A remark about notation: I will use t instead of y.

The χ_y genus is the integral of the Hirzebruch class T_y (and also of the motivic Chern class):

$$\chi_y(X) = \pi_* \, T_y(X) = \pi_! \, \mathrm{mc}(X) \in \mathbb{Z}[t]$$

where $\pi : X \to$ pt is the collapsing map.

Similarly as the Hirzebruch class unifies 3 classes, the χ_y genus unifies three characteristics:

$$\chi_y(X) = \begin{cases} \chi(X) & t \mapsto -1 & \text{Euler characteristic} \\ \chi(X, \mathcal{O}_X) & t \mapsto 0 & \text{arithmetic genus}^8 \\ \mathrm{sign}(X) & t \mapsto +1 & \text{signature} \end{cases}$$

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[7] The Euler characteristic of coincident root loci is more-or-less trivial

[8] the Hirzebruch arithmetic number, *not* the Severi arithmetic genus p_a

Generating function for the χ_y genus

For the χ_y genus, we know all the plugins: The "pushforwards" are trivial (that is, the identity of $\mathbb{Z}[t]$), and for the projective space we have

$$\chi_y(\mathbb{P}^n) = 1 - t + t^2 - t^3 + \cdots \pm t^n$$

Let's encode all the χ_y genera of all $D(\underline{n})$ loci into one big generating function:

$$\mathcal{F}(\underline{x}) = \sum \underline{x}^{\underline{n}} \cdot \chi_y(D(\underline{n})) = Z[t][[x_1, x_2, x_3, \ldots]]$$

where the sum runs over all infinite sequences $\underline{n} \in \mathbb{N}^{\mathbb{N}}$ with finite sum (that is, finitely many nonzero elements).

This is a *symmetric function* in infinitely many variables, that is, an element of the ring of symmetric functions: $\mathcal{F} \in \Lambda_{\mathbb{Z}[t]}$.

The equations for the χ_y genus

We need to express the equations

$$\chi_y(D(p)) \cdot \chi_y(D(\underline{n})) = \sum \chi_y(D(k, \underline{d}, \underline{e})) \qquad (*)$$

$$\chi_y(\mathbb{P}^n) = \sum \chi_y(D(\underline{e})) \qquad (**)$$

in the language of generating functions.

This we can do easily if we simply follow what exactly we sum over. In the first equation (*), we sum over triples $(k, \underline{d}, \underline{e})$ such that $p = k + \sum_i e_i$ and $n_j = d_j + e_j$. In the second equation (**), we sum over $\underline{e}$ such that $n = 1e_1 + 2e_2 + 3e_3 + \dots$. Packing these together for all $(p, \underline{n})$ and for all n, respectively, we get:

$$\mathcal{F}(q) \cdot \mathcal{F}(\underline{x}) = \mathcal{F}(q, \underline{x}, q \cdot \underline{x}) \qquad (*)$$

$$\frac{1}{(1-q)(1+qt)} = \sum q^n \cdot \chi_y(\mathbb{P}^n) = \mathcal{F}(q, q^2, q^3, \dots) \qquad (**)$$

Solving the χ_y equations

So we have these equations for the gf. of χ_y genera:

$$\mathcal{F}(q)\cdot\mathcal{F}(\underline{x}) = \mathcal{F}(q,\underline{x},q\cdot\underline{x}) \qquad (*)$$

$$\frac{1}{(1-q)(1+qt)} = \mathcal{F}(q,q^2,q^3,\dots) \qquad (**)$$

How to solve such equations? I have *no idea* whatsoever, in general!

However, we know *a priori* that these equations have a unique solutions, since they represent an actual algorithm to compute the solution. So it's enough to somehow find one solution, and we are done.

Observation: The symmetric polynomial $p_k(\underline{x}) = x_1^k + x_2^k + x_3^k + \dots$ behaves nicely wrt. the substitution in the first equation. In fact, $F_k(\underline{x}) = 1 + p_k(\underline{x})$ is a solution of the *first* equation:

$$\underbrace{(1+q^k)}_{F_k(q)}\underbrace{(1+x_1^k+x_2^k+\dots)}_{F_k(\underline{x})} = \underbrace{1+q^k+x_1^k+x_2^k+\dots+(qx_1)^k+(qx_2)^k+\dots}_{F_k(q,\underline{x},q\,\underline{x})}$$

Solving the χ_y equations, page II.

We can then try the following *ansatz*:

$$\sum \underline{x}^{\underline{n}} \cdot \chi_y(D(\underline{n})) = \mathcal{F}(\underline{x}) = \prod_{k=1}^{\infty} (1 + p_k(\underline{x}))^{a_k(t)/k}$$

where $a_k(t) \in \mathbb{Z}[t]$ are some polynomials of t. This is basically just a linear combination (after taking the logarithm) of the family of solutions we noticed, and thus clearly satisfies the first equation.

Now we can use our knowledge of the χ_y genera for small $D(\underline{n})$, and try to recursively work out the polynomials $a_k(t)$. Based on that, we can conjecture the solution:

$$a_k(t) = \begin{cases} 1 - t & k = 1 \\ \sum_{d|k} \mu(d) \cdot (-t)^{k/d} & k \geq 2 \end{cases}$$

where $\mu(d)$ is the Möbius function (from elementary number theory), and the sum runs overs the (positive) divisors of k.

Finishing the proof

So now we only have to prove that our conjectured solution satisfies the second equation, too, and we are ready!

$$1 + p_k(q, q^2, q^3, \dots) = 1 + q^k + q^{2k} + q^{3k} + \cdots = \frac{1}{1-q^k}$$

$$\log(1 + p_k) = -\log(1 - q^k) = \sum_{i\geq 1} \frac{q^{ki}}{i}$$

$$\log(\mathcal{F}(q, q^2, q^3, \dots)) = \log(1 + p_1) + \sum_{k\geq 1} \log(1 + p_k) \frac{1}{k} \sum_{d|k} \mu(d)(-t)^{k/d} =$$

$$= -\log(1 - q) + \sum_{k\geq 1} \sum_{i\geq 1} \frac{q^{ki}}{ki} \sum_{d|k} \mu(d)(-t)^{k/d} =$$

$$= -\log(1 - q) + \sum_{m\geq 1} \frac{q^m}{m} \underbrace{\sum_{k|m} \sum_{d|k} \mu(d)(-t)^{k/d}}_{(-t)^m} = -\log(1 - q) - \log(1 + qt)$$

QED.

Example χ_y computation

How to extract the χ_y genus of a particular loci from the GF? Consider for example $\mu = (4,4,2,2,1,1,1)$. Rewriting into exponential form: $\mu = (1^3, 2^2, 3^0, 4^2)$ Hence $\chi_y(X_\mu) = \chi_y(D(3,2,0,2))$. Thus we only need to differentiate a few times:

$$\chi_y(X_\mu) = \frac{1}{3! \cdot 2! \cdot 2!} \left[\frac{\partial^7 \mathcal{F}(x_1, x_2, x_3, x_4)}{\partial x_1^3 \;\; \partial x_2^2 \;\; \partial x_4^2} \right] (x_i \mapsto 0)$$

Note that while $\mathcal{F}$ is an infinite product, only a finite part of that affects any particular coefficient, since the exponents of the x_i variables increase.

Now any computer algebra software (eg. Mathematica) can compute:

$$\chi_y(X_{(4,4,2,2,1,1,1)}) = 6t + 6t^2 - 2t^3 - 3t^4 - 3t^5 - 3t^6 - t^7$$

```
xx[i_] := Subscript[x,i]; pk[k_] := Sum[xx[i]^k,{i,1,4}]
apoly[k_] := If[k == 1, 1 - t,
   Sum[MoebiusMu[d]*(-t)^(k/d), {d, Divisors[k]}]]
chiyGF = Product[(1 + pk[k])^(apoly[k]/k), {k, 1, 3}];
D[chiyGF, {xx[1], 3}, {xx[2], 2}, {xx[4], 2}]/ 3!/2!/2!;
Expand[% /. Table[xx[i] -> 0, {i, 1, 4}]]
```

Motivic classes in (co)homology

The next logical step is to consider classes in (co)homology. These are: the Chern(-SM) class, Todd class, the L-class and the Hirzebruch T_y class. In fact the latter generalizes all them:

$$T_y(X) = \begin{cases} c_{\mathrm{SM}}(X) & y \mapsto -1 \\ \mathrm{td}(X) & y \mapsto 0 \\ L(X) & y \mapsto +1 \end{cases}$$

but it is still illustrative to consider the specialized cases.

Remark: These classes originally live in homology, and it turns out that indeed the formulas look better in homology, but we are in smooth compact ambient spaces so we have Poincaré duality, and I'm very used to work in cohomology instead of homology, so these will be mixed freely.

Generating functions for (co)homology classes

First of all, what does it mean to consider the generating function

$$\sum \underline{x}^{\underline{n}} \cdot m(D(\underline{n})) \ = ???$$

These classes all live in different rings!

$$m(D(\underline{n})) \in H^*(\mathbb{P}^{\underline{n}}) = \mathbb{Z}[\underline{u}] \ / \ (u_i^{n_i+1} = 0)$$

Here $u_i = -c_1(L_i)$ denotes the positive hyperplane generator of $H^*(\mathbb{P}^{n_i})$.

Our (preliminary) answer is to take the *normal form* of these classes: they can be written uniquely as linear combinations of $\underline{u}^{\underline{k}}$ with $0 \leq \underline{k} \leq \underline{n}$, and then take the formal sum over $\underline{n}$:

$$\mathcal{G}(\underline{x}\,;\,\underline{u}) = \sum_{\underline{n},\underline{k}} a_{\underline{n},\underline{k}} \cdot \underline{x}^{\underline{n}} \underline{u}^{\underline{k}}$$

such that

$$m(D(\underline{n})) = \sum_{\underline{k}} a_{\underline{n},\underline{k}} \cdot \underline{u}^{\underline{k}}$$

Homology exponential generating functions

Instead of what I just described, we will use a slightly modified version: 1) homology indexing instead of cohomology; and 2) exponential in the homology direction. That is

$$\mathcal{F}(\underline{x}\,;\,\underline{v}) = \sum_{\underline{n},\underline{k}} \frac{a_{\underline{n},\underline{k}}}{k_1!k_2!k_3!\cdots} \cdot x^{\underline{n}} v^{\underline{k}}$$

such that

$$m(D(\underline{n})) = \sum_{\underline{k}} a_{\underline{n},\underline{k}} \cdot u^{\underline{n}-\underline{k}}$$

It turns out that this version is much more convenient to work with.

To help notation, let us introduce a hat version of all motivic classes, which conforms to this convention, so we can (somewhat informally) write

$$\mathcal{F}(\underline{x}\,;\,\underline{v}) = \sum_{\underline{n}} \underline{x}^{\underline{n}} \cdot \widehat{m}(D(\underline{n}))$$

while understanding that this means the above convention.

The pushforwards in cohomology

The pushforwards along our basic maps are quite easy to work out:

$$\begin{array}{rcll}
(\Delta^m)_*\, u^j & = & s_{(n^{m-1},j)}(v_1, v_2, \ldots v_k) & \mathbb{P}^n \to \mathbb{P}^n \times \cdots \times \mathbb{P}^n \\
(\Psi_{n,m})_*\, u^i v^j & = & \binom{n+m-i-j}{n-i} w^{i+j} & \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m} \\
(\Omega^d)_*\, u^j & = & d^{(n-j)} \cdot v^{(d-1)n+j} & \mathbb{P}^n \to \mathbb{P}^{dn}
\end{array}$$

where s_λ are the Schur polynomials. The multiplication map Ψ can be naturally generalized to any number of inputs, and there will be a multinomial coefficient on the RHS.

Now we have to translate this to our (homology, exponential) generating functions, and that's where the magic happens:

$$\begin{array}{ll}
(\Delta^2_*\,\mathcal{F})(\underline{y}, \underline{z}\,;\, \underline{v}, \underline{w}) & = \mathcal{F}(\underline{y}\cdot\underline{z}\,;\, \underline{v} + \underline{w}) \\
(\Psi_*\,\mathcal{F})(\underline{z}\,;\, \underline{w}) & = \mathcal{F}(\underline{z}, \underline{z}\,;\, \underline{w}, \underline{w}) \\
(\Omega^d_*\,\mathcal{F})(\underline{y}\,;\, \underline{v}) & = \mathcal{F}(\underline{y}^d\,;\, d\cdot\underline{v})
\end{array}$$

which is why we chose this particular convention for our GFs.

The equations in (co)homology

We can now easily write down the (co)homology version of our equations:

$$\mathcal{F}(q\,;z)\cdot\mathcal{F}(\underline{x}\,;\underline{u}) = \mathcal{F}(q,\underline{x},q\cdot\underline{x}\;;\;z,\underline{u},z+\underline{u}) \qquad (*)$$

$$\sum_n q^n\cdot\widehat{m}(\mathbb{P}^n) = \mathcal{F}(q,q^2,q^3,\dots;\;z,2z,3z,\dots) \qquad (**)$$

Not surprisingly, we cannot solve these (more complicated) equations directly either, however, again we have some simple solutions for the first equation:

$$F_k(\underline{x};\underline{u}) = 1+\sum_i x_i^k = 1+p_k(\underline{x})$$

$$G_k(\underline{x};\underline{u}) = \exp\left[\frac{\sum_i x_i^k u_i}{1+p_k(\underline{x})}\right]$$

It is elementary algebraic manipulation to check that these satisfy the first equation.

The ansatz

In lack of better ideas, we can again write down an ansatz, and try to figure out the unknown coefficients such that the second equation is satisfied, too.

Our ansatz (for the Hirzebruch class T_y) will be

$$\log(\mathcal{F}(\underline{x};\underline{u})) = \sum_{k\geq 1} \frac{a_k(t)}{k} \cdot \log(1+p_k(\underline{x})) + \sum_{k\geq 1} \frac{b_k(t)}{k} \cdot \frac{\sum_i x_i^k u_i}{1+p_k(\underline{x})}$$

where $a_k(t)$ and $b_k(t)$ are some polynomials in $\mathbb{Z}[t]$ (for the other classes, they are just numbers).

It turns out that we are lucky, and solution has indeed this form.

Motivic (co)homology classes of $\mathbb{P}^n$

One remaining piece of information we need are the classes of $\mathbb{P}^n$; more precisely, their generating functions.

Theorem:

$$
\begin{aligned}
\sum q^n \cdot \widehat{c_{\mathrm{SM}}}(\mathbb{P}^n) &= (1-q)^{-2} \cdot \exp[qz/(1-q)] \\
\sum q^n \cdot \widehat{\mathrm{td}}(\mathbb{P}^n) &= (1-q)^{-1-z} \\
\sum q^n \cdot \widehat{L}(\mathbb{P}^n) &= (1-q)^{-1-z/2} \cdot (1+q)^{-1+z/2} \\
\sum q^n \cdot \widehat{T}_y(\mathbb{P}^n) &= (1-q)^{-1-z/(1+t)} \cdot (1+tq)^{-1+z/(1+t)}
\end{aligned}
$$

The last one implies the others, by specializing t to $-1, 0, +1$, respectively. A further lemma:

$$
\mathrm{td}(\mathbb{P}^n) = \sum_{k=0}^{n} \frac{k!}{n!} \cdot |s(n+1, k+1)| \cdot u^{n-k} \in H^*(\mathbb{P}^n) = \mathbb{Z}[u] \,/\, (u^{n+1} = 0)
$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Proof for $\mathrm{td}(\mathbb{P}^n)$ (Hirzebruch is the same)

$$[u^{n-k}]\,\mathrm{td}(\mathbb{P}^n) = [u^{n-k}]\,\left(\frac{u}{1-e^{-u}}\right)^{n+1} = \mathrm{Res}_{u=0}\,\frac{u^k}{(1-e^{-u})^{n+1}} =$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{u^k}{(1-e^{-u})^{n+1}}\,\mathrm{d}u$$

Applying the change of variables:

$$\frac{1}{x} = \frac{1}{1-e^{-u}} \qquad u = -\log(1-x) \qquad \mathrm{d}u = \frac{\mathrm{d}x}{1-x}$$

we get

$$[u^{n-k}]\,\mathrm{td}(\mathbb{P}^n) = \frac{1}{2\pi i}\int_{\gamma'} \frac{(-\log(1-x))^k}{x^{n+1}(1-x)}\,\mathrm{d}x = [x^n]\,\frac{(-\log(1-x))^k}{(1-x)}$$

Summing over k:

$$\sum_{k=0}^{\infty} y^k \cdot [u^{n-k}]\mathrm{td}(\mathbb{P}^n) = [x^n]\,\frac{1}{(1-x)(1+y\log(1-x))}$$

Exponential vs. ordinary GFs

You can switch between ordinary and exponential generating functions using Laplace transform. This is a neat trick: If we have

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad G(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

then they are related by

$$\begin{aligned} F(x) &= x^{-1} \cdot \mathcal{L}[G](x^{-1}) \\ G(t) &= \mathcal{L}^{-1}\left[x \mapsto x^{-1} F(x^{-1})\right](t) \end{aligned}$$

Applying this to variable y in

$$\sum x^n \cdot \mathrm{td}(\mathbb{P}^n) = \frac{1}{(1-x)(1+y\log(1-x))}$$

we get

$$\sum x^n \cdot \widehat{\mathrm{td}}(\mathbb{P}^n) = (1-x)^{-1-z} \quad \Longleftrightarrow \quad \widehat{\mathrm{td}}(\mathbb{P}^n) = \frac{(z+n)_n}{n!}$$

The solution for the Hirzebruch class

The (homology, exponential) GF of the Hirzebruch classes, $\sum \underline{x}^{\underline{n}} \cdot \widehat{T}_y(D(\underline{n}))$, is:

$$\exp\left[\underbrace{\sum_{k\geq 1} \frac{a_k(t)}{k} \cdot \log(1 + p_k(\underline{x}))}_{\chi_y \text{ part}} + \sum_{k\geq 1} \frac{b_k(t)}{k} \cdot \frac{\sum_i x_i^k u_i}{1 + p_k(\underline{x})}\right]$$

where $a_k(t)$ and $b_k(t)$ are polynomials in $\mathbb{Z}[t]$, defined by

$$a_k(t) = \begin{cases} 1 - t & k = 1 \\ \sum_{d|k} \mu(d) \cdot (-t)^{k/d} & k \geq 2 \end{cases}$$

$$b_k(t) = \sum_{d|k} \mu(d) \cdot d \cdot \frac{1 - (-t)^{k/d}}{1 + t}$$

You can get the GF for the X_μ loci by substituting $u_i \mapsto i \cdot v$.

The three specializations

It is interesting to look at the specializations $t \mapsto -1, 0, +1$, too. Denote by

$$E_k(\underline{x};\underline{u}) = \frac{\sum_i x_i^k u_i}{1 + p_k(\underline{x})}$$

Then the EGF of the CSM, Todd and L classes of $D(\underline{n})$ are:

$$\widehat{c_{\mathrm{SM}}} = (1 + p_1(\underline{x}))^2 \cdot \exp\left[E_1(\underline{x};\underline{u})\right]$$

$$\widehat{\mathrm{td}} = (1 + p_1(\underline{x})) \cdot \prod_{k=1}^{\infty} \exp\left[\left(\sum_{d|k} \frac{d \cdot \mu(d)}{k}\right) \cdot E_k(\underline{x};\underline{u})\right]$$

$$\widehat{L} = (1 + p_2(\underline{x})) \cdot \prod_{k=1}^{\infty} \exp\left[\left(\sum_{d|k} \frac{d \cdot \mu(d)}{k}\right)\left(E_k(\underline{x};\underline{u}) - \frac{1}{2}E_{2k}(\underline{x};\underline{u})\right)\right]$$

Btw, $\sum_{d|k} d \cdot \mu(d) = \prod_{p|k}(1 - p)$ is the Dirichlet inverse of Euler's totient $\phi(k)$.

Single-variable specializations

Another interesting specialization is the single-variable case, which corresponds to the EGF for $D(n) = X_{1^n}$. These we can get by

$$x_i \mapsto \begin{cases} x & i = 1 \\ 0 & i \geq 2 \end{cases}$$

The results are:

$$\begin{aligned}
\chi_y &= (1+x) \cdot (1+tx^2) \cdot (1+tx)^{-1} \\
\widehat{c_{\mathrm{SM}}} &= (1+x)^2 \cdot \exp[zx/(1+x)] \\
\widehat{\mathrm{td}} &= (1-x)^z \cdot (1+x)^{2z-1} \\
\widehat{L} &= (1+x^2)^{1-z} \cdot (1-x)^{z/2} \cdot (1+x)^{3z/2} \\
\widehat{T_y} &= (1-x)^{z/(1+t)} \cdot (1+x)^{1+2z/(1+t)} \cdot \\
&\qquad \cdot (1+tx)^{-1+z/(1+t)} \cdot (1+tx^2)^{1-2z/(1+t)}
\end{aligned}$$

Proofs

We only have to prove that these solutions satisfy the second (normalization) equation:

$$\mathcal{F}(q, q^2, q^3, \dots ; z, 2z, 3z, \dots) = \sum_n q^n \cdot \widehat{m}(\mathbb{P}^n)$$

These boil down to combinatorial identities like this:

Lemma. We have the following identities:

$$\sum_{k=1}^{\infty} \sum_{d|k} \mu(d) \frac{d}{k} \frac{y^k}{1-y^k} = -\log\left[(1-y)\right] = \sum_k \frac{y^k}{k}$$

$$\sum_{k=1}^{\infty} \sum_{d|k} \mu(d) \frac{d}{k} \frac{y^k}{1+y^k} = \log\left[(1+y)^2(1-y)\right] = 2\sum_k \frac{y^{2k}}{k} - \sum_k \frac{y^k}{k}$$

There are more formulas like these. I haven't proved all of them, but I believe they shouldn't be *too* hard.

Motivic classes in K-theory

The main example we have in K-theory is the 'motivic Chern class' of Brasselet-Schürmann-Yokura, which is a class $\mathrm{mc}(X) \in K^0(X)[t]$.

First of all, where do these classes live?

$$\begin{array}{lll} \mathrm{mc}(X_\mu) & \in K^0(\mathbb{P}^n)[t] & = \mathbb{Z}[t][H] \;/\; (H^{n+1} = 0) \\ \mathrm{mc}(D(\underline{n})) & \in K^0(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r})[t] & = \mathbb{Z}[t][H_1, \ldots, H_r] \;/\; (H_i^{n_i+1} = 0) \end{array}$$

where $L = [L]$ is the class of the tautological line bundle L, and $H = 1 - L$ is the hyperplane class.

The algorithm to compute $\mathrm{mc}(D(\underline{n}))$ works perfectly well, we just need to supply the "plugins": the motivic Chern classes of $\mathbb{P}^n$, and the pushforwards along the basic maps:

$$\Delta_!^k : K^0(\mathbb{P}^n) \to K^0(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)$$
$$\Psi_! : K^0(\mathbb{P}^n \times \mathbb{P}^m) \to K^0(\mathbb{P}^{n+m})$$

Motivic Chern class of $\mathbb{P}^n$

For smooth compact varieties the mc class is defined to be the total lambda class λ_t of the cotangent bundle:

$$\mathrm{mc}(\mathbb{P}^n) = \lambda_t\left[T^*\mathbb{P}^n\right] = \sum_{i=0}^{n}\left[\Lambda^i(T^*\mathbb{P}^n)\right]\cdot t^i =$$
$$= \frac{(1+tL)^{n+1}}{1+t} = \frac{(1+t(1-H))^{n+1}}{1+t} \in \mathbb{Z}[t][H]/(H^{n+1}=0)$$

where again $H = 1-L$ is the hyperplane class.

From this it is easy to derive the generating functions:

$$\text{OGF:} \qquad \sum x^n\cdot\mathrm{mc}(\mathbb{P}^n) = \frac{1}{(1+txH)(1-x-tx+txH)}$$
$$\text{EGF:} \qquad \sum q^n\cdot\widehat{\mathrm{mc}}(\mathbb{P}^n) = \frac{1}{(1+tq)^2}\cdot\exp\left[J\cdot\frac{(1+t)q}{1+tq}\right]$$

where we use J instead of H in the exponential GF to distinguish between the two. Rule of thumb: $J^k/k! = H^{n-k} = (1-L)^{n-k}$.

The pushforwards in K-theory

We can compute the pushforwards using Grothendieck-Riemann-Roch[9] and our knowledge of the cohomology pushforwards. For $\pi : X \to Y$ and $\alpha \in K^0(X)$:

$$\mathrm{ch}_Y(\pi_!\,\alpha) \cdot \mathrm{td}(Y) = \pi_*\Big[\mathrm{ch}_X(\alpha) \cdot \mathrm{td}(X)\Big]$$

where $\mathrm{ch}_X : K^0(X) \to \oplus_k H^{2k}(X)$ is the Chern character.

It's not very hard to conjecture the following formulas based on the numbers:

$$\Delta_!^m(H^k) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} s_{(n^{m-1},k+i)}(H_1, H_2, \ldots, H_m)$$

$$\Psi_!(H_1^i \cdot H_2^j) = \sum_{k=0}^{p} (-1)^k \cdot \binom{n+m-i-j-k}{p}\binom{p}{k} \cdot H^{i+j+k}$$

$$p = \min(n-i, m-j)$$

where s_λ are the Schur polynomials. Observation: These formulas are *deformations* of the corresponding cohomology formulas.

[9] unfortunately I don't know any other tool...

The pushforwards in generating function language

We can translate the pushforward formulas into the language of generating functions.

Motivated by the fact that the K-theory versions are deformations of the cohomology version, we opt for the same type of generating functions: homology indexing, and exponential (though the choice is less clear here?)

With this convention, we have:

$$(\Delta_!^k\,\mathcal{F})(x_1,\ldots,x_k;J_1,\ldots,J_k) = \left[1-\frac{\partial}{\partial J}\right]^{k-1}(\mathcal{F})\left(\prod x_i\;;\;\sum J_i\right)$$

$$(\Psi_!\,\mathcal{F})(x;J) = \exp\left[-J\cdot\frac{\partial^2}{\partial J_1\,\partial J_2}\right](\mathcal{F})\,(x,x\;;\;J,J)$$

The equations in K-theory?

In principle now we could write down the equations for the generating function in K-theory.

In practice this looks very complicated, and I have *really no idea* about the form of the solutions! (apart from $1 + p_k(\underline{x})$). So we have a working algorithm (good for testing), but no formulas.

However, we know that the Chern character is an isomorphism, and (after multiplying by the Todd class of the ambient space) it maps the motivic Chern class into the (unnormalized) Hirzebruch class. Therefore we can try the "reverse engineer" the MC class from this connection.

Extracting the Hirzebruch class from the MC class

You can compute the normalized Hirzebruch class T_y from the motivic Chern class using the following steps of "yoga":

- take the Chern character (that is, $H \mapsto 1 - \exp(-u)$)
- multiply by the Todd class of the ambient variety
- what you have now is the unnormalized Hirzebruch class $\widetilde{T}_y$
- then substitute $u \mapsto (1+t)u$
- and divide by $(1+t)^d$ where d is the ambient dimension

It's a bit hard to follow, but in the generating function language this translates to

$$H^{n-k} = \frac{J^k}{k!} \quad \longmapsto \quad \widehat{\mathrm{td}}_{(1+t)^{-1}}(\mathbb{P}^k)$$

where by the RHS I mean that the (homology, exponential) variable is twisted by $(1+t)^{-1}$.

Interlude: the unnormalized Hirzebruch class

The Hirzebruch classes are defined (for smooth X) by

$$T_y(X) = \prod_i Q_y(\alpha_i) \qquad \text{normalized}$$
$$\widetilde{T}_y(X) = \prod_i \widetilde{Q}_y(\alpha_i) \qquad \text{unnormalized}$$

where α_i are the Chern roots of the tangent bundle TX, and

$$Q_y(\alpha) = \frac{\alpha(1+t)}{1-e^{-\alpha(1+t)}} - \alpha t$$
$$\widetilde{Q}_y(\alpha) = \frac{\alpha(1+te^{-\alpha})}{1-e^{-\alpha}}$$

which are related by:

$$Q_y(\alpha) = \frac{\widetilde{Q}_y\big(\alpha(1+t)\big)}{1+t}.$$

Observation: For the trivial line bundle we have $T_y(\mathbf{1}) = 1$, but $\widetilde{T}_y(\mathbf{1}) = 1 + t \neq 1$!

Reverse engineering the MC class

Recall that

$$H^{n-k} = G^k = \frac{J^k}{k!} \quad \longmapsto \quad \widehat{\mathrm{td}}_{(1+t)^{-1}}(\mathbb{P}^k)$$

Summing these together for different k, we get that:

$$\underbrace{A \cdot \exp\left[\beta \cdot J\right]}_{\text{EGF of mc(something)}} \quad \longmapsto \quad \underbrace{A \cdot (1-\beta)^{-1-z/(1+t)}}_{\text{EGF of } T_y\text{(something)}}$$

Notes: We could move the $(1-\beta)$ factor to the LHS; and we could also convert any or both sides into the OGF version. For example:

$$\underbrace{A \cdot \frac{(1-\beta)}{(1-G\beta)}}_{\text{OGF of mc(something)}} \quad \longmapsto \quad \underbrace{\frac{A \cdot (1+t)}{(1+t) + z\log(1-\beta)}}_{\text{OGF of } T_y\text{(something)}}$$

But the forms where the RHS is exponential are more handy.

Motivic Chern classes of $D(n) = X_{(1^n)}$

The single variable case is quite easy. Recall that

$$\sum x^n \cdot \widehat{T}_y(D(n)) = (1-x)^{z/(1+t)} \cdot (1+x)^{1+2z/(1+t)} \cdot$$
$$\cdot (1+tx)^{-1+z/(1+t)} \cdot (1+tx^2)^{1-2z/(1+t)} =$$

$$= \underbrace{\frac{(1+x)(1+tx^2)}{(1+tx)}}_{A} \exp\left\{ \frac{z}{(1+t)} \underbrace{\log\left[\frac{(1-x)(1+x)^2(1+tx)}{(1+tx^2)^2}\right]}_{-\log[1-\beta]} \right\}$$

From this it follows directly that

$$\sum_n x^n \cdot \widehat{\mathrm{mc}}(D(n)) = \frac{(1+tx^2)^3}{(1-x)(1+x)(1+tx)^2} \cdot$$
$$\cdot \exp\left[\frac{(1+t)Jx(1-x-x^2-tx^3)}{(1-x)(1+x)^2(1+tx)}\right]$$

An ugly formula for the MC classes of $D(\underline{n})$

We can do the exact same thing as in the single variable case:

$$\widehat{T}_y = \underbrace{\exp\left[\sum_{k\geq 1}\frac{a_k(t)}{k}\cdot\log(1+p_k(\underline{x}))\right]}_{A\ (\,=\text{gf. of }\chi_y\,)}\cdot\underbrace{\exp\left[\sum_{k\geq 1}\frac{b_k(t)}{k}\cdot\frac{\sum_i x_i^k u_i}{1+p_k(\underline{x})}\right]}_{\prod_i(1-\beta_i)^{-u_i/(1+t)}}$$

that is,

$$\sum_{k\geq 1}\frac{b_k(t)}{k}\cdot\frac{x_i^k}{1+p_k(\underline{x})} = -\frac{\log(1-\beta_i)}{(1+t)}$$

from which (btw the $(1+t)$ term cancels with the $(1+t)^{-1}$ in $b_k(t)$)

$$\beta_i = 1-\exp\left\{-(1+t)\sum_{k\geq 1}\frac{b_k(t)}{k}\cdot\frac{x_i^k}{1+p_k(\underline{x})}\right\}$$

From this, we have the homology OGF resp. EGF of the MC classes:

$$A\cdot\prod_{i=1}^{\infty}\frac{1-\beta_i}{1-G_i\beta_i} \qquad\qquad A\cdot\prod_{i=1}^{\infty}(1-\beta_i)\exp(J_i\beta_i)$$

where $G_i^k = H_i^{n_i-k}$.

Future work

Some future work remains:

- better formulas for the motivic Chern classes?
- equivariant classes
- applications
- stability
- closures of the strata (?)
- positivity?

Note: I have an algorithm for the equivariant classes in cohomology (that is, I can compute the pushforwards algorithmically), but haven't implemented it yet.