PARTIAL FLAG VARIETIES, STABLE ENVELOPES AND WEIGHT FUNCTIONS

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Abstract. We consider the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of a GL_n partial flag variety, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, $|\boldsymbol{\lambda}| = \sum_i \lambda_i = n$, and the torus $T = (\mathbb{C}^\times)^{n+1}$ equivariant cohomology $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. In [MO], a Yangian module structure was introduced on $\oplus_{|\boldsymbol{\lambda}|=n} H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We identify this Yangian module structure with the Yangian module structure introduced in [GRTV]. This identifies the operators of quantum multiplication by divisors on $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, described in [MO], with the action of the dynamical Hamiltonians from [TV2, MTV1, GRTV]. To construct these identifications we provide a formula for the stable envelope maps, associated with the partial flag varieties and introduced in [MO]. The formula is in terms of the Yangian weight functions introduced in [TV1], c.f. [TV3, TV4], in order to construct q-hypergeometric solutions of qKZ equations.

To the memory of I.M. Gelfand (1913-2009)

Contents

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1. Introduction

In [MO], D. Maulik and A. Okounkov develop a general theory connecting quantum groups and equivariant quantum cohomology of Nakajima quiver varieties, see [N1, N2]. In this paper, we consider the constructions and results of that general theory applied to the cotangent bundles of GL_n partial flag varieties. We identify the objects and results from [MO] with known objects and results associated with the Yangian $Y(\mathfrak{gl}_N)$, see [TV1, TV2, TV3, TV4, MTV1, GRTV].

More precisely, we consider the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of a GL_n partial flag variety, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N), |\boldsymbol{\lambda}| = \sum_i \lambda_i = n$, and the torus $T = (\mathbb{C}^\times)^{n+1}$ equivariant cohomology $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. In [MO], a Yangian module structure was introduced on $\oplus_{|\boldsymbol{\lambda}|=n} H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We identify this Yangian module structure with the Yangian module structure introduced in [GRTV]. This identifies the operators of quantum multiplication by divisors on $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, described in [MO], with the action of the dynamical Hamiltonians from [TV2, MTV1, GRTV]. To construct these identifications we provide a formula for the stable envelope maps, associated with the partial flag varieties and introduced in [MO]. The formula is in terms of the Yangian weight functions introduced in [TV1], c.f. [TV3, TV4], in order to construct q-hypergeometric solutions of qKZ equations.

In Section 2, we follow [MO] and define the stable envelope maps associated with partial flag varieties, $\mathrm{Stab}_\sigma : (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[z_1, \dots, z_n; h] \to H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, $\sigma \in S_n$. In Section 3, we introduce weight functions and the cohomological weight function maps $[W_\sigma] : (\mathbb{C}^N)^{\otimes n} \otimes$

$\mathbb{C}[z_1,\dots,z_n;h] \to H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$, $\sigma \in S_n$. In Section 4 we prove our main result, Theorem 4.1, which relates the stable envelope maps and the cohomological weight function maps,

$$[W_\sigma] = c_{\boldsymbol\lambda} \circ \mathrm{Stab}_\sigma,$$

where $c_{\boldsymbol\lambda}$ is the operator of multiplication by an element $c_{\boldsymbol\lambda}(\boldsymbol\Theta) \in H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ defined in (4.1). The element $c_{\boldsymbol\lambda}(\boldsymbol\Theta)$ is not a zero-divisor. The inverse maps to the stable envelope maps for GL_n partial flag varieties were considered in [GRTV, Formulas (5.9), (5.10)]. One of those maps ν was reintroduced in Section 6.6 where we prove that

$$\nu \circ \mathrm{Stab}_{\mathrm{id}} = \mathrm{Id},$$

here $\mathrm{id} \in S_n$ is the identity.

In Section 5, we describe the orthogonality relations for the stable envelope maps $\mathrm{Stab}_{\mathrm{id}}$ and Stab_{σ_0}, where $\sigma_0 \in S_n$ is the longest permutation. The orthogonality relations are analogues of the orthogonality relations for Schubert cycles corresponding to dual Young diagrams.

In Section 6, our Theorem 6.3 and Corollary 6.4 say that the Yangian module structure on $\oplus_{|\boldsymbol\lambda|=n} H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$, introduced in [MO], coincides with the Yangian structure introduced in [GRTV] and Section 6.4. We introduce the dynamical Hamiltonians and trigonometric dynamical connection in Section 7. In Corollary 7.6, we identify the operators of quantum multiplication by divisors $D_i, i = 1,\dots,N$, on $H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ with the action of the dynamical Hamiltonian $X^q_{\boldsymbol\lambda,i}, i = 1,\dots,N$. This identifies the quantum connection on $H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ with the trigonometric dynamical connection. In Section 7, we discuss also the qKZ difference connection, compatible with the trigonometric dynamical connection, see [TV2]. The qKZ difference connection on $H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ corresponds to the shift operator difference connection introduced in [MO].

This paper is motivated by two goals. The first is to relate the stable envelope maps and the cohomological weight function maps. The second goal is to identifies the quantum connection on $H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ with the trigonometric dynamical connection. The flat sections of the trigonometric dynamical connection and the associated qKZ difference connection were constructed in [SV, MV, TV2] in the form of multidimensional hypergeometric integrals. The results of this paper allow us to construct flat sections of the quantum connection and the shift operator difference connection on $H_T^*(T^*\mathcal{F}_{\boldsymbol\lambda})$ in the form of multidimensional hypergeometric integrals. Such a presentation of flat sections manifests the Landau-Ginzburg mirror symmetry for the cotangent bundles of partial flag varieties.

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2. Stable envelopes

In this section we follow [MO] and define stable envelopes associated with partial flag varieties.

2.1. **Partial flag varieties.** Fix natural numbers N, n. Let $\boldsymbol\lambda \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol\lambda| = \lambda_1 + \dots + \lambda_N = n$. Consider the partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing chains of subspaces

$$0 \,=\, F_0 \subset F_1 \subset \ldots \subset F_N = \, \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \dots, N$. Denote by $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ the cotangent bundle of $\mathcal{F}_{\boldsymbol{\lambda}}$, and let $\pi : T^*\mathcal{F}_{\boldsymbol{\lambda}} \to \mathcal{F}_{\boldsymbol{\lambda}}$ be the projection of the bundle. Denote

$$\mathcal{X}_n = \cup_{|\boldsymbol{\lambda}|=n} T^*\mathcal{F}_{\boldsymbol{\lambda}}.$$

Example. If $n = 1$, then $\boldsymbol{\lambda} = (0, \dots, 0, 1_i, 0, \dots, 0)$, $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is a point and $\mathcal{X}_1$ is the union of N points.

If $n = 2$ then $\boldsymbol{\lambda} = (0, \dots, 0, 1_i, 0, \dots, 0, 1_j, 0, \dots, 0)$ or $\boldsymbol{\lambda} = (0, \dots, 0, 2_i, 0, \dots, 0)$. In the first case $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is the cotangent bundle of projective line, in the second case $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is a point. Thus $\mathcal{X}_2$ is the union of N points and $N(N-1)/2$ copies of the cotangent bundle of projective line.

Let $I = (I_1, \dots, I_N)$ be a partition of $\{1, \dots, n\}$ into disjoint subsets $I_1, \dots, I_N$. Denote $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all partitions I with $|I_j| = \lambda_j$, $j = 1, \dots N$.

Let $u_1, \dots, u_n$ be the standard basis of $\mathbb{C}^n$. For any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, let $x_I \in \mathcal{F}_{\boldsymbol{\lambda}}$ be the point corresponding to the coordinate flag $F_1 \subset \dots \subset F_N$, where F_i is the span of the standard basis vectors $u_j \in \mathbb{C}^n$ with $j \in I_1 \cup \ldots \cup I_i$. We embed $\mathcal{F}_{\boldsymbol{\lambda}}$ in $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ as the zero section and consider the points x_I as points of $T^*\mathcal{F}_{\boldsymbol{\lambda}}$.

2.2. **Schubert cells.** For any $\sigma \in S_n$, we consider the coordinate flag in $\mathbb{C}^n$,

$$V^\sigma \;:\; 0 \,=\, V_0 \subset V_1 \subset \dots \subset V_n \,=\, \mathbb{C}^n$$

where V_i is the span of $u_{\sigma(1)}, \dots, u_{\sigma(i)}$. For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ we define the *Schubert cell*

$$\Omega_{\sigma,I} = \{F \in \mathcal{F}_{\boldsymbol{\lambda}} \mid \dim(F_p \cap V^\sigma_q) = \#\{i \in I_1 \cup \ldots \cup I_p \mid \sigma^{-1}(i) \leqslant q\} \;\forall p \leqslant N, \forall q \leqslant n\}.$$

Lemma 2.1. *The Schubert cell $\Omega_{\sigma,I}$ is an affine space of dimension*

$$\ell_{\sigma,I} = \#\{(i,j) \in \{1, \dots, n\}^2 \mid \sigma(i) \in I_a, \sigma(j) \in I_b, a < b, i > j\}.$$

For a fixed σ the flag manifold is the disjoint union of the cells $\Omega_{\sigma,I}$. We have $x_I \in \Omega_{\sigma,I}$.

Proof. The structure of Schubert cells is well known, see e.g. [FP, Sect.2.2]. □

2.3. **Equivariant cohomology.** Denote $G = GL_n(\mathbb{C}) \times \mathbb{C}^\times$. Let $A \subset GL_n(\mathbb{C})$ be the torus of diagonal matrices. Denote $T = A \times \mathbb{C}^\times$ the subgroup of G.

The groups $A \subset GL_n$ act on $\mathbb{C}^n$ and hence on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$. Let the group $\mathbb{C}^\times$ act on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ by multiplication in each fiber. We denote by $-h$ its $\mathbb{C}^\times$-weight.

We consider the equivariant cohomology algebras $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$ and

$$H^*_T(\mathcal{X}_n) = \oplus_{|\boldsymbol{\lambda}|=n} H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C}).$$

Denote by $\Gamma_i = \{\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}\}$ the set of the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber F_i/F_{i-1}. Let $\boldsymbol{\Gamma} = (\Gamma_1; \dots; \Gamma_N)$. Denote by $\boldsymbol{z} = \{z_1, \dots, z_n\}$ the Chern roots corresponding to the factors of the torus T. Then

$$H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \,=\, \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} \otimes \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}[h] \Big/ \Big\langle \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) \,=\, \prod_{a=1}^{n} (u - z_a) \Big\rangle.$$

The cohomology $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ is a module over $H^*_T(pt; \mathbb{C}) = \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}[h]$.

Example. If $n = 1$, then

$$H^*_T(\mathcal{X}_1) = \oplus_{i=1}^N H^*_T(T^*\mathcal{F}_{(0,\dots,0,1_i,0,\dots,0)})$$

is naturally isomorphic to $\mathbb{C}^N \otimes \mathbb{C}[z_1; h]$ with basis $v_i = (0, \dots, 0, 1_i, 0, \dots, 0)$, $i = 1, \dots, N$.

For $i = 1, \dots, N$, denote $\lambda^{(i)} = \lambda_1 + \dots + \lambda_i$. Denote $\Theta_i = \{\theta_{i,1}, \dots, \theta_{i,\lambda^{(i)}}\}$ the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber F_i. Let $\boldsymbol{\Theta} = (\Theta_1, \dots, \Theta_N)$. The relations

$$\prod_{j=1}^{\lambda^{(i)}} (u - \theta_{i,j}) = \prod_{\ell=1}^{i} \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}), \qquad i = 1, \dots, N,$$

define the homomorphism

$$\mathbb{C}[\boldsymbol{\Theta}]^{S_{\lambda^{(1)}} \times \dots \times S_{\lambda^{(N)}}} \otimes \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}[h] \to H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}).$$

2.4. **Fixed point sets.** The set $(T^*\mathcal{F}_{\boldsymbol{\lambda}})^A$ of fixed points of the torus A action is $(x_I)_{I \in \mathcal{I}_{\boldsymbol{\lambda}}}$. We have

$$(\mathcal{X}_n)^A = \mathcal{X}_1 \times \dots \times \mathcal{X}_1.$$

The cohomology algebra $H^*_T((\mathcal{X}_n)^A)$ is naturally isomorphic to

$$(\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[\boldsymbol{z}; h].$$

This isomorphism sends the identity element $1_I \in H^*_T(x_I)$ to the vector

$$v_I = v_{i_1} \otimes \dots \otimes v_{i_n}, \tag{2.1}$$

where $i_j = i$ if $i_j \in I_i$.

2.5. **Chamber decomposition.** Let $\mathfrak{a}$ be the Lie algebra of A. The cocharacters $\eta : \mathbb{C}^\times \to A$ form a lattice of rank n. We define

$$\mathfrak{a}_{\mathbb{R}} = \mathrm{Cochar}(A) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{a}.$$

Each weight of A defines a hyperplane in $\mathfrak{a}_{\mathbb{R}}$.

Let $z_1, \dots, z_n$ be the standard basis of the dual space $\mathfrak{a}^*$, as in Section 2.3. Then the *torus roots* are the A-weights $\alpha_{i,j} = z_i - z_j$ for all $i \neq j$. The root hyperplanes partition $\mathfrak{a}_{\mathbb{R}}$ into open chambers

$$\mathfrak{a}_{\mathbb{R}} - \cup \alpha_{i,j}^\perp = \cup_{\sigma \in S_n} \mathfrak{C}_\sigma.$$

The chamber $\mathfrak{C}_\sigma$ consists of points $p \in \mathfrak{a}_{\mathbb{R}}$ such that $z_{\sigma(1)}(p) > \dots > z_{\sigma(n)}(p)$.

2.6. **Stable leaves.** Let $\mathfrak{C}$ be a chamber. We say that $x \in \mathcal{X}_n$ is $\mathfrak{C}$-stable if the limit $\lim_{z \to 0} \eta(z) \cdot x \in (\mathcal{X}_n)^A$ exists for one (equivalently, all) cocharacters $\eta \in \mathfrak{C}$. This limit is independent of the choice of $\eta \in \mathfrak{C}$ and will be denoted by $\lim_{\mathfrak{C}} x$.

Given a point $x_I \in (\mathcal{X}_n)^A$, we denote by $\mathrm{Leaf}_{\mathfrak{C},I} = \{x \mid \lim_{\mathfrak{C}} x = x_I\}$ the stable leaf of x_I. For $\sigma \in S_n$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, we denote by $C\Omega_{\sigma,I} \subset T^*\mathcal{F}_{\boldsymbol{\lambda}}$ the conormal bundle of the Schubert cell $\Omega_{\sigma,I}$.

Lemma 2.2. *We have* $\mathrm{Leaf}_{\mathfrak{C}_\sigma,I} = C\Omega_{\sigma,I}$.

Proof. Consider the natural A-invariant identification of $\pi^{-1}(\Omega_{\sigma,I})$ with $\mathbb{C}^{\ell_{\sigma,I}} \oplus \mathbb{C}^{\dim \mathcal{F}_{\boldsymbol{\lambda}}}$ mapping x_I to the origin. The weights on the first component $\mathbb{C}^{\ell_{\sigma,I}}$ are

$$z_{\sigma(j)} - z_{\sigma(i)} \qquad \text{for} \qquad \sigma(i) \in I_a, \sigma(j) \in I_b, a < b, i > j, \tag{2.2}$$

and the weights on the second component $\mathbb{C}^{\dim \mathcal{F}_{\boldsymbol{\lambda}}}$ are

$$z_{\sigma(i)} - z_{\sigma(j)} \qquad \text{for} \qquad \sigma(i) \in I_a, \sigma(j) \in I_b, a < b. \tag{2.3}$$

Consider the splitting

$$\mathbb{C}^{\ell_{\sigma,I}} \oplus \mathbb{C}^{\dim \mathcal{F}_{\boldsymbol{\lambda}}} = \underbrace{\mathbb{C}^{\ell_{\sigma,I}} \oplus T_1^*}_{C\Omega_{\sigma,I}} \oplus T_2^*,$$

where T_1^* is the sum of weight subspaces with weights

$$z_{\sigma(i)} - z_{\sigma(j)} \qquad \text{for} \qquad \sigma(i) \in I_a, \sigma(j) \in I_b, a < b, i < j, \tag{2.4}$$

and T_2^* is the sum of weight subspaces with weights

$$z_{\sigma(i)} - z_{\sigma(j)} \qquad \text{for} \qquad \sigma(i) \in I_a, \sigma(j) \in I_b, a < b, i > j. \tag{2.5}$$

For a cocharacter $\eta \in \mathfrak{C}_\sigma$ the weights in (2.2) and (2.4) are all positive and the weights in (2.5) are all negative.

Therefore, a point in $\pi^{-1}(\Omega_{\sigma,I})$ has $\lim_{\mathfrak{C}} x = x_I$ if it belongs to $C\Omega_{\sigma,I}$ and is not $\mathfrak{C}$-stable if it does not belong to $C\Omega_{\sigma,I}$. Applying the same argument for all other $J \in \mathcal{I}_\lambda$, we see that $\mathrm{Leaf}_{\mathfrak{C}_{\sigma},I} \subset \pi^{-1}(\Omega_{\sigma,I})$, and moreover, $\mathrm{Leaf}_{\mathfrak{C}_{\sigma},I} = C\Omega_{\sigma,I}$. ☐

For $\sigma \in S_n$, we define the *geometric* partial ordering on the set $\mathcal{I}_{\boldsymbol{\lambda}}$. For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, we say that $J \leqslant_g I$ if x_J lies in the closure of $\mathrm{Leaf}_{\sigma,I}$.

We also define the *combinatorial* partial ordering. For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, let

$$\sigma^{-1}(\cup_{\ell=1}^k I_\ell) = \{a_1^k < \dots < a_{\lambda^{(k)}}^k\}, \qquad \sigma^{-1}(\cup_{\ell=1}^k J_\ell) = \{b_1^k < \dots < b_{\lambda^{(k)}}^k\}$$

for $k = 1, \dots, N-1$. We say that $J \leqslant_c I$ if $b_i^k \leqslant a_i^k$ for $k = 1, \dots, N-1$, $i = 1, \dots, \lambda^{(k)}$.

Lemma 2.3. *The geometric and combinatorial partial orderings are the same.*

Proof. This is the so-called "Tableau Criterion" for the Bruhat (i.e. geometric) order, see e.g. [BB, Thm. 2.6.3]. ☐

In what follows we will denote both partial orderings by $\leqslant_\sigma$.

Lemma 2.4. *For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, $I \neq J$, there exists $\sigma \in S_n$ such that $J \not\leqslant_\sigma I$.*

Proof. The group S_n has an obvious action on $\mathcal{I}_\lambda$ as well. Observe that $J \leqslant_\sigma I$ is equivalent to $\sigma^{-1}(J) \leqslant_{\mathrm{id}} \sigma^{-1}(I)$. Hence the requirement of the Lemma is achieved by choosing σ such that $\sigma^{-1}(I)$ is the $\leqslant_{\mathrm{id}}$-smallest element of $\mathcal{I}_\lambda$, namely $(\{1, \dots, \lambda^{(1)}\}, \{\lambda^{(1)}+1, \dots, \lambda^{(2)}\}, \dots)$. ☐

For $\sigma \in S_n$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, we define $\mathrm{Slope}_{\sigma,I} = \cup_{J \leqslant_\sigma I} \mathrm{Leaf}_{\sigma,J}$. The $\mathrm{Slope}_{\sigma,I}$ is a closed subset of $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ by [MO, Lemma 3.2.7].

2.7. **Stable envelopes.** Given a closed T-invariant subset $Y \subset T^*\mathcal{F}_{\boldsymbol{\lambda}}$ and a class $E \in H_T^*(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, we say that E is supported in Y if $E|_{T^*\mathcal{F}_{\boldsymbol{\lambda}}-Y} = 0$.

For given I and σ, we define the following classes in $H_T^*(pt)$

$$e^{hor}_{\sigma,I,+} = \prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\prod_{i>j}(z_{\sigma(j)}-z_{\sigma(i)}), \qquad e^{hor}_{\sigma,I,-} = \prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\prod_{i<j}(z_{\sigma(j)}-z_{\sigma(i)}),$$

$$e^{ver}_{\sigma,I,+} = \prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\prod_{i<j}(z_{\sigma(i)}-z_{\sigma(j)}-h), \quad e^{ver}_{\sigma,I,-} = \prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\prod_{i>j}(z_{\sigma(i)}-z_{\sigma(j)}-h).$$

These are the products of the positive ("+") and negative ("−") T-weights (w.r.t. to $\mathfrak{C}_\sigma$) at x_I in the tangent to $\mathcal{F}_{\boldsymbol{\lambda}}$ direction ("hor") and fiber direction ("ver").

Let $e_{\sigma,I,-} = e^{hor}_{\sigma,I,-}\cdot e^{ver}_{\sigma,I,-}$. Let $\mathrm{sgn}_{\sigma,I} = (-1)^{\mathrm{codim}(\Omega_{\sigma,I}\subset\mathcal{F}_{\boldsymbol{\lambda}})} = \deg(e^{hor}_{\sigma,I,-})$.

Theorem 2.5. *For any $\sigma\in S_n$, there exists a unique map of $H_T^*(pt)$-modules*

$$\mathrm{Stab}_\sigma : H_T^*((\mathcal{X}_n)^A) \to H_T^*(\mathcal{X}_n)$$

such that for any $\boldsymbol{\lambda}$ with $|\boldsymbol{\lambda}| = n$ and any $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, the stable envelope $E_{\sigma,I} = \mathrm{Stab}_\sigma(1_I)$ satisfies:

(i) supp $E_{\sigma,I}\subset \mathrm{Slope}_{\sigma,I}$,
(ii) $E_{\sigma,I}|_{x_I} = \mathrm{sgn}_{\sigma,I}\cdot e_{\sigma,I,-}$,
(iii) $\deg_{\boldsymbol{z}} E_{\sigma,I}|_{x_J} < \dim\mathcal{F}_{\boldsymbol{\lambda}} = \sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j$ *for any $J\in\mathcal{I}_{\boldsymbol{\lambda}}$ with $J<_\sigma I$.*

This is Theorem 3.3.4 in [MO] applied to GL_n partial flag varieties. The choice of sign in (ii) is called a polarization in [MO]. We will fix the polarization $\mathrm{sgn}_{\sigma,I}$ in the whole paper.

2.8. **Geometric R-matrices.** The maps Stab_σ become isomorphisms after inverting the elements $(e_{\sigma,I,-})_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$. For $\sigma',\sigma\in S_n$, we define the R-matrix

$$R_{\sigma',\sigma} = \mathrm{Stab}^{-1}_{\sigma'}\circ\mathrm{Stab}_\sigma \in \mathrm{End}(H_T((\mathcal{X}_n)^A))\otimes\mathbb{C}(\boldsymbol{z};h) = \mathrm{End}((\mathbb{C}^N)^{\otimes n})\otimes\mathbb{C}(\boldsymbol{z};h), \tag{2.6}$$

where $\mathbb{C}(\boldsymbol{z};h)$ is the algebra of rational functions in $\boldsymbol{z},h$.

Example [MO, Example 4.1.2]. Let $n=2$. The group S_2 consists of two elements: id and the transposition s. After the identification $H_T^*((\mathcal{X}_n)^A) = (\mathbb{C}^N)^{\otimes 2}\otimes\mathbb{C}[\boldsymbol{z};h]$, the R-matrix is given by

$$R_{s,\mathrm{id}} = R(z_1-z_2), \tag{2.7}$$

where we define

$$R(u) = \frac{u\,\mathrm{Id} - hP}{u-h},$$

and $P\in\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)$ is the permutation of tensor factors.

For the convenience of the reader we show the calculation leading to (2.7). The space $\mathcal{X}_2$ is the union of $N(N-1)/2$ copies of $T^*\mathbb{P}^1$ and N points. The space $\mathcal{X}_2^A$ thus has $N(N-1)$ points in the $N(N-1)/2$ copies of $T^*\mathbb{P}^1$ together with the N isolated points of $\mathcal{X}_2$. On H_T^*

of the isolated points of $\mathcal{X}_2$ both sides of (2.7) act as identity. Let $i<j$. Consider the $T^*\mathbb{P}^1$ component corresponding to $\lambda=(0,\dots,0,1_i,0,\dots,0,1_j,0,\dots,0)$. Then

$$H_T^*(T^*\mathbb{P}^1)=\mathbb{C}[\gamma_{i,1},\gamma_{j,1}]\otimes\mathbb{C}[z_1,z_2;h]/\left\langle(u-\gamma_{i,1})(u-\gamma_{j,1})=(u-z_1)(u-z_2)\right\rangle.$$

The two fixed points x_I and x_J in this $T^*\mathbb{P}^1$ component are indexed by $I=(I_1,\dots,I_N)$ such that $I_i=\{1\}$ and $I_j=\{2\}$ and $I_m=\varnothing$ for all other indices, and $J=(I_1,\dots,I_N)$ such that $I_i=\{2\}$ and $I_j=\{1\}$ and $I_m=\varnothing$ for all other indices. Let F_I and F_J denote the fibers over x_I and x_J in $T^*\mathbb{P}^1$. We have

$$\begin{aligned}
\mathrm{Stab}_{\mathrm{id}}\quad &:\quad 1_I\mapsto-[F_I]=\gamma_{i,1}-z_2, && 1_J\mapsto[\mathbb{P}^1]+[F_I]=\gamma_{i,1}-z_1-h,\\
\mathrm{Stab}_s\quad &:\quad 1_I\mapsto[\mathbb{P}^1]+[F_J]=\gamma_{i,1}-z_2-h, && 1_J\mapsto-[F_J]=\gamma_{i,1}-z_1.
\end{aligned}$$

Here the geometric statements (e.g. $\mathrm{Stab}_{\mathrm{id}}(1_I)=-[F_I]$) can be checked by verifying the conditions of Theorem 2.5, and the calculation (e.g. $-[F_I]=\gamma_{i,1}-z_2$) can be verified by equivariant localization. Therefore we have

$$\begin{aligned}
\mathrm{Stab}_s^{-1}\circ\mathrm{Stab}_{\mathrm{id}}(1_I)&=\mathrm{Stab}_s^{-1}(\gamma_{i,1}-z_2)\\
&=\mathrm{Stab}_s^{-1}\left(\frac{z_1-z_2}{z_1-z_2-h}(\gamma_{i,1}-z_2-h)+\frac{-h}{z_1-z_2-h}(\gamma_{i,1}-z_1)\right)\\
&=\frac{z_1-z_2}{z_1-z_2-h}\cdot 1_I+\frac{-h}{z_1-z_2-h}\cdot 1_J.
\end{aligned}\tag{2.8}$$

This, together with a similar calculation for 1_J proves the claim (2.7) for the fixed points in $T^*\mathbb{P}^1$.

It is enough to consider R-matrices corresponding to pairs of chambers separated by a wall. Such a pair has the following form. For $i=1,\dots,n-1$, let $s_i\in S_n$ be the transposition $(i,i+1)$. Any chamber $\mathfrak{C}_\sigma=\{p\in\mathfrak{a}_{\mathbb{R}}\,|z_{\sigma(1)}(p)>\dots>z_{\sigma(n)}(p)\}$ is separated by a wall from exactly $n-1$ chambers. They are $\mathfrak{C}_{\sigma s_i}=\{p\in\mathfrak{a}_{\mathbb{R}}\mid z_{\sigma(1)}(p)>\dots>z_{\sigma(i-1)}>z_{\sigma(i+1)}>z_{\sigma(i)}>z_{\sigma(i+2)}>\dots>z_{\sigma(n)}(p)\}$, $i=1,\dots,n-1$.

Theorem 2.6 (Section 4.1.6 in [MO]).

$$R_{\sigma s_i,\sigma}=R^{(\sigma(i),\sigma(i+1))}(z_{\sigma(i)}-z_{\sigma(i+1)})\in\mathrm{End}((\mathbb{C}^N)^{\otimes n})\otimes\mathbb{C}(\boldsymbol{z};h),\tag{2.9}$$

where the superscript means that the R-matrix of formula (2.7) *operates in the $\sigma(i)$-th and $\sigma(i+1)$-th tensor factors.*

3. Weight functions

3.1. **Weight functions W_I.** For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, we define the weight functions $W_I(\boldsymbol{t};\boldsymbol{z};h)$, c.f. [TV1, TV4].

Recall $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$. Denote $\lambda^{(i)}=\lambda_1+\ldots+\lambda_i$ and $\lambda^{\{1\}}=\sum_{i=1}^{N-1}\lambda^{(i)}=\sum_{i=1}^{N-1}(N-i)\,\lambda_i$. Recall $I=(I_1,\dots,I_N)$. Set $\bigcup_{k=1}^{j}I_k=\{i_1^{(j)}<\ldots<i_{\lambda^{(j)}}^{(j)}\}$. Consider the variables $t_a^{(j)}$, $j=1,\dots,N$, $a=1,\dots,\lambda^{(j)}$, where $t_a^{(N)}=z_a$, $a=1,\dots,n$. Denote $t^{(j)}=(t_k^{(j)})_{k\leqslant\lambda^{(j)}}$ and $\boldsymbol{t}=(t^{(1)},\dots,t^{(N-1)})$.

The weight functions are

$$W_I(\boldsymbol{t};\boldsymbol{z};h)\,=\,(-h)^{\lambda^{\{1\}}}\,\mathrm{Sym}_{t_1^{(1)},\dots,t_{\lambda^{(1)}}^{(1)}}\,\ldots\,\mathrm{Sym}_{t_1^{(N-1)},\dots,t_{\lambda^{(N-1)}}^{(N-1)}}\,U_I(\boldsymbol{t};\boldsymbol{z};h)\,,\tag{3.1}$$

$$U_I(\boldsymbol{t};\boldsymbol{z};h) = \prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\Bigg(\prod_{\substack{c=1\\ i_c^{(j+1)}<i_a^{(j)}}}^{\lambda^{(j+1)}}(t_a^{(j)}-t_c^{(j+1)}-h)\prod_{\substack{d=1\\ i_d^{(j+1)}>i_a^{(j)}}}^{\lambda^{(j+1)}}(t_a^{(j)}-t_d^{(j+1)})\prod_{b=a+1}^{\lambda^{(j)}}\frac{t_a^{(j)}-t_b^{(j)}-h}{t_a^{(j)}-t_b^{(j)}}\Bigg).$$

In these formulas for a function $f(t_1,\dots,t_k)$ of some variables we denote

$$\mathrm{Sym}_{t_1,\dots,t_k} f(t_1,\dots,t_k) = \sum_{\sigma\in S_k} f(t_{\sigma_1},\dots,t_{\sigma_k}).$$

Example. Let $N=2$, $n=2$, $\boldsymbol{\lambda}=(1,1)$, $I=(\{1\},\{2\})$, $J=(\{2\},\{1\})$. Then

$$W_I(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_2), \qquad W_J(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_1-h).$$

Example. Let $N=2$, $n=3$, $\boldsymbol{\lambda}=(1,2)$, $I=(\{2\},\{1,3\})$. Then

$$W_I(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_1-h)(t_1^{(1)}-z_3).$$

For a subset $A\subset\{1,\dots,n\}$, denote $\boldsymbol{z}_A=(z_a)_{a\in A}$. For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, denote $\boldsymbol{z}_I=(\boldsymbol{z}_{I_1},\dots,\boldsymbol{z}_{I_N})$. For $f(t^{(1)},\dots,t^{(N)})\in\mathbb{C}[t^{(1)},\dots,t^{(N)}]^{S_{\lambda^{(1)}}\times\dots\times S_{\lambda^{(N)}}}$, we define $f(\boldsymbol{z}_I)$ by replacing $t^{(j)}$ with $\cup_{k=1}^{j}\boldsymbol{z}_{I_k}$. Denote

$$c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I) = \prod_{a=1}^{N-1}\prod_{i,j\in\cup_{b=1}^{a}I_b}(z_i-z_j-h)\,. \tag{3.2}$$

Lemma 3.1.

(i) *For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, the polynomial $W_I(\boldsymbol{z}_J;\boldsymbol{z};h)$ is divisible by $c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)$.*

(ii) *For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,*

$$W_I(\boldsymbol{z}_I;\boldsymbol{z};h) = c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)\prod_{a<b}\prod_{\substack{i\in I_a\\ j\in I_b}}\Big(\prod_{i<j}(z_i-z_j)\prod_{i>j}(z_i-z_j-h)\Big).$$

(iii) *For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, $J<_{\mathrm{id}}I$*

$$\deg_{\boldsymbol{z}} W_I(\boldsymbol{z}_J;\boldsymbol{z};h) < \deg_{\boldsymbol{z}} W_I(\boldsymbol{z}_I;\boldsymbol{z};h;\boldsymbol{z}_I).$$

(iv) *For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, we have $W_I(\boldsymbol{z}_J;\boldsymbol{z};h)=0$ unless $J\leqslant_{\mathrm{id}}I$.*

Lemma 3.1 is proved in Section 8.

3.2. **Weight functions $W_{\sigma,I}$.** For $\sigma\in S_n$ and $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, we define

$$W_{\sigma,I}(\boldsymbol{t};\boldsymbol{z};h) = W_{\sigma^{-1}(I)}(\boldsymbol{t};z_{\sigma(1)},\dots,z_{\sigma(n)};h),$$

where $\sigma^{-1}(I)=(\sigma^{-1}(I_1),\dots,\sigma^{-1}(I_N))$.

Example. Let $N=2$, $n=2$, $\boldsymbol{\lambda}=(1,1)$, $I=(\{1\},\{2\})$, $J=(\{2\},\{1\})$. Then

$$W_{\mathrm{id},I}(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_2), \qquad W_{\mathrm{id},J}(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_1-h),$$
$$W_{s,I}(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_2-h), \qquad W_{s,J}(\boldsymbol{t};\boldsymbol{z};h) = -h\,(t_1^{(1)}-z_1).$$

Lemma 3.2. *For any $\sigma\in S_n$, we have the following statements:*

(i) *For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, the polynomial $W_{\sigma,I}(\boldsymbol{z}_J;\boldsymbol{z};h)$ is divisible by $c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)$.*

(ii) *For* $I \in \mathcal{I}_{\boldsymbol{\lambda}}$,

$$W_{\sigma,I}(\boldsymbol{z}_I;\boldsymbol{z};h) = c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)\prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\Big(\prod_{i<j}(z_{\sigma(i)}-z_{\sigma(j)})\prod_{i>j}(z_{\sigma(i)}-z_{\sigma(j)}-h)\Big).$$

(iii) *For* $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, $J <_\sigma I$,

$$\deg_{\boldsymbol{z}} W_{\sigma,I}(\boldsymbol{z}_J;\boldsymbol{z};h) < \deg_{\boldsymbol{z}} W_{\sigma,I}(\boldsymbol{z}_I;\boldsymbol{z};h).$$

(iv) *For* $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, *we have* $W_{\sigma,I}(\boldsymbol{z}_J;\boldsymbol{z};h) = 0$ *unless* $J \leqslant_\sigma I$.

Lemma 3.2 follows from Lemma 3.1.

Lemma 3.3. *For any* $\sigma \in S_n$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, $i = 1,\dots,n-1$, *we have*

$$W_{\sigma s_{i,i+1},I} = \frac{z_{\sigma(i)}-z_{\sigma(i+1)}}{z_{\sigma(i)}-z_{\sigma(i+1)}+h}W_{\sigma,I} + \frac{h}{z_{\sigma(i)}-z_{\sigma(i+1)}+h}W_{\sigma,s_{\sigma(i),\sigma(i+1)}(I)}, \tag{3.3}$$

where $s_{i,j} \in S_n$ *is the transposition of* i *and* j.

Lemma 3.3 is proved in Section 8.

Let $\sigma_0 \in S_n$ be the longest permutation, that is, $\sigma_0 : i \mapsto n+1-i$, $i = 1,\dots,n$.

Lemma 3.4. *For* $J, K \in \mathcal{I}_{\boldsymbol{\lambda}}$, *we have*

$$\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{W_{\mathrm{id},J}(\boldsymbol{z}_I;\boldsymbol{z};h)W_{\sigma_0,K}(\boldsymbol{z}_I;\boldsymbol{z};h)}{R(\boldsymbol{z}_I)\,Q(\boldsymbol{z}_I)\,c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)^2} \;=\; \delta_{J,K}\,,$$

where

$$R(\boldsymbol{z}_I) \;=\; \prod_{1\leqslant a<b\leqslant N}\;\prod_{i\in I_a}\;\prod_{j\in I_b}(z_i-z_j)\,,\qquad Q(\boldsymbol{z}_I) \;=\; \prod_{1\leqslant a<b\leqslant N}\;\prod_{i\in I_a}\;\prod_{j\in I_b}(z_i-z_j-h)\,.$$

Lemma 3.4 is proved in Section 8.

3.3. R**-matrices.** Consider $\mathbb{C}[\boldsymbol{t};\boldsymbol{z};h]\otimes_{\mathbb{C}[\boldsymbol{z};h]}\mathbb{C}(\boldsymbol{z};h)$ as a $\mathbb{C}(\boldsymbol{z};h)$-module. Denote by $M_{\boldsymbol{\lambda}}$ the $\mathbb{C}(\boldsymbol{z};h)$-submodule generated by the polynomials $(W_I(\boldsymbol{z};h;\boldsymbol{t}))_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$. Denote $M_n = \oplus_{|\boldsymbol{\lambda}|=n}M_{\boldsymbol{\lambda}}$.

Lemma 3.5. *The module* $M_{\boldsymbol{\lambda}}$ *is free of rank* $|\mathcal{I}_{\boldsymbol{\lambda}}|$ *with the basis* $(W_I)_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$.

Proof. The lemma follows from parts (ii) and (iv) of Lemma 3.1. □

Lemma 3.6. *For any* $\sigma \in S_n$ *the polynomials* $(W_{\sigma,I})_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ *form a basis of* $M_{\boldsymbol{\lambda}}$.

Proof. The fact that $W_{\sigma,I}$ belongs to M_λ follows from Lemma 3.3 by induction on the length of σ. The independence of $(W_{\sigma,I})_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ follows from parts (ii) and (iv) of Lemma 3.2. Alternatively, the independence of $(W_{\sigma,I})_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ follows by induction from Lemma 3.3 with σ replaced by σs_i (this is equivalent to inverting the formula in Lemma 3.3). □

For $\sigma \in S_n$, we define the *algebraic weight function map*

$$W_\sigma : (\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}(\boldsymbol{z};h) \to M_n,\quad v_I \mapsto W_{\sigma,I}(\boldsymbol{t};\boldsymbol{z};h).$$

For $\sigma', \sigma \in S_n$, we define the R-matrix

$$\tilde{R}_{\sigma',\sigma} = W_{\sigma'}^{-1}\circ W_\sigma \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})\otimes\mathbb{C}(\boldsymbol{z};h).$$

Theorem 3.7 ([TV4]). *For any $\sigma \in S_n$ and $i = 1, \dots, n-1$, we have*

$$\tilde{R}_{\sigma s_i,\sigma} = R^{(\sigma(i),\sigma(i+1))}(z_{\sigma(i)} - z_{\sigma(i+1)}) \in \mathrm{End}((\mathbb{C}^N)^{\otimes n}) \otimes \mathbb{C}(\boldsymbol{z}; h), \tag{3.4}$$

where the superscript means that the R-matrix of formula (2.7) *operates in the $\sigma(i)$-th and $\sigma(i+1)$-th tensor factors.*

Proof. The theorem follows from Lemma 3.3. □

For $\sigma \in S_n$, we define the *cohomological weight function map*

$$[W_\sigma] : H_T^*((\mathcal{X}_n)^A) = (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[\boldsymbol{z}; h] \to H_T^*(\mathcal{X}_n), \quad v_I \mapsto [W_{\sigma,I}(\boldsymbol{\Theta}; \boldsymbol{z}; h)],$$

where $W_{\sigma,I}(\boldsymbol{\Theta}; \boldsymbol{z}; h)$ is the polynomial $W_{\sigma,I}(\boldsymbol{t}; \boldsymbol{z}; h)$ in which variables $t_i^{(j)}$ are replaced with $\theta_{j,i}$ and $[W_{\sigma,I}(\boldsymbol{\Theta}; \boldsymbol{z}; h)]$ is the cohomology class represented by $W_{\sigma,I}(\boldsymbol{\Theta}; \boldsymbol{z}; h)$.

Let $\mathrm{Loc} : H_T^*(\mathcal{X}_n) \to H_T^*((\mathcal{X}_n)^A)$ be the localization map $w \mapsto (w|_{x_I})_{x_I \in (\mathcal{X}_n)^A}$. According to Lemma 3.2, the composition $\mathrm{Loc} \circ [W_\sigma]$ is upper triangular with respect to the order $\leqslant_\sigma$. Therefore – after tensoring with $\mathbb{C}(\boldsymbol{z}; h)$ – the map $\mathrm{Loc} \circ [W_\sigma]$ is invertible. We obtain that for $\sigma', \sigma \in S_n$, we can define the R-matrix

$$\bar{R}_{\sigma',\sigma} = [W_{\sigma'}]^{-1} \circ [W_\sigma] \in \mathrm{End}((\mathbb{C}^N)^{\otimes n}) \otimes \mathbb{C}(\boldsymbol{z}; h),$$

and that

$$\bar{R}_{\sigma s_i,\sigma} = \tilde{R}_{\sigma s_i,\sigma}.$$

Corollary 3.8. *For any $\sigma \in S_n$ and $i = 1, \dots, n-1$, we have*

$$\bar{R}_{\sigma s_i,\sigma} = R^{(\sigma(i),\sigma(i+1))}(z_{\sigma(i)} - z_{\sigma(i+1)}) \in \mathrm{End}((\mathbb{C}^N)^{\otimes n}) \otimes \mathbb{C}(\boldsymbol{z}; h), \tag{3.5}$$

where the superscript means that the R-matrix of formula (2.7) *operates in the $\sigma(i)$-th and $\sigma(i+1)$-th tensor factors.*

Remark. The maps $W_\sigma : (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(\boldsymbol{z}; h) \to M_n$, $\sigma \in S_n$, form what is called in [V] local tensor coordinates on M_n. The stable envelope maps as well as cohomological weight function maps are basically other examples of that structure.

4. Stable envelope maps and weight function maps

4.1. **Main theorem.** Denote

$$c_{\boldsymbol{\lambda}}(\boldsymbol{\Theta}) = \prod_{a=1}^{N-1} \prod_{i=1}^{\lambda^{(a)}} \prod_{j=1}^{\lambda^{(a)}} (\theta_{a,i} - \theta_{a,j} - h) \in H_T^*(T^*\mathcal{F}_{\boldsymbol{\lambda}}). \tag{4.1}$$

Observe that $c_{\boldsymbol{\lambda}}(\boldsymbol{\Theta})$ is the equivariant Euler class of the bundle $\oplus_{a=1}^{N-1} \mathrm{Hom}(F_a, F_a)$ if we make $\mathbb{C}^\times$ act on it with weight $-h$. Here, by a slight abuse of notation, we wrote F_a for the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber F_a.

Let $c_n(\boldsymbol{\Theta}) = (c_{\boldsymbol{\lambda}}(\boldsymbol{\Theta}))_{|\boldsymbol{\lambda}|=n} \in H_T^*(\mathcal{X}_n)$. Note that $c_n(\boldsymbol{\Theta})$ is not a zero-divisor in $H_T^*(\mathcal{X}_n)$, because none of its fixed point restrictions is zero.

Theorem 4.1. *For any $\sigma \in S_n$, the maps $\mathrm{Stab}_\sigma : H_T^*((\mathcal{X}_n)^A) \to H_T^*(\mathcal{X}_n)$ and $[W_\sigma] : H_T^*((\mathcal{X}_n)^A) \to H_T^*(\mathcal{X}_n)$ are related:*

$$[W_\sigma] = c_n \circ \mathrm{Stab}_\sigma, \tag{4.2}$$

where c_n denotes the operator of multiplication by $c_n(\boldsymbol{\Theta})$.

Theorem 4.1 is proved in Section 4.2.

Remark. In [RTV] we defined cohomology classes κ_I and κ'_I in the cohomology ring of the cotangent bundle of a Grassmannian. The classes were suggested as candidates for the equivariant fundamental cohomology class of the cotangent bundle of the Schubert variety $\bar{\Omega}_{\mathrm{id},I}$, which is singular in general. Theorem 4.1 implies that the classes κ_I, κ'_I defined in [RTV] are supported on $\mathrm{Slope}_{\mathrm{id},I}$.

4.2. **Triangularity.** The $\mathbb{C}(\boldsymbol{z};h)$-module $H^*_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z};h)$ has a basis $(w_I(\boldsymbol{\Gamma};\boldsymbol{z};h))_{I\in\mathcal{I}_{\boldsymbol{\lambda}},|\boldsymbol{\lambda}|=n}$,

$$w_I(\boldsymbol{\Gamma};\boldsymbol{z};h) = \prod_{1\leqslant i<j\leqslant N}\ \prod_{x\in\Gamma_i}\prod_{k\in I_i}\prod_{\ell\in I_j}\frac{x-z_\ell}{z_k-z_\ell}.$$

We have

$$w_I(\boldsymbol{z}_J;\boldsymbol{z};h) = \delta_{I,J}, \qquad I,J\in\mathcal{I}_{\boldsymbol{\lambda}}.$$

Thus we have the distinguished basis $(w_I(\boldsymbol{\Gamma};\boldsymbol{z};h))_{I\in\mathcal{I}_{\boldsymbol{\lambda}},|\boldsymbol{\lambda}|=n}$ of the $\mathbb{C}(\boldsymbol{z};h)$-module $H^*_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z};h)$ and the distinguished basis $(1_I)_{I\in\mathcal{I}_{\boldsymbol{\lambda}},|\boldsymbol{\lambda}|=n}$ of the $\mathbb{C}(\boldsymbol{z};h)$-module $H^*_T((\mathcal{X}_n)^A)$.

We denote the matrices of the $\mathbb{C}(\boldsymbol{z};h)$-module isomorphisms Stab_σ and $[W_\sigma]$ with respect to these bases by $\mathcal{A}_\sigma$ and $\mathcal{B}_\sigma$ respectively.

Lemma 4.2. *The matrix $\mathcal{A}_\sigma$ is upper-triangular with respect to the partial ordering $\leqslant_\sigma$, that is, if $(\mathcal{A}_\sigma)_{J,I}\neq 0$, then $J\leqslant_\sigma I$.*

Proof. The lemma follows from part (i) of Theorem 2.5. □

Lemma 4.3. *The matrix $\mathcal{B}_\sigma$ is upper-triangular with respect to the partial ordering $\leqslant_\sigma$, that is, if $(\mathcal{B}_\sigma)_{J,I}\neq 0$, then $J\leqslant_\sigma I$.*

Proof. The lemma follows from part (iv) of Lemma 3.2. □

Define the $\mathbb{C}(\boldsymbol{z};h)$-module isomorphism

$$f_\sigma = [W_\sigma]\circ(\mathrm{Stab}_\sigma)^{-1} : H^*_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z};h)\to H^*_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z};h).$$

Corollary 4.4. *For any $\sigma\in S_n$, the matrix of f_σ is upper-triangular with respect to the partial ordering $\leqslant_\sigma$.*

Lemma 4.5. *The operator f_σ does not depend on $\sigma\in S_n$.*

Proof. For $\sigma',\sigma\in S_n$ we have

$$R_{\sigma',\sigma} = [W_{\sigma'}]^{-1}\circ[W_\sigma] = \mathrm{Stab}_{\sigma'}^{-1}\circ(f_{\sigma'}^{-1}\circ f_\sigma)\circ\mathrm{Stab}_\sigma, \qquad R_{\sigma',\sigma} = \mathrm{Stab}_{\sigma'}^{-1}\circ\mathrm{Stab}_\sigma$$

Hence $f_{\sigma'}^{-1}\circ f_\sigma = 1$ and $f = f_\sigma$ does not depend on σ. □

Lemma 4.6. *The matrix of the operator f is diagonal.*

Proof. For any $\sigma\in S_n$ the matrix of f is upper-triangular by Corollary 4.4. Lemma 2.4 implies that the matrix is diagonal. □

To prove Theorem 4.1 it is enough to evaluate the diagonal entries of the matrix of f and of the matrices $\mathcal{A}_\sigma,\mathcal{B}_\sigma$ and then check that the diagonal entries satisfy (4.2). The diagonal entries of $\mathcal{A}_\sigma$ are given by part (ii) of Theorem 2.5 and the diagonal entries of $\mathcal{B}_\sigma$ are given by part (ii) of Lemma 3.2. Observing that $c_{\boldsymbol\lambda}(\boldsymbol\Theta)$ restricted to the fixed point x_I is $c_{\boldsymbol\lambda}(\boldsymbol z_I)$, and that

$$\prod_{a<b}\prod_{\substack{\sigma(i)\in I_a\\ \sigma(j)\in I_b}}\Big(\prod_{i<j}(z_{\sigma(i)}-z_{\sigma(j)})\prod_{i>j}(z_{\sigma(i)}-z_{\sigma(j)}-h)\Big)=\big(\mathrm{sgn}_{\sigma,I}\,e^{hor}_{\sigma,I,-}\big)\cdot e^{ver}_{\sigma,I,-}$$

we obtain that the diagonal entries indeed satisfy (4.2). Theorem 4.1 is proved.

5. Orthogonality

5.1. **Orthogonality on $\mathcal F_{\boldsymbol\lambda}$.** Consider the bilinear form on $H^*_A(\mathcal F_{\boldsymbol\lambda})$ defined by $(f,g)_{\mathcal F_{\boldsymbol\lambda}}=\int_{\mathcal F_{\boldsymbol\lambda}} fg$, where the equivariant integral on $\mathcal F_{\boldsymbol\lambda}$ can be expressed via localization by

$$\int_{\mathcal F_{\boldsymbol\lambda}}\alpha(\boldsymbol\Gamma;\boldsymbol z)=\sum_{I\in\mathcal I_\lambda}\frac{\alpha(\boldsymbol z_I;\boldsymbol z)}{\prod_{a<b}\prod_{i\in I_a,j\in I_b}(z_j-z_i)}.$$

Note that the denominator is the equivariant Euler class of the tangent space to $\mathcal F_{\boldsymbol\lambda}$ at the fixed point x_I.

Let σ_0 be the longest permutation in S_n, that is, $\sigma_0 : i\mapsto n+1-i$. It is well known in Schubert calculus that

$$([\bar\Omega_{\mathrm{id},J}],[\bar\Omega_{\sigma_0,K}])_{\mathcal F_{\boldsymbol\lambda}}=\delta_{J,K},$$

where $\bar\Omega_{\sigma,I}$ is the closure of the Schubert cell $\Omega_{\sigma,I}$. In Theorem 5.1 below we will show the $T^*\mathcal F_{\boldsymbol\lambda}$ version of this orthogonality statement.

5.2. **Orthogonality on $T^*\mathcal F_{\boldsymbol\lambda}$.** Consider the bilinear form on $H^*_T(T^*\mathcal F_{\boldsymbol\lambda})$ defined by

$$(f,g)_{T^*\mathcal F_{\boldsymbol\lambda}}=\int_{T^*\mathcal F_{\boldsymbol\lambda}} fg,$$

where the equivariant integral on $T^*\mathcal F_{\boldsymbol\lambda}$ is *defined* via localization by

$$\int_{T^*\mathcal F_{\boldsymbol\lambda}}\alpha(\boldsymbol\Gamma;\boldsymbol z;h)=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{\alpha(\boldsymbol z_I;\boldsymbol z;h)}{\prod_{a<b}\prod_{i\in I_a,j\in I_b}(z_j-z_i)(z_i-z_j-h)}.$$

Note that the denominator is the equivariant Euler class of the tangent space to $T^*\mathcal F_{\boldsymbol\lambda}$ at the fixed point x_I. This bilinear form takes values in $\mathbb C(\boldsymbol z;h)$.

Theorem 5.1. *We have*

$$(\mathrm{Stab}_{\mathrm{id}}(1_J),\mathrm{Stab}_{\sigma_0}(1_K))_{T^*\mathcal F_{\boldsymbol\lambda}}=\delta_{J,K}\cdot(-1)^{\dim\mathcal F_{\boldsymbol\lambda}}.$$

Proof. We have

$$(\mathrm{Stab}_{\mathrm{id}}(1_J),\mathrm{Stab}_{\sigma_0}(1_K))_{T^*\mathcal F_{\boldsymbol\lambda}}=(-1)^{\dim\mathcal F_{\boldsymbol\lambda}}\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{(\mathrm{Stab}_{\mathrm{id}}(1_J)\mathrm{Stab}_{\sigma_0}(1_K))|_{x_I}}{R(z_I)Q(z_I)}=$$

$$=(-1)^{\dim\mathcal F_{\boldsymbol\lambda}}\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{W_{\mathrm{id},J}(\boldsymbol z_I;\boldsymbol z;h)W_{\sigma_0,K}(\boldsymbol z_I;\boldsymbol z,h)}{R(\boldsymbol z_I)\,Q(\boldsymbol z_I)\,c_{\boldsymbol\lambda}(\boldsymbol z_I)^2}=\delta_{J,K}\cdot(-1)^{\dim\mathcal F_{\boldsymbol\lambda}}.$$

where the first equality is by definition, the second by Theorem 4.1 and the third by Lemma 3.4. ∎

6. Yangian actions

6.1. Yangian $Y(\mathfrak{gl}_N)$. The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T_{i,j}^{\{s\}}$ for $i,j=1,\dots,N$, $s\in\mathbb{Z}_{>0}$, subject to relations

$$(u-v)\big[T_{i,j}(u)\,,T_{k,l}(v)\big] \,=\, T_{k,j}(v)\,T_{i,l}(u)-T_{k,j}(u)\,T_{i,l}(v)\,,\qquad i,j,k,l=1,\dots,N\,, \tag{6.1}$$

where
$$T_{i,j}(u)=\delta_{i,j}+\sum_{s=1}^{\infty}T_{i,j}^{\{s\}}\,u^{-s}\,.$$

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct $\Delta: Y(\mathfrak{gl}_N)\to Y(\mathfrak{gl}_N)\otimes Y(\mathfrak{gl}_N)$ given by $\Delta\big(T_{i,j}(u)\big)=\sum_{k=1}^N T_{k,j}(u)\otimes T_{i,k}(u)$ for $i,j=1,\dots,N$. The Yangian $Y(\mathfrak{gl}_N)$ contains, as a Hopf subalgebra, the universal enveloping algebra $U(\mathfrak{gl}_N)$ of the Lie algebra $\mathfrak{gl}_N$. The embedding is given by $e_{i,j}\mapsto T_{j,i}^{\{1\}}$, where $e_{i,j}$ are standard standard generators of $\mathfrak{gl}_N$.

Notice that $\big[T_{i,j}^{\{1\}},T_{k,l}^{\{s\}}\big]=\delta_{i,l}T_{k,j}^{\{s\}}-\delta_{j,k}T_{i,l}^{\{s\}}$ for $i,j,k,l=1,\dots,N$, $s\in\mathbb{Z}_{>0}$, which implies that the Yangian $Y(\mathfrak{gl}_N)$ is generated by the elements $T_{i,i+1}^{\{1\}}, T_{i+1,i}^{\{1\}}$, $i=1,\dots,N-1$, and $T_{1,1}^{\{s\}}$, $s>0$.

6.2. Algebra $\widetilde{Y}(\mathfrak{gl}_N)$. In this section we follow [GRTV, Section 3.3]. In formulas of that Section 3.3 we replace h with $-h$.

Let $\widetilde{Y}(\mathfrak{gl}_N)$ be the subalgebra of $Y(\mathfrak{gl}_N)\otimes\mathbb{C}[h]$ generated over $\mathbb{C}$ by $\mathbb{C}[h]$ and the elements $(-h)^{s-1}T_{i,j}^{\{s\}}$ for $i,j=1,\dots,N$, $s>0$. Equivalently, the subalgebra $\widetilde{Y}(\mathfrak{gl}_N)$ is generated over $\mathbb{C}$ by $\mathbb{C}[h]$ and the elements $T_{i,i+1}^{\{1\}}, T_{i+1,i}^{\{1\}}$, $i=1,\dots,N-1$, and $(-h)^{s-1}T_{1,1}^{\{s\}}$, $s>0$.

For $p=1,\dots,N$, $\boldsymbol{i}=\{1\leqslant i_1<\dots<i_p\leqslant N\}$, $\boldsymbol{j}=\{1\leqslant j_1<\dots<j_p\leqslant N\}$, define
$$M_{\boldsymbol{i},\boldsymbol{j}}(u)=\sum_{\sigma\in S_p}(-1)^\sigma\,T_{i_1,j_{\sigma(1)}}(u)\dots T_{i_p,j_{\sigma(p)}}(u-p+1)\,.$$

Introduce the series $A_1(u),\dots,A_N(u)$, $E_1(u),\dots,E_{N-1}(u)$, $F_1(u),\dots,F_{N-1}(u)$:

$$A_p(u)\,=\,M_{\boldsymbol{i},\boldsymbol{i}}(-u/h)\,=\,1+\sum_{s=1}^{\infty}(-h)^s\,A_{p,s}\,u^{-s}\,, \tag{6.2}$$

$$E_p(u)\,=\,-h^{-1}M_{\boldsymbol{j},\boldsymbol{i}}(-u/h)\big(M_{\boldsymbol{i},\boldsymbol{i}}(-u/h)\big)^{-1}\,=\,\sum_{s=1}^{\infty}(-h)^{s-1}E_{p,s}\,u^{-s}\,, \tag{6.3}$$

$$F_p(u)\,=\,-h^{-1}\big(M_{\boldsymbol{i},\boldsymbol{i}}(-u/h)\big)^{-1}M_{\boldsymbol{i},\boldsymbol{j}}(-u/h)\,=\,\sum_{s=1}^{\infty}(-h)^{s-1}F_{p,s}\,u^{-s}\,,$$

where in formulas (6.2) and (6.3) we have $\boldsymbol{i}=\{1,\dots,p\}$, $\boldsymbol{j}=\{1,\dots,p-1,p+1\}$. Observe that $E_{p,1}=T_{p+1,p}^{\{1\}}$, $F_{p,1}=T_{p,p+1}^{\{1\}}$ and $A_{1,s}=T_{1,1}^{\{s\}}$, so the coefficients of the series $E_p(u)$, $F_p(u)$ and $h^{-1}\big(A_p(u)-1\big)$ together with $\mathbb{C}[h]$ generate $\widetilde{Y}(\mathfrak{gl}_N)$. In what follows we will describe actions of the algebra $\widetilde{Y}(\mathfrak{gl}_N)$ by using series (6.2), (6.3).

6.3. $\widetilde{Y}(\mathfrak{gl}_N)$**-action on** $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$**.** Let $\mathbb{C}[\boldsymbol{z};h]$ act on $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$ by multiplication. Set

$$L(u) \,=\, (u-z_n-h\,P^{(0,n)})\dots(u-z_1-h\,P^{(0,1)})\,, \tag{6.4}$$

where the factors of $\mathbb{C}^N\otimes(\mathbb{C}^N)^{\otimes n}$ are labeled by $0,1,\dots,n$. $L(u)$ is a polynomial in $u,\boldsymbol{z},h$ with values in $\mathrm{End}(\mathbb{C}^N\otimes(\mathbb{C}^N)^{\otimes n})$. We consider $L(u)$ as an $N\times N$ matrix with $\mathrm{End}(V)\otimes\mathbb{C}[u;\boldsymbol{z};h]$-valued entries $L_{i,j}(u)$.

Proposition 6.1 (Proposition 4.1 in [GRTV])**.** *The assignment*

$$\phi\bigl(T_{i,j}(-u/h)\bigr) \,=\, L_{i,j}(u)\,\prod_{a=1}^{n}\,(u-z_a)^{-1} \tag{6.5}$$

defines the action of the algebra $\widetilde{Y}(\mathfrak{gl}_N)$ *on* $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$*. Here the right-hand side of* (6.5) *is a series in* u^{-1} *with coefficients in* $\mathrm{End}((\mathbb{C}^N)^{\otimes n})\otimes\mathbb{C}[\boldsymbol{z};h])$*.*

Under this action, the subalgebra $U(\mathfrak{gl}_N)\subset\widetilde{Y}(\mathfrak{gl}_N)$ acts on $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$ in the standard way: any element $x\in\mathfrak{gl}_N$ acts as $x^{(1)}+\dots+x^{(n)}$.

The action ϕ was denoted in [GRTV] by ϕ^+. After the identification $H^*_T((\mathcal{X}_n)^A) = (\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$, the action ϕ defines the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure on $H^*_T((\mathcal{X}_n)^A)$. This $\widetilde{Y}(\mathfrak{gl}_N)$-module structure on $H^*_T((\mathcal{X}_n)^A)$ coincides with the Yangian module structure on $H^*_T((\mathcal{X}_n)^A)$ introduced in [MO, Section 5.2.6].

6.4. $H^*_T(\mathcal{X}_n)$ **as a** $\widetilde{Y}(\mathfrak{gl}_N)$**-module according to** [GRTV]**.** We define the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ on $H^*_T(\mathcal{X}_n)$ by formulas (6.6), (6.8), (6.9). Notice that this $\widetilde{Y}(\mathfrak{gl}_N)$-module structure was denoted in [GRTV] by ρ^- and h in [GRTV] is replaced with $-h$. We define $\rho\bigl(A_p(u)\bigr) : H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\to H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ by

$$\rho\bigl(A_p(u)\bigr)\,:\,[f]\,\mapsto\,\Bigl[\,f(\boldsymbol{\Gamma};\boldsymbol{z};h)\,\prod_{a=1}^{p}\prod_{i=1}^{\lambda_p}\Bigl(1-\frac{h}{u-\gamma_{p,i}}\Bigr)\Bigr]\,, \tag{6.6}$$

for $p=1,\dots,N$. In particular,

$$\rho(X_i^\infty)\,:\,[f]\,\mapsto\,[\,(\gamma_{i,1}+\dots+\gamma_{i,\lambda_i})\,f(\boldsymbol{\Gamma};\boldsymbol{z};h)]\,,\qquad i=1,\dots,N\,. \tag{6.7}$$

Let $\alpha_1,\dots,\alpha_{N-1}$ be simple roots, $\alpha_p=(0,\dots,0,1,-1,0,\dots,0)$, with $p-1$ first zeros. We define

$$\rho\bigl(E_p(u)\bigr)\,:\,H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}-\alpha_p})\,\mapsto\,H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\,,$$

$$\rho\bigl(E_p(u)\bigr)\,:\,[f]\,\mapsto\,\Biggl[\,\sum_{i=1}^{\lambda_p}\,\frac{f(\boldsymbol{\Gamma}'^{i};\boldsymbol{z};h)}{u-\gamma_{p,i}}\,\prod_{\substack{j=1\\ j\neq i}}^{\lambda_p}\,\frac{1}{\gamma_{p,j}-\gamma_{p,i}}\,\prod_{k=1}^{\lambda_{p+1}}(\gamma_{p,i}-\gamma_{p+1,k}-h)\Biggr]\,, \tag{6.8}$$

$$\rho\bigl(F_p(u)\bigr)\,:\,H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}+\alpha_p})\,\mapsto\,H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\,,$$

$$\rho\big(F_p(u)\big) : [f] \;\mapsto\; \Big[\, \sum_{i=1}^{\lambda_{p+1}} \frac{f(\mathbf{\Gamma}^{i\prime};\boldsymbol{z};h)}{u-\gamma_{p+1,i}} \prod_{\substack{j=1\\ j\neq i}}^{\lambda_{p+1}} \frac{1}{\gamma_{p+1,i}-\gamma_{p+1,j}} \prod_{k=1}^{\lambda_p} (\gamma_{p,k}-\gamma_{p+1,i}-h) \Big]\,, \tag{6.9}$$

where

$$\mathbf{\Gamma}^{\prime i} = (\Gamma_1;\dots;\Gamma_{p-1};\Gamma_p-\{\gamma_{p,i}\};\Gamma_{p+1}\cup\{\gamma_{p,i}\};\Gamma_{p+2};\dots;\Gamma_N)\,,$$

$$\mathbf{\Gamma}^{i\prime} = (\Gamma_1;\dots;\Gamma_{p-1};\Gamma_p\cup\{\gamma_{p+1,i}\};\Gamma_{p+1}-\{\gamma_{p+1,i}\};\Gamma_{p+2};\dots;\Gamma_N)\,.$$

Theorem 6.2 (Theorem 5.10 in [GRTV]). *These formulas define a $\widetilde{Y}(\mathfrak{gl}_N)$-module structure on $H^*_T(\mathcal{X}_n)$.*

The topological interpretation of this action see in [GRTV, Theorem 5.16].

The $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ on $H^*_T(\mathcal{X}_n)$ is a Yangian version of representations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}_N})$ considered in [Vas1, Vas2].

6.5. **Stable envelopes and Yangian actions.** As we know, formula (6.5) defines the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ϕ on $H^*_T((\mathcal{X}_n)^A) = (\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}(\boldsymbol{z};h)$, and formulas (6.6), (6.8), (6.9) define the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ on $H^*_T(\mathcal{X}_n)$.

Theorem 6.3. *For the identity element* $\mathrm{id}\in S_n$, *the map* $\mathrm{Stab}_{\mathrm{id}} : H^*_T((\mathcal{X}_n)^A)\to H^*_T(\mathcal{X}_n)$ *is a homomorphism of $\widetilde{Y}(\mathfrak{gl}_N)$-modules.*

Corollary 6.4. *The $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ on $H^*_T(\mathcal{X}_n)$ coincides with the Yangian module structure on $H^*_T(\mathcal{X}_n)$ introduced in* [MO].

Proof of Corollary 6.4. As we know, the Yangian module structure on $H^*_T((\mathcal{X}_n)^A)$, introduced in [MO], coincides with the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ϕ. In [MO], the Yangian module structure on $H^*_T(\mathcal{X}_n)$ is induced from the Yangian module structure on $H^*_T((\mathcal{X}_n)^A)$ by the map $\mathrm{Stab}_{\mathrm{id}} : H^*_T((\mathcal{X}_n)^A)\to H^*_T(\mathcal{X}_n)$, see [MO, Sections 4 and 5]. Now Theorem 6.3 implies the corollary. □

6.6. **Proof of Theorem 6.3.** Define operators $\tilde{s}_1,\dots,\tilde{s}_{n-1}$ acting on $(\mathbb{C}^N)^{\otimes n}$-valued functions of $\boldsymbol{z},h$ by

$$\tilde{s}_i\, f(z_1,\dots,z_n,h) = \frac{(z_i-z_{i+1})\,P^{(i,i+1)}+h}{z_i-z_{i+1}+h}\, f(z_1,\dots,z_{i+1},z_i,\dots,z_n,h)\,.$$

Lemma 6.5 (Lemma 2.3 in [GRTV]). *The assignment $s_i\mapsto\tilde{s}_i$, $i=1,\dots,n-1$, defines an action of S_n.* □

For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, introduce $\xi_I\in(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}(\boldsymbol{z};h)$ by the formula

$$\xi_I = \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_{\sigma_0,J}(\boldsymbol{z}_I;\boldsymbol{z};h)}{Q(\boldsymbol{z}_I)\,c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)} v_J, \tag{6.10}$$

where $\sigma_0\in S_n$ is the longest permutation and v_I is defined in (2.1). Notice that $\frac{W_{\sigma_0,J}(\boldsymbol{z}_I;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)}$ is a polynomial for every J, by Lemma 3.2. Let

$$D = \prod_{1\leqslant i<j\leqslant n} (z_j-z_i-h)\,.$$

Define $I^{\min} \in \mathcal{I}_{\boldsymbol{\lambda}}$ by $I^{\min} = \big(\{1,\dots,\lambda_1\},\{\lambda_1+1,\dots,\lambda_1+\lambda_2\},\ \dots\ ,\{n-\lambda_N+1,\dots,n\}\big)$.

Lemma 6.6 (C.f. Proposition 2.14 in [GRTV]). *The elements $\xi_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}$, are unique elements of $(\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[\boldsymbol{z};h;D^{-1}]$ such that $\xi_{I^{\min}} = v_{I^{\min}}$ and $\xi_{s_i(I)} = \tilde{s}_i \xi_I$ for every $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ and $i = 1,\dots,n-1$.*

Proof. The fact that $\xi_{I^{\min}} = v_{I^{\min}}$ follows from Lemma 3.2. The property $\xi_{s_i(I)} = \tilde{s}_i \xi_I$ follows from Lemma 3.3. ☐

By comparing Lemma 6.6 and [GRTV, Proposition 2.14] we conclude that the elements $\xi_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}$, coincide with the elements $\xi_I^+, I \in \mathcal{I}_{\boldsymbol{\lambda}}$, of Proposition 2.14 in which h is replaced with $-h$.

Consider the map

$$\nu = \oplus_{|\boldsymbol{\lambda}|=n} \nu_{\boldsymbol{\lambda}} \ :\ \oplus_{|\boldsymbol{\lambda}|=n} H_T^*(\mathcal{X}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(\boldsymbol{z};h) \ \to\ (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(\boldsymbol{z};h),$$

where $\nu_{\boldsymbol{\lambda}}$ is defined by the formula

$$[f(\boldsymbol{\Theta};\boldsymbol{z};h)] \mapsto \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{f(\boldsymbol{z}_I;\boldsymbol{z};h)}{R(\boldsymbol{z}_I)} \xi_I\,,$$

see [GRTV, Formula (5.9)].

Lemma 6.7. *We have $\nu \circ \mathrm{Stab}_{\mathrm{id}} = \mathrm{Id}$.*

Proof. We have

$$\nu \circ \mathrm{Stab}_{\mathrm{id}}(1_I) = \nu\Big(\frac{[W_{\mathrm{id},I}(\boldsymbol{\Theta};\boldsymbol{z};h)]}{c_{\boldsymbol{\lambda}}(\boldsymbol{\Theta})}\Big) = \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_{\mathrm{id},I}(\boldsymbol{z}_J;\boldsymbol{z};h)}{R(\boldsymbol{z}_J)c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)}\,\xi_J =$$

$$= \sum_{J,K\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_{\mathrm{id},I}(\boldsymbol{z}_J;\boldsymbol{z};h)\,W_{\sigma_0,K}(\boldsymbol{z}_J;\boldsymbol{z};h)}{R(\boldsymbol{z}_J)Q(\boldsymbol{z}_J)c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)^2}\,v_K = v_I = 1_I\,,$$

where the next to the last equality follows from Lemma 3.4. ☐

Theorem 5.10 in [GRTV] says that ν is a homomorphism of the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ on $H_T^*(\mathcal{X}_n) \otimes \mathbb{C}(\boldsymbol{z};h)$ to the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ϕ on $(\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(\boldsymbol{z};h)$. This proves Theorem 6.3.

7. Dynamical Hamiltonians and quantum multiplication

7.1. **Dynamical Hamiltonians.** Assume that $q_1,\dots,q_N$ are distinct numbers. Define the elements $X_1^q,\dots,X_N^q \in \widetilde{Y}(\mathfrak{gl}_N)$ by the rule

$$X_i^q \ =\ -h\,T_{i,i}^{\{2\}} + \frac{h}{2}\,T_{i,i}^{\{1\}}\big(T_{i,i}^{\{1\}}-1\big) - h\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{q_j}{q_i-q_j}\,G_{i,j}\,,$$

where $G_{i,j} = T_{i,j}^{\{1\}}T_{j,i}^{\{1\}} - T_{j,j}^{\{1\}} = T_{j,i}^{\{1\}}T_{i,j}^{\{1\}} - T_{i,i}^{\{1\}}$. Notice that

$$T_{i,i}^{\{1\}} = e_{i,i}\,, \qquad G_{i,j} \,=\, e_{i,j}\,e_{j,i} - e_{i,i} \,=\, e_{j,i}\,e_{i,j} - e_{j,j}\,.$$

By taking the limit $q_{i+1}/q_i \to 0$ for all $i = 1, \dots, N-1$, we define the elements $X_1^\infty, \dots, X_N^\infty \in \widetilde{Y}(\mathfrak{gl}_N)$,

$$X_i^\infty = -h T_{i,i}^{\{2\}} + \frac{h}{2} e_{i,i} \bigl(e_{i,i} - 1\bigr) + h \left(G_{i,1} + \ldots + G_{i,i-1}\right),$$

see [GRTV]. We call the elements X_i^q, X_i^∞, $i = 1, \dots, N$, the *dynamical Hamiltonians*. Observe that

$$X_i^q = X_i^\infty - h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{i,j} - h \sum_{j=i+1}^{n} \frac{q_j}{q_i - q_j} G_{i,j} .$$

Given $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, set $G_{\boldsymbol{\lambda},i,j} = e_{j,i} e_{i,j}$ for $\lambda_i \geqslant \lambda_j$ and $G_{\boldsymbol{\lambda},i,j} = e_{i,j} e_{j,i}$ for $\lambda_i < \lambda_j$. Define the elements $X^q_{\boldsymbol{\lambda},1}, \dots, X^q_{\boldsymbol{\lambda},N} \in \widetilde{Y}(\mathfrak{gl}_N)$,

$$X^q_{\boldsymbol{\lambda},i} = X_i^\infty - h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{\boldsymbol{\lambda},i,j} - h \sum_{j=i+1}^{n} \frac{q_j}{q_i - q_j} G_{\boldsymbol{\lambda},i,j} .$$

Let $\kappa \in \mathbb{C}^\times$. The formal differential operators

$$\nabla_{\boldsymbol{q},\kappa,i} = \kappa\, q_i \frac{\partial}{\partial q_i} - X_i^q, \qquad i = 1, \dots, N, \tag{7.1}$$

pairwise commute and, hence, define a flat connection for any $\widetilde{Y}(\mathfrak{gl}_N)$-module, see [GRTV].

Lemma 7.1 (Lemma 3.5 in [GRTV])**.** *The connection* $\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa}$ *defined by*

$$\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa,i} = \kappa\, q_i \frac{\partial}{\partial q_i} - X^q_{\boldsymbol{\lambda},i} ,$$

$i = 1, \dots, N$, *is flat for any* κ.

Proof. The connection $\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa,i}$ is gauge equivalent to connection (7.1),

$$\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa,i} = (\Upsilon_{\boldsymbol{\lambda}})^{-1}\, \nabla_{\boldsymbol{q},\kappa,i}\, \Upsilon_{\boldsymbol{\lambda}} , \qquad \Upsilon_{\boldsymbol{\lambda}} = \prod_{1 \leqslant i < j \leqslant n} (1 - q_j/q_i)^{h\, \varepsilon_{\boldsymbol{\lambda},i,j}/\kappa} ,$$

where $\varepsilon_{\boldsymbol{\lambda},i,j} = e_{j,j}$ for $\lambda_i \geqslant \lambda_j$, and $\varepsilon_{\boldsymbol{\lambda},i,j} = e_{i,i}$ for $\lambda_i < \lambda_j$. □

Connection (7.1) was introduced in [TV2], see also Appendix B in [MTV1], and is called the *trigonometric dynamical connection*. Later the definition was extended from $\mathfrak{sl}_N$ to other simple Lie algebras in [TL] under the name of the trigonometric Casimir connection.

The trigonometric dynamical connection is defined over $\mathbb{C}^N$ with coordinates $q_1, \dots, q_N$, it has singularities at the union of the diagonals $q_i = q_j$. In the case of a tensor product of evaluation $\widetilde{Y}(\mathfrak{gl}_N)$-modules, the trigonometric dynamical connection commutes with the associated *qKZ* difference connection, see [TV2]. Under the $(\mathfrak{gl}_N, \mathfrak{gl}_n)$ duality, the trigonometric dynamical connection and the associated *qKZ* difference connection are respectively identified with the trigonometric *KZ* connection and the dynamical difference connection, see [TV2].

7.2. ***qKZ* difference connection.** Recall the $\widetilde{Y}(\mathfrak{gl}_N)$-action ϕ on $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z};h]$ introduced in Section 6.3. Let

$$R^{(i,j)}(u) \,=\, \frac{u-hP^{(i,j)}}{u-h}\,, \qquad i,j=1,\dots,n\,,\quad i\neq j\,.$$

For $\kappa\in\mathbb{C}^\times$, define operators $K_1,\dots,K_n\in\operatorname{End}((\mathbb{C}^N)^{\otimes n})\otimes\mathbb{C}[\boldsymbol{z};h]$,

$$\begin{aligned}K_i(\boldsymbol{q};\kappa) \,=\, &R^{(i,i-1)}(z_i-z_{i-1})\,\dots\,R^{(i,1)}(z_i-z_1)\,\times\\ &\times\, q_1^{e^{(i)}_{1,1}}\dots\,q_N^{e^{(i)}_{N,N}}\,R^{(i,n)}(z_i-z_n-\kappa)\,\dots\,R^{(i,i+1)}(z_i-z_{i+1}-\kappa)\,.\end{aligned}$$

Consider the difference operators $\widehat{K}_{\boldsymbol{q},\kappa,1},\dots,\widehat{K}_{\boldsymbol{q},\kappa,n}$ acting on $(\mathbb{C}^N)^{\otimes n}$-valued functions of $\boldsymbol{z},\boldsymbol{q},h$,

$$\widehat{K}_{\boldsymbol{q},\kappa,i}\,f(z_1,\dots,z_n,h) \,=\, K_i(\boldsymbol{q};\kappa)\,f(z_1,\dots,z_{i-1},z_i-\kappa,z_{i+1},\dots,z_n)\,.$$

Theorem 7.2 ([FR])**.** *The operators* $\widehat{K}_{\boldsymbol{q},\kappa,1},\dots,\widehat{K}_{\boldsymbol{q},\kappa,n}$ *pairwise commute.*

Theorem 7.3 ([TV2])**.** *The operators* $\widehat{K}_{\boldsymbol{q},\kappa,1},\dots,\widehat{K}_{\boldsymbol{q},\kappa,n}$*,* $\phi(\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa,i}),\dots,\phi(\nabla_{\boldsymbol{\lambda},\boldsymbol{q},\kappa,i})$ *pairwise commute.*

The commuting difference operators $\widehat{K}_{\boldsymbol{q},\kappa,1},\dots,\widehat{K}_{\boldsymbol{q},\kappa,n}$ define the *rational qKZ difference connection.* Theorem 7.3 says that the rational *qKZ* difference connection commutes with the trigonometric dynamical connection.

7.3. **Dynamical Hamiltonians $X^q_{\boldsymbol{\lambda},i}$ on $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$.** Recall the $\widetilde{Y}(\mathfrak{gl}_N)$-module structure ρ defined on $H^*_T(\mathcal{X}_n)=\oplus_{|\boldsymbol{\lambda}|=n}H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ in Section 6.4. For any $\boldsymbol{\mu}=(\mu_1,\dots,\mu_N)\in\mathbb{Z}^N_{\geqslant0}$, $|\boldsymbol{\mu}|=n$, the action of the dynamical Hamiltonians $X^q_{\boldsymbol{\mu},i}$ preserve each of $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$.

Lemma 7.4. *For any* $\boldsymbol{\lambda}$ *and* $i=1,\dots,n$*, the restriction of* $X^q_{\boldsymbol{\lambda},i}$ *to* $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ *has the form:*

(7.2)

$$\begin{aligned}\rho(X^q_{\boldsymbol{\lambda},i}) \;&=\; (\gamma_{i,1}+\dots+\gamma_{i,\lambda_i})-h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,\rho(G_{\boldsymbol{\lambda},i,j})\,-\,h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,\rho(G_{\boldsymbol{\lambda},i,j})=\\ &=\; (\gamma_{i,1}+\dots+\gamma_{i,\lambda_i})-h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,\rho(e_{j,i}e_{i,j})\,-\,h\sum_{j=i+1}^{n}\frac{q_j}{q_i-q_j}\,\rho(e_{i,j}e_{j,i})\,+C,\end{aligned}$$

where $(\gamma_{i,1}+\dots+\gamma_{i,\lambda_i})$ *denotes the operator of multiplication by the cohomology class* $\gamma_{i,1}+\dots+\gamma_{i,\lambda_i}$*, the operator* C *is a scalar operator on* $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$*, and for any* $i\neq j$ *the element* $\rho(G_{\boldsymbol{\lambda},i,j})$ *annihilates the identity element* $1_{\boldsymbol{\lambda}}\in H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$*.*

Proof. The first equality in (7.2) follows from (6.7). The operator C is scalar since $e_{j,i}e_{i,j}-G_{\boldsymbol{\lambda},i,j}$ and $e_{i,j}e_{j,i}-G_{\boldsymbol{\lambda},i,j}$ lie in the Cartan subalgebra and act on $H^*_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ as scalars. By Theorem 6.3, in order to show that $\rho(G_{\boldsymbol{\lambda},i,j})$ annihilates the identity element $1_{\boldsymbol{\lambda}}$ it is enough to show that $\phi(G_{\boldsymbol{\lambda},i,j})$ annihilates the element $\nu(1_{\boldsymbol{\lambda}})=\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{1}{R(\boldsymbol{z}_I)}\xi_I$ and that is the statement of [GRTV, Lemma 2.20], see also [RTVZ]. □

7.4. **Quantum multiplication by divisors on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$.** In [MO], the quantum multiplication by divisors on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ is described. The fundamental equivariant cohomology classes of divisors on $T^*\mathcal{F}_{\boldsymbol\lambda}$ are linear combinations of $D_i = \gamma_{i,1}+\dots+\gamma_{i,\lambda_i}$, $i=1,\dots,N$. The quantum multiplication $D_i *_{\tilde{\boldsymbol q}}$ depends on parameters $\tilde{\boldsymbol q} = (\tilde q_1,\dots,\tilde q_N)$.

Theorem 7.5 (Theorem 10.2.1 in [MO])**.** *For $i=1,\dots,N$, the quantum multiplication by D_i is given by the formula:*

(7.3)

$$D_i *_{\tilde{\boldsymbol q}} \;=\; (\gamma_{i,1}+\dots+\gamma_{i,\lambda_i}) + h\sum_{j=1}^{i-1}\frac{\tilde q_j/\tilde q_i}{1-\tilde q_j/\tilde q_i}\,\rho(e_{j,i}e_{i,j}) - h\sum_{j=i+1}^{n}\frac{\tilde q_i/\tilde q_j}{1-\tilde q_i/\tilde q_J}\,\rho(e_{i,j}e_{j,i}) \,+C,$$

where C is a scalar operator on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ fixed by the requirement that the purely quantum part of $D_i *_{\tilde{\boldsymbol q}}$ annihilates the identity $1_{\boldsymbol\lambda}$.

Corollary 7.6. *For $i=1,\dots,N$, the operator $D_i *_{\tilde{\boldsymbol q}}$ of quantum multiplication by D_i on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ equals the action $\rho(X^q_{\boldsymbol\lambda,i})$ on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ of the dynamical Hamiltonian $X^q_{\boldsymbol\lambda,i}$ if we put $(q_1,\dots,q_N)=(\tilde q_1^{-1},\dots,\tilde q_N^{-1})$.*

The quantum connection $\nabla_{\mathrm{quant},\boldsymbol\lambda,\tilde{\boldsymbol q},\kappa}$ on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ is defined by the formula

$$\nabla_{\mathrm{quant},\boldsymbol\lambda,\tilde{\boldsymbol q},\kappa,i} \;=\; \kappa\,\tilde q_i\frac{\partial}{\partial \tilde q_i} \,-\, D_i *_{\tilde{\boldsymbol q}}\,, \qquad i=1,\dots,N, \tag{7.4}$$

where $\kappa\in\mathbb{C}^\times$ is a parameter of the connection, see [BMO]. By Corollary 7.6, we have

$$\nabla_{\mathrm{quant},\boldsymbol\lambda,\tilde{\boldsymbol q},\kappa,i} = \rho(\nabla_{\boldsymbol\lambda,\tilde q_1^{-1},\dots,\tilde q_N^{-1},-\kappa}), \qquad i=1,\dots,N. \tag{7.5}$$

By Theorem 7.3, the difference operators

$$\mathrm{Stab}_{\mathrm{id}}\circ \widehat K_{\tilde q_1^{-1},\dots,\tilde q_N^{-1},-\kappa,1}\circ\nu,\quad \dots,\quad \mathrm{Stab}_{\mathrm{id}}\circ \widehat K_{\tilde q_1^{-1},\dots,\tilde q_N^{-1},-\kappa,n}\circ\nu$$

and the differential operators $\nabla_{\mathrm{quant},\boldsymbol\lambda,\tilde{\boldsymbol q},\kappa,1},\dots,\nabla_{\mathrm{quant},\boldsymbol\lambda,\tilde{\boldsymbol q},\kappa,N}$ pairwise commute. The difference operators form the rational qKZ difference connection on $H^*_T(T^*\mathcal{F}_{\boldsymbol\lambda})$. This difference connection is discussed in [MO] under the name of the shift operators.

8. Proofs of lemmas on weight functions

8.1. **Proof of Lemma 3.1.** Parts (ii-iv) of Lemma 3.1 are proved by inspection of the definition of weight functions.

For $N=2$, part (i) of Lemma 3.1 is proved in [RTV], see Lemma 3.6 and Theorem 4.2 in [RTV]. The general case $N>2$ is proved as follows.

Let $U_{I,\boldsymbol\sigma}(\boldsymbol t;\boldsymbol z;h)$ be the term in the symmetrization in (3.1) obtained by permuting the variables $t^{(i)}_a$ by an element $\boldsymbol\sigma\in S_{\lambda^{(1)}}\times\dots\times S_{\lambda^{(N-1)}}$. Set

$$U_{I,J,\boldsymbol\sigma}(\boldsymbol z;h) = U_{I,\boldsymbol\sigma}(\boldsymbol z_J;\boldsymbol z;h).$$

We show that each term $U_{I,J,\boldsymbol\sigma}(\boldsymbol z;h)$ is divisible by $c_{\boldsymbol\lambda}(\boldsymbol z_J)$.

Recall that $\cup_{b=1}^a I_a = I^{(a)} = \{i^{(a)}_1<\dots<i^{(a)}_{\lambda^{(a)}}\}$. Similarly, let $\cup_{b=1}^a J_a = J^{(a)} = \{j^{(a)}_1 < \dots < j^{(a)}_{\lambda^{(a)}}\}$.

The substitution $\boldsymbol{t} = \boldsymbol{z}_J$ implies $t_a^{(N-1)} = z_{j_a^{(N-1)}}$. Denote by σ the component of $\boldsymbol{\sigma}$ in the last factor $S_{\lambda^{(N-1)}}$. Consider the factor $f_{c,d} = z_{j_c^{(N-1)}} - z_{j_d^{(N-1)}} - h$ in $c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)$ for $c \neq d$.

Let $a = \sigma^{-1}(c)$. If $j_d^{(N-1)} < i_a^{(N-1)}$, then $f_{c,d}$ divides $U_{I,J,\boldsymbol{\sigma}}$ due to the factor $t_{\sigma(a)}^{(N-1)} - z_{j_d^{(N-1)}} - h$ in $U_{I,\boldsymbol{\sigma}}$.

Let $b = \sigma^{-1}(d)$. If $i_a^{(N-1)} \leqslant j_{\sigma(b)}^{(N-1)} = j_d^{(N-1)} \leqslant i_b^{(N-1)}$, then $a < b$, because $i_a^{(N-1)} \leqslant i_b^{(N-1)}$ and $a \neq b$. Then $f_{c,d}$ divides $U_{I,J,\boldsymbol{\sigma}}$ due to the factor $t_{\sigma(a)}^{(N-1)} - t_{\sigma(b)}^{(N-1)} - h$ in $U_{I,\boldsymbol{\sigma}}$.

If $i_b^{(N-1)} < j_d^{(N-1)}$, then $U_{I,J,\boldsymbol{\sigma}} = 0$ due to the factor $t_{\sigma(a)}^{(N-1)} - z_{j_d^{(N-1)}}$ in $U_{I,\boldsymbol{\sigma}}$.

Once the substitution $t_a^{(N-1)} = z_{j_a^{(N-1)}}$ is done, the consideration of the factors $z_{j_c^{(N-2)}} - z_{j_d^{(N-2)}} - h$ in $c_{\boldsymbol{\lambda}}(\boldsymbol{z}_J)$ is similar.

8.2. **Proof of Lemma 3.3.** It is enough to prove that for $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, $i = 1, \dots, n-1$, we have

$$W_{s_i,I}(\boldsymbol{t};\boldsymbol{z};h) = \frac{z_i - z_{i+1}}{z_i - z_{i+1} + h} W_{\mathrm{id},I}(\boldsymbol{t};\boldsymbol{z};h) + \frac{h}{z_i - z_{i+1} + h} W_{\mathrm{id},s_i(I)}(\boldsymbol{t};\boldsymbol{z};h). \tag{8.1}$$

Moreover, it is straightforward to see from (3.1) that it suffices to prove relations (8.1) for $n = 2$, $i = 1$, and the following two cases for I. The first case is $I = (I_1, \dots, I_N)$, where $I_1 = \{1, 2\}$, and $I_2, \dots, I_N$ are empty. The second case is $I = (I_1, \dots, I_N)$, where $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3, \dots, I_N$ are empty. In each of the two cases, formula (8.1) is proved by straightforward verification. All other cases of formula (8.1) can be deduced from these two by picking up a suitable subexpression and an appropriate change of notation.

8.3. **Proof of Lemma 3.4.** In addition to vectors $\xi_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}$, defined in (6.10), we introduce the vectors

$$\xi_{\sigma_0,I} = \sum_{J \in \mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_{\mathrm{id},J}(\boldsymbol{z}_I;\boldsymbol{z};h)}{Q(\boldsymbol{z}_I)\, c_{\boldsymbol{\lambda}}(\boldsymbol{z}_I)}\, v_J. \tag{8.2}$$

Let $\mathcal{S}$ be the $\mathbb{C}(\boldsymbol{z};h)$-bilinear form on $(\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(\boldsymbol{z};h)$ such that the basis (v_J) is orthonormal. Then the statement of the lemma is equivalent to the statement

$$\mathcal{S}(\xi_I, \xi_{\sigma_0,J}) = \delta_{I,J}\, \frac{R(\boldsymbol{z}_I)}{Q(\boldsymbol{z}_I)},$$

which is the statement of [GRTV, Theorem 2.18].

References

[BB] Anders Bjorner, Francesco Brent, *Combinatorics of Coxeter Groups*, Graduate Texts in Math., vol. 231, Springer, New York, 2005

[BMO] A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Preprint (2010), 1–35, arXiv:1001.0056

[FR] I. Frenkel, N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60

[FP] W. Fulton, P. Pragacz, *Schubert Varieties and Degeneracy Loci*, Springer LNM **1689**, 1998

[GRTV] V. Gorbounov, R. Rimányi, V. Tarasov, A. Varchenko, *Cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra*, `arXiv:1204.5138`, 1–44

[KS] P. P. Kulish, E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes in Phys., **151** (1982), 61–119

[M] A. Molev, *Yangians and classical Lie algebras*, Math. Surveys and Monographs, vol. 143, AMS, 2007

[MO] D. Maulik, A. Okounkov, *Quantum Groups and Quantum Cohomology*, Preprint (2012), 1–276, `arXiv:1211.1287`

[MTV1] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. (2006), no. 8, P08002, 1–44

[MV] Y. Markov and A. Varchenko, *Hypergeometric solutions of trigonometric KZ equations satisfy dynamical difference equations*, Adv. Math. **166** (2002), no. 1, 100–147

[N1] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac- Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416

[N2] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560

[NT] M. Nazarov, V. Tarasov, *Representations of Yangians with Gelfand-Zetlin bases*, J. Reine Angew. Math. **496** (1998) 181–212

[RTV] R. Rimányi, V. Tarasov, A. Varchenko: *Cohomology classes of conormal bundles of Schubert varieties and Yangian weight functions*, Preprint (2012), 1–21, `arXiv:1204.4961`

[RTVZ] R. Rimányi, V. Tarasov, A. Varchenko, P. Zinn-Justin, *Extended Joseph polynomials, quantized conformal blocks, and a q-Selberg type integral*, 1–28; to appear in J. of Geometry and Physics `arXiv:1110.2187`

[SV] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139–194

[TL] V. Toledano Laredo, *The trigonometric Casimir connection of a simple Lie algebra*, J. Algebra **329** (2011), 286–327

[TV1] V. Tarasov and A. Varchenko, *Jackson Integral Representations for Solutions to the Quantized Knizhnik-Zamolodchikov Equation*, Preprint (1993), 1–50, `arXiv:hep-th/9311040`

[TV2] V. Tarasov and A. Varchenko, *Duality for Knizhnik-Zamolodchikov and dynamical equations*, Acta Appl. Math. **73** (2002), no. 1-2, 141–154

[TV3] V. Tarasov and A. Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, Invent. Math. **128** (1997), no. 3, 501–588

[TV4] V. Tarasov and A. Varchenko, *Combinatorial formulae for nested Bethe vectors*, Preprint (2007), 1–31, `arXiv:math/0702277`

[V] A. Varchenko, *Quantized Knizhnik-Zamolodchikov equations, quantum Yang-Baxter equation, and difference equations for q-hypergeometric functions*, Comm. Math. Phys. **162** (1994), no. 3, 499–528

[Vas1] E. Vasserot, *Représentations de groupes quantiques et permutations*, Ann. Sci. ENS **26** (1993), 747–773

[Vas2] E. Vasserot, *Affine quantum groups and equivariant K-theory*, Transformation groups **3** (1998), 269–299