CLASSES OF DEGENERACY LOCI FOR QUIVERS—THE THOM POLYNOMIAL POINT OF VIEW

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1. Introduction

The goal of this paper is to demonstrate the usefulness of the theory of Thom polynomials for group actions developed by Maxim Kazarian in [Kaz95] and [Kaz97] by calculating some formulas of degeneracy loci. Our calculations are based on our method, the restriction equations ([FRa]), and a beautiful chapter of algebra: the representation theory of quivers. The paper is intended to be self contained except some technical details on the existence of the Poincaré dual and standard facts from the representation theory of quivers.

Certain types of Thom polynomials were studied under different names:

The name Thom polynomial comes from singularity theory where René Thom proposed the following question: Given a smooth map $f : M \to N$ what is the cohomology class $[\eta(f)] \in H^*(M)$ defined via Poincaré duality by the closure of $\eta(f) \subset M$—the points of M where f has a singularity of type η. As Thom observed this class can be expressed as a polynomial of characteristic classes of the vector bundles TM and f^*TN.

In homotopy theory an extensively studied question is whether a fiber bundle admits a section. One obstruction is the so called first obstruction which is a cohomology class of the base of the fiber bundle measuring the non-existence of a section. Thom polynomials for group actions are first obstructions.

In algebraic topology these questions can be translated to questions on equivariant cohomology theory.

In algebraic geometry Thom polynomials are called classes or formulas of degeneracy loci. The earliest example—which was also the first example in singularity theory—is the following:

Example 1.1. Let E and F are complex vector bundles of complex dimension n and p over the manifold M and $s : E \to F$ is a vector bundle homomorphism—i.e. a section of $\mathrm{Hom}(E, F)$. Let $\Sigma_k(s)$ denote the set of $m \in M$ such that the linear map $s(m)$ has corank k. We are looking for an expression for the cohomology class $[\Sigma_k(s)]$—the Poincaré dual of the closure of $\Sigma_k(s)$.

The corresponding situation in singularity theory is that if we have a smooth map $f : M \to N$ then we look for the set of points in M where df—the Jacobian of f—has corank k. It is a special case of Example 1.1—for real vector bundles—for the bundles $E = TM,\ F = f^*TN$, and the section df.

The homotopy theory approach to the same problem would be to look at the subspace $\Sigma_{<k} \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ containing matrices of corank smaller than k. $\Sigma_{<k}$ possesses a group action:

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It is an invariant subset for the group $G = GL(n,\mathbb{C}) \times GL(p,\mathbb{C})$ acting on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$. So for any G-principal bundle P we can associate a fiber bundle $P \times_G \Sigma_{<k}$. It is not too difficult to prove that the first obstruction is exactly $[\Sigma_k(s)]$ (in particular it doesn't depend on the section s provided it is 'generic' see 2.6).

The formulas were proved by Ian Porteous:

Theorem 1.2 ([Por71])**.**

$$[\Sigma_k(s)] = \det(A),$$

where $A_{ij} = c_{i-j+k}$ *for* $i, j = 1, \dots, k+p-n$ *and* $c_l = c_l(F \ominus E)$ *are the Chern classes of the difference bundle* $F \ominus E$.

Generalized Thom polynomial theory is a powerful tool to study a wide variety of questions. However, to calculate the actual values of the coefficients of these polynomials was a notoriously difficult problem. Until recently the only known method was the method of resolutions. For example to calculate the Thom polynomial of Σ_k it was necessary to find a resolution of the singular variety $\overline{\Sigma}_k$.

In [Rim01] the second author found a different method to calculate Thom polynomials of singularities, which was easy to generalize ([FRa]). This method provides the Thom polynomials as—for a wide variety of cases—the unique solution of a system of linear equations, called the restriction equations. Roughly speaking the method of resolutions finds the Thom polynomial as the *image of a pushforward* map and restriction equation method finds it in the *kernel of pullback* maps.

In this paper we would like to demonstrate this method on a case which was studied intensively by algebraic geometers. This is a straightforward generalization of Example 1.1:

Problem 1.3. Take now several vector bundles $E_1, \dots, E_n$ over a manifold M and vector bundle maps $\varphi_{ij} : E_i \to E_j$ for some pairs (i,j). To keep track of these pairs we can consider the oriented graph Q with vertices $Q_0 = \{1, \dots, n\}$ and arrows $Q_1 = \{(i,j)$: if we have a map $\varphi_{ij} : E_i \to E_j\}$. (So we can make sense of multiple arrows and loops as well.) And we can ask questions like what is the cohomology class $[\Omega_r]$ defined by the degeneracy locus Ω_r where $r : Q_1 \to \mathbb{N}$ and

$$\Omega_r = \{m \in M : \text{the rank of } \varphi_{ij}(m) = r(i,j) \text{ for all } (i,j) \in Q_1)\}.$$

In fact in Section 3 we will see what the good question is.

In this context these oriented graphs are called *quivers*. The quiver of Example 1.1 is $\bullet \to \bullet$ (called A_2). The cohomology formulas $[\Omega_r]$ were calculated by Buch and Fulton in [Buc99] and [BF99] for the quivers $\overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \to \overset{n}{\bullet}$ (called A_n). They use the resolution method.

Our method works for a wider class of quivers—the so called representation-finite quivers—and we believe that it is conceptually simpler.

The *way* our method works is the following. For any *fixed* orbit of a *fixed* quiver representation we build a system of linear equations whose unique solution is the sought Thom polynomial. Although this is a definite algorithm, it may seem somewhat implicit. However, the authors think that such a description can give at least as much insight into the behaviour of these polynomials as any other algorithmic description. We have a growing evidence on this, see [FRc], [BFR]. On the other hand our method has the disadvantage of not producing formulas for infinite series of Thom polynomials (or at least additional work is needed for them as in the papers quoted above). For example our algorithm works for *any* orbit of the D_4 quiver (see section 6 for examples), but we can at present not give a formula to cover *all* orbits of D_4.

In Section 2 we outline the general theory of Thom polynomials for group actions following [Kaz97].

In Section 3 we show how to apply the theory for representation-finite quivers. To get some feeling of this algebraic machinery we demonstrate it on the case of Example 1.1.

In Section 4 we calculate the formulas for quivers of type A_n (with the usual orientation).

In Section 5 we calculate a slightly more complicated example (for an A_3-type representation) than Example 1.1 to illustrate the method. We compare the result with the result of the Buch-Fulton algorithm.

In Section 6 we calculate some formulas for the quiver D_4, and make some comments on other quivers.

We are grateful to Tamás Hausel for drawing our attention to quivers and to Mátyás Domokos for very valuable discussions on the algebraic theory of quivers: we learned all the material in section 3 from him.

2. Thom polynomials for group actions

In this section we give a heuristic introduction to the theory of Thom polynomials for group actions. The approach relies on a generalization of the Poincaré dual. In Remark 2.6 we sketch the technical points. Details can be found in [Kaz97] and about the restriction equation method in [FRa].

Though the theory can be formulated in a more general context we restrict our attention to the following situation: Let $\rho : G \to GL(V)$ be a linear representation of the complex Lie group G on a complex vector space V. Then for any principal bundle $P \to M$ we can associate a vector bundle $E = P \times_\rho V$. If η is an orbit of the G-action on V and $s : M \to E$ is a section, we can ask: At which points of M does the section s belong to the orbit η? It turns out that the cohomology class defined by this set depends only on the G-characteristic classes of the bundle P. In other words it defines a cohomology class $\mathrm{Tp}(\eta) \in H^*(BG)$, called the Thom polynomial of η.

Being a crossed product E has a well defined map ω to the orbit space V/G. In this paper we assume that V/G consists of finitely many points.

Definition 2.1.

$$\eta(E) := \omega^{-1}(\eta) \text{ and } \eta(s) := s^{-1}(\eta(E)).$$

We are interested in $[\eta(s)] \in H^*(M) = H^*(M;\mathbb{Q})$ the Poincaré dual of the closure of $\eta(s)$. (In this paper we work with rational cohomology. It is possible to work with integral cohomology see [FRb].) Example 1.1 is a special case for $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ and $G = GL(n,\mathbb{C}) \times GL(p,\mathbb{C})$ acting on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ by $\rho(A,B)X := BXA^{-1}$. The orbits of this action are $\Sigma_k = \{v \in V : \mathrm{corank}(v) = k\}$.

Proposition 2.2. $[\eta(s)] = s^*[\eta(E)]$ *for a generic section* $s : M \to E$*, where* $s^* : H^*(E) \to H^*(M)$ *is the induced map in cohomology.*

The class $[\eta(E)]$ is the generalized Poincaré dual of the *closure* of $\eta(E)$ and generic means transversal to $\xi(E)$ for all $\xi \in V/G$. See Remark 2.6 for more details.

Proposition 2.2 implies that the class $[\eta(s)]$ is the same for any generic section s since $s^*: H^*(E) \to H^*(M)$ is the same map for any section s. In fact it is an isomorphism.

Next we show that it is enough to calculate one case—the universal one. Let $k: M \to BG$ be the classifying map of the principal G-bundle $P \to M$. Then k induces a map $k_E: E \to BV$, where $BV = EG \times_\rho V$ is the universal V-bundle (we consider V as a G-space and suppress ρ from the notation) and $EG \simeq *$ is the universal principal G-bundle over BG. The B in BV refers to *Borel construction*.

Proposition 2.3.

$$[\eta(E)] = k_E^*[\eta(BV)].$$

The space BV is infinite dimensional. However we can still make sense of $[\eta(BV)]$, see Remark 2.6. We can think of $\operatorname{Tp}(\eta)$ as the G-equivariant Poincaré dual of η in V.

$\eta(BV)$ as a set is nothing else but $B\eta = EG \times_\rho \eta$. It is an easy exercise to show that $B\eta \simeq B\operatorname{Stab}_\eta$, where Stab_η is the stabilizer subgroup of the orbit η (more precisely of a point of η). It is usually more convenient to work with a maximal compact subgroup G_η of Stab_η. Since $BG_\eta \simeq B\operatorname{Stab}_\eta$ it doesn't effect the calculations.

Proposition 2.3 means that $[\eta(s)]$ can be expressed in terms of G-characteristic classes of P. Since $H^*(BG)$ is a subring of a polynomial ring we call the element $[\eta(BV)] \in H^*(BV) \cong H^*(BG)$ the *Thom polynomial* of η.

$[\eta(BV)]$ shares some properties of the ordinary Poincaré dual. In particular

(i) restricted to the complement of the closure $\overline{\eta(BV)}$ is zero, and
(ii) restricted to itself we get the top Chern class—we will also use the name Euler class—of the normal bundle of $\eta(BV)$ in BV.

These imply the following:

Theorem 2.4 ([FRa, Thm.2.9]). *Suppose that θ is an orbit of ρ with* $\operatorname{codim}\theta \leq \operatorname{codim}\eta$ *and $j_\theta: B\theta \to BV$ is induced by the inclusion $\theta \subset V$.*

Then

$$j_\theta^* \operatorname{Tp}(\eta) = \begin{cases} e(\nu_\eta) & \text{if } \theta = \eta \qquad \text{‘principal equation'} \\ 0 & \text{if } \theta \neq \eta \qquad \text{‘homogeneous equations'} \end{cases},$$

where ν_η is the normal bundle of $B\eta$ in BV and e denotes the top Chern class or Euler class.

We can see that $\nu_\eta \cong EG_\eta \times_{\rho_\eta} N_\eta$, where $N_\eta = T_xV/T_x\eta$ is the normal space of η at a point $x \in \eta$. We refer to the equations above as *restriction equations*.

Theorem 2.4 seems to be an innocent observation, but it is enough to calculate $\operatorname{Tp}(\eta)$:

Theorem 2.5 ([FRa]). *Let $\rho: G \to GL(V)$ be a linear representation on a complex vector space V with finitely many orbits. Suppose that for every orbit η we have $e(\nu_\eta) \neq 0$. Then the restriction equations have a unique solution.*

The proof of this theorem is based on an induction using Mayer-Vietoris and Gysin sequences.

So we will use Theorem 2.4 to calculate Thom polynomials for group actions of quiver type in Section 3. It remains now to sketch how the various Poincaré duals used above can be defined. In the next remark we sketch the general approach of Kazarian, although a simpler construction works in our case, since the orbits are complex algebraic varieties, see [FRa].

Remark 2.6. In [Kaz97] Kazarian uses the codimension filtration $\mathcal{F}(V)$ of V—i.e. $\mathcal{F}_d = \bigcup\{\text{orbits of codimension} \leq d\}$ to get a filtration $\mathcal{F}(E)$ on the total space of any V-bundle E. A filtration $\mathcal{F}$ of a space X defines a relative cohomology spectral sequence $E_*^{*,*}(\mathcal{F})$ converging to $H^*(X)$. In our case the orbits are complex manifolds, so their codimension is even. It implies that $E_1^{0,2k+1} = 0$ therefore $E_1^{0,*} = E_2^{0,*}$. Also, the complex structure defines an orientation of the normal bundles of the orbits, which defines an element $o(\eta) \in E_1^{0,d}$ for every orbit η with codimension d. Composing the above isomorphism with the edge homomorphism $e : E_2^{0,*} \to H^*(E)$ we get elements $[\eta(E)] := e(o(\eta))$. If we have a filtration preserving map it induces a map between the spectral sequences. In particular we get such maps for pullbacks of V-bundles and for sections $s : M \to E$ if s is transversal to every orbit (transversality implies that $s^{-1}(\eta(E))$ is a manifold with the right codimension).

3. Geometry of quiver representations

In this section we return to Problem 1.3. Using the theory developed in Section 2 first we define a certain class—quiver type—of representations.

Let $Q = (Q_0, Q_1)$ be an oriented graph where Q_0 is the set of vertices and Q_1 is the set of arrows $e = (e', e'') \in Q_0 \times Q_0$ (multiple arrows and loops allowed). Given a nonnegative integer function d on Q_0 called *dimension vector*, we associate a representation $\rho(Q, d)$ of $G = GL(d) = \times_{i\in Q_0} GL(d(i))$ on the vector space $V = V(Q, d) = \bigoplus_{e\in Q_1} \mathrm{Hom}(\mathbb{C}^{d(e')}, \mathbb{C}^{d(e'')})$. In this context Q is called a *quiver* and V is called the *space of representations of* Q.

Now we want to answer the following questions:

(i) What are the orbits of $\rho(Q, d)$?
(ii) For which quivers Q are the conditions of Theorem 2.5 (finitely many orbits and nonzero normal Euler classes) satisfied?
(iii) How to calculate the necessary input of the restriction equations: the stabilizers and the actions on the normal spaces?

All these questions can be answered with algebraic methods. We happily realized that these type of questions were studied in the theory of representations of quivers. We are grateful to Mátyás Domokos who provided all the information we needed and explained us this beautiful chapter of representation theory unknown to us before. As a general reference for the section we recommend [ARS95].

The basic idea of applying algebra is that orbits of the action $\rho(Q, d)$ can be identified with certain modules over an algebra. Then a dictionary can be developed connecting geometry with algebra. For example the stabilizer of an orbit can be identified with the automorphism group of the corresponding module.

Definition 3.1. *The* path algebra $\mathbb{C}Q$ *of the quiver* Q *is the* $\mathbb{C}$*-algebra generated by the oriented paths of* Q*, including for every* $i \in Q_0$ *the trivial path* ψ_i *starting and ending at* i*. Multiplication corresponds to the concatenation of paths. If they don't fit then the product is zero.*

We can see that the ψ_i's are the minimal idempotents in $\mathbb{C}Q$ and $\sum \psi_i = 1$.

An element v of $V(Q, d)$ can be considered as a functor from the category Q to the category of finite dimensional vector spaces $Vect$ such that $v(i) = \mathbb{C}^{d(i)}$ for $i \in Q_0$. We will frequently use the following equivalence (which is not too difficult to verify):

Theorem 3.2 ([ARS95]). *The category of functors $Q \to Vect$ is equivalent to the category of finite dimensional (right) $\mathbb{C}Q$-modules.*

In particular for a $\mathbb{C}Q$-module M we can recover the dimension vector d_M via $d_M(i) := \dim(M(i))$ where $M(i) := M\psi_i$.

To get some familiarity with these abstract notions we turn to Example 1.1: The corresponding quiver is A_2 and $\mathbb{C}A_2$ is generated by the elements ψ_1, ψ_2 and a, where ψ_i are the trivial paths corresponding to the two vertices and a is the only nontrivial path. Multiplication is defined by $\psi_1^2 = \psi_1,\ \psi_2^2 = \psi_2,\ \psi_1 a = a\psi_2 = a$ and all other products are zero. Suppose now that we have a map $\varphi \in V(A_2) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ then we can define a $\mathbb{C}A_2$-module M_φ on the vector space $\mathbb{C}^n \oplus \mathbb{C}^p$ by the rule:

$$(v_1, v_2)\psi_1 = (v_1, 0), \quad (v_1, v_2)\psi_2 = (0, v_2), \quad (v_1, v_2)a = (0, \varphi(v_1)).$$

On the other hand from the multiplication table it follows that $M = M\psi_1 \oplus M\psi_2$ and $M\psi_1 a = Ma\psi_2 \subset M\psi_2$ so multiplication by a defines a map $\varphi_a : M\psi_1 \to M\psi_2$. It is also not too difficult to show that $M_\varphi \cong M_{\varphi'}$ if and only if $\varphi' = B\varphi A^{-1}$ for some invertible linear maps A and B.

The most useful feature of the language of modules is that we don't have to fix the dimension vectors: we can take direct sums of $\mathbb{C}Q$-modules. Geometrically these modules correspond to orbits of different representations, so this "additive" structure is hidden. In particular we cannot see geometrically the importance of the basic blocks—the *indecomposable* $\mathbb{C}Q$-modules. The next theorem on indecomposable $\mathbb{C}Q$-modules is the key to our results:

Theorem 3.3 (P. Gabriel [BGP73]). *$\mathbb{C}Q$ admits finitely many indecomposable modules if and only if Q is a quiver of Dynkin type (more precisely a Dynkin diagram with simple arrows i.e. of type A_i, D_i, E_6, E_7 or E_8). The indecomposable modules are determined by their dimension vectors and correspond to the positive roots $R(Q)$ of Q.*

The proof is based on the observation that you can define a bilinear form on the dimension vectors of Q called the *Euler form*:

Definition 3.4.
$$E_Q(a, b) := \sum_{i \in Q_0} a(i)b(i) - \sum_{e \in Q_1} a(e')b(e'').$$

It can be shown that $\mathbb{C}Q$ admits finitely many indecomposable modules if and only if E_Q is positive definite, and a is a dimension vector of an indecomposable module if and only if $E_Q(a, a) = 1$ and $a(i) \geq 0$ for all $i \in Q_0$ i.e. a is a *positive root* of Q.

Translating this theorem into the language of group representations these are the cases when the representation $\rho(Q, d)$ has finitely many orbits. This answers half of our questions raised at the beginning of this section. It is not impossible to calculate Thom polynomials for other quivers—see Section 7—but it requires different methods.

Our next goal is to calculate the maximal compact subgroup G_v of the stabilizer group of a $v \in V(Q, d)$. Using the equivalence of categories we have $G_v \cong \mathrm{Aut}_{\mathbb{C}Q}(M_v)$, where M_v is the $\mathbb{C}Q$-module corresponding to v. We also use the notation G_M for G_v. First we write M_v as a sum of indecomposable modules. Since the Remak-Krull-Schmidt theorem holds in this category ([ARS95, Thm. 2.2]) we have the following:

Theorem 3.5. *For every finite dimensional $\mathbb{C}Q$-module M in the decomposition $M \cong \bigoplus_{r\in R(Q)} \mu_r l_r$, where l_r is the indecomposable $\mathbb{C}Q$-module with dimension vector d, the multiplicities $\{\mu_r\}$ are uniquely defined.*

Proposition 3.6.

$$G_M \cong \bigtimes_{r\in R(Q)} U(\mu_r).$$

It may help the reader to look at the case of the quiver A_2 again: The module M_φ is indecomposable if and only if φ is indecomposable, i.e. cannot be written as $\varphi = \varphi' \oplus \varphi''$ in a nontrivial way. Gauss elimination shows that we have three indecomposable maps: $l_{1,1} = \mathrm{Id} : \mathbb{C} \to \mathbb{C}$, $l_{1,0} : \mathbb{C} \to 0$ and $l_{0,1} : 0 \to \mathbb{C}$. Also, if we have a map $\varphi \in V(A_2) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ of corank k, then we have the decomposition:

$$M_\varphi \cong (n-k)l_{1,1} \oplus kl_{1,0} \oplus (p-n+k)l_{0,1},$$

so Proposition 3.6 specializes to the classical fact (see e.g. [FRa]) that $G_\varphi \cong U(n-k) \times U(k) \times U(p-n+k)$.

For the proof of Proposition 3.6 we introduce the *Auslander-Reiten graph* $A(Q)$ of a quiver Q which contains most of the information we need for our calculations.

Definition 3.7. *$A(Q)_0 := R(Q)$ and there is an arrow $(l,m) \in A(Q)_1$ if and only if there is an irreducible homomorphism in* $\mathrm{Hom}(l,m)$.

Proposition 3.8.

$$G_M \cong MaxCpt\Big(\bigtimes_{r\in R(Q)} \mathrm{Aut}(\mu_r l_r)\Big).$$

Proof. Since Q is Dynkin, $A(Q)$ contains no closed oriented path (as it is easy to see from the construction in [Gab79, §6.5]). Hence it defines a partial ordering of $R(Q)$. Let $<$ be an ordering extending this partial ordering. If $X \in \mathrm{Aut}(M)$ then $X = (X_{ij})$ where $X_{ij} : \mu_i l_i \to \mu_j l_j$ because of the unicity of the decomposition. Using the ordering above $X_{ij} = 0$ if $i > j$ i.e. $X = (X_{ij})$ is "upper triangular". □

Proposition 3.9. $\mathrm{Aut}(l) \cong \mathbb{C}^\times$ *for $l \in R(Q)$.*

Sketch of proof: As we mentioned after Theorem 3.3 the dimension vector of an indecomposable module l is a root for the Euler form, i.e.

$$1 = E_Q(l,l) = \sum_{i\in Q_0} d_l^2(i) - \sum_{e\in Q_1} d_l(e')d_l(e'') = \dim GL(d_l) - dimV(Q,d_l).$$

It shows that if $d_M = d_l$ then the orbit of M is open iff $\dim(\mathrm{Aut}(M)) = 1$. By Theorem 3.3 the module l is the only indecomposable module with this dimension vector and clearly $\dim(\mathrm{Aut}(M)) > 1$ for a decomposable module M. So the orbit of l has to be the only open orbit. □

Proposition 3.6 is a direct consequence of Propositions 3.8 and 3.9. The proof also describes the homomorphism $G_M \to G$, so we can calculate the maps $j_M^* : H^*(BG) \to H^*(BG_M)$: We have $\dim M(v) = \sum \mu_r \dim l_r(v)$. Let $\pi(v,\cdot)$ be a bijection of sets corresponding to this equation. For example an ordering on $R(Q)$ defines such a π. Let $\{\alpha_{v,k} : v \in Q_0,\ k \leq \dim_{\mathbb{C}} M(v)\}$ denote the Chern roots of $H^*(BG)$ and $\{\alpha_{r,j} : r \in R(Q),\ j \leq \mu_r\}$ denote the Chern roots of $H^*(BG_M)$.

Proposition 3.10. $j_M^*\alpha_{v,k} = \alpha_{\pi(v,k)}$.

To see the map j_M^* more explicitly suppose that we have an ordering on $R(Q)$. It defines a bijection $R(Q) \to \{1, \ldots, R := |R(Q)|\}$. Using this numbering to index the elements of $R(Q)$ and the notation $d_r(v) := \dim l_r(v)$ we get:

Chern roots of BG	$\cdots$	$\alpha_{v,1}$	$\alpha_{v,2}$	$\cdots$	$\alpha_{v,d_1(v)}$	$\alpha_{v,d_1(v)+1}$	$\cdots$	$\alpha_{v,2d_1(v)}$
j_M^*	$\cdots$	$\alpha_{1,1}$	$\alpha_{1,1}$	$\cdots$	$\alpha_{1,1}$	$\alpha_{1,2}$	$\cdots$	$\alpha_{1,2}$

$\alpha_{v,2d_1(v)+1}$	$\cdots$	$\alpha_{v,\mu_1 d_1(v)}$	$\alpha_{v,\mu_1 d_1(v)+1}$	$\cdots$	$\alpha_{v,\sum_{i=1}^R \mu_i d_i(v)}$	$\cdots$
$\alpha_{1,3}$	$\cdots$	α_{1,μ_1}	$\alpha_{2,1}$	$\cdots$	α_{R,μ_R}	$\cdots$

A different choice of π leads to the same map j_M^* since we look at only the symmetric polynomials of the roots.

To calculate the principal equation of Theorem 2.4 we need the normal space N_M and the action of G_M on N_M:

Proposition 3.11. *$N_M \cong \mathrm{Ext}_{\mathbb{C}Q}(M, M)$ as G_M-representations.*

This is a version of the *Voight-lemma*, see [Voi77]. It was first observed in [LP90] that the Voight-lemma can be applied to quivers. Proposition 3.11 is stated in [DZ01] for simple modules, but the argument is the same for non simple modules.

Using the bilinearity of the functor Ext we get:

Corollary 3.12. *Suppose that $M \cong \bigoplus_{r\in R(Q)} \mu_r l_r$ is the decomposition of the $\mathbb{C}Q$-module M into indecomposables. Then*

$$\mathrm{Ext}_{\mathbb{C}Q}(M, M) \cong \bigoplus_{r,s\in R(Q)} \mathrm{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})^{m_{rs}}$$

as G_M-representations, where $m_{rs} = \dim \mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s)$.

The action of $G_M \cong \times_{r\in R(Q)} U(\mu_r)$ on the right hand side is that only the $U(\mu_r)$ factor acts from the left and the $U(\mu_s)$ from the right on the summand $\mathrm{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})$.

Corollary 3.13.

$$e(\nu_\eta) = \prod_{r,s\in R, i\le\mu_r, j\le\mu_s} (\alpha_{s,j} - \alpha_{r,i})^{m_{rs}},$$

where ν_η is the normal bundle of $\eta(BV)$ and $M \cong \bigoplus_{r\in R(Q)} \mu_r l_r$ is the $\mathbb{C}Q$-module corresponding to the orbit η and $\alpha_{r,i}$ are the Chern roots of the universal bundle over BG_η.

So we need an algorithm to calculate $\mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s)$ for indecomposable modules l_r and l_s. The essential tool for this is a partial self map τ of $R(Q)$, the *Auslander-Reiten translation*. For the definition and calculation of τ see [ARS95] (they use the notation $D\,\mathrm{Tr}$) and [Gab79, §6.5]. We need the following lemmas to calculate the numbers m_{rs}:

Proposition 3.14.

(i) $\dim \mathrm{Ext}_{\mathbb{C}Q}(M, N) = \dim \mathrm{Hom}_{\mathbb{C}Q}(N, \tau M)$ *or 0 if τM is not defined.* [ARS95] *(Auslander-Reiten formula)*

(ii) $\mathrm{Ext}_{\mathbb{C}Q}(M, N) \cong \mathrm{Ext}_{\mathbb{C}Q}(\tau M, \tau N)$ *if τ is defined on M and N.* [ARS95]

(iii) For quivers of Dynkin type every indecomposable module can be translated via τ into a projective module. [BGP73]

Corollary 3.15. *Any representation of a quiver of Dynkin type satisfies the assumption of Theorem 2.5, therefore the restriction equations have a unique solution.*

Proof. We have to show that $e(\nu_\eta) \neq 0$. By Corollary 3.13 it is enough to show that $\mathrm{Ext}_{\mathbb{C}Q}(l_r, l_r) = 0$. Which is a consequence of Proposition 3.14.(ii) and (iii) and the fact that $\mathrm{Ext}_{\mathbb{C}Q}(P, \cdot) = 0$ for a projective module P. □

Now it remains to calculate $\mathrm{Hom}_{\mathbb{C}Q}(P, N)$ where P is a projective indecomposable module. The following statements are easy to verify:

Proposition 3.16.

(1) The projective indecomposable modules are $\{P_i = \psi_i \mathbb{C}Q : i \in Q_0\}$.
(2) $\mathrm{Hom}_{\mathbb{C}Q}(P_v, M) \cong M(v)$ for the projective module corresponding to $v \in Q_0$.

For the quiver A_2 we can easily calculate the two indecomposable projective modules:

$$P_1 = \psi_1 \mathbb{C}A_2 = \langle \psi_1, a \rangle = l_{1,1}, \qquad P_2 = \psi_2 \mathbb{C}A_2 = \langle \psi_2 \rangle = l_{0,1}.$$

It shows that for a linear map φ the corresponding $\mathbb{C}A_2$-module M_φ is projective if and only if $\varphi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ is injective.

The only Auslander-Reiten translation is: $\tau(l_{1,0}) = l_{0,1}$. So by Corollary 3.12 and Proposition 3.14 we get that if φ has corank k then:

$$N_\varphi \cong \mathrm{Ext}(M_\varphi, M_\varphi) \cong \mathrm{Hom}(\mathbb{C}^{p-n+k}, \mathbb{C}^k) \cong \mathrm{Hom}(\mathrm{Coker}\,\varphi, \ker\varphi).$$

The rest of the calculation of the Thom-Porteous formulas along these lines can be found in [FRa].

We would like to demonstrate in the next sections that the calculations of this section are very simple, and only the last step when a system of linear equations has to be solved requires computer.

4. Quivers of type A_n

In this section we study the quivers $A_n = \overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \to \overset{n}{\bullet}$. This is the case where Buch annd Fulton calculated the Thom polynomials in [BF99]. So $(A_n)_0 = \{v_1, \dots, v_n\}$ and $(A_n)_1 = \{(v_1, v_2), \dots, (v_{n-1}, v_n)\}$. The vector space $V = V(A_n) = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(V_i, V_{i+1})$ and $G = \mathsf{X}_{i=1}^n GL(V_i)$.

The positive roots are $R(A_n) = \{l_{ij} : 1 \leq i \leq j \leq n\}$ where $l_{ij} := (0, \dots, 0, \overset{i}{1}, \dots, \overset{j}{1}, 0, \dots, 0)$. We use the same notation for the corresponding indecomposable modules.

The reader may ask that given an element $\varphi = (\varphi_{12}, \dots, \varphi_{n-1,n}) \in V$ how can we decide which orbit φ belongs to. In other words how to decompose φ (we use the same notation for φ and the corresponding $\mathbb{C}A_n$-module) into indecomposable modules? To answer this we define $\varphi_{ij} : V_i \to V_j$ taking compositions:

Lemma 4.1. *Let $\varphi = \bigoplus \mu_{ij} l_{ij}(\varphi) \in V$ and suppose that the map $\varphi_{ij} : V_i \to V_j$ has rank $r_{ij}(\varphi)$. Then the multiplicity of l_{ij} in φ is*

$$\mu_{ij}(\varphi) = r_{i+1,j-1} - r_{i,j-1} - r_{i+1,j} + r_{i,j}$$

Proof. Using that $r_{ij}(\varphi_1 \oplus \varphi_2) = r_{ij}(\varphi_1) + r_{ij}(\varphi_2)$ we get that

$$r_{ij}(l_{km}) = \begin{cases} 1 & \text{if } k \leq i < j \leq m, \\ 0 & \text{otherwise} \end{cases},$$

so $r_{ij}(\varphi) = \sum_{k \leq i < j \leq m} \mu_{k,m}$. This system of equations can be solved for $\mu_{k,m}$ using Figure 1. □

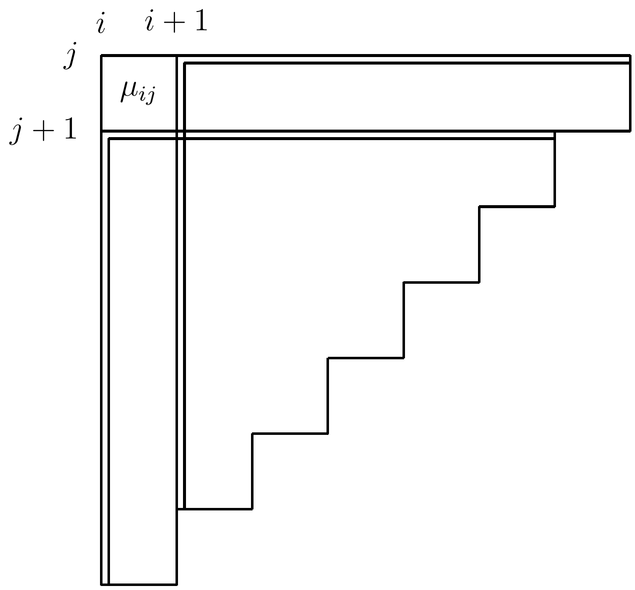


FIGURE 1. The multiplicity of l_{ij}

Remark 4.2.

(i) Lemma 4.1 shows that in the case of A_n the orbits can be described using rank conditions. This is not true for every quiver. For example if you change the orientation of some arrows in A_n, you get a quiver, where you don't have enough compositions. However the Thom polynomial method works equally well for these cases. The dimension vectors of the indecomposable modules don't change, but the Auslander-Reiten translation will be different. It also changes the behaviour of the Thom polynomials, see Remark 4.6.

(ii) The decomposition into indecomposable modules is encoded in the "diagram of dots connected with lines" in [BF99, Sect. 2.3]. Also the numbers $\mu_{ij}(\varphi) = r_{i+1,j-1} - r_{i,j-1} - r_{i+1,j} + r_{i,j}$ play a central role in their calculations.

Proposition 3.6 implies that $G_\varphi \cong \mathsf{X}\, U(\mu_{ij}(\varphi))$. Next we calculate the Euler class of the normal bundle ν_φ:

Lemma 4.3.

$$\mathrm{Hom}(l_{ij}, l_{km}) \cong \begin{cases} \mathbb{C} & \textit{if } i \leq k \leq j \leq m, \\ 0 & \textit{otherwise.} \end{cases}$$

It is also easy to see that $\tau l_{ij} = l_{i+1,j+1}$ and the projective indecomposable corresponding to the vertex v_i is $P_i = l_{in}$ (see Remark 6.1 where we explain how to calculate these) so we can calculate the coefficients m_{rs} of Theorem 3.12:

Lemma 4.4.

$$m_{ij,km} = \dim \operatorname{Ext}(l_{ij}, l_{km}) = \dim \operatorname{Hom}(l_{km}, l_{i+1,j+1}) = \begin{cases} 1 & \textit{if } i+1 \le k \le j+1 \le m, \\ 0 & \textit{otherwise.} \end{cases}$$

Now we can calculate the Euler class:

Proposition 4.5. *For $\varphi \in V(A_n)$:*

$$e(\nu_\varphi) = \prod_{i+1\le k\le j+1\le m} \quad \prod_{a\le \mu_{ij}(\varphi),\, b\le \mu_{km}(\varphi)} (\alpha_{ija} - \alpha_{kmb}),$$

where the α's are the Chern roots of G_φ.

We also have to calculate the maps $j_\varphi^* : H^*(BG) \to H^*(BG_\varphi)$. From Proposition 3.10 we know that $j_M^*\alpha_{v,k} = \alpha_{\pi(v,k)}$ for an appropriate map π on the indices of the Chern roots. We can see that

$$\dim V_k = \dim \varphi(v_k) = r_{kk}(\varphi) = \sum \mu_{ij} \dim(l_{ij}(v_k)) = \sum_{i\le k\le j} \mu_{ij}.$$

So one way to fix π is:

$$\pi(v_k, m) := (i, j, l) \qquad \text{if} \qquad \sum_{a=1}^{i-1}\sum_{b=k}^{n} \mu_{ab} + \sum_{b=k}^{j-1} \mu_{ib} + l = m \text{ and } l \le \mu_{ij}.$$

This choice of π corresponds to the lexicographic ordering of the roots $\{(i,j) : 1 \le i \le j \le n\}$.

Remark 4.6. Since l_{1n} is projective and injective at the same time, it is not difficult to see that $\operatorname{Ext}(M + l_{1n}, M + l_{1n}) = \operatorname{Ext}(M, M)$, which implies that:

$$\operatorname{Tp}(M) = \operatorname{Tp}(M + l_{1n}).$$

(Use that $j_{M+l_{1n}}^*$ factors through j_M^* to show that $\operatorname{Tp}(M + l_{1n})$ satisfies all the equations for $\operatorname{Tp}(M)$.) In other words we can recover the fact in [BF99] that $\operatorname{Tp}(M)$ is a polynomial of Chern classes of differences of universal bundles. However if we change the orientation of some of the arrows, then there is no module projective and injective at the same time and similar statement doesn't hold.

Remark 4.7. If the dimension vector has the form $d = (1, 2, \dots, k-1, k, k, k-1, \dots, 2, 1)$, then the Thom polynomials of certain orbits are the so called *double Schubert polynomials*, see e.g. [BF99] or [FP98]. The theory of Thom polynomials for group actions allows us to change the representation and it turns out that the double Schubert polynomials are Thom polynomials for a smaller group $G = B^+ \times B^-$ acting on the vector space $\operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^k)$ where B^+ (B^-) is the group of invertible upper (lower) triangular $k \times k$ matrices. This approach leads quickly to the Lascoux-Schützenberger definition of the double Schubert polynomials (see [FRc]).

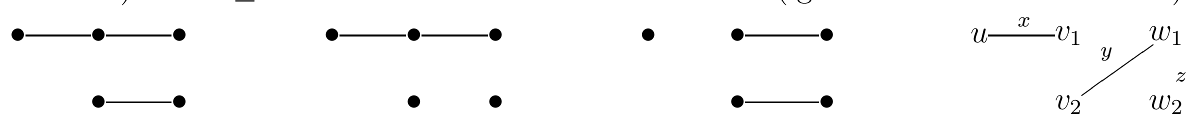
5. How does it all work in a concrete case?

In this section we explicitly calculate an example:

Example 5.1. Suppose we have complex vector bundles E_1, E_2, E_3 over the manifold X with fibers $\mathbb{C}^1, \mathbb{C}^2, \mathbb{C}^2$ respectively, and bundle maps $\varphi_{12} : E_1 \to E_2$, $\varphi_{23} : E_2 \to E_3$ and $\varphi_{13} := \varphi_{23} \circ \varphi_{12} : E_1 \to E_3$. Assuming that φ_{ij} is generic, what is the Poincaré dual $[\Omega_{110}(\varphi)]$ of the degeneracy locus $\overline{\Omega}_{110}(\varphi)$, where

$$\overline{\Omega}_{110}(\varphi) = \{x \in X : \operatorname{rank} \varphi_{12}(x) \leq 1, \operatorname{rank} \varphi_{23}(x) \leq 1, \operatorname{rank} \varphi_{13}(x) = 0\}?$$

In the language of Section 4 we look at the A_3 quiver (arrows are oriented to the right) with dimension vector $(1, 2, 2)$, i.e. we consider the group $GL(1) \times GL(2) \times GL(2)$ acting on the vector space $\operatorname{Hom}(\mathbb{C}^1, \mathbb{C}^2) \times \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^2)$ by $(U, V, W) \cdot (\varphi, \psi) := (V\varphi U^{-1}, W\psi V^{-1})$. This action has a unique codim 0 orbit (A), a unique codim 1 orbit (B), two codim 2 orbits $(C1, C2)$ and higher codimensional ones. (The problem of determining the orbits is solved by using the language of the representations of the quiver algebra where this problem reduces to partitioning the dimension vector $(1, 2, 2)$ into dimension vectors of indecomposable modules of $\mathbb{C}Q(A_3)$. The codimension of orbits are calculated in Proposition 3.11.) The 'diagram of dots connected with lines' (where the connected components correspond to the indecomposable summands) of the ≤ 3 codimensional orbits are as follows (ignore the labels on $C2$ now)



A B $C1$ $C2$

Our task is to compute the Thom polynomial of $C2$ since $\operatorname{Tp}(C2)(E_1, E_2, E_3) = [\Omega_{110}(\varphi)]$. By definition $\operatorname{Tp}(C2) \in H^4\big(B(GL(1) \times GL(2) \times GL(2))\big)$. Denoting the 'Chern root generators' of $H^*(BGL(1))$, $H^*(BGL(2))$, $H^*(BGL(2))$ by u, v_1, v_2 and w_1, w_2 respectively (α_{12}, α_{22}, α_{23}, α_{32}, α_{33} with the notation of Proposition 3.10 but to avoid double indices we renamed the vertices of the quiver to u, v and w), the sought Thom polynomial is a degree two polynomial in the variables u, v_1, v_2, w_1, w_2, symmetric in v_1, v_2 and w_1, w_2 respectively. So it must have the form

$$\begin{aligned}(1)\quad \operatorname{Tp}(C2) = au^2 + b(v_1+v_2)^2 + cv_1v_2 + d(w_1+w_2)^2 + \\ ew_1w_2 + fu(v_1+v_2) + gu(w_1+w_2) + h(v_1+v_2)(w_1+w_2).\end{aligned}$$

for some coefficients $a, b, \dots, h$. Our task is to determine these coefficients. The method of section 2, i.e. Theorem 2.4 gives linear equations on these coefficients. Let us start with the 'principal equation'. According to this we know that the value of $\operatorname{Tp}(C2)$ under the substitution

$$u \to x, \qquad v_1 \to x, \qquad v_2 \to y, \qquad w_1 \to y, \qquad w_2 \to z$$

must be $(y-x)(z-x)$. (The substitution (that is j^*_{C2}) is computed in Proposition 3.10—see also the enhanced diagram above—and the right hand side is computed in Corollary 3.13 or Proposition 4.5.)

So we have:

$$\begin{aligned}(2)\quad ax^2 + b(x+y)^2 + cxy + d(y+z)^2 + eyx + fx(x+y) + gx(y+z) + h(x+y)(y+z) = \\ (y-x)(z-x).\end{aligned}$$

The three variables x, y and z have $\binom{4}{2} = 6$ quadratic monomials so (2) gives six linear equations for the coefficients $a, b, \dots, h$. Similarly the homogeneous equations for A, B and $C1$ give $3+6+6$ equations, so it is an overdetermined system of linear equations with 21 equations and 8 unknowns. Theorem 2.5 guarantees the unique solution:

$$\operatorname{Tp}(C2) = u(v_1+v_2) - u(w_1+w_2) - v_1v_2 + w_1w_2.$$

The Buch-Fulton algorithm gives the result in the form:

(3)

$$s_{(1)}\left(\frac{1-u}{1-(v_1+v_2)+v_1v_2}\right) s_{(1)}\left(\frac{1-(v_1+v_2)+v_1v_2}{1-(w_1+w_2)+w_1w_2}\right) + s_{(1,1)}\left(\frac{1-(v_1+v_2)+v_1v_2}{1-(w_1+w_2)+w_1w_2}\right) =$$

$$(v_1+v_2-u)(w_1+w_2-v_1-v_2)+$$

$$\det\begin{pmatrix} w_1+w_2-v_1-v_2 & v_1v_2-(v_1+v_2)(w_1+w_2)-w_1w_2+(w_1+w_2)^2 \\ 1 & w_1+w_2-v_1-v_2 \end{pmatrix},$$

where the s's are the supersymmetric Schur polynomials (which simply means to substitute the Taylor terms of the rational fuction into the corresponding ordinary Schur polynomial). One can see that the two approaches are quite different, both having their own advantages—although clearly both needs a computer to be effectively computable. The comparison of the two approaches, i.e. showing linear connections between the coefficients of Schur and Schubert type polynomials might turn out to be interesting in the future.

In the next section an advantage of our approach is presented: its applicability to Dynkin-graphs different from A_n.

6. The quiver D_4

The Auslander-Reiten graph of D_4 is:

$$\begin{tikzcd}
 & P_2 = \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \arrow[dr] & & R_2 = \begin{smallmatrix}0&\\1&1\\1&\end{smallmatrix} \arrow[dr] & & I_2 = \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix} \\
P_1 = \begin{smallmatrix}0&\\0&1\\0&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & P_3 = \begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix} \arrow[r] & R_1 = \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & R_3 = \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \arrow[r] & I_1 = \begin{smallmatrix}1&\\1&1\\1&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & I_3 = \begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix} \\
 & P_4 = \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix} \arrow[ur] & & R_4 = \begin{smallmatrix}1&\\1&1\\0&\end{smallmatrix} \arrow[ur] & & I_4 = \begin{smallmatrix}0&\\0&0\\1&\end{smallmatrix}
\end{tikzcd}$$

where P_i are the projective modules and I_i are the injective modules. The Auslander-Reiten translation is translating two steps to the left, i.e. $\tau(R_i) = P_i$ and $\tau(I_i) = R_i$.

Remark 6.1. We can demonstrate on this example how to calculate the Auslander-Reiten graph of of a quiver:

(1) Draw the opposite graph with the indecomposable projective modules P_j of the corresponding vertices.

(2) Calculate the dimension vectors of P_j: $d_i(P_j)$ =the number of oriented paths from j to i.
(3) Calculate the dimension vectors of the cokernels of the existing maps i.e.:

$$d(R_1) = d(P_2) + d(P_3) + d(P_4) - d(P_1), \qquad d(R_2) = d(R_1) - d(P_2)$$

and so on.
(4) Stop when you get negative numbers.

To demonstrate how easy to read off the restriction equations from the Auslander-Reiten graph we look at a small dimensional example $\rho(D_4, \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix})$ in details. This is the smallest case where all the indecomposable modules show up.

By adding up the dimension vectors we can see that there are 15 orbits (modules), e.g:

$$R_1,\ P_1 + R_2 + I_2,\ 2P_1 + I_2 + I_3 + I_4,\ P_1 + I_1,\ P_2 + P_3 + I_4,$$

and so on.

Let us calculate some of the maps j_M^*: Let us denote the Chern roots of $G = GL(\begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix})$ by $\begin{smallmatrix}k&\\l&s,t\\m&\end{smallmatrix}$ and the Chern roots of $M = 2P_1 + I_2 + I_3 + I_4$ by p_1, p_1', i_2, i_3, i_4. Then:

$$j_M^* : \begin{smallmatrix}k&\\l&s,t\\m&\end{smallmatrix} \to \begin{smallmatrix}i_2&\\i_3&p_1,p_1'\\i_4&\end{smallmatrix}.$$

Similarly if $M = P_2 + P_3 + I_4$ then the Chern roots are p_2, p_3, i_4 and

$$j_M^* : \begin{smallmatrix}k&\\l&s,t\\m&\end{smallmatrix} \to \begin{smallmatrix}p_2&\\p_3&p_2,p_3\\i_4&\end{smallmatrix},$$

i.e. we put a p_2, where the dimension vector of P_2 is 1 and so on.

For the principal equations we need the Euler classes, so we have to calculate the Ext-groups. Using that Ext is τ invariant and that Ext of a projective module is 0, we get e.g. for $M = P_1 + I_1$:

$$\operatorname{Ext}(M, M) \cong \operatorname{Ext}(I_1, P_1) \cong \mathbb{C}^{\dim \operatorname{Hom}(P_1,R_1)} \cong \mathbb{C}^{d_{R_1}(1)} = \mathbb{C}^2,$$

so $e_M = (i_1 - p_1)^2$.

For $M = 2P_1 + I_2 + I_3 + I_4$:

$$\operatorname{Ext}(M, M) \cong 2 \bigoplus_{j=2,3,4} \operatorname{Ext}(I_j, P_1) \cong 2 \bigoplus_{j=2,3,4} \mathbb{C}^{\dim \operatorname{Hom}(P_1,R_j)} \cong \bigoplus_{j=2,3,4} \mathbb{C}^{2d_{R_j}(1)} = \mathbb{C}^6$$

and $e_M = (i_2 - p_1)(i_3 - p_1)(i_4 - p_1)(i_2 - p_1')(i_3 - p_1')(i_4 - p_1')$, which shouldn't be a surprise, since $2P_1 + I_2 + I_3 + I_4$ corresponds to the orbit 0 in $V(D_4, \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}) \cong \mathbb{C}^6$.

Such a way we can write down all the restriction equations. We should warn the adventurous reader to use computer at this point since even at this small example the number of equations can reach 50. Some examples for Thom polynomials:

$$\operatorname{Tp}(P_2 + P_3 + I_4) = k^2 - (s + t)k + st = c_2(E_1 \ominus E_4),$$

where E_i is the universal bundle corresponding to the i^{th} vertex.

$$\operatorname{Tp}(P_2 + R_2) = -l - m + (s + t),$$

which is an example that the Thom polynomial is not necessarily a polynomial of Chern classes of differences of universal bundles, as in the case of the previous section.

7. OTHER EXAMPLES

Example 7.1. The one-loop quiver $\circlearrowleft$—i.e. the adjoint representation has infinitely many indecomposable modules, but the orbittype stratification is a Vassiliev stratification. However it is not difficult to show—we plan to publish it in a different paper—that all the Thom polynomials are zero.

We suspect that for not too complicated quivers as X—corresponding to the problem of quadruple subspaces—and the double arrow the calculations are still possible combining the Dynkin case with the one-loop case.

References

[ARS95] M. Auslander, I. Reiten, and S. O. Smalo. *Representation Theory of Artin Algebras*. Number 36 in Cambridge Studies in Adv. Math. CUP, 1995.

[BF99] A. S. Buch and W. Fulton. Chern class formulas for quiver varieties. *Inv. Math.*, (135):665–687, 1999.

[BFR] G. Bérczi, L. Fehér, and R. Rimányi. Thom polynomials for A_3 singularities in any codimension. in preparation.

[BGP73] I. N. Berstein, I. M. Gel'fand, and V. A. Ponomarev. Coxeter functors and Gabriel's theorem. *Usp. Mat. Nauk*, (28):19–33, 1973.

[Buc99] A. S. Buch. *Combinatorics of degeneracy loci*. PhD thesis, University of Chicago, 1999.

[DZ01] M. Domokos and A.N. Zubkov. Semisimple representations of quivers in characteristic p. *Algebras and Representation Theory*, 2001. to appear.

[FP98] W. Fulton and P. Pragacz. *Schubert varieties and degeneracy loci*, volume 1689 of *Lecture Notes in Math.* Springer, 1998.

[FRa] L. Fehér and R. Rimányi. Calculation of Thom polynomials for group actions. www.cs.elte.hu/analysis/rimanyi/cikkek.

[FRb] L. Fehér and R. Rimányi. Thom polynomials with integer coefficients. www.cs.elte.hu/analysis/rimanyi/cikkek.

[FRc] L. Fehér and R. Rimányi. Schur and Schubert polynomials as Thom polynomials—cohomology of moduli spaces. www.cs.elte.hu/analysis/rimanyi.

[Gab79] P. Gabriel. Auslander-Reiten squences and representation-finite algebras. In V. Dlab and P. Gabriel, editors, *Representation Theory I*, number 831 in Lecture Notes in Mathematics, pages 1–71. Springer-Verlag, 1979.

[Kaz95] M. É. Kazarian. Characteristic classes of Lagrange and Legendre singularities. *Uspekhi Math. Nauk*, 50(304):45–70, 1995. (Russian).

[Kaz97] M. É. Kazarian. Characteristic classes of singularity theory. In V. I. Arnold et al., editors, *The Arnold-Gelfand mathematical seminars: geometry and singularity theory*, pages 325–340, 1997.

[LP90] L. Le Bruyn and C. Procesi. Semisimple representations of quivers. *Trans. Amer. Math. Soc.*, (317):585–598, 1990.

[Por71] I. Porteous. Simple singularities of maps. In *Liverpool Singularities — Symposium I*, number 192 in SLNM, pages 286–307, 1971.

[Rim01] R. Rimányi. Thom polynomials, symmetries and incidences of singularities. *Inv. Math.*, (143):499–521, 2001.

[Voi77] D. Voigt. Endliche algebraische Gruppen. Number 592 in Lecture Notes in Math. Springer-Verlag, 1977.

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