DOCTORAL DISSERTATION

Generalized Pontrjagin-Thom Construction for Singular Maps

Richárd Rimányi

Eötvös Loránd University
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Prof. András Szűcs

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Introduction

Pontrjagin and later Thom considered cobordism classes of those n dimensional submanifolds in a fixed $n+k$-manifold P, whose normal bundles have structure group $G \leq O(k)$, a fixed Lie group. The Pontrjagin-Thom construction provides a bijection — in certain cases a group isomorphism — between such cobordism classes and the homotopy classes $[P, MG]$, where MG is the Thom space of the universal vector bundle with structure group G. In fact, Pontrjagin considered the case $G = \{1\}$, and Thom considered the general case.

Another way to look at these results is the following. There is a subspace BG in MG, and an embedding $BG \longrightarrow MG$ — although BG and MG are usually not manifolds but the map $BG \longrightarrow MG$ "looks like" an embedding between manifolds. This map is a *universal embedding* in the sense that all k codimensional embeddings (with normal bundle a G-bundle) can be "pulled back" from this one in a unique way (up to cobordism). Keeping in mind this "universality" of the map $BG \longrightarrow MG$ we shall call the Thom space MG the *classifying space for embeddings* (with a reduction of the structure group of the normal bundle to G).

The subject of this paper is to construct analogous universal maps and classifying spaces for singular maps. Let us fix a set $\mathcal{T}$ of k-codimensional stable map germs $(\mathbb{R}^n, \{x_1, \ldots, x_r\}) \longrightarrow (\mathbb{R}^{n+k}, 0)$ where k is a positive integer. Suppose also that if $\eta \in \mathcal{T}$ then its suspension $Susp(\eta) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k+1}$, $(x,t) \mapsto (\eta(x), t)$ is also in $\mathcal{T}$. A smooth map $f : N \longrightarrow P$ is called a $\mathcal{T}$-map if for every $y \in P$ the germ of f at $f^{-1}(y)$ is (equivalent to) one from $\mathcal{T}$. A cobordism between $\mathcal{T}$-maps $f_i : N_i \longrightarrow P$ is an abstract cobordism W between N_1 and N_2 as well as a $\mathcal{T}$-map $F : W \longrightarrow P \times [0,1]$ whose restrictions to N_1 and N_2 are f_1 and f_2. One can easily define $\mathcal{T}$ so that $\mathcal{T}$-maps are *embeddings*, *immersions*, Σ^1*-maps*, etc.

The goal of this paper is to construct a space — a CW complex — $X\mathcal{T}$ which is a *classifying space for* $\mathcal{T}$*-maps*. That is, we will have a bijection between cobordism classes of $\mathcal{T}$-maps to P and the homotopy classes $[P, X\mathcal{T}]$. We will also present a *universal* $\mathcal{T}$*-map* $Y\mathcal{T} \longrightarrow X\mathcal{T}$ (an analogue of the universal embedding $BG \longrightarrow MG$).

The construction of $X\mathcal{T}$ goes by induction. Suppose we have already constructed $X\mathcal{T}$ and we want to add a singularity type η to $\mathcal{T}$. Consider an arbitrary $\mathcal{T} \cup \{\eta\}$-map $N \longrightarrow P$. Cut out a tubular neighbourhood T of the submanifold $K_\eta \subset P$ consisting of those y's for which $f : (N, f^{-1}(y)) \longrightarrow (P, y)$ is η. Roughly speaking the rest is classified by $X\mathcal{T}$. We have to figure out what to glue to $X\mathcal{T}$ in order to classify $\mathcal{T} \cup \{\eta\}$-maps. The tubular neighbourhood T, as a bundle is classified by the relevant universal bundle. Gluing the total space of this universal bundle to $X\mathcal{T}$ in an appropriate way we obtain the space $X(\mathcal{T} \cup \{\eta\})$ which classifies $\mathcal{T} \cup \{\eta\}$-maps.

So what we have to know is the smallest group, to which the vector bundle $T \longrightarrow K_\eta$ can be reduced. The structure group of this bundle is *a priori* the automorphism group of a singularity type η' (the "root of η"[1]) — thus an infinite dimensional group. If this automorphism group were a finite dimensional Lie

[1] the root η' of η has the smallest source (and target) dimension in η's stable equivalence class

group, then we could reduce the structure group to its maximal compact subgroup. The main property of the maximal compact subgroups that would make this reduction possible is that the quotient space modulo a maximal compact subgroup is contractible.

The direct imitation of the finite dimensional case for the automorphism group of η' does not work (since there is no convenient topology on the automorphism group of a germ). Instead, without defining topology on the automorphism group of η',we will define its "maximal compact subgroups" (following Jänich and Wall). They will share some properties with the maximal compact subgroups of Lie groups. The best news is that it will have a (difficultly-phrased) property which will allow us to reduce the $T \longrightarrow K_\eta$ bundle to this "maximal compact subgroup". We will call this property "contractibility". The tool we will use here is the so called *homotopy method* of Thom — that is, with the machinery of ordinary differential equations we will solve a "continuous version" of the problem. Similar task is solved by Jänich ([J]) for functions instead of maps. The difference between functions and maps is not just technical here. The main reason for this is that when we are dealing with functions then infinitesimal properties are usually expressed by (in)equalities between modules over the same ring. In the case of maps this is not so, therefore Nakayama type arguments can not be used here directly. Deeper algebraic and analytical results are needed here like the Malgrange preparation theorem and some algebraic lemmas of Mather.

Of course, there remains the problem of computing the "maximal compact subgroup" of the automorphism group of a singularity. In section 1.5 we will reduce this "infinite dimensional" problem to a finite dimensional one, which will be solved in certain cases, in fact in all cases up to $III_{2,2}$-singularities. (For the hierarchy of $\Sigma^{2,0}$ germs see the Appendix.)

With all these at hand we can construct the classifying spaces $X\mathcal{T}$ of $\mathcal{T}$-maps (chapter 2). As in the case of embeddings, this surely will have many consequences to differential topology.

For example one can make explicit algebraic topological computations to gain information on

- orientability of K_η
- cobordism groups of singular maps;
- elimination of singularities by cobordisms;
- cobordism groups of non-singular maps like embeddings and immersions (in the non-stable range) by projecting these non-singular maps to some subspaces and studying the cobordism groups of the obtained singular maps (see [Sz4], [Sz5]).

In chapter 3 a handful of examples are presented for some of these computations.

The idea of the program just sketched (and presented in the paper), i. e. the construction of classifying spaces for singular maps and the study of homotopical, homological properties of these classifying spaces is due to A. Szűcs (who developed an idea of M. Gromov). Using geometrical methods he constructed and studied classifying spaces of immersions and Morin singularities.

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Preface by the supervisor

The contribution of the present paper can be summarized as follows: algebra fights back.

The history of Algebraic Topology is that of permanent algebraization perturbed by efforts of geometrization. For the latter the best example is Pontrjagin's approach to the computation of homotopy groups of spheres. This geometric approach has been algebraized by Thom, who used the connection established by the Pontrjagin-Thom construction from algebra to geometry.

Another effort of geometrization was the extension of the Pontrjagin-Thom construction to Morin maps using geometric considerations ([Sz1] *).*

The moral of Rimányi's work is that in this theory again geometry should be replaced by algebra. Cobordisms of complicated singular maps can be attacked with success only by studying local algebras and automorphism groups, instead of trying to understand the geometry of these singular maps.

So these algebraic considerations finally give nice geometric results.

András Szűcs

Chapter 1

Symmetry of singularities

1.1. Preliminaries

In this section we summarize the results of singularity and stability theory needed in the paper. For more detailed study and proofs the reader should look up any of the text books or papers on this topic, for example [M1]-[M6], [W1], [GWPL] or [GG].

• Stability of maps

Let N and P be manifolds of dimensions n and p respectively. The set of smooth maps from N to P will be called $C^\infty(N,P)$. As it is usual in singularity theory, we will consider the Whitney C^∞ topology on $C^\infty(N,P)$. To define it first we need to define *jet spaces*.

Let f and g be smooth maps from N to P and let x be a point in N. Suppose that $f(x)=g(x)=y$ (they have 0'th order contact at x). We say that f has first order contact with g at x if $(df)_x=(dg)_x$ as mappings of $T_xN \longrightarrow T_yP$. Now, f and g have k'th order contact at x if $df: TN \longrightarrow TP$ has $k-1$'st order contact with $dg: TN \longrightarrow TP$. The relation of "having k'th order contact at x" is clearly an equivalence relation, let the equivalence classes be called *jets at* x, and let their set be denoted by $J^k(N,P)_{x,y}$. The equivalence class of f is denoted by $j^k_x(f)$. The disjoint union $\cup_{x\in N, y\in P} J^k(N,P)_{x,y}$ is denoted by $J^k(N,P)$. Its elements are called *jets*. With the only reasonable definition $J^k(N,P)$ is a smooth manifold of dimension

$$n+p+p\sum_{i=1}^{k}\binom{n}{i}.$$

The map $J^k(N,P) \longrightarrow N\times P$ assigning to a jet its source and its target point is a bundle map with fibre a Euclidean space (but not a vector bundle). Another important map is the *jet map*: $j^kf: N \longrightarrow J^k(N,P)$: $x \mapsto j^k_x(f)$.

1.1.1. Definition. *The topology on $C^\infty(N,P)$ whose basis is*

$$\{\ \{f\in C^\infty(N,P) \mid j^kf(N)\subset U\} \mid U\subset J^k(N,P) \text{ open }\}$$

is called the Whitney C^k topology. If W_k is the set of the open sets in this C^k topology then the topology whose basis is $\cup_k W_k$ is called the Whitney C^∞ topology on $C^\infty(N,P)$.

The nicest feature of the Whitney C^∞ topology is that it makes $C^\infty(N,P)$ a Baire space. Also the map $j^k : C^\infty(N,P) \longrightarrow C^\infty(N, J^k(N,P)); f \mapsto j^k f$ is continuous. For other properties of this topological space see for example [GG].

Now we are going to define *stable* maps with respect to the Whitney topology.

1.1.2. Definition. *The maps $f, g : N \longrightarrow P$ are equivalent if there exist diffeomorphisms ψ and ϕ of N and P respectively, so that the following diagram is commutative:*

$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\phi} \\ N & \xrightarrow{f} & P. \end{array}$$

1.1.3. Definition. *The map $f : N \longrightarrow P$ is stable if there is a neighbourhood U of f in $C^\infty(N,P)$ (in the Whitney C^∞ topology) such that for all g in U the maps f and g are equivalent.*

If $f : N \longrightarrow P$ is a map then the vector space of *vector fields along* f will be called θ_f. When f is the identity map of a manifold M then we write θ_M for θ_{id_M}. Following Mather we define the maps:

$$\begin{array}{lcl} \omega f : \theta_P \longrightarrow \theta_f & \text{by} & \omega f(h) = h \circ f, \\ tf : \theta_N \longrightarrow \theta_f & \text{by} & tf(h) = df \circ h. \end{array}$$

1.1.4. Definition. *The map $f : N \longrightarrow P$ is infinitesimally stable if*

$$\theta_f = tf(\theta_N) + \omega f(\theta_P).$$

*Remark.*This definition can be motivated. Suppose that a Lie group G acts on a finite dimensional manifold M. Let us call a point in the manifold stable if it is an interior point of an orbit. For $x \in M$ let the map $l_x : G \longrightarrow M$ be defined by $l_x(g) = g \cdot x$. Since $l_x(G)$ is a submanifold of M the stability condition for x can be rephrased (by the implicit function theorem): a point x is stable if and only if $(dl_x)_e : T_eG \longrightarrow T_xM$ is surjective (where e is the neutral element of G).

Now suppose the same holds for infinite dimensional manifolds, too. Then let M be $C^\infty(N,P)$ and let G be the product of the diffeomorphism group of N and that of P. The action of $(\psi, \phi) \in G$ is $f \mapsto \phi \circ f \circ \psi^{-1}$. The stability of maps defined above and the stability under this group action is the same. Then using the (*not existing*) implicit function theorem we would get that f is stable if and only if $(dl_x)_e : T_eG \longrightarrow T_fC^\infty(N,P)$. We can identify $T_fC^\infty(N,P)$ with θ_f and T_eG with $\theta_N \times \theta_P$. Also the linear map $(dl_x)_e$ can be seen to have the form tf on θ_N and ωf on θ_P.

So if we had the generalization of the implicit function theorem for infinite dimensional manifolds then we could conclude that $f \in C^\infty(N,P)$ is stable if and only if it is infinitesimally stable. Unfortunately, the implicit function theorem fails to be true in infinite dimension. Still, Mather has proved that the mentioned corollary holds:

1.1.5. THEOREM. *The map* $f \in C^\infty(N, P)$ *is stable if and only if it is infinitesimally stable.* $\square$

• Stability of map germs

In this section we are going to introduce some basic notions for map germs, such as: stable map germs, $\mathcal{A}$- and $\mathcal{K}$-equivalence, local algebra, finite $\mathcal{A}$- and $\mathcal{K}$-determinacy and unfolding.

Let N and P be smooth manifolds and fix a point x in N. Consider smooth maps $f : U_f \longrightarrow P$, where U_f is a neighbourhood of x in N. Two such maps f and g are said to be equivalent if there exists a neighbourhood $V \subset U_f \cap U_g$ of x in N such that $f|_V = g|_V$. The equivalence classes are called map germs $(N, x) \longrightarrow P$. The equivalence class of f will sometimes be denoted by $\tilde{f}$, but usually — if it does not cause misunderstanding — f will denote a map and its germ, too. Since smooth manifolds locally look like Euclidean spaces, it is no restriction to consider only the case $N = \mathbb{R}^n$, $P = \mathbb{R}^p$, $x = 0$.

The set of germs $(\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$ is called $\mathcal{E}(n, p)$. We can define addition of germs (by pointwise addition) and we can multiply a germ by scalars (pointwise multiplication) which makes $\mathcal{E}(n, p)$ a vector space over $\mathbb{R}$. If $p = 1$, pointwise multiplication can also be defined on $\mathcal{E}(n) := \mathcal{E}(n, 1)$, which makes it an algebra. Also, $\mathcal{E}(n, p)$ is a module over $\mathcal{E}(n)$, in fact a free module of rank p so we will sometimes identify $\mathcal{E}(n, p)$ with $\mathcal{E}(n)^p$. As a ring $\mathcal{E}(n)$ is local, i. e. it has a unique maximal ideal $\mathfrak{m}(n)$ consisting of the germs that map $0 \in \mathbb{R}^n$ to $0 \in \mathbb{R}^p$. If $m \leq n$ then the standard embedding $\mathbb{R}^m \subset \mathbb{R}^n$ induces an inclusion $\mathcal{E}(m) \subset \mathcal{E}(n)$ and we will not distinguish $\mathcal{E}(m)$ from its image under this inclusion.

If we want to define the stability of map germs, we could follow three ways:

(a) defining topology on $\mathcal{E}(n, p)$ and an equivalence relation corresponding to the one defined for maps above;
(b) generalizing the notion of *infinitesimal stability* for germs;
(c) declaring germs of stable maps stable.

If these definitions are made with care they all yield the same notion of stable map germs. To follow idea (a) is however a bit technical and authors usually talk about the far less technical "unfoldings" (see below) instead. Here we follow (b).

Let $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$ be a germ, and f a representative of it. Take the germs at 0 of the vector fields along f. Their vector space is denoted by $\theta_{\tilde{f}}$. This is a correct definition, i. e. does not depend on the representative chosen. Notice that $\theta_{\tilde{f}}$ is an $\mathcal{E}(n)$-module and using the linear structure on $\mathbb{R}^p$ the space $\theta_{\tilde{f}}$ can be identified with $\mathcal{E}(n, p)$ and this identification is consistent with the $\mathcal{E}(n)$-module structures. Like in the case of maps we shorten the notation by putting $\theta_M := \theta_{\tilde{id}_M}$, and we can define the "germ versions" of tf and ωf, too.

1.1.6. DEFINITION. *The map germ* $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$ *is infinitesimally stable if*

$$\theta_{\tilde{f}} = t\tilde{f}(\theta_N) + \omega\tilde{f}(\theta_P).$$

This notion of infinitesimal stability of germs is in the "right relation" with the (infinitesimal) stability of maps (equivalent to the correct definition on the path (c)). We will not need this fact here; the reader can consult e.g. [GWPL]. From now on in this paper we take *stable* as a shorthand expression for *infinitesimally stable*.

Now we are going to define useful equivalence relations on map germs. First, let $\mathcal{A}$ be $Diff(\mathbb{R}^n) \times Diff(\mathbb{R}^p)$, where $Diff(\mathbb{R}^m)$ is the group of diffeomorphism germs at 0 of $\mathbb{R}^m$ fixing 0. The group $\mathcal{A}$ acts on $\mathcal{E}(n,p)$ (by convention) on the left:

$$(\psi, \phi) \cdot \tilde{f} := \phi \circ \tilde{f} \circ \psi^{-1} \qquad (\psi, \phi) \in \mathcal{A}, \tilde{f} \in \mathcal{E}(n,p).$$

If $\tilde{f}$ and $\tilde{g}$ are in the same orbit, they are called *$\mathcal{A}$-equivalent* or simply *equivalent* (c. f. equivalence on maps).

Remark. In some sense the space $\theta_{\tilde{f}}$ is the tangent space of $\mathcal{E}(n,p)$ at $\tilde{f}$, and $\theta_N \times \theta_P$ is the tangent space of $\mathcal{A}$ at $1 \in \mathcal{A}$. The infinitesimal stability of map germs can be interpreted as it was done for maps above.

Let $\mathcal{R}$ be $Diff(\mathbb{R}^n)$ and let its action on $\mathcal{E}(n,p)$ be defined by:

$$\psi \cdot \tilde{f} := \tilde{f} \circ \psi^{-1} \qquad \psi \in \mathcal{R}, \tilde{f} \in \mathcal{E}(n,p).$$

Observe that although we have not defined topology on $Diff(\mathbb{R}^p)$ the smoothness of a map $M : \mathbb{R}^n \longrightarrow Diff(\mathbb{R}^p)$ can be defined: it is smooth if the map $\mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n \times \mathbb{R}^p$ given by $(x,y) \mapsto (x, M(x)(y))$ is smooth for some representative. Let $\mathcal{C}$ be the group of smooth map germs $(\mathbb{R}^n, 0) \longrightarrow Diff(\mathbb{R}^p)$ — the multiplication is defined pointwise: $(MN)(x) := M(x)N(x)$. The group $\mathcal{C}$ acts on $\mathcal{E}(n,p)$ (from the left):

$$(M \cdot \tilde{f})(x) := M(x)(\tilde{f}(x)) \qquad M \in \mathcal{C}, \tilde{f} \in \mathcal{E}(n,p).$$

Now form the semidirect product of $\mathcal{R}$ and $\mathcal{C}$ corresponding to the homomorphism: $\mathcal{R} \longrightarrow Aut(\mathcal{C})$: $\psi \mapsto (M \mapsto M \circ \psi^{-1})$. This product group — the *contact group*, $\mathcal{K}$ — also acts on $\mathcal{E}(n,p)$ from the left:

$$((\psi, M) \cdot \tilde{f})(x) := M(x)(f(\psi^{-1}(x))) \qquad \psi \in \mathcal{R}, M \in \mathcal{C}, \tilde{f} \in \mathcal{E}(n,p).$$

If $\tilde{f}$ and $\tilde{g}$ are in the same orbit under the action of $\mathcal{K}$ then we call them $\mathcal{K}$-equivalent or contact equivalent.

Remark. In fact $\mathcal{K}$ is the subgroup of $Diff(\mathbb{R}^n \times \mathbb{R}^p)$ consisting of the elements fixing $\mathbb{R}^n \times 0$ and commuting with the projection to $\mathbb{R}^n$. The action of $\mathcal{K}$ on $\mathcal{E}(n,p)$ is just the action on the graph.

Remark. It is easy to see that $\mathcal{A} \leq \mathcal{K}$, and that if $\tilde{f}$ is stable then any other germ $\mathcal{A}$-equivalent to it is also stable. This is not the case with $\mathcal{K}$.

A germ $\tilde{f} \in \mathcal{E}(n,p)$ induces an algebra homomorphism $\tilde{f}^* : \mathcal{E}(p) \longrightarrow \mathcal{E}(n)$ by $\tilde{h} \mapsto \tilde{h} \circ \tilde{f}$.

1.1.7. Definition. *For a germ $\tilde{f} \in \mathcal{E}(n,p)$ let its local algebra be*

$$Q_{\tilde{f}} := \mathcal{E}(n)/f^*\mathfrak{m}(p) \cdot \mathcal{E}(n).$$

Mather proved the following powerful theorems.

1.1.8. Theorem. *The germs $\tilde{f}, \tilde{g} \in \mathcal{E}(n,p)$ are $\mathcal{K}$-equivalent if and only if their local algebras are isomorphic.*

1.1.9. Theorem. *The stable germs $\tilde{f}, \tilde{g} \in \mathcal{E}(n,p)$ are $\mathcal{A}$-equivalent if and only if their local algebras are isomorphic.*

If we take the k-jet at 0 of a map $\mathbb{R}^n \longrightarrow \mathbb{R}^p$, it clearly depends only on its germ at 0. So we can define the map $j^k := j_0^k : \mathcal{E}(n,p) \longrightarrow J^k(\mathbb{R}^n, \mathbb{R}^p)$. Now we define some other useful properties of germs. Let $\mathcal{H}$ be either $\mathcal{A}$ or $\mathcal{K}$. A germ $\tilde{f} \in \mathcal{E}(n,p)$ is called finitely $\mathcal{H}$-determined if for some k all germs having the same k-jet are $\mathcal{H}$-equivalent to it. (If the group $\mathcal{H}$ is clear from the context, we will just say that $\tilde{f}$ is finitely determined or k-determined.) Mather proved that finite $\mathcal{H}$-determinacy is equivalent to appropriate algebraic conditions.

1.1.10. Theorem. *The germ $\tilde{f} \in \mathcal{E}(n,p)$ is finitely $\mathcal{A}$-determined if and only if*

$$\mathfrak{m}(n)^l \theta_{\tilde{f}} \subset t\tilde{f}(\theta_{\mathbb{R}^n}) + \omega\tilde{f}(\theta_{\mathbb{R}^p})$$

for some positive integer l.

In fact the finite $\mathcal{A}$-determinacy of $\tilde{f}$ implies the condition

$$\mathfrak{m}(n)^{l'} \theta_{\tilde{f}} \subset tf(\mathfrak{m}(n)^2\theta_{\mathbb{R}^n}) + \omega f(\mathfrak{m}(p)^2\theta_{\mathbb{R}^p}) \tag{1}$$

for some (larger) l', see [M3].

1.1.11. Theorem. *The germ $\tilde{f} \in \mathcal{E}(n,p)$ is finitely $\mathcal{K}$-determined if and only if*

$$\mathfrak{m}(n)^l \theta_{\tilde{f}} \subset t\tilde{f}(\theta_{\mathbb{R}^n}) + \tilde{f}^*(\mathfrak{m}(p))\theta_{\tilde{f}}$$

for some positive integer l.

Remark. For a finitely $\mathcal{A}$-determined germ the local algebra is isomorphic to

$$\mathbb{R}[[x_1, \dots, x_n]]/(y_1 \circ f^*, \dots, y_p \circ f^*),$$

where y_j's are the coordinate functions of $\mathbb{R}^p$.

Now we are going to give the definition of "miniversal unfolding". (Although unfolding theory has many nice results and applications, here we will not deal with it generally, but restrict ourselves to the small part of it needed in the subsequent chapters.)

Consider a finitely $\mathcal{K}$-determined map germ $f \in \mathcal{E}(n,p)$ whose differential at 0 vanishes. Using the last theorem we see that the quotient

$$\mathcal{N}_f = \theta_f \;/\; tf(\theta_{\mathbb{R}^n}) + f^*(\mathfrak{m}(p))\theta_f$$

is a finite dimensional vector space. Choose a complement $V \cong \mathbb{R}^r$ in $\mathfrak{m}(n)\theta_f$ of $tf(\theta_{\mathbb{R}^n}) + f^*(\mathfrak{m}(p))\theta_f \subset \mathfrak{m}(n)\theta_f$. Using the linear structure of $\mathbb{R}^p$ we can identify the elements of θ_f (and in particular, the elements of V) with map germs $\mathbb{R}^n \longrightarrow \mathbb{R}^p$. So the following formula makes sense:

$$F : \mathbb{R}^n \times V \longrightarrow \mathbb{R}^p \times V$$

$$(x, \psi) \mapsto (f(x) + \psi(x), \psi).$$

We call the germ $F : (\mathbb{R}^{n+r}, 0) \longrightarrow \mathbb{R}^{p+r}$ a miniversal unfolding of f. A theorem of Mather says that the miniversal unfolding, as a map germ is stable. It is not difficult to verify that different miniversal unfoldings of f are $\mathcal{A}$-equivalent and miniversal unfoldings of $\mathcal{K}$-equivalent unfoldings are also $\mathcal{A}$-equivalent. The converse is also true: if $F, F' \in \mathcal{E}(n+r, p+r)$ are $\mathcal{A}$-equivalent germs and are miniversal unfoldings of $f, f' \in \mathcal{E}(n,p)$ with $(df)_0$ and $(df')_0$ identically 0, then f and f' are $\mathcal{K}$-equivalent.

Remark. The word "miniversal" comes from *mini*mal dimensional uni*versal* unfolding. The prefix "uni" is omitted because the universal unfolding is not unique in general. However, as stated above, the $\mathcal{A}$-equivalence class of a miniversal unfolding is unique.

This theory allows us to derive normal forms for stable singularities with a given algebra Q. Present Q as

$$\mathbb{R}[[x_1, \ldots, x_n]]/(r_1, \ldots, r_p),$$

with $r_j = r_j(x_1, \ldots, x_n) \in \mathfrak{m}(n)^2$. Then the germ $f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$ mapping $(x_1, \ldots, x_n)$ to $(r_1, \ldots, r_p)$ clearly has local algebra Q. It is not necessarily stable, though, but $(df)_0 = 0$. Its miniversal unfolding is a stable germ. In fact, we can obtain all *isolated* stable germs in this way — that is every stable germ is equivalent to one of the form

$$H : \mathbb{R}^{n+r} \times \mathbb{R}^t \longrightarrow \mathbb{R}^{p+r} \times \mathbb{R}^t$$

$$H : (x, u) \mapsto (F(x), u),$$

where t is maximal and F (the isolated germ) can be obtained from its local algebra (which is the same as the local algebra of H) the way just described.

We make two more remarks which will be useful later.

The map $\omega f : \theta_{\mathbb{R}^p} \longrightarrow \theta_f$ induces a map

$$\overline{\omega}_f : \theta_{\mathbb{R}^p}/\mathfrak{m}(p) \longrightarrow \mathcal{N}_f,$$

which is an isomorphism if f is an isolated stable singularity (a miniversal unfolding of some germ). Although we will not need it, the converse of this last statement is also true.

Let $F \in \mathcal{E}(n+r, p+r)$ be a miniversal unfolding of $f \in \mathcal{E}(n,p)$, and let h be an element of θ_f which is now identified with $\mathcal{E}(n,p)$. Then we define $\overline{\rho}_{f,F}(h) \in \theta_F$ by

$$\overline{\rho}_{f,F}(h) : (\mathbb{R}^n \times \mathbb{R}^r, 0) \longrightarrow \mathbb{R}^p \times \mathbb{R}^r$$

$$(x, u) \mapsto (h(x), 0).$$

It can be verified that the map $\overline{\rho}_{f,F} : \theta_f \longrightarrow \theta_F$ induces a vector space isomorphism $\rho_{f,F} : \mathcal{N}_f \longrightarrow \mathcal{N}_F$.

1.2. Existence and uniqueness of maximal compact subgroups

Let $\mathcal{H}$ be one of the groups $\mathcal{A}$ or $\mathcal{K}$. Remember that these groups act on $\mathcal{E}(n, p)$ from the left.

1.2.1. Definition. *The stabiliser of f in $\mathcal{H}$ is denoted by $Aut_{\mathcal{H}} f$.*

Since we have not defined topology on $\mathcal{A}$ and $\mathcal{K}$, we do not have topology on these stabilisers either. Still we would like these groups to share some properties with Lie groups (with finitely many components). Namely, we will define *compact*ness of some subgroups of $Aut_{\mathcal{H}} f$ and prove that
1) every compact subgroup is contained in a maximal one,
2) the maximal compact subgroups are conjugate and that
3) the quotient by a maximal compact subgroup is contractible in a generalized sense.

In fact, the first two of these goals have been reached by Jänich [J] and Wall [W2]. We will just summarize their results in this section. In the next section we prove 3) (i. e. the generalized contractibility property) and in the following section we compute the maximal compact subgroups.

1.2.2. Definition. *A subgroup G of $Aut_{\mathcal{H}} f$ is called compact if it is conjugate in $\mathcal{H}$ to a compact linear group.*

The following theorem is proved by Jänich [J] and Wall [W2], for completeness we give a summary of their proof.

1.2.3. Theorem. *Let $f : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$ be a finitely $\mathcal{H}$-determined germ. Then*
1) every compact subgroup of $Aut_{\mathcal{H}} f$ is contained in a maximal compact subgroup,
2) any two maximal compact subgroups are conjugate in $Aut_{\mathcal{H}} f$.

Proof. The proof of the theorem will be done simultaneously for $\mathcal{H} = \mathcal{A}$ and $\mathcal{H} = \mathcal{K}$. We can do it because the following two lemmas are true for both groups, and in the proof of the theorem we will not use other properties of the group but these lemmas.

Lemma A. *Suppose $G \leq \mathcal{H}$ is compact, and the k-jet of every element of G is linear. Then there is a $\phi \in \mathcal{H}$ for which $\phi^{-1} G \phi$ is a linear group and whose k-jet is the identity.*

Lemma B. *Let the map germ f be k-determined and let $G \leq Aut_{\mathcal{H}} f$ be compact and linear. Then f is k-G-determined, that is, if $G \leq Aut_{\mathcal{H}} g$ for another germ and $j^k g = j^k f$ then there exists a G-equivariant $\phi \in \mathcal{H}$ such that $g = \phi \cdot f$.*

Lemma A is proved as Bochner's classical theorem [Bo] is and the proof of Lemma B can be found in [W2]. We will sketch the proof of the theorem using these two lemmas.

Start with part 1). Since f is finitely $\mathcal{H}$-determined, there is a k large enough for which any other germ having the same k-jet is $\mathcal{H}$-equivalent to f. Let H^k denote the k-jets of the elements of $\mathcal{H}$, and $H^k_f = \{z \in H^k \mid z \cdot j^k f = j^k f\}$. The following diagram may help the reader to follow the proof. (For shortness let $\mathcal{H}_f$ denote the group $Aut_{\mathcal{H}} f$ in this proof.)

$$\begin{array}{ccccccc}
G_0 & \leq & G & \leq & \mathcal{H}_f & & \\
\Big\downarrow{\scriptstyle j^k} & & & & \Big\downarrow{\scriptstyle j^k} & & \\
j^k(G_0) & \leq & U & \leq & H^k_f & & \\
\Big\downarrow{\scriptstyle j^1} & & & & \Big\downarrow{\scriptstyle j^1} & & \\
j^1(G_0) & \leq & V & \leq & j^1\mathcal{H}_f & \leq & GL_{n+p}(\mathbb{R}),
\end{array}$$

A trivial corollary of Lemma A is that the map $j^1 : \mathcal{H}_f \longrightarrow GL_{n+p}(\mathbb{R})$ is injective on compact subgroups. Then notice that $j^1\mathcal{H}_f$ is a closed subgroup of $GL_{n+p}(\mathbb{R})$ with finitely many components (for an exact proof see [J]), therefore — using Iwasawa's theorem — the existence (and uniqueness) of maximal compact subgroup is true for $j^1\mathcal{H}_f$. In view of this the following statement implies the statement of the theorem: *If $G_0 \leq \mathcal{H}_f$ is compact, and V is a compact subgroup of $j^1\mathcal{H}_f$ containing $j^1(G_0)$ then there is a compact subgroup G of $\mathcal{H}_f$ containing G_0 which maps onto V under the map j^1.*

First we can try to lift V to H^k_f — that is finding a subgroup U of H^k_f whose 1-jet is V. In fact, the maximal compact subgroup of the inverse image of V under j^1 is such a U.

Now there exists an element $\psi \in \mathcal{H}$ whose 1-jet is the identity, and which simultaneously linearizes G_0 and U. Indeed, we can first linearize U. As the k-jet of elements of G_0 will be linear, we can further linearize it (using lemma A) with an element whose k-jet is the identity, so it will not have effect on the (by now) linear U. Suppose we can find a compact subgroup of $\mathcal{H}_{\psi \cdot f}$ mapping to $V \leq j^1\mathcal{H}_{\psi \cdot f}$. Then conjugating back with ψ we have the group G sought. Therefore we suppose from now on that U and G_0 are linear.

Let $g = \int_U u \cdot f \, du$ which has the same k-jet as f because $U \leq H^k_f$. As g is also invariant under the $\mathcal{H}$ action of G_0, f and g satisfy the conditions of Lemma B. Therefore we have a G_0-equivariant ϕ transforming f to g. Putting $G = \phi^{-1} U \phi$ we obtained the compact subgroup of $\mathcal{H}_f$ that we were looking for (remark, that $\phi^{-1} G_0 \phi = G_0$ so $G_0 \leq G$).

Now we turn to the proof of part 2), i.e. we sketch the proof of the following: *If G_1 and G_2 are maximal compact subgroups of $\mathcal{H}_f$ then they are conjugate in $\mathcal{H}_f$.*

Let $\psi_1 \in \mathcal{H}$ and $\psi_2 \in \mathcal{H}$ be linearizing elements of G_1 and G_2 respectively (we can also assume that $j^1\psi_1 = j^1\psi_2 = id$). As $j^1\mathcal{H}_f$ is a Lie group and $j^1(G_1)$ and

$j^1(G_2)$ are maximal compact subgroups of it (see the first half of the proof), they are conjugate:

$$G = j^1(G_1) = \psi_1 G_1 \psi_1^{-1} \ \sim_{a \in j^1(\mathcal{H}_f)} \ \psi_2 G_2 \psi_2^{-1} = j^1(G_2),$$

or in other words

$$a(j^1 G_2)a^{-1} = j^1 G_1.$$

Denote the germs $\psi_1 \cdot f$ and $a\psi_2 \cdot f$ by g_1 and g_2, they are both $\mathcal{H}$-equivalent to f. It is easy to see that g_1 and g_2 are G-invariant.

Suppose that there is an element ϕ in the normalizer of G in $\mathcal{H}$ that transforms g_2 to g_1. Then we have $\psi_1 f = \phi a \psi_2 f$ and therefore

$$\psi_1^{-1} \phi a \psi_2 \in \mathcal{H}_f.$$

If we conjugate G_2 by this product we get G_1:

$$\psi_1^{-1}\phi a \psi_2 G_2 \psi_2^{-1} a^{-1} \phi^{-1} \psi_1 = \psi_1^{-1} \phi a j^1(G_2) a^{-1} \phi^{-1} \psi_1 =$$

$$= \psi_1^{-1} \phi G \phi^{-1} \psi_1 = \psi_1^{-1} G \psi_1 = G_1,$$

what we wanted. The rest of the proof is to show the existence of ϕ in the normalizer group of G in $\mathcal{H}$ for which $g_1 = \phi \cdot g_2$.

First look for an element β in the normalizer of G in $\mathcal{H}$, for which g_1 and $\beta \cdot g_2$ have the same k-jet (let k be large enough compared with the determinacy of f). Since the linear group G is maximal compact in $j^1\mathcal{H}_{g_i}$ it is also maximal compact in $H^k_{g_i}$. With $g_1 = \lambda g_2$ we have that G and $j^k\lambda G j^k \lambda^{-1}$ are maximal compact in H^k_f, and hence conjugate in this group. Choose an element $\gamma \in \mathcal{H}$ with $j^k\gamma \in H^k_f$ and $j^k\gamma\lambda G j^k \lambda^{-1}\gamma^{-1} = G$ in $H^k_{g_1}$. We have $j^k g_1 = j^k \gamma\lambda g_2$, but $G' = \gamma\lambda G \lambda^{-1}\gamma^{-1} \leq \mathcal{H}$ may differ from G and hence $\gamma\lambda$ need not be in the normalizer of G. However G' is linear on the k-jet level and its linear part is G, so we may linearize it to G by some $\mu \in \mathcal{H}$ with $j^k\mu = id$ (see Lemma A), and then $\beta := \mu\gamma\lambda$ is in the normalizer of G in $\mathcal{H}$ and $j^k g_1 = j^k \beta g_2$ as we wanted.

Now Lemma B shows that there is an equivariant $\chi \in \mathcal{H}$ transforming βg_2 to g_1, and hence $\phi := \chi\beta$ has the required properties. $\square$

The maximal compact subgroup of $Aut_{\mathcal{H}} f$ — as an abstract group — will be denoted by $MC\ Aut_{\mathcal{H}} f$. Let the germs f and g be $\mathcal{H}$-equivalent. Then $Aut_{\mathcal{H}} f$ and $Aut_{\mathcal{H}} g$ are clearly isomorphic — in fact conjugate in $\mathcal{H}$ —, and this (not necessarily unique) conjugation takes compact subgroups to compact subgroups. So $MC\ Aut_{\mathcal{H}} f \cong MC\ Aut_{\mathcal{H}} g$.

1.3. Contractibility of the quotient

If M is a smooth manifold then let $Diff_M(M \times \mathbb{R}^n)$ denote the fibrewise diffeomorphism germs of $M \times \mathbb{R}^n$ at the zero section. The following proposition is an analogue of the main lemma in [J; p. 150] (where it is stated for function germs instead of map germs) but the proof here is a bit more complicated.

Let us fix a finitely $\mathcal{A}$-determined map germ $\eta : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$.Then there exists an $l \in \mathbb{N}$ large enough for which the following proposition holds.

1.3.1. Proposition. *Let M be an r-dimensional manifold with boundary and let $f, g : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^p$ be map germs at $M \times 0$ with the following properties*

$$f \circ pr_M = pr_M \circ f, \qquad g \circ pr_M = pr_M \circ g,$$

$$f|_{\partial M \times \mathbb{R}^n} = g|_{\partial M \times \mathbb{R}^n},$$

$$j^l(f|_{u \times \mathbb{R}^n}) = j^l(g|_{u \times \mathbb{R}^n}) = j^l\eta \qquad \text{for all } u \in M.$$

Then there exist $(\psi, \phi) \in Diff(M \times \mathbb{R}^n) \times Diff(M \times \mathbb{R}^p)$ such that $g = \phi \circ f \circ \psi^{-1}$ and

$$\psi|_{\partial M \times \mathbb{R}^n} = id \qquad \phi|_{\partial M \times \mathbb{R}^n} = id$$

$$j^1(\psi|_{u \times \mathbb{R}^n}) = id \qquad j^1(\phi|_{u \times \mathbb{R}^p}) = id$$

for all $u \in M$.

Proof. Instead of constructing only ψ and ϕ we prove the existence of two one-parameter families of diffeomorphisms. This method is sometimes called the *homotopy method* of Thom.

Let $F : M \times \mathbb{R}^n \times \mathbb{R} \longrightarrow M \times \mathbb{R}^p \times \mathbb{R}$ be the map germ at $M \times 0 \times \mathbb{R}$ defined by

$$(u, x, t) \mapsto (u, (1-t)f(u, x) + tg(u, x), t).$$

From now on $u = (u_i)$, $x = (x_i)$, $y = (y_i)$ and t will always denote coordinates of M, $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}$ respectively; and e. g. F_y will denote the composition $pr_{\mathbb{R}^p} \circ F$.

We will construct two flows, i. e. families of curves. The first family contains curves $\gamma_{u,x} : [0, 1] \longrightarrow M \times \mathbb{R}^n \times \mathbb{R}$ (for all $u \in M$, $x \in \mathbb{R}^n$) starting in $(u, x, 0)$ and ending somewhere in $M \times \mathbb{R}^n \times 1$. Suppose also that the 3rd (t) coordinate of $\gamma_{u,x}(t)$ is t. The second family contains similar curves $\delta_{u,y} : [0, 1] \longrightarrow M \times \mathbb{R}^p \times \mathbb{R}$.

We want these two flows to be "compatible" with F, that is

$$F(\gamma_{u,x}(\tau)) = \delta_{f(u,x)}(\tau). \tag{2}$$

Putting $\tau = 1$ we see that the maps $\psi : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n$, $(u, x) \mapsto \gamma_{u,x}(1)$, and $\phi : M \times \mathbb{R}^p \longrightarrow M \times \mathbb{R}^p$, $(u, y) \mapsto \delta_{u,y}(1)$ satisfy $g = \phi \circ f \circ \psi^{-1}$.

We will define the two flows by their systems of differential equations. This will assure that ψ and ϕ are diffeomorphisms (at least near $M \times 0$). We will also pay attention to the other conditions on ψ and ϕ.

Suppose therefore that we are given two vector field germs: $X : M \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $Y : M \times \mathbb{R}^p \times \mathbb{R} \longrightarrow \mathbb{R}^p$ satisfying

$$\sum_{i=1}^{n} \frac{\partial F_{y_j}}{\partial x_i}(u, x, t) X_i(u, x, t) + \frac{\partial F_{y_j}}{\partial t}(u, x, t) = Y_j(F(u, x, t)) \quad j = 1, \ldots, p \tag{3}$$

$$X|_{M \times 0 \times \mathbb{R}} = 0 \qquad Y|_{M \times 0 \times \mathbb{R}} = 0 \tag{4}$$

$$(5) \qquad \frac{\partial X}{\partial x_i}(u,0,t) = 0 \quad i = 1,\ldots,n \qquad\qquad \frac{\partial Y}{\partial y_j}(u,0,t) = 0 \quad j = 1,\ldots,n$$

$$(6) \qquad X|_{\partial M \times \mathbb{R}^n \times \mathbb{R}} = 0 \qquad\qquad Y|_{\partial M \times \mathbb{R}^p \times \mathbb{R}} = 0.$$

Then let us consider the trajectories of the differential equations

$$\bar{X} : M \times \mathbb{R}^n \times \mathbb{R} \longrightarrow TM \times T\mathbb{R}^n \times T\mathbb{R}, \quad (u,x,t) \mapsto (0, X(u,x,t), 1),$$

$$\bar{Y} : M \times \mathbb{R}^p \times \mathbb{R} \longrightarrow TM \times T\mathbb{R}^p \times T\mathbb{R}, \quad (u,x,t) \mapsto (0, Y(u,x,t), 1).$$

These trajectories exist at least in a neighbourhood of $M \times 0 \times \mathbb{R}$, because of (4). The equation (3) is just the condition of $\bar{X}$ being the derivative of a flow satisfying (2). The maps ψ, ϕ assigned to the trajectories as above will have the properties

$$\psi|_{\partial M \times \mathbb{R}^n} = id \qquad\qquad \phi|_{\partial M \times \mathbb{R}^p} = id$$

$$j^1(\psi|_{u \times \mathbb{R}^n}) = id \qquad\qquad j^1(\phi|_{u \times \mathbb{R}^p}) = id,$$

because of (6) and (5).

It means that we reduced the problem of finding ψ and ϕ to the existence of X and Y satisfying (3)–(6). It is enough to prove the existence of X and Y locally (near a point in $M \times [0,1]$) and to use *partition of unity* to “add up” these solutions.

(I) First we solve the local problem near a point $(u,t) \in int\ M \times [0,1]$. In this case condition (6) is vacuous and the others can be summarized in the condition:

$$\left(\frac{\partial F_{y_j}}{\partial t}(u,x,t)\right)_{j=1,\ldots,p} \in \left\langle \left(\frac{\partial F_{y_1}}{\partial x_i}, \ldots, \frac{\partial F_{y_p}}{\partial x_i}\right) \ i = 1,\ldots,n \right\rangle_{\mathcal{E}(r+n+1)} \mathfrak{m}(n)^2 +$$

$$+ F^* \left(\mathfrak{m}(p)^2 \mathcal{E}(r+p+1)\right)^p.$$

A coordinate of the left hand side is however

$$\frac{\partial F_{y_j}}{\partial t}(u,x,t) = g_{y_j}(u,x) - f_{y_j}(u,x) \in \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$$

(because f and g have the same l-jets in every fiber), so it is enough to show that

$$(7) \quad \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p \subset (tF)\left(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\right) + (\omega F)\left(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)^p\right)$$

(on the right hand side we just used the definition of tF and ωF). We know that the finite determinacy of the germ η implies a very similar inclusion (see (1)):

$$(8) \qquad \mathfrak{m}(n)^k\mathcal{E}(n)^p \subset (t\eta)\left(\mathfrak{m}(n)^2\mathcal{E}(n)^n\right) + (\omega\eta)\left(\mathfrak{m}(p)^2\mathcal{E}(p)^p\right).$$

The rest of the proof is showing (7) with the aid of (8). First we have to compare the corresponding terms on the right hand side of (7) and (8), that is to control the non-commutativity of the following diagram:

$$\begin{array}{ccccc} \mathcal{E}(n)^n & \xrightarrow{t\eta} & \mathcal{E}(n)^p & \xleftarrow{\omega\eta} & \mathcal{E}(p)^p \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}(r+n+1)^n & \xrightarrow{tF} & \mathcal{E}(r+n+1)^p & \xleftarrow{\omega F} & \mathcal{E}(r+p+1)^p, \end{array}$$

where the vertical arrows are just the natural inclusions (which we will omit from the formulas). This is done in the following two lemmas.

1.3.2. Lemma. *If $h \in \mathcal{E}(p)^p$ then $\omega F(h) - \omega\eta(h) \in \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$.*

Proof. The jth coordinate of $\omega F(h) - \omega\eta(h)$ is $h_j(F_y(u,x,t)) - h_j(\eta(x))$. We will show that it is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$. Let h_j be written in the form $p(y)+q(y)$ where p is a polynomial of degree l and $q \in \mathfrak{m}(p)^{l+1}$. Then

$$h_j(F_y(u,x,t)) - h_j(\eta(x)) = \Big(p(F_y(u,x,t)) - p(\eta(x))\Big) + q(F_y(u,x,t)) - q(\eta(x)).$$

The second and third terms are in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$ so it remains to show that the first term is there too. It is no restriction to consider only the case when p is a monomial. We will use induction on the degree of p. If p is a constant, then the statement is evident. If the degree is bigger, we can write $p = y_i p'(y)$ for some i and a monomial p' with smaller degree. In this case

$$p(F_y(u,x,t)) - p(\eta(x)) = F_{y_i}(u,x,t)p'(F_y(u,x,t)) - \eta_i(x)p'(\eta(x)) =$$

$$= p'(\eta(x))\Big(F_{y_i}(u,x,t) - \eta_i(x)\Big) + F_{y_i}(u,x,t)\Big(p'(F_y(u,x,t)) - p'(\eta(x))\Big),$$

and the elements in both brackets are in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$, the first by definition and the second by the induction hypotheses. $\square$

1.3.3. Lemma. *If $h \in \mathcal{E}(n)^n$ then $tF(h) - t\eta(h) \in \mathfrak{m}(n)^l\mathcal{E}(r+n+1)^p$.*

Proof. The jth coordinate of $tF(h) - t\eta(h)$ is $\sum_{i=1}^n (\frac{\partial F_{y_j}}{\partial x_i}(u,x,t) - \frac{\partial \eta_j}{\partial x_i}(x))h_i(x)$. This must be in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)$ since $F_{y_j}(u,x,t) - \eta_j(x)$ is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$. $\square$

Now let us denote by U the intersection

$$(\omega\eta)^{-1}\Big(t\eta(\mathfrak{m}(n)^2\mathcal{E}(n)^n) + \mathfrak{m}(n)^k\mathcal{E}(n)^p\Big) \;\cap\; \mathfrak{m}(p)^2\mathcal{E}(p)^p.$$

This is an $\mathcal{E}(p)$-submodule of $\mathcal{E}(p)^p$. Let V be the $\mathcal{E}(r+p+1)$-submodule of $\mathcal{E}(r+p+1)^p$ generated by the image of U under the natural inclusion $\mathcal{E}(p)^p \longrightarrow \mathcal{E}(r+p+1)^p$.

Now we claim that the following equality holds:

$$\omega F(V) + tF\Big(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\Big) + \mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p =$$

$$= tF\Big(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\Big) + \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p. \tag{9}$$

To prove the $\subset$ part, it is enough to show that $\omega F(U) \subset$ the right hand side, since the right hand side is an $\mathcal{E}(r+p+1)$-submodule. Let $v \in U$ — so $v = (\omega\eta)^{-1}(t\eta(\xi)+\zeta)$ where $\xi \in \mathfrak{m}(n)^2\mathcal{E}(n)^n$ and $\zeta \in \mathfrak{m}(n)^k\mathcal{E}(n)^p$. Then

$$\omega F(v) = \Big(\omega F(v) - \omega\eta(v)\Big) + \Big(t\eta(\xi) - tF(\xi)\Big) + \zeta + tF(\xi).$$

According to the lemmas above the element in the first bracket is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$ and the element in the second bracket is in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)^p$. So (choosing l bigger than k) we have

$$\omega F(v) \in \mathfrak{m}(n)\mathcal{E}(r+n+1)^p + tF\left(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\right),$$

which implies LHS $\subset$ RHS in (9).

Now it has to be shown that LHS $\supset$ RHS in (9). The only thing to be proved is that $\mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p$ is part of the LHS. Let $\rho \in \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p$, then

$$\rho(u,x,t) = x^k[h_0(x) + s_1(t,u)h_1(x) + s_2(t,u)h_2(x) + \ldots + s_{l-k}(t,u)h_{l-k}(u,x,t)],$$

where $s_i(t,u)$ is a polynomial in $t, u_1, \ldots, u_r$ of degree i, and the h_i's are smooth maps $\mathbb{R}^n \longrightarrow \mathbb{R}^p$ for $i < l-k$ and h_{l-k} is a smooth map $\mathbb{R}^{r+n+1} \longrightarrow \mathbb{R}^p$. The last term is in the LHS of (9) by definition and since the LHS of (9) is closed to the multiplication by t and u, it is enough to show that $x^k h_0(x) \in$ LHS of (9).

Because of (8) we can write $x^k h_0(x)$ in the form $t\eta(\xi) + \omega\eta(\zeta)$ for some $\xi \in \mathfrak{m}(n)^2\mathcal{E}(n)^n$ and $\zeta \in \mathfrak{m}(p)^2\mathcal{E}(p)^p$. Therefore

$$x^k h_0(x) = \Big(t\eta(\xi) - tF(\xi)\Big) + \Big(\omega\eta(\zeta) - \omega F(\zeta)\Big) + tF(\xi) + \omega F(\zeta),$$

and the elements in the two brackets are in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)$ and $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$ respectively (see the two lemmas above), and the remaining two terms are also in the LHS of (9) by definition. So the proof of (9) is complete.

Having the formula (9) we now want to prove (7). This will be a so called Nakayama-type argument — although it needs a more sophisticated lemma than that of Nakayama, namely the following.

1.3.4. Lemma. [M3; p. 135] *Let $G : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be a smooth map germ. Suppose A is a finitely generated $\mathcal{E}(p)$-module; B and C are $\mathcal{E}(n)$-modules (C is finitely generated); $\beta : B \longrightarrow C$ is an $\mathcal{E}(n)$-module homomorphism and $\alpha : A \longrightarrow C$ is a homomorphism over $G^* : \mathcal{E}(p) \longrightarrow \mathcal{E}(n)$. Let a be the dimension of the vector space $A/m(p)A$ over $\mathbb{R}$. Then*

$$\alpha(A) + \beta(B) + (G^*(\mathfrak{m}(p)) + \mathfrak{m}(n)^{a+1})C = C$$

implies

$$\alpha(A) + \beta(B) = C.$$

Remark. The proof of this lemma is based on Nakayama's lemma and the Malgrange preparation theorem.

A trivial consequence of this lemma is the following. Using the same notation as above, if $D \subset C$ satisfies

$$\alpha(A) + \beta(B) + D = C \tag{10}$$

and

$$D \subset (G^*(\mathfrak{m}(p)) + \mathfrak{m}(n)^{a+1})C, \tag{11}$$

then

$$\alpha(A) + \beta(B) = C. \tag{12}$$

We will use this lemma with the following substitution:

$$\begin{array}{rcc} G & = & F \\ A & = & V \\ B & = & \mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n \\ C & = & tF(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n) + \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p \\ D & = & \mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p \\ \alpha & = & \omega F \\ \beta & = & tF. \end{array}$$

We have to check (10) and (11). In fact (10) is exactly (9) which we have just proved, and (11) is

$$\mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p \subset$$

$$\subset \Big(F^*(\mathfrak{m}(r+p+1)) + \mathfrak{m}(r+n+1)^{a+1}\Big) \cdot$$

$$\cdot \Big(tF(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n) + \mathfrak{m}(n)^k + \mathcal{E}(r+n+1)^p\Big).$$

If we chose l at least $a+k+1$ then this is clearly true (see the product of the second terms in both brackets). Therefore we have (12) in our situation, which is exactly (7) what we wanted to prove.

(II) Now we want to solve the same local problem as in (I) but near a point in $\partial M \times [0,1]$. It will turn out that this problem can be reduced to the case studied in (I).

Indeed, extend F from $\mathbb{R}^r_+ \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^r_+ \times \mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}$ still satisfying the conditions required for F in the theorem. This time we have to show that

$$\Big(\frac{\partial F_{y_1}}{\partial t}, \ldots, \frac{\partial F_{y_p}}{\partial t}\Big) \in \Big\langle \Big(\frac{\partial F_{y_1}}{\partial x_i}, \ldots, \frac{\partial F_{y_p}}{\partial x_i}\Big) \mid i = 1, \ldots, n \Big\rangle_{\mathcal{E}(r+n+1)} \mathfrak{m}(n)^2\mathfrak{m}(1) +$$

$$+F^*\Big(\mathfrak{m}(p)^2\mathfrak{m}(1)\mathcal{E}(r+p+1)\Big)^p. \tag{13}$$

(Note that $\mathfrak{m}(1)$ here refers to the ideal generated by the first local coordinate u_1 of M, where the boundary ∂M is defined by the equation $u_1 = 0$.) Because of condition (6) now the left hand side is in $\mathfrak{m}(1)\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$, so it is enough to show that this submodule is part of the right hand side of (13). If we multiply the inclusion (7) by $\mathfrak{m}(1)$ we get

$$\mathfrak{m}(1)\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p \subset$$

$$\subset \Big\langle \ \Big(\frac{\partial F_{y_1}}{\partial x_i}, \dots, \frac{\partial F_{y_p}}{\partial x_i}\Big) \ \mid i = 1, \dots, n \ \Big\rangle_{\mathcal{E}(r+n+1)} \mathfrak{m}(n)^2\mathfrak{m}(1) +$$

$$+\mathfrak{m}(1)F^* \Big(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)\Big)^p. \tag{14}$$

So it is enough to prove that

$$\mathfrak{m}(1)F^* \Big(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)\Big)^p \subset F^* \Big(\mathfrak{m}(p)^2\mathfrak{m}(1)\mathcal{E}(r+p+1)\Big)^p,$$

which is clearly true, since a coordinate of the left hand side can be written in the form of $u_1 F_y^2(u,x,t)h(u,x,t)$ and a coordinate of the right hand side has the form $F_y^2(u,x,t)\ F_{u_1}(u,x,t)\ h(u,x,t)$ — but $F_{u_1}(u,x,t) = u_1$ and therefore these two sets are in fact equal.

The proof of the proposition is complete. $\square$

For shortness denote $Aut_{\mathcal{A}}\eta$ by $\mathcal{A}_\eta$ and denote by A^l the Lie group of l-jets of the elements of $\mathcal{A}$, and put

$$A^l_\eta = \{\ z = (z_1, z_2) \in A^l \mid z_2 \circ j^l\eta \circ z_1^{-1} = j^l\eta\ \}.$$

From the proof of the theorem above it is also clear that for l large enough the image of a maximal compact subgroup of $\mathcal{A}_\eta$ under the map j^l is a maximal compact subgroup (in the classical sense) of A^l_η. In fact, if η is well chosen from its $\mathcal{A}$-equivalence class then the maximal compact subgroup G of $\mathcal{A}_\eta$ is linear, so $j^1(G) = G$.

Let M be a differentiable manifold with boundary and let G be a subgroup of $\mathcal{A}$. Call a map $q : M \longrightarrow \mathcal{A}/G$ differentiable if M can be covered by open sets U, on which q can be represented by pairs of local diffeomorphisms ($U \times \mathbb{R}^n \longrightarrow U \times \mathbb{R}^n$ and $U \times \mathbb{R}^p \longrightarrow U \times \mathbb{R}^p$) (in fact germs at the zero section), which map all the fibres $u \times \mathbb{R}^n$ and $u \times \mathbb{R}^p$ into themselves.

1.3.5. Definition. *Let G be a subgroup of $\mathcal{A}_\eta$. We call $\mathcal{A}_\eta/G$ contractible if for every smooth manifold M with boundary any differentiable map $q : \partial M \longrightarrow \mathcal{A}_\eta/G$ can be extended to a differentiable map $M \longrightarrow \mathcal{A}_\eta/G$.*

The main theorem of this section is the following.

1.3.6. Theorem. *If η is finitely determined and $G \leq \mathcal{A}_\eta$ is a maximal compact subgroup then $\mathcal{A}_\eta/G$ is contractible.*

Proof. What we have to show is that a differentiable map $f : \partial M \longrightarrow \mathcal{A}_\eta/G$ extends to a differentiable map $\bar{f} : M \longrightarrow \mathcal{A}_\eta/G$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_\eta & \xrightarrow{j^l} & A^l_\eta \\ \big\downarrow \pi & & \big\downarrow \pi^l \\ \mathcal{A}_\eta \,/\, G & \xrightarrow{\overline{j^l}} & A^l_\eta \,/\, G. \end{array}$$

There is a section σ^l of π^l and it induces a section σ of π. It is easy to check that π and σ are differentiable — in the sense that for a differentiable map $q_1 : N \longrightarrow \mathcal{A}_\eta$ the composition $\pi \circ q_1$ is also differentiable, and for a differentiable map $q_2 : N \longrightarrow \mathcal{A}_\eta/G$ the composition $\sigma \circ q_2$ is also differentiable.

We want to prove that the differentiable map $k = \sigma \circ f : \partial M \longrightarrow \mathcal{A}_\eta$ extends to a map $\bar{k} : M \longrightarrow \mathcal{A}_\eta$. This implies the theorem since the composition of $\bar{f} = \pi \circ \bar{k}$ will extend f.

The composition $g = \overline{j^l} \circ f$ extends to $\bar{g} : M \longrightarrow A^l_\eta/G$ since G is a maximal compact subgroup (in the classical sense) of a Lie group, so the quotient is contractible. Composing g and $\bar{g}$ with σ^l we get maps h and $\bar{h}$. It is clear that $j^l \circ k = h$, our task is to construct $\bar{k}$ such that $j^l \circ \bar{k} = \bar{h}$. We will do it in two steps. First we extend k in a bigger group then $\mathcal{A}_\eta$. Namely, let

$$\mathcal{A}_{j^l\eta} := \{ \ \psi = (\psi_1, \psi_2) \in \mathcal{A} \mid j^l\psi \in A^l_\eta \ \} \supset \mathcal{A}_\eta.$$

We will construct a map $k' : M \longrightarrow \mathcal{A}_{j^l\eta}$ which extends k and satisfies $j^l \circ k' = \bar{h}$. This is equivalent to the following problem: we seek diffeomorphism germs at $M \times 0$

$$F_1 : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n,$$

and

$$F_2 : M \times \mathbb{R}^p \longrightarrow M \times \mathbb{R}^p$$

if they are given in $\partial M \times \mathbb{R}^n$ and $\partial M \times \mathbb{R}^p$, and their l-jets are given everywhere. We show the existence of F_1, the existence of F_2 can be proved in the same way. First we show an F_1 locally and then use *partition of unity* to "add up" these solutions. The local problem near a point in *int* M is trivial (let F_1 coincide with the given l-jet), and for a coordinate function of F_1 the local problem near a point in ∂M is the following. Given a polynomial P of degree l in the variables $x_1, \dots, x_n$ with coefficients from $\mathcal{E}(r)$ and a function $\alpha_0 : \mathbb{R}^{r-1+n} \longrightarrow \mathbb{R}$ such that

$$j^l_x \alpha_0 = P(0, u_2, \dots, u_r, x_1, \dots, x_n)$$

(as usual the coordinates of $\mathbb{R}^n$ are denoted by x and the local coordinates of M are denoted by u and ∂M is given by $u_1 = 0$) a function $\alpha : \mathbb{R}^{r+n} \longrightarrow \mathbb{R}$ is needed with the properties that

$$\alpha|_{u_1=0} = \alpha_0,$$

$$j^l_x \alpha = P(u_1, \dots, u_r, x_1, \dots, x_n).$$

The following function satisfies these conditions

$$\alpha(u, x) = \alpha_0(u_2, \dots, u_r, x) - P(0, u_2, \dots, u_r, x) + P(u_1, \dots, u_r, x).$$

So we proved the existence of k'. The next step is to prove the existence of $\bar{k}$. Let the map k' be represented by the pair (F_1, F_2) of diffeomorphism germs (of $M \times \mathbb{R}^n$ and $M \times \mathbb{R}^p$ respectively). If the *fibrewise* map germs $a, b : (M \times \mathbb{R}^n, M) \longrightarrow (M \times \mathbb{R}^p, M)$ are defined by

$$pr_{\mathbb{R}^p} \circ a = \eta \circ pr_{\mathbb{R}^n}$$

$$pr_{\mathbb{R}^p} \circ F_2 \circ b = \eta \circ pr_{\mathbb{R}^n} \circ F_1,$$

then clearly $a \circ F_1 = F_2 \circ b$. Further a and b coincide over ∂M and their l-jets coincide over the whole M. So they satisfy the conditions of the proposition above, and therefore there exist diffeomorphism germs ψ and ϕ (of $M \times \mathbb{R}^n$ and $M \times \mathbb{R}^p$ respectively) such that $a \circ \psi = \phi \circ b$.

If we denote $F_1 \circ \psi$ by $\bar{\psi}$ and $F_2 \circ \phi$ by $\bar{\phi}$ then the pair $(\bar{\psi}, \bar{\phi}) : M \longrightarrow \mathcal{A}_\eta$ represents k, because

$$a \circ \bar{\psi} = a \circ F_1 \circ \psi = F_2 \circ b \circ \psi = F_2 \circ \phi \circ b = \bar{\phi} \circ b.$$

The proof of the theorem is complete. $\square$

1.4. Computation of the maximal compact subgroup

Let F be a stable map germ. In this section we reduce the "infinite dimensional" problem of finding $MC\ Aut_{\mathcal{A}}F$ to a "finite dimensional" problem of finding the maximal compact subgroup of a Lie group (in fact the automorphism group of a finite dimensional algebra).

- $MC\ Aut_{\mathcal{A}}F \cong MC\ Aut_{\mathcal{K}}f$

Remember that if F is an isolated stable map germ then it is a miniversal unfolding of a germ $f \in \mathcal{E}(n, p)$ whose differential at 0 vanishes. Let the *unfolding dimension* be r, i. e. let F map from $(\mathbb{R}^{n+r}, 0)$ to $\mathbb{R}^{p+r}$. Our first goal is to prove that the maximal compact subgroup of $Aut_{\mathcal{A}}F$ is the same as the maximal compact subgroup of $Aut_{\mathcal{K}}f$. This result is stated in [W2].

Remark. The following trivial remarks will be useful:

$$Aut_{\mathcal{A}}f \leq Aut_{\mathcal{K}}f,$$

$$Aut_{\mathcal{K}}f \cap GL_{n+p}(\mathbb{R}) \leq Aut_{\mathcal{A}}f.$$

1.4.1. Lemma. *Let G be a compact linear subgroup of $Aut_{\mathcal{A}}f$. Then there is a compact subgroup of $Aut_{\mathcal{A}}F$ which is isomorphic to G.*

Proof. Recall that the miniversal unfolding of f is

$$F : (\mathbb{R}^n \times V, 0) \longrightarrow \mathbb{R}^p \times V$$

$$(x, \phi) \mapsto (f(x) + \phi(x), \phi),$$

where $V = \mathbb{R}^r$ is a complement of

$$tf(\theta_{\mathbb{R}^n}) + f^*\mathfrak{m}(p)\theta_f$$

if $\mathfrak{m}(n)\theta_f$. The expression $\phi(x)$ makes sense since using the linear structure of $\mathbb{R}^p$, the space θ_f can be identified with $\mathcal{E}(n,p)$.

Notice that G has a natural linear action on θ_f:

$$(\alpha, \beta) \cdot \phi := \beta \circ \phi \circ \alpha^{-1}, \qquad (\alpha, \beta) \in G \leq GL_n(\mathbb{R}) \times GL_p(\mathbb{R}), \phi \in \theta_f = \mathcal{E}(n,p).$$

To see this action geometrically consider θ_f as smooth vertical vector fields along the graph of f in $\mathbb{R}^n \times \mathbb{R}^p$. The group G acts on $\mathbb{R}^n \times \mathbb{R}^p$ linearly keeping the graph of f invariant, so it transforms vertical vector fields along *graph* f to other vertical vector fields along it. It is easy to see that $\mathfrak{m}(n)\theta_f$ is an invariant subspace of θ_f. This geometric interpretation helps us to make the essential observation: the subspaces $tf(\theta_{\mathbb{R}^n})$ and $f^*\mathfrak{m}(p)\theta_f$ are invariant under this G-action. Indeed, $tf(\theta_{\mathbb{R}^n})$ consists of vertical projections of *tangent* vector fields of *graph* f. The group G sends *graph* f into itself, so it sends tangent vector fields along it into tangent vector fields, therefore $tf(\theta_f)$ is an invariant subspace of θ_f. Observe also, that $f^*\mathfrak{m}(p)\theta_f$ consists of restrictions to *graph* f of vertical vector fields on $\mathbb{R}^n \times \mathbb{R}^p$ vanishing on $\mathbb{R}^n \times 0$. As G transforms such a vector field on $\mathbb{R}^n \times \mathbb{R}^p$ into another such one, $f^*\mathfrak{m}(p)\theta_f$ is also an invariant subspace of θ_f under the action of G.

Remark. The following may help to understand the G action on θ_f and the invariant subspaces of it. Think of θ_f as the tangent space at f to the infinite dimensional manifold $\mathcal{E}(n,p)$. Let $\mathcal{K}f$ denote the set of points in $\mathcal{E}(n,p)$ contact equivalent to f. This is a "submanifold" of $\mathcal{E}(n,p)$. As $G \leq \mathcal{K}$, and fixes f, it has an action on $\mathcal{E}(n,p)$ fixing f. The group G will send a germ $\mathcal{K}$-equivalent to f into another germ $\mathcal{K}$-equivalent to f (because $G \leq \mathcal{K}$). Therefore the submanifold $\mathcal{K}f$ is kept invariant. Hence the *differential* of the G-action which is again a G-action but now on the tangent space at f (i. e. on θ_f) leaves the tangent space of $\mathcal{K}f$ invariant. It is not hard to identify $tf(\theta_{\mathbb{R}^n}) + f^*\mathfrak{m}(p)\theta_f$ as the tangent space of $\mathcal{K}f$ in θ_f. In fact, $\mathcal{K}$ is locally a direct product of $\mathcal{R}$ and $\mathcal{C}$, and $\mathcal{R}f = tf(\theta_{\mathbb{R}^n})$, $\mathcal{C}f = f^*\mathfrak{m}(p)\theta_f$.

Now choose $\bar{V} = \mathbb{R}^r$ to be a G-invariant complement of $tf(\theta_{\mathbb{R}^n}) + f^*\mathfrak{m}(p)\theta_f$. This can be done because G is compact. Define the action on $\mathbb{R}^n \times \bar{V}$ and on $\mathbb{R}^p \times \bar{V}$ as follows ($x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $\phi \in \bar{V}$)

$$(\alpha, \beta) \cdot (x, \phi) := ((\alpha, \beta) \cdot x, (\alpha, \beta) \cdot \phi) = (\alpha(x), (\alpha, \beta) \cdot \phi),$$

$$(\alpha, \beta) \cdot (y, \phi) := ((\alpha, \beta) \cdot y, (\alpha, \beta) \cdot \phi) = (\beta(y), (\alpha, \beta) \cdot \phi).$$

This action on the source and the target space of

$$\bar{F} : \mathbb{R}^n \times \bar{V} \longrightarrow \mathbb{R}^p \times \bar{V}$$

$$(x, \phi) \mapsto (f(x) + \phi(x), \phi)$$

makes $G \leq Aut_{\mathcal{A}}\bar{F}$. Indeed, for $(\alpha, \beta) \in G$, $(x, \phi) \in \mathbb{R}^n \times V$:

$$\bar{F}((\alpha, \beta) \cdot (x, \phi)) = \bar{F}(\alpha(x), (\alpha, \beta) \cdot \phi) = (f(\alpha(x)) + (\alpha, \beta) \cdot \phi(\alpha(x)), (\alpha, \beta) \cdot \phi) =$$

$$= (\beta(f(x)) + (\beta\phi\alpha^{-1})(\alpha(x)), (\alpha, \beta) \cdot \phi) =$$

$$= (\beta(f(x) + \phi(x)), (\alpha, \beta) \cdot \phi) = (\alpha, \beta) \cdot \bar{F}(x, \phi).$$

Here we used that G is linear and that it is a subgroup of $Aut_{\mathcal{A}}f$. This shows that $G \leq Aut_{\mathcal{A}}\bar{F}$. Since F and $\bar{F}$ are miniversal unfoldings of the same germ, they are $\mathcal{A}$-equivalent. Therefore $Aut_{\mathcal{A}}F$ has to contain a compact group isomorphic to G. $\square$

1.4.2. THEOREM. *If $G \leq Aut_{\mathcal{K}}f$ is compact then there is a compact subgroup of $Aut_{\mathcal{A}}F$ which is isomorphic to G.*

Proof. By definition G is conjugate in $\mathcal{K}$ to a linear group, which being in $\mathcal{K}$ must have the form $G_0 \leq GL_n(\mathbb{R}) \times GL_p(\mathbb{R})$. Let $k \in \mathcal{K}$ such that $kGk^{-1} = G_0$, and let $f_0 = k \cdot f$. Then obviously $G_0 \leq Aut_{\mathcal{K}}f_0$, moreover by the trivial remark before the theorem $G_0 \leq Aut_{\mathcal{A}}f_0$. Since G_0 is linear we can apply the lemma above to G_0 and f_0 and conclude that there is a compact subgroup $H_0 \leq Aut_{\mathcal{A}}F_0$ isomorphic to G_0 where F_0 is a miniversal unfolding of f_0. Since f and f_0 are $\mathcal{K}$-equivalent, the germs F and F_0 are $\mathcal{A}$-equivalent and therefore $Aut_{\mathcal{A}}F_0 \cong Aut_{\mathcal{A}}F$ and this isomorphism takes compact subgroups to compact subgroups. $\square$

Let $f \in \mathcal{E}(n, p)$ and $F \in \mathcal{E}(n + r, p + r)$ be as above: F is a miniversal unfolding of f. Suppose that $g \in Aut_{\mathcal{A}}F$ has the form $(\psi, \phi, \lambda, \mu) \in GL_n(\mathbb{R}) \times GL_r(\mathbb{R}) \times GL_p(\mathbb{R}) \times GL_r(\mathbb{R})$ and suppose that $(\psi, \lambda) \in Aut_{\mathcal{A}}f$. Then the map $\overline{\rho}_{f,F}$ defined at the end of section 1.1 commutes with the G actions. Indeed, for $h \in \theta_f$

$$g \cdot \overline{\rho}(h)(x, u) = (\lambda, \mu)\overline{\rho}(h)(\psi^{-1}(x), \phi^{-1}(u)) =$$

$$= (\lambda, \mu)(h\psi^{-1}(x), 0) = (\lambda h\psi^{-1}(x), 0) = \overline{\rho}(g \cdot h)(x).$$

We will use this in the proof of the next theorem.

1.4.3. THEOREM. *If $G \leq Aut_{\mathcal{A}}F$ is compact then there is a compact subgroup of $Aut_{\mathcal{K}}f$ which is isomorphic to G.*

Proof. By the definition of the compactness of G there is a $h \in \mathcal{A}$ such that $G_0 := hGh^{-1} \leq GL_{n+r}(\mathbb{R}) \times GL_{p+r}(\mathbb{R})$. If $F_0 = h \cdot F$ then $G_0 \leq Aut_{\mathcal{A}}F_0$ because

$$(hgh^{-1})graph\ F_0 = (hg)graph\ F = h\ graph\ F = graph\ F_0$$

for any $g \in G$. Since G_0 is linear and $graph\ F_0$ is G_0-invariant the tangent space $(dF_0)_0 = T_0(graph\ F_0) \subset \mathbb{R}^{n+r} \times \mathbb{R}^{p+r}$ is also G_0-invariant. It follows that $A := (dF_0)_0 \cap \mathbb{R}^{n+r} \times \{0\}$ and $C := pr_{\mathbb{R}^{p+r}}((dF_0)_0)$ are also G_0-invariant subspaces of dimensions n and r respectively. Choose G_0-invariant complements of A and C in $\mathbb{R}^{n+r}$ and $\mathbb{R}^{p+r}$: B and D. Therefore $A = \mathbb{R}^n$, $B = \mathbb{R}^r$, $C = \mathbb{R}^r$, $D = \mathbb{R}^p$. Denote the map $(pr_D \circ F_0)|_A : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ by f_0. It can be proved that F_0 is a miniversal unfolding of f_0. Since F_0 and F are $\mathcal{A}$-equivalent f_0 and f must be $\mathcal{K}$-equivalent (see section 1.1).

If we project the group $G_0 \leq GL(A) \times GL(B) \times GL(C) \times GL(D)$ to $GL(A) \times GL(D)$ — let this projection be π — then the resulting group is clearly in $Aut_{\mathcal{A}} f_0$. What we want to show is that this projection is injective on G_0, that is we want to prove that the action of G_0 on A and D determines its action on B and C (both are isomorphic to $\mathbb{R}^r$). It is enough to deal with the action on C, because it (together with the actions on A and D) determines the action on B.

Now consider the diagram (for definitions of the maps see the end of section 1.1)

$$\theta_{\mathbb{R}^{p+r}}/\mathfrak{m}(p+r)\theta_{\mathbb{R}^{p+r}} \xrightarrow{\varpi F_0} \mathcal{N}_{F_0} \xleftarrow{\rho_{f_0,F_0}} \mathcal{N}_{f_0}.$$

There is a naturally defined G_0 action on all three spaces involved here: the space $\theta_{\mathbb{R}^{p+r}}/\mathfrak{m}(p+r)\theta_{\mathbb{R}^{p+r}}$ is naturally identified with $\mathbb{R}^{p+r} \cong C \times D$, so it has a G_0 action. Since $G_0 \leq Aut_{\mathcal{A}} F_0$ it has an action on $\mathcal{N}_{F_0}$ described in the proof of the preceding theorem. The group G_0 also operates as $\mathcal{A}$-automorphism group on f_0 through π, so it has an action on $\mathcal{N}_{f_0}$, too. The discussion before the theorem says that the map ρ_{f_0,F_0} is G_0-equivariant. The other map ϖF_0 is trivially G_0-equivariant. Since both maps ϖF_0 and ρ_{f_0,F_0} are G_0-equivariant isomorphisms we obtain that the action of G_0 on $\mathcal{N}_{f_0}$ determines the action of G_0 on C (and D). But the action on $\mathcal{N}_{f_0}$ depends only on the π-image of G_0, so $\pi|_{G_0}$ is injective.

The end of the proof is a routine: $G_0 \leq Aut_{\mathcal{A}} f_0$ compact, so $G_0 \leq Aut_{\mathcal{K}} f_0$ compact. Since f and f_0 are $\mathcal{K}$-equivalent, there is a compact subgroup of $Aut_{\mathcal{K}} f$ isomorphic (conjugate in $\mathcal{K}$) to G_0. As $G_0 \cong G$, the theorem is proved. □

Theorem 1.4.2. and 1.4.3. together proves the following

1.4.4. Theorem. *If F is a miniversal unfolding of f then*

$$MC\ Aut_{\mathcal{A}} F \cong MC\ Aut_{\mathcal{K}} f.$$

□

• $Aut_{\mathcal{K}} f \leq Aut\ Q_f \times O(k-d)$

Let us fix a finitely $\mathcal{K}$-determined germ $f \in \mathcal{E}(n, n+k)$ for which $(df)_0$ is constant 0. Present its local algebra as

$$\mathbb{R}[[x_1, \ldots, x_n]]/(r_1, \ldots, r_p),$$

where $p - n$ is minimal (n is also minimal because of the condition on df). Let us call this minimal $p - n$ the *defect* of the algebra, and denote it by $d = d(Q_f)$. The germ f has to be $\mathcal{K}$-equivalent to

$$g : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p \times \mathbb{R}^{k-d}$$

$$(x_1, \ldots, x_n) \mapsto (r_1, \ldots, r_p,\ 0, \ldots, 0).$$

Let us denote by h the germ $h : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$,

$$(x_1, \dots, x_n) \mapsto (r_1, \dots, r_p).$$

The proof of the following lemma is trivial.

1.4.5. Lemma. *$MC\ Aut_{\mathcal{K}} g \cong MC\ Aut_{\mathcal{K}} h \times O(k-d)$.* $\square$

Now we turn to the study of $Aut_{\mathcal{K}} h$. If $\eta \in \mathcal{E}(n,p)$ then I_η will denote the image of $\eta^* \mathfrak{m}(p)\mathcal{E}(n) \leq \mathcal{E}(n)$ in $\mathbb{R}[[x_1, \dots, x_n]]$.

1.4.6. Theorem. *Let $G \leq Aut_{\mathcal{K}} h$ be compact. Then there is a compact subgroup H of $Aut\ Q_h$ isomorphic to G.*

Proof. Without loss of generality we can suppose that G acts linearly on $\mathbb{R}^n \times \mathbb{R}^p$, because if not, then we change h by an appropriate element of $\mathcal{K}$ (which linearizes G).

Now let $(\psi, \phi) \in G \leq GL_n(\mathbb{R}) \times GL_p(\mathbb{R})$ then ψ acts on $\mathbb{R}[[x_1, \dots, x_n]]$ and it induces an action

$$\bar{\psi} : Q_h \longrightarrow Q_h$$

on the local algebra Q_h. To see this we only have to check that $\psi \cdot I_h \subset I_h$. Since $(\psi, \phi) \in G$ we have $h(x) = \phi \circ h \circ \psi^{-1}(x)$. The matrix ϕ is invertible, so $I_h = I_{\psi \circ h} = \psi I_h$, which means that $\bar{\psi}$ is well defined.

The map $\bar{\psi}$ is an automorphism for all $(\psi, \phi) \in G$. To see this, we have to check that it is an injective and surjective homomorphism. All the three of these properties are easily verified.

Now consider the homomorphism $G \longrightarrow Aut\ Q_h$ mapping (ψ, ϕ) to $\bar{\psi}$. We are going to show that it is injective. Suppose that $\bar{\psi}$ is the identity. Then every element of Q_h is mapped to an element I_h-equivalent to it. Especially $[x_i]$ and $[\sum \psi_{ij} x_j]$ are I_h equivalent. But I_h is contained in the square of the maximal ideal of $\mathbb{R}[[x_1, \dots, x_n]]$, so these linear polynomials can only be I_h equivalent if they are equal. This means that $\psi = id$.

It remains to show that ϕ is also the identity. Usually, if the action of an element of $Aut_{\mathcal{A}} h$ is given on the source space, then the action of the same element on the target space is not determined. It is determined only on $pr_{\mathbb{R}^p}(graph\ h)$ — and therefore on the linear space $W_h := span\ pr_{\mathbb{R}^p}(graph\ h)$ it generates. However we prove that in our case $W_h = \mathbb{R}^p$. Suppose that W_h is smaller than $\mathbb{R}^p$. Then h has the form (after appropriate coordinate changes):

$$(x_1, \dots, x_n) \mapsto (r'_1, \dots, r'_{p-1}, 0),$$

which means that Q_h can be presented as

$$\mathbb{R}[[x_1, \dots, x_p]]/(r'_1, \dots, r'_{p-1})$$

which contradicts to the condition that the defect of Q_h is $d = p - n$.

Therefore the map $(\psi, \phi) \mapsto \bar{\psi}$ is injective so we have proved the theorem. $\square$

Putting all these together we have the following

1.4.7. THEOREM. *If F is an isolated stable singularity then*

$$MC\ Aut_{\mathcal{A}}F \leq MC\ Aut\ Q_F \times O(k-d),$$

where k is the codimension of F and d is the defect of its algebra.

Remark. Let us study whether the converse of this statement is true, that is whether $MC\ Aut_{\mathcal{A}}F \cong MC\ Aut\ Q_F \times O(k-d)$. We know that $MC\ Aut_{\mathcal{A}}F \cong MC\ Aut_{\mathcal{K}}f$, where F is a miniversal unfolding of f, and clearly it is enough to consider the case $k = d$, i.e. the case of $f = h$ (see above). Let $G \leq Aut\ Q_h$ be a compact subgroup. The question is whether there is a compact subgroup $H \leq Aut_{\mathcal{K}}f$ isomorphic to G. If we want to read the proof above backwards, then it would have two essential steps.

First we should lift G into the automorphism group of $\mathcal{E}(n)$. Of course these automorphisms would leave $J_f := f^*\mathfrak{m}(p)\mathcal{E}(n)$ invariant. This lifting is possible, using Nakayama's lemma, and we can even guarantee that these automorphisms operate on the coordinate functions x_i linearly (after possibly changing f to a $\mathcal{K}$-equivalent germ). This linear automorphism assigned to g will be called ψ_g.

Secondly we should further "lift" the compact subgroup H of

$$\{\phi : \mathcal{E}(n) \longrightarrow \mathcal{E}(n) \text{ automorphism } \mid \phi J_f = J_f\}$$

to $Aut_{\mathcal{K}}f$. Every element $g \in H$ can be easily lifted: the coordinate functions of f are generators of J_f so there is a matrix $M_g \in \mathcal{E}(n)^{p\times p}$ for which

$$g \cdot f = M_g f.$$

(In fact there is a lot of choices to find M_g even if we want M_g to be invertible.) The element $(\psi_g, M_g) \in \mathcal{C} \times \mathcal{R}$ is a lift of g to $Aut_{\mathcal{K}}f$. The problem is that we want to lift the whole compact subgroup together, not just its elements individually. In other words, we want the lifts to form a subgroup of $Aut_{\mathcal{K}}f$. As this remains an open problem, the converse of theorem 1.4.7 remains a conjecture.

The question which — the author thinks — is at the heart of this open problem is the following: *suppose we are given an ideal J of $\mathbb{R}[[x_1, \dots, x_n]]$ which can be generated by p elements but can not be generated by smaller number of its elements. A generator set $r_1, \dots, r_p$ of J is called nice if for every $\psi \in GL_n(\mathbb{R})$ (considered as acting on $\mathbb{R}[[x_1, \dots, x_n]]$) which maps J onto J, there is matrix $M \in \mathbb{R}^{p\times p}$ satisfying $(\psi(r_1), \dots, \psi(r_p)) = M \cdot (r_1, \dots, r_p)$. Is there a nice generator set for every ideal (of finite codimension)?*

In the next section we will see examples, and we will observe that in all these examples the group $MC\ Aut\ Q_F \times O(k-d)$ is isomorphic to a subgroup of $Aut_{\mathcal{A}}F$. So theorem 1.4.7 says that this group coincides with the maximal compact subgroup of $Aut_{\mathcal{A}}F$.

1.5. Examples

The aim of this section is to compute the maximal compact subgroup of the $\mathcal{A}$-automorphism group of some stable germs. This means that we will compute $Aut_{\mathcal{A}}F$ as an abstract group and its representations λ_1 and λ_2 in the source $\mathbb{R}^n$ and in the target $\mathbb{R}^p$ spaces. In fact we will carry out the computations for the "simplest" singularities: those of type Σ^1 and the simplest type of $\Sigma^{2,0}$. (For the definition of (Thom-Boardman-) *type* of a germ, see e.g. [GG].)

Note that if the germ $F : (\mathbb{R}^{n+t}, 0) \longrightarrow \mathbb{R}^{p+t}$ has the form $(x, u) \mapsto (h(x), u)$, then $Aut_{\mathcal{A}}F \cong Aut_{\mathcal{A}}h \times O(t)$. Therefore it is enough to deal with *isolated* singularities.

For easier reference let us summarize the procedure we will carry out for Σ^1 and $\Sigma^{2,0}(III_{2,2})$ germs.

• Summary of the procedure of finding $Aut_{\mathcal{A}}F$

1) *Given a germ F.* A germ can be given by a representative (a normal form) of it or — using Mather's result — by its local algebra Q_F (or what is almost the same by a germ $f : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ whose miniversal unfolding is F). In our examples we will only use the local algebra. This means that in our procedure we will not need to write up a particular representative of the germ — we consider this an advantage of this procedure (c. f. the complexity of local algebras and normal forms in the Appendix).
2) *Determining $MC\ Aut\ Q_f$.* This is an algebraic computation. In the following examples these calculations are trivial.
3) *Determining $MC\ Aut_{\mathcal{K}}f$.* By theorem 1.4.6 the group $MC\ Aut\ Q_f \times O(k-d)$ is an upper bound for $MC\ Aut_{\mathcal{K}}f$. In the examples below we are lucky: these two groups coincide. At this point we have already solved the first task, i. e. determined $MC\ Aut_{\mathcal{A}}F$. Indeed, by theorem 1.4.4 this group is isomorphic to $MC\ Aut_{\mathcal{K}}f$. Now we seek its representations λ_1 and λ_2 on the source and target spaces.
4) *Finding a nice representative in the $\mathcal{K}$-equivalence class of f* for which $G := MC\ Aut_{\mathcal{K}}f$ acts linearly (and so as an $\mathcal{A}$-automorphism group). We will continue to call it f.
5) *Finding a G-invariant complement V of* $\mathfrak{m}(n)\theta_f / tf(\theta_{\mathbb{R}^n}) + f^*(\mathfrak{m}(p))\theta_f$. Here we only have to understand the G action on θ_f.
6) *Determining λ_1 and λ_2.* We know that $\lambda_1 = \mu_1 \oplus \mu_V$ and $\lambda_2 = \mu_2 \oplus \mu_V$ where μ_1, μ_2 are the actions of G on the source and target spaces of f, and μ_V is the action of G on V.

Before turning to the Σ^1 and $\Sigma^{2,0}$ cases note that if F is a germ of an isolated embedding (a "Σ^0 singularity") then it is equivalent to $(\mathbb{R}^0, 0) \longrightarrow \mathbb{R}^k$, $0 \mapsto 0$. Here we have that $Aut_{\mathcal{A}}F \cong O(k)$ and the action on $\mathbb{R}^k$ is the usual $O(k)$-action.

Notation. In what follows ρ_l will always mean the usual representation of $O(l)$ on $\mathbb{R}^l$. If ρ_l is written as a representation of a group $O(l) \times H$ then ρ_l is really meant to be $\rho_l \circ pr_{O(l)}$.

• Σ^1 germs

For all codimension k and for all $r \in \{1, 2, \dots\}$ there is an isolated stable singularity $F_{r,k}$ of type Σ^1. The one corresponding to r and k (the "isolated Morin singularity of type Σ^{1_r} in codimension k") has local algebra $Q_{F_{r,k}} = \mathbb{R}[[x]]/(x^{r+1})$ (defect=0), so it is the miniversal unfolding of $f_{r,k} : (\mathbb{R}, 0) \longrightarrow \mathbb{R}^{k+1}$, $x \mapsto (x^{r+1}, 0, \dots, 0)$. A trivial argument shows that the maximal compact subgroup of $Aut\ Q_{F_{r,k}}$ is $\mathbb{Z}_2$. Therefore, according to section 1.4 $MC\ Aut_{\mathcal{K}} f_{r,k}$ is a subgroup of $G := \mathbb{Z}_2 \times O(k) = O(1) \times O(k)$. On the other hand $G \leq MC\ Aut_{\mathcal{A}} f_{r,k} \leq MC\ Aut_{\mathcal{K}} f_{r,k}$. Indeed, we can define the action of G as

$$\mu_1 := \rho_1 \qquad \text{on the source}$$

and

$$\mu_2 := \rho_1^{r+1} \oplus \rho_k \qquad \text{on the target.}$$

This shows that

$$MC\ Aut_{\mathcal{A}} F_{r,k} = (MC\ Aut_{\mathcal{K}} f_{r,k}) = O(1) \times O(k).$$

Now we are going to find the representation of G on the source and on the target spaces of $F_{r,k}$. First we need a G-invariant complement of the G-invariant subspace $tf_{r,k}(\theta_{\mathbb{R}^1}) + f_{r,k}^* \mathfrak{m}(k+1)\theta_{f_{r,k}}$ in $\mathfrak{m}(1)\theta_{f_{r,k}}$ — denote it by V — and determine the action of G on this V. By probation (or by the usual integration method) we can find a G-invariant V generated by $A_1, \dots, A_{r-1}, B_{1,1}, \dots, B_{r,k}$ defined by the map germs $x \mapsto$

$$\begin{array}{cccc}
(x, 0, \dots, 0)[A_1], & (0, x, 0, \dots, 0)[B_{1,1}], & \dots & (0, \dots, 0, x)[B_{1,k}] \\
\vdots & \vdots & & \vdots \\
(x^{r-1}, 0, \dots, 0)[A_{r-1}], & (0, x^{r-1}, 0, \dots, 0)[B_{r-1,1}], & \dots & (0, \dots, 0, x^{r-1})[B_{r-1,k}] \\
 & (0, x^r, 0, \dots, 0)[B_{r,1}], & \dots & (0, \dots, 0, x^r)[B_{r,k}].
\end{array}$$

As usual, we identified $\theta_{f_{r,k}}$ with $\mathcal{E}(1, k+1)$. Remark that by finding V we determined the source and the target space of $F_{r,k}$: they are $\mathbb{R}^{rk+r}$ and $\mathbb{R}^{rk+r+k}$, respectively.

Recall that the G action on V is the following

$$g \cdot \phi = \mu_2(g) \circ \phi \circ \mu_1(g^{-1})$$

for $g \in G$ and $\phi \in V$ (see the proof of lemma 1.4.1.). Computing this action on the basis of V given, we get that for $g = (\varepsilon, \psi) \in O(1) \times O(k) = G$

$$(\varepsilon, \psi) \cdot A_i = \varepsilon^{r+1+i} A_i \qquad \text{for } i = 1, \dots, r-1,$$

and

$$(\varepsilon, \psi) \cdot B_{i,j} = \varepsilon^i (\psi_{1,j} B_{i,1} + \dots + \psi_{k,j} B_{i,k}) \qquad \text{for } i = 1, \dots, r,\ j = 1, \dots, k.$$

So the G-representation on V is

$$\mu_V := \Big(\sum_{l=r+2}^{2r} \rho_1^{\otimes l}\Big) \oplus \Big(\sum_{i=1}^{r} \rho_k \otimes \rho_1^{\otimes i}\Big) =$$

$$= \left\lceil \frac{r-1}{2} \right\rceil 1 \oplus \left\lfloor \frac{r-1}{2} \right\rfloor \rho_1 \oplus \left\lfloor \frac{r}{2} \right\rfloor \rho_k \oplus \left\lceil \frac{r}{2} \right\rceil \rho_1 \otimes \rho_k.$$

So we have proved the following theorem.

1.5.1. THEOREM. *If F is a k-codimensional isolated germ of type Σ^{1_r} then $MC\ Aut_{\mathcal{A}}F \cong O(1) \times O(k)$ with the representations on the source and target spaces:*

$$\lambda_1 = \mu_1 \oplus \mu_V, \qquad \lambda_2 = \mu_2 \oplus \mu_V.$$

□

We have gained our goal of determining $MC\ Aut_{\mathcal{A}}F$ without explicitly writing up a *normal form* for F. One can however do it and verify that the given group with the given action is really an $\mathcal{A}$-automorphism group of F.

Remark. All the results on the maximal compact subgroup of Σ^1 (Morin) singularities given above have already been achieved by A. Szűcs [Sz6], [Sz7] by a completely different, geometrical approach.

• The simplest $\Sigma^{2,0}$ germs

Mather proved in [M6] that there are five infinite sequences of algebras corresponding to $\Sigma^{2,0}$ singularity types:

$I_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a + y^b)$	$b \geq a \geq 2$
$II_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a - y^b)$	$b \geq a \geq 2$ both even
$III_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a, y^b)$	$b \geq a \geq 2$
IV_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a)$	$a \geq 3$
V_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a, yx^{a-1})$	$a \geq 3$

Now we will compute $MC\ Aut_{\mathcal{A}}F$ (and its representations λ_1, λ_2 on the source and on the target spaces) for stable map germs corresponding to the algebras $III_{2,2}$. In some sense this is the simplest among the $\Sigma^{2,0}$ singularities — c. f. Appendix.

The algebra $III_{2,2}$

Denote by ρ_2^l the map $O(2) \longrightarrow O(2)$ which sends

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \cos l\alpha & -\sin l\alpha \\ \sin l\alpha & \cos l\alpha \end{pmatrix}$$

and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to itself.

We will compute the maximal compact subgroup of a k-codimensional isolated stable singularity F of type $III_{2,2}$. The germ F is then a miniversal unfolding of $f : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}^{k+2}$, $(x, y) \mapsto (x^2, y^2, xy, 0, \dots, 0)$.

First we have to compute $MC\ Aut\ \mathbb{R}[[x, y]]/(x^2, y^2, xy)$. With some algebraic machinery one can show that this group is $O(2)$ (the action of $O(2)$ is the extension of the usual action on the $\{(x, y)\}$ plane to the algebra $\mathbb{R}[[x, y]]/(x^2, y^2, xy)$). Fortunately $G := O(2) \times O(k-1)$ acts on f as a compact $\mathcal{K}$-automorphism group and therefore $MC\ Aut_{\mathcal{K}} f = O(2)\times O(k-1)$. It's easier to see this $O(2)\times O(k-1)$-action not on f but on the $\mathcal{K}$-equivalent germ:

$$\bar{f} : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}^{k+2}$$

$$(x, y) \mapsto (x^2 + y^2, x^2 - y^2, 2xy, 0, \dots, 0).$$

In fact the action of $O(2) \times O(k-1)$ is now an $\mathcal{A}$-action with representations

$$\mu_1 := \rho_2 \qquad \text{and} \qquad \mu_2 = 1 \oplus \rho_2^2 \oplus \rho_{k-1}$$

on the source and target spaces. One can verify it by explicit calculation or by comparing $\bar{f}$ with the complex map $z \mapsto z^2$.

A G-invariant complement V of $t\bar{f}(\theta_{\mathbb{R}^2}) + \bar{f}^*\mathfrak{m}(k+2)\theta_{\bar{f}}$ in $\mathfrak{m}(2)\theta_{\bar{f}}$ is spanned by $A, B, C, D, E_{1,1}, \dots, E_{2,k-1}$ defined by the germs: $(x, y) \mapsto$

$$\begin{array}{llll}
(0,\ x,\ y, 0, \dots, 0)\ [A], & (0,0,0,x,0,\dots,0)\ [E_{1,1}], & \dots & (0,\dots,0,x)\ [E_{1,k-1}] \\
(0, -y,\ x, 0, \dots, 0)\ [B], & (0,0,0,y,0,\dots,0)\ [E_{2,1}], & \dots & (0,\dots,0,y)\ [E_{2,k-1}] \\
(0,\ x, -y, 0, \dots, 0)\ [C], & & & \\
(0,\ y,\ x, 0, \dots, 0)\ [D]. & & &
\end{array}$$

Remark that by finding V we determined the dimensions of the source and the target spaces of F: $2k + 4$ and $3k + 4$.

Using the G action on V

$$g \cdot \phi = \mu_2(g) \circ \phi \circ \mu_1(g^{-1})$$

for $g \in G$ and $\phi \in V$ we can compute the $O(2) \times O(k-1)$ action on the basis of V given. We obtain that $O(2) \times O(k-1)$ acts on the plane spanned by A and B by ρ_2, it acts on the plane spanned by C and D by ρ_2^3, and it acts on the subspace spanned by $E_{1,1}, \dots, E_{2,k-1}$ by $\rho_2 \otimes \rho_{k-1}$. So the G representation on V is

$$\mu_V := \rho_2 \oplus \rho_2^3 \oplus (\rho_2 \otimes \rho_{k-1}).$$

We have proved the following theorem:

1.5.2. THEOREM. *If F is a k-codimensional isolated germ of type $III_{2,2}$ then $MC\ Aut_{\mathcal{A}} F \cong O(2) \times O(k-1)$ with the representations on the source and target spaces:*

$$\lambda_1 = \mu_1 \oplus \mu_V, \qquad \lambda_2 = \mu_2 \oplus \mu_V.$$

□

1.6. Multisingularities

1.6.1. Definition. *Let S be a finite subset of $\mathbb{R}^n$. The set of germs near S of smooth maps $(R^n, S) \longrightarrow (\mathbb{R}^p, 0)$ will be called $\mathcal{E}_S(n,p)$.*

As only the cardinality s of S counts, we will usually take $S = \{(i, 0 \ldots, 0) \mid i = 1, \ldots, |S|\}$. Let $\mathcal{A}_{n,p;s} := Diff_s(\mathbb{R}^n) \times Diff(\mathbb{R}^p)$, where $Diff_s\mathbb{R}^n$ is the group of germs of smooth maps $(\mathbb{R}^n, S) \longrightarrow (\mathbb{R}^n, S)$ that are diffeomorphisms near the points of S. This group acts on $\mathcal{E}_S(n,p)$ as in the case $s = 1$. The germs in the same orbit are called $\mathcal{A}$-equivalent.

1.6.2. Definition. *If $\eta \in \mathcal{E}_S(n,p)$ then its suspension $Susp(\eta) \in \mathcal{E}_S(n+1, p+1)$ is the germ defined by $(x,t) \mapsto (\eta(x), t)$, (for $x \in \mathbb{R}^n, t \in \mathbb{R}$).*

The suspension of $\mathcal{A}$-equivalent germs are also $\mathcal{A}$-equivalent. Generate an equivalence relation on

$$\bigcup_{s,n<p} \mathcal{E}_S(n,p)/\mathcal{A}_{n,p;s}$$

by $Susp()$. Call the equivalence classes (multi-)singularity *kind*s; and let the kind of η be denoted by $[\eta]$ (regardless of η being a germ or an $\mathcal{A}$ equivalence class). Every singularity kind contains a unique $\mathcal{A}$ equivalence class η whose multiple suspensions form $[\eta]$. We will call such an η the root of its kind. Recall that in case $s = 1$ we called roots *isolated singularities* in the previous sections.

Examples.

• Let $\eta : \mathbb{R}^0 \longrightarrow \mathbb{R}^k$, $0 \mapsto 0$. Then $[\eta]$ consists of all k-codimensional embeddings, and η is the root of $[\eta]$. Let $Emb(k) := [\eta]$.

• Let $\eta : (\mathbb{R}^k, \{x_1, x_2\}) \longrightarrow (\mathbb{R}^{2k}, 0)$ satisfy:

$$x \mapsto (x, 0) \in \mathbb{R}^k \times \mathbb{R}^k \text{ near } x_1 \qquad x \mapsto (0, x) \in \mathbb{R}^k \times \mathbb{R}^k \text{ near } x_2.$$

Then η is the root of its kind $Imm^2(k) := [\eta]$ which contains double point immersion germs.

• Let $\eta : (\mathbb{R}^{(r-1)k}, \{x_1, \ldots, x_r\}) \longrightarrow (\mathbb{R}^{rk}, 0)$ satisfy

$$(\mathbb{R}^k)^{r-1} \ni (x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{r-1}) \in (\mathbb{R}^k)^r$$

near x_i. The germ η is the root of its kind $Imm^r(k) := [\eta]$ which contains all r-tuple immersions.

• Let $\eta \in \mathcal{E}(rk+r, rk+r+k)$ be the miniversal unfolding of $(\mathbb{R}, 0) \longrightarrow (\mathbb{R}^{k+1}, 0)$, $x \mapsto (x^{r+1}, 0, \ldots, 0)$. Then η is the root of its kind $\Sigma^{1_r}(k) := [\eta]$ which contains all maps of Thom-Boardman type Σ^{1_r}.

• Let $\eta_i \in \mathcal{E}(n_i, n_i + k)$, $i = 1, \ldots, r$ be roots of k codimensional singularity kinds. Then define the germ:

$$\sum_{i=1}^r \eta_i : (\mathbb{R}^{\sum n_i + (r-1)k}, \{x_1, \ldots, x_r\}) \longrightarrow \mathbb{R}^{\sum (n_i+k)}$$

which maps

$$(x_1, \dots, x_n, t_1, \dots, t_{r-1}) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_r} \times (\mathbb{R}^k)^{r-1}$$

to

$$(x_1, \dots, x_{i-1}, pr_{\mathbb{R}^{n_i}} \circ \eta_i(x_i), x_{i+1}, \dots, x_n, t_1, \dots, t_{i-1}, pr_{\mathbb{R}^k} \circ \eta_i(x_i), t_i, \dots, t_{r-1})$$

$$\in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_r} \times (\mathbb{R}^k)^r$$

near x_i. This germ is the root of its kind, named $\sum \eta_i$. Observe that in this sense $Imm^r(k) = \sum_{i=1}^r Emb(k)$.

Remark also that every multisingularity kind is the sum of some simple ($s = 1$) singularities.

We can define (infinitesimal) stability of η, compact subgroups of $Aut_{\mathcal{A}}\eta$, contractibility of quotients of $Aut_{\mathcal{A}}\eta$ for a multisingularity η, as it was defined in the special case $s = 1$. The theorems of the previous sections easily extend to multisingularities.

1.6.3. THEOREM. *Let $\eta_i \in \mathcal{E}(n_i, n_i + k)$ be non-equivalent stable germs for $i = 1, \dots, r$ and let they all be roots of their kinds. Denote by η their linear combination:*

$$\sum c_i \eta_i : \mathbb{R}^{\sum c_i n_i + k(\sum c_i - 1)} \longrightarrow \mathbb{R}^{\sum c_i n_i + k \sum c_i}.$$

Then

- *every compact subgroup of $Aut_{\mathcal{A}}\eta$ is contained in a maximal one;*
- *any two maximal compact subgroups are conjugate in $Aut_{\mathcal{A}}\eta$;*
- *the quotient by a maximal compact subgroup is contractible;*
-
$$MC\ Aut_{\mathcal{A}}\eta = \prod_{i=1}^{r} \left(S_{c_i} \ltimes (MC\ Aut_{\mathcal{A}}\eta_i)^{c_i} \right),$$

where the semidirect product is determined by $\pi(g_1, \dots, g_{c_i}) \mapsto (g_{\pi(1)}, \dots, g_{\pi(c_i)})$.

One can also find the representations of the maximal compact subgroup on the source and the target spaces. If λ_1^i and λ_2^i are the representations of $MC\ Aut_{\mathcal{A}}\eta_i$ on the source and the target spaces, then the representation of $MC\ Aut_{\mathcal{A}}\eta$ on the target space $\mathbb{R}^{\sum c_i n_i + k \sum c_i} = \oplus(\mathbb{R}^{n_i+k})^{c_i}$ can be seen as follows. The group $(MC\ Aut_{\mathcal{A}}\eta_i)^{c_i}$ acts on it diagonally with the aid of λ_2^i and S_{c_i} permutes the subspaces $\mathbb{R}^{n_i+k}$. To find the action on the source space (near S) is not difficult either, but the formal description would make it that.

Chapter 2
The generalized Pontrjagin-Thom construction

2.1. $\mathcal{T}$-maps and their cobordism classes

Fix a positive integer k and a set $\mathcal{T}$ of singularity kinds all of codimension k.

2.1.1. Definition. *A smooth map $f : N \longrightarrow P$ is called a $\mathcal{T}$-map if for every $y \in f(N)$ the kind of the germ of f at $f^{-1}(y)$ is from $\mathcal{T}$. If N is a manifold with boundary then we also suppose that f behaves nicely near ∂N, i. e. $f(\partial N) \subset \partial P$ and for a collar C of ∂N: $f|_C = Susp(f|_{\partial N})$.*

If $\mathcal{T} = \{Emb(k)\}$ then $\mathcal{T}$-maps are the k-codimensional embeddings. If $\mathcal{T} = \{Emb(1), Imm^2(1), Imm^3(1), \Sigma^{1_1}(1)\}$ then $\mathcal{T}$-maps are dense among the maps $N^2 \longrightarrow P^3$. In general, if n and p are fixed and the pair (n, p) is *nice* (see [M6]) then there is a finite $\mathcal{T}$ containing stable (multi-)singularities for which $\mathcal{T}$-maps are dense among the maps $N^n \longrightarrow P^p$.

2.1.2. Definition. *The $\mathcal{T}$-maps $f_1 : N_1^n \longrightarrow P^p$ and $f_2 : N_2^n \longrightarrow P^p$ (N_1 and N_2 are closed) are called $\mathcal{T}$-cobordant if there is a manifold W with boundary the disjoint union of N_1 and N_2, and a $\mathcal{T}$-map $f : W \longrightarrow P \times [0, 1]$ such that $f|_{N_1} = f_1$, $f|_{N_2} = f_2$.*

$\mathcal{T}$-cobordism is clearly an equivalence relation, its equivalence classes are called ($\mathcal{T}$-)cobordism classes, their set is denoted by $Cob_n(P^p; \mathcal{T})$. If $P = S^p$ then we can define addition on it by "remote disjoint union", which makes it an Abelian group. The evidently defined oriented version of $Cob_n(P; \mathcal{T})$ is denoted by $Cob_n^{SO}(P; \mathcal{T})$.

Our goal is to define a classification space $X\mathcal{T}$ for $\mathcal{T}$-maps, i. e. a space for which there is a bijection (in certain cases a group isomorphism) between

$$Cob_n(P; \mathcal{T}) \qquad \text{and} \qquad [P, X\mathcal{T}],$$

where $[A, B]$ denotes the homotopy classes of maps $A \longrightarrow B$.

If $\eta \notin \mathcal{T}$ then we say that η suits to $\mathcal{T}$ if the kind of any germ sufficiently close to η is in $\mathcal{T}$ — or more precisely, there is at least one $\mathcal{T} \cup \{[\eta]\}$-map $f : N \longrightarrow P$ whose germ $(N, f^{-1}(y)) \longrightarrow (P, y)$ at some y is η. We will build up $\mathcal{T}$ as follows. Start with $\mathcal{T} = \{Emb(k)\}$ and add a kind $[\eta_1]$ to it for which η_1 suits $\mathcal{T}$. (In fact, here in the first step there is only one such $[\eta_1]$: $Imm^2(k)$.) Now, add $[\eta_2]$ which suits to $\{Emb(k), [\eta_1]\}$, and so on, successively adding such new kinds to $\mathcal{T}$ that suit to the already defined $\mathcal{T}$. We will consider only such $\mathcal{T}$'s that can be built up this way. The others are not interesting in our applications. This condition on $\mathcal{T}$ is equivalent to saying that $\mathcal{T}$ is "ascending" in the hierarchy of germs — see also the Appendix.

2.2. The main theorem

2.2.1. THEOREM. *If $\mathcal{T}$ is as above, then there exist topological spaces $X\mathcal{T}$ and $Y\mathcal{T}$ and a continuous map $f\mathcal{T} : Y\mathcal{T} \longrightarrow X\mathcal{T}$ for which:*

(A) if $P^{n+k} \xrightarrow{g} X\mathcal{T}$ is continuous and $\bar{f} : M^{n-1} \longrightarrow \partial P$ is a $\mathcal{T}$-map making the diagram

$$\begin{array}{ccc} \partial P & \xrightarrow{g|_{\partial P}} & X\mathcal{T} \\ \Big\uparrow{\scriptstyle \bar{f}} & & \Big\uparrow{\scriptstyle f\mathcal{T}} \\ M & \xrightarrow{h} & Y\mathcal{T} \end{array}$$

commutative for some continuous h, then (after possibly a small perturbation[2] of g) there is a manifold N^n with boundary M and maps $f : N \longrightarrow P$, $\bar{g} : N \longrightarrow Y\mathcal{T}$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & X\mathcal{T} \\ \Big\uparrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle f\mathcal{T}} \\ N & \xrightarrow{\bar{g}} & Y\mathcal{T} \end{array}$$

commutative, and $\bar{g}|_{\partial N} = h$.

(B) If $f : N^n \longrightarrow P^{n+k}$ is a $\mathcal{T}$-map between manifolds with boundary, and a commutative diagram

$$\begin{array}{ccc} \partial P & \xrightarrow{h} & X\mathcal{T} \\ \Big\uparrow{\scriptstyle f|_{\partial N}} & & \Big\uparrow{\scriptstyle f\mathcal{T}} \\ \partial N & \xrightarrow{\bar{h}} & Y\mathcal{T} \end{array}$$

is given, then h and $\bar{h}$ extend to maps g and $\bar{g}$, making the following diagram commutative

$$\begin{array}{ccc} P & \xrightarrow{g} & X\mathcal{T} \\ \Big\uparrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle f\mathcal{T}} \\ N & \xrightarrow{\bar{g}} & Y\mathcal{T}. \end{array}$$

Before proving the theorem prove the most important corollary of it.

2.2.2. COROLLARY. *The space $X\mathcal{T}$ is a classifying space for $\mathcal{T}$-maps in the following sense. There is a bijection between*

$$Cob_n(P^{n+k}; \mathcal{T}) \qquad \text{and} \qquad [P^{n+k}, X\mathcal{T}].$$

Proof of the Corollary. Let N and P be closed manifolds and let $f : N^n \longrightarrow P^{n+k}$ be a $\mathcal{T}$-map. The homotopy class of g assigned to f in part (B) of the theorem is in $[P, X\mathcal{T}]$. If f_1 is $\mathcal{T}$-cobordant to f then the cobordism between them is a $\mathcal{T}$-map $F : W \longrightarrow P \times [0,1]$. Then the map $G : P \times [0,1] \longrightarrow X\mathcal{T}$ assigned to F in part (B) is a homotopy between g and g_1 (g_1 is assigned to f_1 in part (B)).

[2]by a small perturbation we mean a homotopy. Although the map homotopic to g can be chosen close to g in some sense (that's why "small perturbation"; see [MS, ch. 18]), we will not need this fact.

So $\phi : Cob_n(P;\mathcal{T}) \longrightarrow [P, X\mathcal{T}]$ is well defined. The surjectivity of ϕ is proved by part (A). To prove injectivity of ϕ suppose we have $\mathcal{T}$-maps $f : N \longrightarrow P$ and $f_1 : N_1 \longrightarrow P$, and the maps assigned to them in part (B) are g and g_1. Let $g, g_1 : P \longrightarrow X\mathcal{T}$ be homotopic by $G : P \times [0,1] \longrightarrow X\mathcal{T}$. Then the map assigned to G by part (A) is a cobordism between f and f_1.[3] $\square$

Proof of the theorem. The proof will proceed by induction: suppose we know the theorem for $\mathcal{T}'$ and we want to prove it for $\mathcal{T} = \mathcal{T}' \cup \{[\eta]\}$ where $[\eta]$ suits to $\mathcal{T}'$. Suppose also that $\eta : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+k}$ is the root of $[\eta]$.

Let G denote the maximal compact subgroup of $Aut_{\mathcal{A}}\eta$ with the representations λ_1 and λ_2 on the source and the target spaces. The vector bundles associated to $EG \longrightarrow BG$ using these representations will be called $\bar{\xi_\eta}$ and ξ_η respectively. There is also a well defined fibrewise map $f_\eta : E(\bar{\xi_\eta}) \longrightarrow E(\xi_\eta)$ which is $\mathcal{A}$-equivalent to η in each fibre (see [Sz3]). Since $[\eta]$ suits to $\mathcal{T}'$, the restriction of f_η to the boundary of $D(\bar{\xi_\eta})$ is a $\mathcal{T}'$-map so by the induction hypothesis there are maps $\rho, \bar{\rho}$ making the diagram

$$\begin{array}{cccc} f_\eta|_{\partial(D\bar{\xi_\eta})}: & \partial(D\bar{\xi_\eta}) & \longrightarrow & \partial(D\xi_\eta) \\ & \big\downarrow \bar{\rho} & & \big\downarrow \bar{\rho} \\ & Y\mathcal{T}' & \xrightarrow{f\mathcal{T}'} & X\mathcal{T}' \end{array}$$

commutative. We will prove that the spaces

$$X\mathcal{T} := X\mathcal{T}' \cup_\rho D\xi_\eta,$$

$$Y\mathcal{T} := Y\mathcal{T}' \cup_{\bar{\rho}} D\bar{\xi_\eta}$$

and the map

$$f\mathcal{T} := f\mathcal{T}' \cup f_\eta$$

satisfy the conditions of the theorem.[4]

Part (A). We will carry out the proof only in the case when P is closed and η is a simple (non-multi-) singularity — the general case is not more difficult, but the notation and the technical machinery would conceal the idea.

The space BG is in $D\xi_\eta$ (as the zero section), so it is in $X\mathcal{T}$ also. After possibly a small perturbation (homotopy) of g we can suppose that g is transversal to BG. Let $K := g^{-1}(BG) \subset P$ and U a closed tubular neighbourhood of K in P. Then the tube U can be identified with $D(g|_K^*\xi_\eta)$. Let $P' := \overline{P - U}$.

---

[3]Although we might change G by a small perturbation when we define F, but we will see that there is no need for this perturbation on the two collars of $P \times [0,1]$ where G coincides with $g \times id$ and $g_1 \times id$, and g, g_1 are assigned to $\mathcal{T}$-maps f, f_1 by part (B).

[4]This makes sense, since $f\mathcal{T}'$ and f_η coincide on $\partial(D\bar{\xi_\eta})$, i. e. $\rho \circ f_\eta|_{\partial(D\bar{\xi_\eta})} = f\mathcal{T}' \circ \bar{\rho}$ by definition.

The following diagram may help the reader to follow the proof.

$$\begin{tikzcd}
P & = & P' \arrow[d, "g|_{P'}"] & \cup & U \arrow[d, "g|_U"] & \supset & K \arrow[d, "g|K"] \\
X\mathcal{T} & = & X\mathcal{T}' & \cup_\rho & D\xi_\eta & \supset & BG \\
Y\mathcal{T} \arrow[u, "f\mathcal{T}"] & = & Y\mathcal{T}' \arrow[u, "f\mathcal{T}'"] & \cup_{\bar\rho} & D\bar\xi_\eta \arrow[u, "f_\eta"] & \supset & BG \arrow[u, equal] \\
N & = & N' \arrow[u, "\bar g_{N'}"] & \cup & V \arrow[u, "\bar g_V"] & \supset & K \arrow[u, "g|_K"]
\end{tikzcd}$$

Let V be the disc bundle of $g|_K^*\bar\xi_\eta$ (this defines the map $\bar g_V : V \longrightarrow D\bar\xi_\eta$) and let us pull back f_η to $\bar f_\eta : V \longrightarrow U$. Now we can use the induction hypothesis (the theorem for $\mathcal{T}'$ and P'), because P' is a manifold with boundary and

$$\begin{tikzcd}
\partial P' = & \partial U \arrow[r, "\rho\circ g|_U"] & X\mathcal{T}' \\
 & \partial V \arrow[u, "\bar f_\eta"] \arrow[r, "\rho\circ\bar g_V"] & Y\mathcal{T}' \arrow[u, "f\mathcal{T}'"]
\end{tikzcd}$$

is given. Therefore it extends to

$$\begin{tikzcd}
P' \arrow[r, "g|_{P'}"] & X\mathcal{T}' \\
N' \arrow[u, "f_{N'}"] \arrow[r, "\bar g_{N'}"] & Y\mathcal{T}'. \arrow[u, "f\mathcal{T}'"]
\end{tikzcd}$$

Then

$$N := N' \cup_{id_{\partial V}} V, \qquad \bar g := \bar g_V \cup \bar g_{N'} \qquad \text{and} \qquad f := \bar f_\eta \cup f_{N'}$$

satisfy the conditions of the theorem. $\square$

Before turning to the proof of part (B) let us derive a consequence of theorem 1.3.6.

Let us fix a maximal compact subgroup G of $Aut_{\mathcal{A}}\eta$, where $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is a stable germ. If E is a set, B is a smooth manifold then a map $p : E \longrightarrow B$ is called a bundle with fibre $Aut_{\mathcal{A}}\eta/G$ provided there is given an open cover $\{U_i\}$ of B such that $p|_{p^{-1}(U_i)}$ is the projection $U_i \times Aut_{\mathcal{A}}\eta/G \longrightarrow U_i$, and the transition maps (along which these product spaces are glued together in E) are smooth maps $U_i \cap U_j \longrightarrow Aut_{\mathcal{A}}\eta$ (remember that $Aut_{\mathcal{A}}\eta$ acts on $Aut_{\mathcal{A}}\eta/G$). A smooth section of such a bundle is a section $s : B \longrightarrow E$ satisfying that $pr_{Aut_{\mathcal{A}}\eta/G} \circ s|_{V_i} : V_i \longrightarrow Aut_{\mathcal{A}}\eta/G$ are smooth maps for some open cover $\{V_i\}$ which is a refinement of $\{U_i\}$.

Theorem 1.3.6. — the contractibility of $Aut_{\mathcal{A}}\eta/G$ — assures that all bundles with fibre $Aut_{\mathcal{A}}\eta/G$ have a smooth section. Indeed, a section which is almost everywhere smooth can be constructed by skeleton induction: the induction step is exactly what we have proved in theorem 1.3.6. This section might not be smooth where the cells of B meet. However the standard smoothing procedure of that kind of sections (see e.g. [H] 2.2.11.) works here with no change.

In what follows we will consider bundle germs with fibre $\mathbb{R}^l$ whose structure group is a subgroup of $Diff(\mathbb{R}^l)$. Note that this kind of bundle germs over smooth base spaces can be defined even if there is no topology on $Diff(\mathbb{R}^l)$, because the smoothness of the transition maps is defined. The usual notion of *equivalence of bundles* also extends to this generalized bundles.

Now consider two bundle germs $\xi_1 : E_1 \longrightarrow B$ and $\xi_2 : E_2 \longrightarrow B$ with fibres $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. Let the structure group of ξ_i be $pr_i(Aut_{\mathcal{A}}\eta)$ (remember that $Aut_{\mathcal{A}}\eta \subset Diff(\mathbb{R}^n) \times Diff(\mathbb{R}^p)$). Also suppose that ξ_1 and ξ_2 are "associated to each other" in the following sense. There is an open cover $\{U_i\}$ of B whose elements are trivializing neighbourhoods of both ξ_1 and ξ_2 for which the transition maps $\phi^1_{ij} : U_i \cap U_j \longrightarrow Diff(\mathbb{R}^n)$ and $\phi^2_{ij} : U_i \cap U_j \longrightarrow Diff(\mathbb{R}^p)$ have the form $\phi^1_{ij} = pr_1 \circ \phi_{ij}$ and $\phi^2_{ij} = pr_2 \circ \phi_{ij}$ for some smooth $\phi_{ij} : U_i \cap U_j \longrightarrow Aut_{\mathcal{A}}\eta$.

Our goal is to reduce the structure group from $Aut_{\mathcal{A}}\eta$ to its maximal compact subgroup, which is a Lie group, so the bundles become vector bundles.

2.2.3. Lemma. *There exist $\xi_1' : E_1' \longrightarrow B$ and $\xi_2' : E_2' \longrightarrow B$ bundle germs which*

- *are equivalent to ξ_1 and ξ_2,*
- *have structure groups $pr_1(G)$ and $pr_2(G)$, and are associated to each other.*

Proof. First associate to ξ_1 and ξ_2 a "bundle" $\xi' : E' \longrightarrow B$ with fibre $Aut_{\mathcal{A}}\eta/G$: if the U_i's are trivializing neighbourhoods of ξ_1 and ξ_2 (with transition maps ϕ^1_{ij}, ϕ^2_{ij} as above) then glue $U_i \times Aut_{\mathcal{A}}\eta/G$'s together by ϕ_{ij}. Denote by p_i the projection of $U_i \times Aut_{\mathcal{A}}\eta/G$ to the second factor. Take a smooth section s of ξ'. Let us recall that in section 1.3. we constructed a a section $\sigma : \mathcal{A}_\eta/G \longrightarrow \mathcal{A}_\eta$ of the "fibration" $\pi : \mathcal{A}_\eta \longrightarrow \mathcal{A}_\eta/G$. Now let $\lambda_i = \sigma \circ p_i \circ s : U_i \longrightarrow Aut_{\mathcal{A}}\eta$ and $\bar\phi_{ji} = \lambda_j^{-1}\phi_{ji}\lambda_i$. Using the new transition maps $pr_1 \circ \bar\phi_{ji}$ and $pr_2 \circ \bar\phi_{ji}$ we can construct bundle germs ξ_1' and ξ_2'. From the form of $\bar\phi_{ij}$ it is clear that ξ_1 and ξ_2 are equivalent to ξ_1' and ξ_2', and $\bar\phi_{ij}(u) \in G$ because if $\pi : Aut_{\mathcal{A}}\eta \longrightarrow Aut_{\mathcal{A}}\eta/G$ is the natural projection then

$$\pi(\bar\phi_{ji}(u)) = \pi(\lambda_j(u)^{-1})\pi(\phi_{ji}(u))\pi(\lambda_i(u)) =$$

$$= (p_j f(u))^{-1}\pi(\phi_{ji}(u))p_i f(u) = (p_j f(u))^{-1} p_j f(u) = \text{the coset of } G.$$

□

Proof of part (B). Suppose that the statement is true for $\mathcal{T}'$ and prove it for $\mathcal{T} = \mathcal{T}' \cup \{[\eta]\}$ where we assume η to be a simple (non-multi) singularity suiting to $\mathcal{T}'$, and that η is the root of $[\eta]$. Also suppose that $\partial N = \partial P = \emptyset$. (The proof for multisingularities and manifolds with boundaries goes along the same line but it makes notation extremely difficult.)

Let $K \subset P$ be the submanifold of y's for which the germ $f : (N, f^{-1}(y)) \longrightarrow (P, y)$ is from $[\eta]$, and let $\bar K := f^{-1}(K)$. (Remark that $f|_{\bar K}$ is a diffeomorphism.) To understand the situation we note that the restriction of f maps a transversal slice of $\bar K$ to a transversal slice of K, and this restriction is $\mathcal{A}$-equivalent to η.

Take tubular neighbourhoods $\bar U$ and U of $\bar K$ and K in N and P respectively. The projection maps

$$\bar U \longrightarrow \bar K,$$

$$U \longrightarrow K$$

are bundles with fibres $\mathbb{R}^n$ and $\mathbb{R}^p$ and have structure groups $pr_1(Aut_{\mathcal{A}}\eta)$ and $pr_2(Aut_{\mathcal{A}}\eta)$ (see also [Sz3]). Further, they are "associated to each other" in the sense used in the discussion before the theorem. Now lemma 2.2.3. states that the structure groups can be reduced to $pr_i(MC\ Aut_{\mathcal{A}}\eta)$, $i = 1, 2$. This means that the bundles $\bar{U} \longrightarrow K$ and $U \longrightarrow K$ are pull-back bundles of $\bar{\xi}_\eta \longrightarrow BG$ and $\xi_\eta \longrightarrow BG$ by some maps $\bar{h} : \bar{K} \longrightarrow BG$ and $h : K \longrightarrow BG$, and this diagram commutes

$$\begin{tikzcd}
\bar{\xi}_\eta \arrow[r] \arrow[rrr, bend left, "f_\eta"] & BG \arrow[r, equal] & BG & \xi_\eta \arrow[l] \\
\bar{U} \arrow[u, "\bar{g}_{\bar{U}}"'] \arrow[r] \arrow[rrr, bend right, "f|_{\bar{U}}"'] & \bar{K} \arrow[u, "\bar{h}"'] \arrow[r, "f|_{\bar{K}}"] & K \arrow[u, "h"'] & U. \arrow[l] \arrow[u, "g_U"']
\end{tikzcd}$$

Because of the induction hypotheses there are maps

$$N' := N - \bar{U} \xrightarrow{\bar{g}_{N'}} Y\mathcal{T}'$$

$$P' := P - U \xrightarrow{g_{P'}} X\mathcal{T}'$$

for which the diagram

$$\begin{array}{ccc} P' & \xrightarrow{g_{P'}} & X\mathcal{T}' \\ \uparrow f|_{N'} & & \uparrow f\mathcal{T}' \\ N' & \xrightarrow{\bar{g}_{N'}} & Y\mathcal{T}' \end{array}$$

is commutative and

$$\bar{g}_{N'}|_{\partial\bar{U}} = \bar{\rho} \circ \bar{g}_{\bar{U}}|_{\partial\bar{U}}, \qquad g_{P'}|_{\partial U} = \rho \circ g_U|_{\partial U}.$$

This means that there are maps

$$g = g_U \cup g_{P'} : P \longrightarrow X\mathcal{T}',$$

$$\bar{g} = \bar{g}_{\bar{U}} \cup \bar{g}_{N'} : N \longrightarrow Y\mathcal{T}';$$

and the commutativity of the diagrams above implies the commutativity of

$$\begin{array}{ccc} P & \xrightarrow{g} & X\mathcal{T} \\ \uparrow f & & \uparrow f\mathcal{T} \\ N & \xrightarrow{\bar{g}} & Y\mathcal{T}. \end{array}$$

□

Now suppose that $P = S^{n+k}$. Then the operation in the homotopy group $\pi_{n+k}(X\mathcal{T}) = [S^{n+k}, X\mathcal{T}]$ clearly corresponds to the (remote) disjoint union operation in $Cob_n(S^{n+k}; \mathcal{T})$. Therefore we have the group isomorphism:

$$\pi_{n+k}(X\mathcal{T}) \cong Cob_n(S^{n+k}; \mathcal{T}).$$

As a special case we obtained the theorem of Thom [T]:

$$\pi_{n+k}(MO(k)) \cong Cob_n(S^{n+k}; \{Emb(k)\}).$$

We can generalize the analogous statement of Thom dealing with oriented cobordisms of oriented embeddings:

$$\pi_{n+k}(MSO(k)) \cong Cob_n^{SO}(S^{n+k}; \{Emb(k)\}).$$

To perform this generalization for $\mathcal{T}$-maps we need some definitions. Denote by $Diff^+(\mathbb{R}^n)$ the subgroup (of index two) in $Diff(\mathbb{R}^n)$ containing the elements whose differentials at 0 have positive determinant and let

$$Diff^-(\mathbb{R}^n) = Diff(\mathbb{R}^n) - Diff^+(\mathbb{R}^n).$$

2.2.4. Definition. *If $G \leq Diff(\mathbb{R}^n) \times Diff(\mathbb{R}^p)$ then*

$$G^{SO} := G \cap \Big(Diff^+(\mathbb{R}^n) \times Diff^+(\mathbb{R}^p) \cup Diff^-(\mathbb{R}^n) \times Diff^-(\mathbb{R}^p) \Big).$$

Now for every simple singularity kind $[\eta]$ in $\mathcal{T}$ change the group $G = MC\ Aut_{\mathcal{A}}\eta$ to G^{SO} in the definition of $X\mathcal{T}$ and $Y\mathcal{T}$ (and perform the analogous changes for multisingularities). Denote the resulting spaces by $X^{SO}\mathcal{T}$, $Y^{SO}\mathcal{T}$. Now it is clear that the oriented cobordism set $Cob_n^{SO}(P, \mathcal{T})$ is in a one-to-one correspondence with $[P, X^{SO}\mathcal{T}]$. In case $P = S^{n+k}$ this correspondence is also a group isomorphism.

Chapter 3

Applications to differential topology

In this chapter some differential topological applications of the generalized Pontrjagin-Thom construction will be presented. We will concentrate on $\Sigma^{2,0}$ singularities, mainly the simplest one ($III_{2,2}$) of that type. The necessary calculations for this germ — i. e. the calculation of the maximal compact subgroup of its symmetry group — can be found in chapter 1. Similar work has been done for Morin singularities by A. Szűcs in [Sz6], [Sz7]. I also thank A. Szűcs for some ideas and techniques used in sections 3.2 and 3.3.

Before turning to the results let us summarize the dimensions of the submanifolds of the points that have the simplest singularities under a map $N^n \longrightarrow P^{n+k}$.

dim	$corank = 0,$	$1,$	$2,$	3
n	Σ^0			
.				
.				
.				
$n-k$	$2\Sigma^0$			
$n-k-1$		Σ^1		
.				
.				
.				
$n-2k$	$3\Sigma^0$			
$n-2k-1$		$\Sigma^0+\Sigma^1$		
$n-2k-2$		$\Sigma^{1,1}$		
$n-2k-3$				
$n-2k-4$			$III_{2,2}$	
.				
.				
.				
$n-3k$	$4\Sigma^0$			
$n-3k-1$		$2\Sigma^0+\Sigma^1$		
$n-3k-2$		$\Sigma^{1,1}+\Sigma^0$; $2\Sigma^1$		
$n-3k-3$		$\Sigma^{1,1,1}$		
$n-3k-4$			$\Sigma^0+III_{2,2}$; $I_{2,2}$; $II_{2,2}$	
$n-3k-5$			$III_{2,3}$	
$n-3k-6$				
$n-3k-7$				
$n-3k-8$				
$n-3k-9$				Σ^3

3.1. Orientability

Fix a stable multisingularity type η. Remember that for almost all smooth maps $f : N^n \longrightarrow P^p$ the the subset $K_\eta \subset P$ containing the points $y \in P$ for which the germ $(N, f^{-1}(y)) \longrightarrow (P, y)$ is $\mathcal{A}$-equivalent to η, is a submanifold. In the most important applications f does not have more complicated singularities than η. In this case K_η is a closed submanifold.

3.1.1. THEOREM. *Let the stable germ $\eta : (\mathbb{R}^n, S) \longrightarrow \mathbb{R}^{n+k}$ be the root of $[\eta]$. The following statements are equivalent:*

a) for every orientable manifold P^{m+k} and smooth map $f : N^m \longrightarrow P^{m+k}$, the submanifold $K_{Susp^{m-n}(\eta)}$ is orientable;

b) $\lambda_2(MC\ Aut_{\mathcal{A}}\eta) \leq Diff^+(\mathbb{R}^{n+k})$.

Proof. The orientability of a submanifold in an orientable manifold is equivalent to the orientability of its normal bundle. In chapter 2 we proved that this normal bundle is the pull-back of the universal bundle with whose structure group $\lambda_2(MC\ Aut_{\mathcal{A}}\eta)$. $\square$

Examples.

- $Imm^r(k)$: If η is the root of $Imm^r(k)$ given in chapter 1, then $MC\ Aut_{\mathcal{A}}\eta \cong S_r \ltimes (O(k))^r$ and the determinant of $\lambda_2(\pi, A_1, \dots, A_r)$ is $(sgn\ \pi)^k \prod_{i=1}^r (\det\ A_i)$. So we obtain that no choice of k can guarantee the orientability of the submanifold of r-tuple points.
- $\Sigma^{1_r}(k)$: The maximal compact subgroup of the root of $\Sigma^{1_r}(k)$ is $O(1) \times O(k)$, and the λ_2 representation of it is

$$\rho_1^{r+1} \oplus \rho_k \oplus \left\lceil \frac{r-1}{2} \right\rceil 1 \oplus \left\lfloor \frac{r-1}{2} \right\rfloor \rho_1 \oplus \left\lfloor \frac{r}{2} \right\rfloor \rho_k \oplus \left\lceil \frac{r}{2} \right\rceil (\rho_1 \otimes \rho_k).$$

If $\varepsilon = \det \rho_1(g)$ and $a = \det \rho_k(g)$ then

$$\det \lambda_2(g) = \varepsilon^{r+1+\lfloor \frac{r-1}{2} \rfloor + k \lceil \frac{r}{2} \rceil} \cdot a^{1+r},$$

and it is positive (for all ε and a) if and only if $r \equiv 1\ mod\ 4$ and k is even.
- $III_{2,2}(k)$: The maximal compact subgroup of the root of $III_{2,2}(k)$ is $O(2) \times O(k-1)$, and the λ_2 representation of it is

$$1 \oplus \rho_2^2 \oplus \rho_{k-1} \oplus \rho_2 \oplus \rho_2^3 \oplus (\rho_2 \otimes \rho_{k-1}).$$

If $\alpha = \det \rho_2(g)$ and $a = \det \rho_{k-1}(g)$ then

$$\det \lambda_2(g) = \alpha^k \cdot a^3,$$

therefore no choice of k can guarantee that this determinant is positive.

We can formulate the corresponding theorem if we suppose the source to be orientable, too:

3.1.2. THEOREM. *Let the stable germ $\eta : (\mathbb{R}^n, S) \longrightarrow \mathbb{R}^{n+k}$ be the root of $[\eta]$. The following statements are equivalent:*

a) for every orientable manifolds N^m and P^{m+k} and smooth map $f : N^m \longrightarrow P^{m+k}$, the submanifold $K_{Susp^{m-n}(\eta)}$ is orientable;

b) $\lambda_2((MC\ Aut_{\mathcal{A}}\eta)^{SO}) \leq Diff^+(\mathbb{R}^{n+k})$.

c) There is no element $g \in MC\ Aut_{\mathcal{A}}\eta$ for which $\lambda_1(g) \in Diff^-(\mathbb{R}^n)$ and $\lambda_2(g) \in Diff^-(\mathbb{R}^{n+k})$.

Proof. The conditions $a)$ and $b)$ are equivalent as in the previous theorem. The conditions $b)$ and $c)$ are equivalent by definition. $\square$

Examples.

- $Imm^r(k)$. To check condition $c)$ we have to see whether $(sgn\ \pi)^k \prod_{i=1}^r (\det\ A_i)$ is positive if every $\det\ A_i$ is positive. Therefore we have that condition $c)$ is guaranteed if and only if k is even.
- $\Sigma^{1_r}(k)$. For the root $\eta \in \mathcal{E}(rk+r, rk+r+k)$ of Σ^{1_r}:

  $$(MC\ Aut_{\mathcal{A}}\eta)^{SO} = SO(1)\times SO(k) \cup NO(1)\times NO(k) \leq O(1)\times O(k) = MC\ Aut_{\mathcal{A}}\eta$$

  if r is odd, and

  $$(MC\ Aut_{\mathcal{A}}\eta)^{SO} = O(1) \times SO(k) \leq O(1) \times O(k) = MC\ Aut_{\mathcal{A}}\eta$$

  if r is even. An easy calculation shows that the condition $c)$ is satisfied if and only if one of the following is: k is even and $r \equiv 0$ or 1 mod 4 or k is odd and r is even.
- $III_{2,2}(k)$. For the root $\eta \in \mathcal{E}(2k+4, 3k+4)$ of $III_{2,2}$:

  $$(MC\ Aut_{\mathcal{A}}\eta)^{SO} = O(2) \times SO(k-1) \leq O(2) \times O(k-1) = MC\ Aut_{\mathcal{A}}\eta.$$

  An easy calculation shows that the condition $c)$ is satisfied if and only if k is even.

Remark that the submanifold of $\Sigma^{2,0}$ points is the closure of the submanifold of $III_{2,2}$ points and the difference of the two sets is a k codimensional submanifold of the first one. The orientability of a manifold can not be spoiled by a k codimensional manifold if $k > 1$, so the orientability results on the manifold of $III_{2,2}$ points are true for the manifold of $\Sigma^{2,0}$ points, too, provided the codimension is at least 2.

3.2. On the cobordism group of $\Sigma^{2,0}$ maps

In this section we prove a theorem on the cobordism group of maps having no worse then $III_{2,2}$ singularities (see Appendix). We will rely on some basic algebraic topology. Using finer algebraic topological tools we can prove sharper result — the theorem presented here is just a simple example of the usage of the generalized Thom construction given in chapter 2.

Let us use the following notations:

$$Imm(k) := \bigcup_{i=1}^{\infty} Imm^r(k),$$

$$\Sigma^1(k) := \bigcup_{i=r}^{\infty} \Sigma^{1_r}(k),$$

$$\Sigma^2(k) := \bigcup \{[\eta] \mid \eta \in \mathcal{E}(?, ? + k) \text{ is of type } \Sigma^2\},$$

$$\mathcal{T}'(k) := \{Emb(k)\} \cup Imm(k) \cup \Sigma^1(k),$$

$$\mathcal{T}(k) := \mathcal{T}'(k) \cup III_{2,2}(k).$$

For a prime p let us denote by $\mathcal{C}_p$ the class of Abelian groups whose exponent is finite and p is not a divisor of this exponent. An Abelian group A is called p-primary if and only if $A \in \mathcal{C}_q$ for all prime q distinct from p. A homomorphism between Abelian groups is called an "isomorphism modulo 2-torsion" if both the kernel and the cokernel is 2-primary.

3.2.1. Theorem. *If k is even then*

$$Cob_n(S^{n+k}; \mathcal{T}(k)) \quad \text{and} \quad Cob_n(S^{n+k}; \mathcal{T}'(k))$$

are isomorphic modulo 2-torsion.

Remark. If $n < 3k + 3$ then $\mathcal{T}$-maps (i. e. maps having no more complicated singularities than the simplest $\Sigma^{2,0}$ ones) are generic. Indeed, the set of points having the simplest $\Sigma^{2,0}$ singularities after $III_{2,2}$ ($I_{2,2}$ and $II_{2,2}$) has dimension $n - 3k - 4$ (see p. 42 or the Appendix), and the set of points having the simplest $\Sigma^{2,1}$ singularities has dimension $n - 4k - 7$. Also in this dimension range stable maps are dense among the smooth maps, therefore

$$Cob_n(S^{n+k}; \mathcal{T}(k)) \cong \mathfrak{N}_n \qquad \text{if } n < 3k + 3,$$

so in the dimension range $n < 3k + 3$ the group $Cob_n(S^{n+k}; \mathcal{T}(k))$ is 2-primary. The same is true for $Cob_n(S^{n+k}; \mathcal{T}'(k))$ (see [Sz7]) so in this dimension range the statement of the theorem is trivial.

Remark also that Koschorke (see [K]) computes the rank of the group

$$Cob_n(S^{n+k}; \Sigma^2(k)),$$

but this group coincides with our $Cob_n(S^{n+k}; \mathcal{T}(k))$ only in the trivial case $n < 3k + 3$ just mentioned.

Proof. The proof will be carried out in 4 steps.

(1) Using the generalized Thom construction of chapter 2 we have that

$$Cob_n(S^{n+k}; \mathcal{T}(k)) \cong \pi_{n+k}(X\mathcal{T}(k)).$$

(2) A portion of the homotopy exact sequence of $(X\mathcal{T}(k), X\mathcal{T}'(k))$ is

$$\pi_{n+k}(X\mathcal{T}'(k)) \longrightarrow \pi_{n+k}(X\mathcal{T}(k)) \longrightarrow \pi_{n+k}(X\mathcal{T}(k), X\mathcal{T}'(k)).$$

The group $\pi_i(X\mathcal{T}'(k))$ is isomorphic to $Cob_{i-k}(S^i; \mathcal{T}'(k))$. Therefore the statement of the theorem follows if we prove that $\pi_{n+k}(X\mathcal{T}(k), X\mathcal{T}'(k))$ is 2-primary (for all n), that is

$$\pi_{n+k}(X\mathcal{T}(k), X\mathcal{T}'(k)) \in \mathcal{C}_p \qquad \text{for all odd prime } p.$$

(3) Fix an odd prime p. The so called "*mod* $\mathcal{C}_p$ approximation theorem" (see e.g. [MT]) states that the condition

$$\pi_{n+k}(X\mathcal{T}(k), X\mathcal{T}'(k)) \in \mathcal{C}_p \qquad \text{for all } n$$

is equivalent to the condition

$$H_{n+k}(X\mathcal{T}(k), X\mathcal{T}'(k); \mathbb{Z}_p) = 0 \qquad \text{for all } n.$$

(4) Let ξ be the universal vector bundle assigned to the Lie group $O(2)\times O(k-1)$ and its representation

$$\lambda_2 = 1 \oplus \rho_2 \oplus \rho_2^2 \oplus \rho_2^3 \oplus \rho_{k-1} \oplus (\rho_2 \otimes \rho_{k-1}).$$

It can be identified with the vector bundle over $BO(2) \times BO(k-1)$

$$\xi = \varepsilon^1 \oplus \pi_2^*(\gamma_2) \oplus \pi_2^*(\gamma_2^2) \oplus \pi_2^*(\gamma_2^3) \oplus \pi_{k-1}^*(\gamma_{k-1}) \oplus (\gamma_2 \times \gamma_{k-1}),$$

where γ_i is the universal $\mathbb{R}^2$-bundle, the maps π_2, π_{k-1} are the projections of $BO(2)\times BO(k-1)$ to its factors, and γ_2^i is the $\mathbb{R}^2$-bundle assigned to the universal $BO(2)$-bundle with the representation ρ_2^i (see p. 29).

Now we compute

$$H_*(X\mathcal{T}(k), X\mathcal{T}'(k); \mathbb{Z}_p) \cong \tilde{H}_*(X\mathcal{T}(k)/X\mathcal{T}'(k); \mathbb{Z}_p) = \tilde{H}_*(T\xi; \mathbb{Z}_p) \cong \tilde{H}^*(T\xi; \mathbb{Z}_p).$$

The pull-back of ξ via the double cover

$$BO(2) \times BSO(k-1) \xrightarrow{\tilde{\kappa}} BO(2) \times BO(k-1)$$

is denoted by $\tilde{\xi}$.

Recall that the determinant of $\lambda_2(\phi, \psi)$ is $(\det \phi)^k \cdot \det \psi$. So if k is even then $\tilde{\xi}$ is an orientable bundle. By the Thom isomorphism theorem

$$\tilde{H}^*(T\tilde{\xi}; \mathbb{Z}_p) \cong U(\tilde{\xi}) \cup \left(\mathbb{Z}_p[p_1(\gamma_2)] \otimes \mathbb{Z}_p[p_1(\gamma_{k-1}), \ldots, p_{[\frac{k-1}{2}]}(\gamma_{k-1})]\right).$$

The involution induced by $\tilde{\kappa}$ on $\tilde{H}^*(T\tilde{\xi};\mathbb{Z}_p)$ maps all the Pontrjagin classes to themselves and it maps $U(\tilde{\xi})$ to $-U(\tilde{\xi})$. Since the ring $\tilde{H}^*(T\xi;\mathbb{Z}_p)$ is the invariant part of $\tilde{H}^*(T\tilde{\xi};\mathbb{Z}_p)$, it is 0.

This completes the proof of the theorem. $\square$

Remark. A theorem of Szűcs ([Sz7]) says that the group

$$Cob_n(S^{n+k};\mathcal{T}'(k))$$

is 2-primary in a wide dimension range. Combining this with our theorem just proved we can conclude that in some dimension range

$$Cob_n(S^{n+k};\mathcal{T}'(k))$$

is also 2-primary.

Remark. In the proof we did not use the condition that k is even until step (4), where we computed the $\mathbb{Z}_p$-cohomology of $T\xi$. Let us compute $\tilde{H}_*(T\xi;\mathbb{Z}_p)$ for k odd. Note that for k odd the bundle $\tilde{\xi}$ is not orientable. Denote the pull-back of $\tilde{\xi}$ via the double cover

$$BSO(2)\times BSO(k-1)\xrightarrow{\bar{\kappa}}BO(2)\times BSO(k-1)$$

by $\bar{\xi}$. Then $\bar{\xi}$ is orientable and its cohomologies are

$$\tilde{H}^*(T\bar{\xi})=U(\bar{\xi})\cup\left(\mathbb{Z}_p[e(\tilde{\gamma}_2)]\otimes\mathbb{Z}_p[p_1(\gamma_{k-1}),\dots,p_{\frac{k-3}{2}}(\gamma_{k-1}),e(\tilde{\gamma}_{k-1})]\right).$$

The involution induced by $\bar{\kappa}$ maps all the Pontrjagin classes into themselves, and it maps

$$e(\tilde{\gamma}_2)\mapsto -e(\tilde{\gamma}_2),\qquad U(\bar{\xi})\mapsto -U(\bar{\xi}).$$

Therefore

$$\tilde{H}^*(T\tilde{\xi};\mathbb{Z}_p)=e(\tilde{\gamma}_2)\cup U(\bar{\xi})\cup\left(\mathbb{Z}_p[p_1(\gamma_2)]\otimes\mathbb{Z}_p[p_1(\gamma_{k-1}),\dots,p_{\frac{k-3}{2}}(\gamma_{k-1}),e(\tilde{\gamma}_{k-1})]\right).$$

The involution on this ring induced by $\tilde{\kappa}$ maps

$$e(\tilde{\gamma}_2)\cup U(\bar{\xi})\mapsto -e(\tilde{\gamma}_2)\cup U(\bar{\xi}),\qquad e(\tilde{\gamma}_{k-1})\mapsto -e(\tilde{\gamma}_{k-1}),$$

and it maps all the Pontrjagin classes into themselves. Therefore

$$\tilde{H}^*(T\xi;\mathbb{Z}_p)=e(\tilde{\gamma}_2)\cup U(\bar{\xi})\cup e(\tilde{\gamma}_{k-1})\cup\left(\mathbb{Z}_p[p_1(\gamma_2)]\otimes\mathbb{Z}_p[p_1(\gamma_{k-1}),\dots,p_{\frac{k-1}{2}}(\gamma_{k-1})]\right).$$

These cohomology groups are not all 0, unlike the case k even. Hence we can only conclude that $\tilde{H}^{n+k}(T\xi;\mathbb{Z}_p)=0$ if

$$n+k<\dim\left(e(\tilde{\gamma}_2)\cup U(\bar{\xi})\cup e(\tilde{\gamma}_{k-1})\right)=2+(3k+4)+(k-1),$$

that is, if $n < 3k+5$. So we can extend the theorem to the case: k odd, $n < 3k+5$. However, this dimension range is only a slightly bigger then the dimension range mentioned in the remark above (before the proof) and for which the statement of the theorem is trivial.

3.3. Removing $\Sigma^{2,0}$ singularities

In this section we present a theorem on removing singularities in cobordism classes. A great variety of this kind of theorems can be proved using the results of chapter 2 — the one given below shows the basic line of their proofs.

Before turning to the theorem we need some definitions. Consider i-dimensional manifolds whose stable normal bundle is split into the direct sum of l stably isomorphic stable bundles. A cobordism between two such one (M and N) is an $i+1$-dimensional manifold with boundary $M \cup N$, whose stable normal bundle is split similarly, and this stable normal bundle restricted to M and N is consistent with the splitting of the stable normal bundles of M and N. The cobordism group is called $\mathfrak{N}_i^{l\gamma}$. Remark that $\mathfrak{N}_i^{1\gamma}$ is simply $\mathfrak{N}_i$.

If $f : N \longrightarrow P$ does not have more complicated singularities than Σ^2, then the points in N where the germ of f is of type Σ^2 is a closed submanifold (for almost all f). Denote this submanifold by $\Sigma^2(f)$.

If the map $f : N \longrightarrow \mathbb{R}^p$ is the composition of an immersion $f' : N \longrightarrow \mathbb{R}^{p+2}$ with the standard projection $\mathbb{R}^{p+2} \longrightarrow \mathbb{R}^p$ then we call f a *prim*-Σ^2 map. Remark that a *prim*-Σ^2 map does not have more complicated singularities than Σ^2 — this explains Σ^2 in the notation and *prim* is the abbreviation of *pr*ojected *im*mersion. One can easily define Σ^2-prim cobordism between Σ^2-prim maps.

3.3.1. Theorem. *Let $n \le 3k$ and $f : N^n \longrightarrow \mathbb{R}^{n+k}$ be a Σ^2-prim map. Then the following statements are equivalent:*

a) there is a Σ^2-prim map $g : M^n \longrightarrow \mathbb{R}^{n+k}$ which is Σ^2-prim cobordant to f and $\Sigma^2(g) = \emptyset$;

b) The manifold $\Sigma^2(f)$ is null-cobordant.

Proof. $a) \to b)$. Because of condition $a)$ there is an $F : W^{n+1} \longrightarrow \mathbb{R}^{n+k} \times [0,1]$ Σ^2-prim map whose restriction to $\partial W = N \cup M$ is $f \cup g$. Since $\Sigma^2(g)$ is empty the manifold $\Sigma^2(f)$ is the boundary of $\Sigma^2(F)$.

$b) \to a)$. Since $n \le 3k$ there is only one type of $\Sigma^{2,0}$ singularity of f, call it η (using the notation of section 1.3 it is called the k codimensional $III_{2,2}$ type — see also the Appendix). A classifying space X can be constructed for Σ^2-prim maps in the sense of chapter 2: this will be $X' \cup_\rho D\xi_t$, where $\xi_t = 3\gamma^{k-1} \oplus \varepsilon^7$ (over the space $BO(k-1)$), X' is a classifying space for those Σ^2-prim maps that have no Σ^2 singularities, and $h : \partial D\xi_t \longrightarrow X'$ is an appropriate map.

Remark. To see this, observe that ξ_t is a vector bundle associated to the universal

$$G = \{(g_1, g_2) \in MC\ Aut_{\mathcal{A}}\eta \mid (dg_1)_0|_{ker\ \eta} = id\}$$

-bundle by the representation $\lambda_2|_G$. The letter "t" in the notation suggests this trivial action on $ker\ \eta$

A portion of the homotopy exact sequence of the pair (X, X') is

$$\begin{array}{ccccc} \pi_{n+k}(X') & \xrightarrow{i_*} & \pi_{n+k}(X) & \xrightarrow{j_*} & \pi_{n+k}(X, X') \\ & & & & || \\ & & & & \pi_{n+k}(T\xi_t). \end{array} \quad \text{since } n < 3k+2$$

What we have to prove is the following. If $[\Sigma^2(f)] = 0 \in \mathfrak{N}_{n-2k-4}$ then $[f] \in \pi_{n+k}(X)$ is in the image of i_*, or what is the same $j_*[f] = 0 \in \pi_{n+k}(T\xi_t)$.

Now let us have a closer look at $\pi_{n+k}(T\xi_t)$. Since $\xi_t = 3\gamma^{k-1} \oplus \varepsilon^7$, the group $\pi_{n+k}(T\xi_t)$ can be identified with the cobordism group of those $n-2k-4$-dimensional submanifolds of S^{n+k} whose normal bundle is of the form $3\zeta^{k-1} \oplus \varepsilon^7$ (and of course the normal bundle of the cobordism manifold $W \longrightarrow \mathbb{R}^{n+k} \times [0,1]$ splits similarly). But because of the dimension setting the normal bundle of these submanifolds are already stable. Therefore we can identify $\pi_{n+k}(T\xi_t)$ with $\mathfrak{N}^{3\gamma}_{n-2k-4}$, and the map j_* with

$$\pi_{n+k}(X) \ni [f] \mapsto [\Sigma^2(f)] \in \mathfrak{N}^{3\gamma}_{n-2k-4}.$$

A theorem of Golubjatnikov [G] states that the forgetful map

$$\mathfrak{N}^{3\gamma}_{n-2k-4} \longrightarrow \mathfrak{N}_{n-2k-4}$$

is an isomorphism. So if $[\Sigma^2(f)] = 0 \in \mathfrak{N}_{n-2k-4}$ then $[\Sigma^2(f)] = 0 \in \mathfrak{N}^{3\gamma}_{n-2k-4}$ which means that $j_*[f] = 0$ and that completes the proof. $\square$

Appendix

The hierarchy of $\Sigma^{2,0}$ germs

Mather has given a complete classification of stable map germs having $\Sigma^{2,0}$ singularities in terms of their local algebras. In this appendix normal forms are given for all these singularities, and also the hierarchy of them is presented, i. e. we determine which other type must occur near a given singularity type. Note that $\Sigma^{2,0}$ is the most difficult type for which the classification is completely known.

To recall the notation we repeat the theorem of Mather [M6] about the classification of stable $\Sigma^{2,0}$ map germs.

4.0.2. Theorem. [M6] *The local algebra of a stable germ of type $\Sigma^{2,0}$ is isomorphic to one from the following list. Furthermore these algebras are all distinct.*

$I_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a + y^b)$	$b \geq a \geq 2$
$II_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a - y^b)$	$b \geq a \geq 2$ *both even*
$III_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a, y^b)$	$b \geq a \geq 2$
IV_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a)$	$a \geq 3$
V_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a, yx^{a-1})$	$a \geq 3$

□

4.1. Local normal forms

In this section we give local normal forms for the algebras above.

4.1.1. Theorem. *If the map germ $\tilde{F} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has singularity type $I_{a,b}$, then it is equivalent to $\tilde{f} \times id$, where id is the identity of $\mathbb{R}^{n-(a+b-1)k-(a+b)}$ and $\tilde{f} : \mathbb{R}^{(a+b-1)k+(a+b)} \longrightarrow \mathbb{R}^{(a+b)k+(a+b)}$ is defined by*

$$\tilde{f} : (x, y, \mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}) \mapsto (g_1(x, y), g_2(x, y, \mathbf{u}, \mathbf{v}), \mathbf{h}(x, y, \mathbf{s}, \mathbf{t}), \mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t})$$

where

$$\mathbf{u} = u_1, \ldots, u_{a-1},\ \mathbf{v} = v_1, \ldots, v_{b-1},\ \mathbf{s} = s_{1,1}, \ldots, s_{k,a},$$

$$\mathbf{t} = t_{1,1}, \ldots, t_{k,b-1},\ \mathbf{h} = h_1, \ldots, h_k$$

and the functions are

$$g_1(x, y) = xy, \qquad g_2(x, y, \mathbf{u}, \mathbf{v}) = x^a + y^b + \sum_{i=1}^{a-1} u_i x^i + \sum_{j=1}^{b-1} v_j y^j,$$

$$h_l(x, y, \mathbf{s}, \mathbf{t}) = \sum_{i=1}^{a} s_{l,i} x^i + \sum_{j=1}^{b-1} t_{l,j} y^j.$$

□

4.1.2. Theorem. *If the map germ $\tilde{F} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has singularity type $II_{a,b}$ then it is equivalent to $\tilde{f} \times id$, where id and $\tilde{f}$ are as above, with the only difference that*

$$g_2(x, y, \mathbf{u}, \mathbf{v}) = x^a - y^b + \sum_{i=1}^{a-1} u_i x^i + \sum_{j=1}^{b-1} v_j y^j.$$

□

4.1.3. Theorem. *If the map germ $\tilde{F} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has singularity type $III_{a,b}$, then it is equivalent to $\tilde{f} \times id$, where id is the identity of $\mathbb{R}^{n-(a+b-2)k-(a+b)}$ and $\tilde{f} : \mathbb{R}^{(a+b-2)k+(a+b)} \longrightarrow \mathbb{R}^{(a+b-1)k+(a+b)}$ is defined by*

$$\tilde{f} : (x, y, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \mathbf{s}, \mathbf{t}) \mapsto$$

$$(g_1(x, y), g_2(x, y, \mathbf{u}, \mathbf{v}), g_3(x, y, \mathbf{w}, \mathbf{z}), \mathbf{h}(x, y, \mathbf{s}, \mathbf{t}), \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \mathbf{s}, \mathbf{t})$$

where

$$\mathbf{u} = u_1, \ldots, u_{a-1}, \; \mathbf{v} = v_1, \ldots, v_{b-1}, \; \mathbf{w} = w_1, \ldots, w_{a-1}, \; \mathbf{z} = z_1, \ldots, z_{b-1},$$

$$\mathbf{s} = s_{1,1}, \ldots, s_{k-1,a-1}, \; \mathbf{t} = t_{1,1}, \ldots, t_{k-1,b-1}, \; \mathbf{h} = h_1, \ldots, h_{k-1}$$

and the functions are

$$g_1(x, y) = xy, \qquad g_2(x, y, \mathbf{u}, \mathbf{v}) = x^a + \sum_{i=1}^{a-1} u_i x^i + \sum_{j=1}^{b-1} v_j y^j,$$

$$g_3(x, y, \mathbf{w}, \mathbf{z}) = y^b + \sum_{i=1}^{a-1} w_i x^i + \sum_{j=1}^{b-1} z_j y^j,$$

$$h_l(x, y, \mathbf{s}, \mathbf{t}) = \sum_{i=1}^{a-1} s_{l,i} x^i + \sum_{j=1}^{b-1} t_{l,j} y^j.$$

□

4.1.4. Theorem. *If the map germ $\tilde{F} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has singularity type IV_a, then it is equivalent to $\tilde{f} \times id$, where id is the identity of $\mathbb{R}^{n-(2a-1)k-(2a)}$ and $f : \mathbb{R}^{(2a-1)k+(2a)} \longrightarrow \mathbb{R}^{(2a)k+(2a)}$ is defined by*

$$\tilde{f} : (x, y, \mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}) \mapsto (g_1(x, y), g_2(x, y, \mathbf{u}, \mathbf{v}), \mathbf{h}(x, y, \mathbf{s}, \mathbf{t}), \mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t})$$

where

$$\mathbf{u} = u_1, \dots, u_{a-1},\ \mathbf{v} = v_1, \dots, v_{a-1},\ \mathbf{s} = s_{1,1}, \dots, s_{k,a-1},\ \mathbf{t} = t_{1,0}, \dots, t_{k,a-1},$$

$$\mathbf{h} = h_1, \dots, h_k$$

and the functions are

$$g_1(x,y) = x^2 + y^2, \qquad g_2(x,y,\mathbf{u},\mathbf{v}) = x^a + \sum_{i=1}^{a-1} u_i x^i + \sum_{j=1}^{a-1} v_j x^{j-1} y,$$

$$h_l(x,y,\mathbf{s},\mathbf{t}) = \sum_{i=1}^{a-1} s_{l,i} x^i + \sum_{j=0}^{a-1} t_{l,j} x^j y.$$

□

4.1.5. Theorem. *If the map germ $\tilde{F} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has singularity type V_a, then it is equivalent to $\tilde{f} \times id$, where id is the identity of $\mathbb{R}^{n-(2a-2)k-(2a)}$ and $\tilde{f} : \mathbb{R}^{(2a-2)k+(2a)} \longrightarrow \mathbb{R}^{(2a-1)k+(2a)}$ is defined by*

$$\tilde{f} : (x, y, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \mathbf{s}, \mathbf{t}) \mapsto$$

$$(g_1(x,y), g_2(x,y,\mathbf{u},\mathbf{v}), g_3(x,y,\mathbf{w},\mathbf{z}), \mathbf{h}(x,y,\mathbf{s},\mathbf{t}), \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \mathbf{s}, \mathbf{t})$$

where

$$\mathbf{u} = u_1, \dots, u_{a-1},\ \mathbf{v} = v_1, \dots, v_{a-1},\ \mathbf{w} = w_1, \dots, w_{a-1},\ \mathbf{z} = z_1, \dots, z_{a-1},$$

$$\mathbf{s} = s_{1,1}, \dots, s_{k-1,a-1},\ \mathbf{t} = t_{1,1}, \dots, t_{k-1,a-1},\ \mathbf{h} = h_1, \dots, h_{k-1}$$

and the functions are

$$g_1(x,y) = x^2 + y^2, \qquad g_2(x,y,\mathbf{u},\mathbf{v}) = x^a + \sum_{i=1}^{a-1} u_i x^i + \sum_{j=1}^{a-1} v_j x^{j-1} y,$$

$$g_3(x,y,\mathbf{w},\mathbf{z}) = x^{a-1} y + \sum_{i=1}^{a-1} w_i x^i + \sum_{j=1}^{a-1} z_j x^{j-1} y,$$

$$h_l(x,y,\mathbf{s},\mathbf{t}) = \sum_{i=1}^{a-1} s_{l,i} x^i + \sum_{j=1}^{a-1} t_{l,j} x^{j-1} y.$$

□

The proofs of the above theorems are based on the procedure prescribed in section 1.1. The only non completely trivial part of the general procedure is to find a basis of a finite dimensional factor of an infinite dimensional space. In our cases these generator sets can easily be found.

4.2. The hierarchy

In this section we state and prove the theorems about the hierarchy of $\Sigma^{2,0}$ map germs. Before doing so, two lemmas from commutative algebra (with addenda) are presented that are used in the proofs later. The proofs of the lemmas are given in the next section. Remark that the degree of a power series is the *smallest* of the degrees of its non zero terms.

Lemma A. *Let $xy \in I$ be an ideal in $\mathbb{R}[[x,y]]$ and suppose that the factor $\mathbb{R}[[x,y]]/I$ is finite dimensional. Then there is an automorphism of $\mathbb{R}[[x,y]]$ which maps I to an ideal of the form*

$$(xy, x^k, y^r) \qquad \textit{or} \qquad (xy, ax^k + by^l).$$

Remark. If $\mathbb{R}[[x,y]]/I$ is the algebra of a $\Sigma^{2,0}$ singularity type then this type is one of the following: $I_{\alpha,\beta}, II_{\alpha,\beta}, III_{\alpha,\beta}$. An upper bound for α, β is given in the following statement.

Addendum. *Let I be generated by xy and a set of power series $\{p_i(x)+q_i(y)\}$. Let $c := min\ deg\ p_i$, and $d := min\ deg\ q_i$. Then $\alpha \le min\{c,d\}$ and $\beta \le max\{c,d\}$.*

Lemma B. *Let $x^2+y^2 \in I$ be an ideal in $\mathbb{R}[[x,y]]$ and suppose that the factor $\mathbb{R}[[x,y]]/I$ is finite dimensional. Then there is an automorphism of $\mathbb{R}[[x,y]]$ which maps I to an ideal of the form*

$$(x^2+y^2, x^k) \qquad \textit{or} \qquad (x^2+y^2, x^k, x^{k-1}y).$$

Remark. If $\mathbb{R}[[x,y]]/I$ is the algebra of a $\Sigma^{2,0}$ singularity type then this type is one of the following: $I_{2,2}$, $III_{2,2}$, IV_α, V_α. An upper bound for α is given in the following statement.

Addendum. *Let I be generated by x^2+y^2 and a set of formal power series $\{p_i(x,y)\}$. Let $c = min\ deg\ p_i$. Then $\alpha \le c$ if $c > 2$ and the algebra $\mathbb{R}[[x,y]]/I$ is isomorphic to the algebra $I_{2,2}$ or $III_{2,2}$ if $c = 2$.*

Now we turn to the investigation of the hierarchy of $\Sigma^{2,0}$ germs.

4.2.1. Definition. *For a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ and a (finite dimensional local) algebra Q let $Q(f)$ be the subset of $\mathbb{R}^n$ consisting of those points x that have the property that the germ $f : (\mathbb{R}^n, x) \longrightarrow (\mathbb{R}^{n+k}, f(x))$ has local algebra isomorphic to Q.*

4.2.2. Definition. *For a map germ $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ and a (finite dimensional local) algebra Q let $\tilde{Q}(\tilde{f})$ denote the germ at 0 of $Q(f)$ where f is a representative of $\tilde{f}$.*

Clearly this definition is correct, i.e. $\tilde{Q}(\tilde{f})$ does not depend on the representative f. The fact that $\tilde{Q}(\tilde{f})$ is non-empty depends only on the equivalence class of $\tilde{f}$. So if Q and P are algebras from the $I_{a,b} - V_a$ list above it is reasonable to ask whether $\tilde{P}(\tilde{f})$ is empty for a germ $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ having local algebra Q. The answer will clearly not depend on n, but it might depend on k.

4.2.3. THEOREM. *If the germ $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has local algebra $I_{a,b}$ or $II_{a,b}$ then considering only $\Sigma^{2,0}$ points only the following set germs will not be empty:*

$$\begin{array}{llll}
I_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b & \textit{except} \ \ (\alpha,\beta) = (a,b) & \textit{if} \ \ Q_{\tilde{f}} = II_{a,b} \\
II_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b \ \ \textit{even} & \textit{except} \ \ (\alpha,\beta) = (a,b) & \textit{if} \ \ Q_{\tilde{f}} = I_{a,b} \\
III_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b & & \textit{if}\ k > 0.
\end{array}$$

Proof. Let us take a representative f of $\tilde{f}$, for example the one given in section 4.2 as "normal form". We have to consider the germs $f_p : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$, $x \mapsto f(x+p) - f(p)$. For an algebra P, $\tilde{P}$ is not empty if and only if we can find p's arbitrarily close to 0 for which the algebra Q_{f_p} is isomorphic to P. In fact we don't have to consider all p's as we are only dealing with $\Sigma^{2,0}$ germs. Taking the derivative of the normal form given in section 4.2 it follows that we can assume the $x, y, u_1, v_1, s_{1,1}, \ldots, s_{k,1}, t_{1,1}, \ldots, t_{k,1}$ coordinates of p to be zero. Now it is an easy computation that the local algebra at 0 of f_p is

$$\mathbb{R}[[x,y]] \ / \ (\ xy, x^a \pm y^b + \sum_{i=2}^{a-1} u_i x^i + \sum_{j=2}^{b-1} v_j y^j,$$

$$\sum_{i=2}^{a} s_{1,i} x^i + \sum_{j=2}^{b-1} t_{1,j} y^j, \ldots, \sum_{i=2}^{a} s_{k,i} x^i + \sum_{j=2}^{b-1} t_{k,j} y^j \ ).$$

According to lemma A this algebra is isomorphic to $I_{\alpha,\beta}$, $II_{\alpha,\beta}$ or $III_{\alpha,\beta}$ for some α, β. The corollary of lemma A yields that the parameters α, β can have only the values given in the theorem. (In fact this corollary — as it is stated — does not prove that there is no $I_{a,b}$ point near a $II_{a,b}$ point and reverse; but this is clear, because the dimension of the submanifolds having $I_{a,b}$ and $II_{a,b}$ singularities are equal.) The fact that the algebras mentioned in the theorem really occur is just the question of setting the parameters $\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}$. □

4.2.4. THEOREM. *If the germ $\tilde{f} : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ has local algebra $III_{a,b}$ then considering only $\Sigma^{2,0}$ points only the following set germs will not be empty:*

$$\begin{array}{lll}
I_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b & \textit{except} \ \ (\alpha,\beta) = (a,b) \\
II_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b \ \ \textit{even} & \textit{except} \ \ (\alpha,\beta) = (a,b) \\
III_{\alpha,\beta} & 2 \le \alpha \le a, 2 \le \beta \le b. &
\end{array}$$

Proof. The proof of this theorem is analogous to the proof of the preceding theorem — it is also based on lemma A and the corollary of it. The only difference is that this lemma and its corollary *a priori* allows $I_{a,b}$ type points near the $III_{a,b}$ point. The codimension of the submanifold of $I_{a,b}$ points is, however, greater then the codimension of the submanifold of $III_{a,b}$ points $((a+b-1)k+(a+b) > (a+b-2)k+(a+b)$ as $k>0)$. $\square$

4.2.5. THEOREM. *If the germ $\tilde{f}:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^{n+k},0)$ has local algebra IV_a then considering only $\Sigma^{2,0}$ points only the following set germs will not be empty:*

$$\begin{array}{ll} IV_\alpha & 3\le\alpha\le a \\ V_\alpha & 3\le\alpha\le a \qquad \textit{except}\quad \alpha=a \quad \textit{if}\quad k=0 \\ I_{2,2} & \\ III_{2,2}. & \end{array}$$

Proof. We use the same method as above. A $\Sigma^{2,0}$ point near 0 has local algebra

$$\mathbb{R}[[x,y]]\ /\ (x^2+y^2, x^a+\sum_{i=2}^{a-1}u_ix^i+\sum_{j=2}^{a-1}v_jx^{j-1}y,$$

$$\sum_{i=2}^{a-1}s_{1,i}x^i+\sum_{j=1}^{a-1}t_{1,j}x^jy,\dots,\sum_{i=2}^{a-1}s_{k,i}x^i+\sum_{j=1}^{a-1}t_{k,j}x^jy).$$

Using lemma B and its addendum it can be seen that this algebra is isomorphic to one of the algebras given in the theorem. We can also see that if $k=0$, then this algebra can not be isomorphic to V_a, because V_a can not be presented with 2 generators and 2 relations (its "defect" is 1).

So it is proved that only the algebras mentioned in the theorem can occur. The fact that they really occur is proven by suitably chosen parameters. $\square$

4.2.6. THEOREM. *If the germ $\tilde{f}:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^{n+k},0)$ has local algebra V_a then considering only $\Sigma^{2,0}$ points only the following set germs will not be empty:*

$$\begin{array}{ll} IV_\alpha & 3\le\alpha\le a-1 \\ V_\alpha & 3\le\alpha\le a \\ I_{2,2} & \\ III_{2,2}. & \end{array}$$

Proof. Using the same method as in the proofs above, we have to consider the algebra:

$$\mathbb{R}[[x,y]]\ /\ (x^2+y^2, x^a+\sum_{i=2}^{a-1}e_ix^i+\sum_{j=2}^{a-1}f_jx^{j-1}y, x^{a-1}y+\sum_{i=2}^{a-1}g_ix^i+\sum_{j=2}^{a-1}h_jx^{j-1}y,$$

$$\sum_{i=2}^{a-1}s_{1,i}x^i+\sum_{j=1}^{a-1}t_{1,j}x^{j-1}y,\dots,\sum_{i=2}^{a-1}s_{k-1,i}x^i+\sum_{j=1}^{a-1}t_{k-1,j}x^jy).$$

Using lemma B and its addendum it is clear that this algebra is isomorphic to one mentioned in the theorem. (Note that IV_a can not occur for dimensional reasons). The singularity types mentioned in the formulation of the theorem really occur for some choice of the parameters. $\square$

General remarks

• The hierarchy of $\Sigma^{2,0}$ map germs essentially does not depend on the codimension k — as the only dependence appears in case $I_{a,b}$ and $II_{a,b}$; and even in this cases the hierarchy depends only on the vanishing of k.

• A singularity type has a codimension: the codimension of the singularity set in the source manifold. In case of Morin singularities this codimension structure determines the hierarchy of the singularities. It turned out that in case of $\Sigma^{2,0}$ singularities this is not so.

• The algerbas $I_{a,b}$ and $II_{a,b}$ are different real forms of the same complex algebra. This fact induces that the corresponding singularities behave similarly in some circumstances. In the above theorems examples and counter-examples can both be found for this impression.

4.3. Proofs of the lemmas from commutative algebra

In this section we prove the lemmas used in the previous section. The arguments are similar to the ones used by Mather [M6] in the classification of the algebras associated to germs of type $\Sigma^{2,0}$.

Proof of lemma A and its addendum. Since $xy \in I$, all other generators of I can be written in the form $f(x)+g(y)$, i.e. each coset in $I/(xy)$ has a representative in that form. Take a generator set of I (which includes xy, and the other elements are in the mentioned form), and let $f_1(x)+g_1(y)$ have the smaller degree (k) among them. Without loss of generality (maybe after swapping x and y) we can assume that it is in the form $ax^k + x^{k+1}h(x) + y^k j(y)$ with $a \neq 0$. This generator can be replaced by $ax^k + y^k j(y)$ since the multiples of x^k and those of $ax^k + x^{k+1}h(x)$ coincide (because $a + xh(x)$ is invertible in $\mathbb{R}[[x]]$). We can treat "the y-side" the same way, and therefore assume that $f_1(x)+g_1(y) = ax^k + by^l$ where $l \geq k$.

If $b = 0$, then I has to be (xy, x^k, y^r), because $\mathbb{R}[[x,y]]$ is finite dimensional and therefore some y-power must be in I, and we can call r the smallest such power.

Let $b \neq 0$. Note that in this case $y^{l+1} \in I$ and the other generators of I can be chosen to have the form $g(y)$. Take the one with the smallest degree: r (there has to be one, otherwise the factor $\mathbb{R}[[x,y]]/I$ would not be finite dimensional). Then it is easy to see that $I = (xy, x^k, y^r)$ if $l \geq r$, and $I = (xy, ax^k + by^l)$ if $l < r$. $\square$

Proof of lemma B and its addendum. Since $x^2+y^2 \in I$, the other elements in a generator set of I can be supposed to have the form: $f(x)+yg(x)$. Assume that $f_1(x)+yg_1(x)$ has the smallest degree (k) in this generator set (not regarding x^2+y^2). Without loss of generality (maybe after swapping x and y) we can assume that its leading term is x^k. The automorphism of $\mathbb{R}[[x,y]]$ induced by

$$x \mapsto x\ cos\ \theta + y\ sin\ \theta$$

$$y \mapsto -x \; sin \; \theta + y \; cos \; \theta$$

takes $ax^k + byx^{k-1}$ into $(a \; cos \; k\theta - b \; sin \; k\theta)x^k + (a \; sin \; k\theta + b \; cos \; k\theta)yx^{k-1}$, and of course it keeps $x^2 + y^2$ fixed. We can clearly choose θ so that the coefficient of yx^{k-1} vanishes (and the coefficient of x^k does not). Therefore we can suppose that $f_1(x) + yg_1(x)$ has the form $x^k + x^{k+1}h(x) + yx^k j(x)$.

Now it is easy to see that x^{k+1} and $x^k y$ are in the ideal, and therefore the $k+1$'st power of the maximal ideal is part of I. Hence the ideal I must be equal to either $(x^2 + y^2, x^k)$ or $(x^2 + y^2, x^k, yx^{k-1})$. $\square$

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