COINCIDENT ROOT LOCI OF BINARY FORMS

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1. Introduction

Coincident root loci are subvarieties of $S^d\mathbb{C}^2$ — the space of binary forms of degree d — labeled by partitions of d. Given a partition λ, let X_λ be the set of forms with root multiplicity corresponding to λ. There is a natural action of $GL_2(\mathbb{C})$ on $S^d\mathbb{C}^2$ and the coincident root loci are invariant under this action. We calculate their equivariant Poincaré duals, generalizing formulas of Hilbert and Kirwan. In the second part we apply these results to present the cohomology ring of the corresponding moduli spaces (in the GIT sense) by geometrically defined relations.

One of the main goals of Geometric Invariant Theory is to calculate the cohomology ring of a geometric quotient. In the case when all semistable point are stable several techniques were developed. But even for very simple representations this condition is not satisfied. In this paper we study the action of $GL(2)$ on the space of binary forms of degree d. In the d odd case methods of [Kir84], [JK95], [Mar99] can be applied, but none of these methods compute the cohomology ring of the moduli space in the d even case. We show how equivariant Poincaré-dual calculations lead to relations for the cohomology ring in both the odd and the even case.

Closely related rings have been computed earlier. The computation for $H^*_G(X^{ss})$ is well-known (since Kirwan's thesis in the case of betti numbers), and the existing procedure is independent of d being even or odd. In the d even case, rational intersection cohomology of the moduli space is also known ([Kir86])—which result we also recover in Remark 4.11.

Our Poincaré-dual (a.k.a. Thom polynomial) calculations are also interesting on their own right since they generalize formulas of Hilbert and Kirwan on coincident root loci. These calculations do not only lead to explicit relations for these cohomology rings but also identify them with the equivariant Poincaré-duals of the simplest unstable coincident root loci.

Consider the d-th symmetric power $S^d\mathbb{C}^2$ of the standard representation of $GL_2(\mathbb{C})$, that is the action of G on the space V_d of degree d homogeneous polynomials in two variables x, y. For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of d (i.e. $\sum_j \lambda_j = d$) we define

$$X_\lambda = \{B(x,y) \in V_d \mid B = \prod_{j=1}^{n} L_j^{\lambda_j} \text{ for some linear forms } L_j\},$$

which is a subvariety invariant under the group action. It is called the *coincident root loci* associated with λ. It is a cone in V_d, let $\mathbb{P}X_\lambda$ be its projectivization in the projective space $\mathbb{P}V_d$. It is more convenient to use a different notation for partitions: $\lambda = (1^{e_1}2^{e_2}\dots r^{e_r})$ will

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mean the partition consisting of e_1 copies of 1, e_2 copies of 2, etc. Then $\sum ie_i = d$, $\sum e_i = n$ and the complex dimension of $\mathbb{P}X_\lambda$ is exactly n.

The study of coincident root loci probably started with Cayley. E.g., the very first question of this type asks the characterization of polynomials B with a double root. The answer is the vanishing of the discriminant which provides in this way an equation for $X_{(1^{d-2}2)}$. For higher codimensional coincident root loci finding the defining equations is very complicated (see [Chi01] for recent results). However, important geometric information can be obtained about these subvarieties. E.g., the starting point of the present paper was Hilbert's formula which calculates the degree of $\mathbb{P}X_\lambda \subset \mathbb{P}V_d$:

$$\deg(\mathbb{P}X_\lambda) = \frac{n!}{\prod_i (e_i!)} \prod_i i^{e_i}.$$

We can interpret this formula as follows: for a generic family of polynomials parametrized by a projective space of dimension equal to the codimension of $\mathbb{P}X_\lambda$ the number of polynomials in the family with root multiplicity λ is $\deg(\mathbb{P}X_\lambda)$.

Generalizing this approach we arrive at the theory of degeneracy loci. Suppose we have a vector bundle $E \to M$ with fiber $S^d\mathbb{C}^2$ and a generic section $s : M \to E$. Let $s^{-1}(X_\lambda)$ be the set of points in M where the value of s is in X_λ. Its Poincaré dual $[s^{-1}(X_\lambda)] \in H^*(M)$ measures the "size" of $s^{-1}(X_\lambda)$. It turns out that for any $S^d\mathbb{C}^2$-bundle, $[s^{-1}(X_\lambda)]$ can be deduced from the corresponding cohomology class of the universal bundle associated with the $GL_2(\mathbb{C})$–representation $S^d\mathbb{C}^2$. This universal invariant is called the $GL_2(\mathbb{C})$-*equivariant Poincaré dual*, or *Thom polynomial* of X_λ in $S^d\mathbb{C}^2$. In section 3 we determine all these polynomials.

Calculating equivariant Poincaré duals for invariant subvarieties of representations has a long history. We can interpret many results of the nineteenth century algebraic geometers in these terms. From the 1970's the main method was a type of resolution of the subvariety, initiated by Porteous [Por71]. The method requires a deep understanding of the geometry of the resolution and can be carried out only in special cases. Most examples can be found in [Ful98]. The first and third author designed a different method (the method of restriction equations, see [FR04]) based on ideas coming from calculating Thom polynomials in singularity theory [Rim01]. However the method of restriction equations works well mainly if the representation has finitely many orbits which is usually not the case (e.g. for $S^d\mathbb{C}^2$ if $d > 3$).

In this paper we return to the technique of resolution, however in a very different way. The main novelty is that our new approach requires only knowledge of some basic cohomological data. Consequently, the method is more flexible. We illustrate this method here by the coincident root loci, but the range of applications is much wider.

Parallel to our work B. Kőműves also provided a presentation of these Poincaré duals in a completely different form [Kőm03]. He worked more in the spirit of the method of restriction equations studying incidences of the coincident root loci with the orbits $X_{(i,d-i)}$.

In section 4 we study the cohomology ring of the moduli space of the representation $S^d\mathbb{C}^2$ (in the Geometric Invariant Theory sense). Following the paper of Atiyah and Bott [AB83] a whole theory for calculating cohomology rings of the moduli space of representations was built up by F. Kirwan; as well as more algebraic methods were successfully applied by e.g. M. Brion [Bri91], S. Martin [Mar99]. However, the application of the general theorems to specific examples is often not easy. Our approach results in explicit presentations of the rational cohomology rings $H^*_G(X^{ss})$, $H^*(X^{ss}//G)$ and $H^*_G(X^s) \cong H^*(X^s//G)$ in terms of generators and relations (if d is

odd then all these rings coincide, but for the even case they are different). We wish to emphasize that a main advantage of our presentation of the cohomology rings is that we attribute to the set of relations deep geometric significance: they are the universal Thom polynomials of some distinguished spaces X_λ.

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2. Review on affine and projective Thom polynomials

Let the group G act on the complex vector space V, and let η be an invariant variety in V, which supports a fundamental class (for more details see [FR04]). Then define the (affine) *Thom polynomial* of η as the Poincaré dual of the fundamental homology class of η in equivariant cohomology:

$$\mathrm{Tp}_\eta = \text{Poincaré dual of } \ [\eta] \in H^*_G(V, \mathbb{Z}).$$

The vector space V is contractible, hence the ring $H^*_G(V, \mathbb{Z})$ is naturally isomorphic to $H^*(BG, \mathbb{Z})$, the ring of G-characteristic classes. The degree of Tp_η is the real codimension $2c$ of η in V, hence $\mathrm{Tp}_\eta \in H^{2c}(BG, \mathbb{Z})$. The direct geometric meaning of Tp_η is the following.

Consider a fiber bundle ξ with fiber V and structure group G over a manifold M. Because of its invariance, the set η can be defined in each fiber, let the union of these be $\eta(\xi)$. Then consider those points where a generic section s of ξ hits $\eta(\xi)$, that is $s^{-1}(\eta(\xi)) \subset M$. By Poincaré duality this set defines a cohomology class in M. Standard arguments show that this class equals $\mathrm{Tp}_\eta(\xi) := f_\xi^* \mathrm{Tp}_\eta$, where $f_\xi : M \to BG$ is a classifying map of ξ.

We will also use the projective version of Thom polynomials (see [FNR]), as follows. Assume that G acts on V in such a way that the scalars are in the image of $G \to GL(V)$. Then the orbits of this action (different from $\{0\}$) are in bijection with the orbits of the induced action of G an $\mathbb{P}V$. Also, the corresponding orbits, η and $\mathbb{P}\eta$ have the same codimension. The equivariant Poincaré dual of $\mathbb{P}\eta$ will be called the *projective Thom polynomial* of η:

$$\mathbb{P}\,\mathrm{Tp}_\eta = \text{Poincaré dual of } \ [\mathbb{P}\eta] \in H^*_G(\mathbb{P}V, \mathbb{Z}) = H^*(BG, \mathbb{Z})[x]/(Q(x)) \quad (\deg(x) = 2),$$

where $Q(x)$ is the product of all $(x + \alpha_j)$'s, where $\alpha_j \in H^2(BG)$ are the weights of the representation of G on V [BT82]. The projective Thom polynomial can be written as $\mathbb{P}\,\mathrm{Tp}_\eta = p_c + p_{c-1}x + \ldots + p_0 x^c$, where $p_i \in H^{2i}(BG)$. By [FNR, Section 6], $p_c = \mathrm{Tp}_\eta$ and p_0 is the degree of the variety $\mathbb{P}\eta$. Seemingly, the projective Thom polynomial contains more information then the "affine" one. This is not the case: $\mathbb{P}\,\mathrm{Tp}_\eta$ can be obtained from Tp_η by a simple substitution, see Theorem 6.1 in [FNR] (although this fact will not be used in the present paper). In particular, the degree p_0 of $\mathbb{P}\eta$ itself can be obtained from Tp_η by a substitution. For this substitution in our specific case, see 3.9(2).

3. Coincident root loci

Consider the d-th symmetric power $V_d = S^d\mathbb{C}^2$ of the standard representation of $G = GL_2(\mathbb{C})$, and the invariant subvariety X_λ associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of d (cf. introduction). In this section we compute its Thom polynomial $\mathrm{Tp}_\lambda \in H^*(BG, \mathbb{Z})$.

Points in the projectivization $\mathbb{P}V_d$ of V_d can be identified with d-tuples of points in $\mathbb{P}^1 = \{(x : y)\}$ (counted with multiplicities). The projectivization $\mathbb{P}X_\lambda$ is then the closure of the set of d-tuples having n distinct points with multiplicities $\lambda_1, \lambda_2, \ldots, \lambda_n$. The variety $\mathbb{P}X_\lambda$ is called the coincident root locus.

Consider also the other notation $\lambda = (1^{e_1}2^{e_2}\ldots r^{e_r})$ with $\sum ie_i = d$ and $\sum e_i = n$ (cf. introduction). Then $\mathbb{P}X_\lambda$ is the image of the map

$$\phi : \mathbb{P}V_{e_1} \times \mathbb{P}V_{e_2} \times \ldots \times \mathbb{P}V_{e_r} \to \mathbb{P}V_d$$

defined (via point-tuples of $\mathbb{P}^1$) by $(D_1, D_2, \ldots, D_r) \mapsto \sum iD_i$. It is readily seen that ϕ is birational onto its image $\mathbb{P}X_\lambda$ (i.e. is a resolution of $\mathbb{P}X_\lambda$). In particular, $\dim(\mathbb{P}X_\lambda) = n$ and Tp_λ is of degree $d - n$ (cf. section 2).

The map ϕ is equivariant under the action of G on the two spaces, hence it makes sense to talk about the maps ϕ^* (induced by ϕ) and $\phi_!$ (the push-forward map of ϕ) in G-equivariant cohomology. The equivariant cohomology rings are as follows (cf. e.g. [BT82], p. 270):

$$H^*_G(\textstyle\prod_i \mathbb{P}V_{e_i}, \mathbb{Z}) = R[x_1, \ldots, x_r]/(Q_{e_1}(x_1), \ldots, Q_{e_r}(x_r)), \text{ resp. } H^*_G(\mathbb{P}V_d, \mathbb{Z}) = R[x]/(Q_d(x)).$$

Here

$$R = H^*(BG, \mathbb{Z}) = \mathbb{Z}[c_1, c_2] = \mathbb{Z}[u, v]^{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ permutes the roots u and v (hence $c_1 = u + v$ and $c_2 = uv$); and the polynomial Q_k ($k \geq 1$) is defined by

$$Q_k(y) = \prod_{\alpha \text{ is a weight of } S^k\mathbb{C}^2} (y + \alpha) = \prod_{j=0}^{k} (y + ju + (k - j)v).$$

The map ϕ^* is a ring homomorphism, it leaves elements of R invariant, and it maps x to

$$\phi^*(x) = \sum_{i=1}^{r} ix_i.$$

The above rings can be described also as finite dimensional modules over R, spanned by $\prod_i x_i^{k_i} (0 \leq k_i \leq e_i)$ and $x^k (0 \leq k \leq d)$, respectively. We say that a representative of an element $[f]$ (in any of these rings) is *reduced* if it is written as an R-linear combination of these monomials. It is denoted by $[f]_{red}$. In this language, the value of the integration maps (along the fibers)

$$\int_{\prod \mathbb{P}V_{e_i}} : H^*_G(\textstyle\prod_i \mathbb{P}V_{e_i}, \mathbb{Z}) \to R, \quad \text{resp.} \quad \int_{\mathbb{P}V_d} : H^*_G(\mathbb{P}V_d, \mathbb{Z}) \to R$$

are the coefficients of the top degree monomials in the corresponding reduced forms: i.e. the coefficient of $\mathbf{x}^e := \prod_i x_i^{e_i}$ in the first case, and the coefficient of x^d in the second case.

Set $q(x) := (Q_d(x) - C_{d+1})/x = x^d + C_1x^{d-1} + \ldots + C_d$, where $Q_d(x) = \sum_{j=0}^{d+1} C_{d+1-j}(c_1, c_2)x^j$ and $C_0 = 1$.

Theorem 3.1. Tp_λ *equals* $\int_{\prod \mathbb{P}V_{e_i}} \phi^*(q)$.

Proof. First we prove that $\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_d}(q \cdot \mathbb{P}\mathrm{Tp}_\lambda)$. Indeed, from the general theory of projective and affine Thom polynomials (cf. section 2) we know that $\mathbb{P}\mathrm{Tp}_\lambda = p_{d-n} + p_{d-n-1}x + \ldots + p_0x^{d-n}$, where $p_j \in \mathbb{Z}[c_1, c_2]$ and $p_{d-n} = \mathrm{Tp}_\lambda$. When we multiply x^jp_{d-n-j} $(1 \leq j \leq d - n)$ with

$q = (Q_d - C_{d+1})/x$ and reduce it modulo $Q_d(x)$, the coefficient of x^d will be 0. So the only contribution comes from qp_{d-n}, which is the coefficient p_{d-n} of $p_{d-n}x^d$.

Now, using the definition of $\mathbb{P}\mathrm{Tp}_\lambda$ and the fact that ϕ is birational, we have $\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_d}(q \cdot \phi_!(1))$ which equals $\int_{\prod \mathbb{P}V_{e_i}} \phi^*(q)$, what we wanted to prove. □

Theorem 3.1 gives the following computational recipe: *Tp_λ is the top coefficient (i.e. the coefficient of $\mathbf{x}^e$) of $\phi^*(q)_{red}$.* Notice that any representative $[f]_{red}$ is automatically computed by computer algebra packages (e.g. [GS]), hence one gets an algorithmic solution of finding the Thom polynomials, see e.g. www.unc.edu/~rimanyi/progs/rootloci.m2. We can, however, give explicit formulae as well.

Formulae for Thom polynomials.

Lemma 3.2. *Set $f \in R[y]$ with class $[f]$ mod $Q_e(y)$. Then the top coefficient of the reduced representative $[f]_{red}$ is*

$$\int_{\mathbb{P}V_e} [f] = \frac{1}{(v-u)^e} \sum_{s=0}^{e} \frac{(-1)^s f\big(-(e-s)u - sv\big)}{s!(e-s)!}. \tag{1}$$

Proof. This is a simple application of the Atiyah-Bott integration formula [AB84, p.9] but we prefer to give a direct proof as follows. The formula is linear in f, hence it is enough to verify it for any $f(y) = y^j$ $(j \geq 0)$. In this case we need A_e where $y^j \equiv A_e y^e + A_{e-1}y^{e-1} + \ldots + A_0$ modulo the ideal $(Q_e(y))$. If we consider this congruence for $y = -eu, -(e-1)u - v, \ldots, -ev$ then we get a system of equations for $A_e, \ldots, A_0$ (since $Q_e(y)$ vanishes at these points). The matrix of this system is a Vandermonde matrix, so by Cramer's rule we get the formula. □

Corollary 3.3. (The "naive" formula) *Let $\sum_{s_1,\ldots,s_r}$ denote the sum over $0 \leq s_i \leq e_i$ for each $1 \leq i \leq r$. Then*

$$\mathrm{Tp}_\lambda = \frac{1}{(v-u)^n} \sum_{j=n}^{d} \sum_{j_1+\ldots+j_r=j} C_{d-j} \binom{j}{j_1,\ldots,j_r} \sum_{s_1,\ldots,s_r} \prod_{i=1}^{r} \frac{(-1)^{s_i}(-i)^{j_i}((e_i-s_i)u + s_i v)^{j_i}}{s_i!(e_i-s_i)!}.$$

Proof. Write $\sum_{j=n}^{d} C_{d-j}(\sum ix_i)^j$ as a linear combination of monomials of type $\prod_{i=1}^{r} x_i^{j_i}$. The polynomial $Q_{e_i}(x_i)$ only contains the variable x_i. Hence to find the top coefficient of the remainder of $\prod_i x_i^{j_i}$, we can simply multiply the top coefficient of the remainders of $x_i^{j_i}$ modulo $Q_{e_i}(x_i)$. Therefore, the formula follows from lemma 3.2 applied one-by-one for each $x_i^{j_i}$. □

One can get a more interesting formula as follows. First notice that $xq + C_{d+1} = Q_d$, hence $(\sum ix_i)\phi^*(q) \equiv -C_{d+1}$ modulo the ideal $\mathcal{I} \subset R[x_1, \ldots, x_r]$ generated by all $Q_{e_i}(x_i)$ $(1 \leq i \leq r)$. We consider the following identities regarding $1/\sum_i ix_i$. Let t be a free variable. Then

$$\frac{1}{-t + \sum_i ix_i} = \frac{1}{-t} \sum_{j \geq 0} \big(\textstyle\sum_i ix_i/t\big)^j = \frac{1}{-t} \sum_{j \geq 0} \sum_{j_1+\ldots+j_r=j} \binom{j}{j_1,\ldots,j_r} \prod_i (ix_i/t)^{j_i}.$$

By lemma 3.2, the top coefficient of the last expression is

$$\frac{1}{-t} \sum_{j \geq 0} \sum_{j_1+\ldots+j_r=j} \binom{j}{j_1,\ldots,j_r} \prod_i \sum_{s_i=0}^{e_i} \frac{(-1)^{s_i}}{(v-u)^{e_i} s_i!(e_i-s_i)!} \big((e_i-s_i)u + s_i v\big)^{j_i} (-i/t)^{j_i}$$

$$= \frac{1}{(-t)(v-u)^n} \sum_{s_1,\dots,s_r} \frac{(-1)^{\sum_i s_i}}{\prod_i s_i!(e_i-s_i)!} \cdot \frac{1}{1+\sum_i i(\,(e_i-s_i)u+s_iv\,)/t}$$

$$= \frac{1}{(v-u)^n} \sum_{s_1,\dots,s_r} \frac{(-1)^{\sum_i s_i}}{\prod_i s_i!(e_i-s_i)!} \cdot \frac{1}{-t-du+(\sum_i is_i)(u-v)}.$$

Let $A(t)$ be this last expression. The above identities show the following congruence (valid for generic t):

$$(-t+\textstyle\sum_i ix_i)(\,A(t)x^e + \text{ lower order terms }) \equiv 1 \ (mod\ \mathcal{I}). \tag{1}$$

Evidently, this is true for $t=0$ as well. On the other hand, notice that there is a unique *reduced* $Y \in R[x_1,\dots,x_r]$ satisfying $(\sum ix_i)Y \equiv -C_{d+1}$ $(mod\ \mathcal{I})$. Indeed, if both Y and Y' satisfy it, then $-C_{d+1}Y' \equiv Y(\sum ix_i)Y' \equiv -C_{d+1}Y$, hence $Y=Y'$. Since $(\sum ix_i)\phi^*(q) \equiv -C_{d+1}$, from (1) (with $t=0$) we get that the top coefficient of $\phi^*(q)_{red}$ is $-C_{d+1}A(0)$. Hence, we proved:

Theorem 3.4. *With the notation $C_{d+1} := C_{d+1}(S^d\mathbb{C}^2) = \prod_{j=0}^d(\,ju+(d-j)v\,)$, one has*

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}}{(v-u)^n} \cdot \sum_{s_1,\dots,s_r} \frac{(-1)^{\sum_i s_i}}{\prod_i s_i!(e_i-s_i)!} \cdot \frac{1}{du-(\sum_i is_i)(u-v)}.$$

This can also be considered as a higher order divided difference formula, cf. 3.7.

Example 3.5. If $\lambda = i^{e_i}$, hence $d = ie_i$, then

$$\mathrm{Tp}_\lambda = i^{e_i} \cdot \prod_{0\le j\le d; i\nmid j} (\,ju+(d-j)v\,).$$

This can be deduced from 3.4 (cf. with the next remark), but one also can argue as follows. Since $ix_i\phi^*(q)+C_{d+1} \equiv 0$ $(mod\ Q_{e_i}(x_i))$, clearly $ix_i\phi^*(q)_{red}+C_{d+1} \equiv 0$ as well. Since $ix_i\phi^*(q)_{red}+C_{d+1}$ and $Q_{e_i}(x_i)$ both have degree e_i+1, one gets that $ix_i\phi^*(q)_{red}+C_{d+1} = C\cdot Q_{e_i}(x_i)$ for some $C\in R$. Comparing the coefficients of $x_i^{e_i+1}$ and x_i^0, one obtains

$$i\,\mathrm{Tp}_\lambda = C_{d+1}(S^d\mathbb{C}^2)/C_{e_i+1}(S^{e_i}\mathbb{C}^2).$$

Remark 3.6. Lemma 3.2 has the following consequence. For some $C\in R$ and $g\in R[y]$, we denote by $[C/g]_{red}$ (or by $\int_{\mathbb{P}V_e}[C/g]$) that reduced element which satisfies $[C/g]_{red}\cdot g \equiv C$ $(mod\ Q_e(y))$ (if it exists). Then one also has:

$$\int_{\mathbb{P}V_e}[C/g] = \frac{1}{(v-u)^e}\sum_{s=0}^{e}\frac{(-1)^s}{s!(e-s)!}\cdot\frac{C}{g\big(-(e-s)u-sv\big)}. \tag{1}$$

Its proof is similar to the proof of 3.4, which, in fact, is a multivariable version of (1) (applied for $-C_{d+1}/\sum ix_i$).

Let us consider again $\lambda = i^{e_i}$. Theorem 3.4 and (1) gives that $\mathrm{Tp}_\lambda = \int_{\mathbb{P}V_{e_i}}[-C_{d+1}(S^d)/ix_i]$. But $x_i(x_i^{e_i}+\cdots)+C_{e_i+1}(S^{e_i}) = Q_{e_i}$, hence $\int_{\mathbb{P}V_{e_i}}[-C_{e_i+1}(S^{e_i})/x_i] = 1$. In particular, $\mathrm{Tp}_\lambda = C_{d+1}(S^d)/iC_{e_i+1}(S^{e_i})$, as it was verified in 3.5.

Example 3.7. Assume that $\lambda = i^{e_i} j^{e_j}$ $(i \neq j)$. Consider the expression given by 3.4 for this λ, and apply in variable x_i the identity 3.6(1). Clearly $du-(is_i+js_j)(u-v) = g(-e_iu+s_i(u-v))$, where $g(x_i) := a - ix_i$ with $a := je_ju - js_j(u-v)$. Therefore

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(v-u)^{e_j}} \sum_{s_j=0}^{e_j} \frac{(-1)^{s_j}}{s_j!(e_j-s_j)!} \cdot \int_{\mathbb{P}V_{e_i}} [1/g(x_i)].$$

Since $Q_{e_i}(x_i) - Q_{e_i}(a/i) = (x_i - a/i)(x_i^{e_i} + \cdots)$ one gets $\int_{\mathbb{P}V_{e_i}} [iQ_{e_i}(a/i)/g(x_i)] = 1$. Hence

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(v-u)^{e_j}} \sum_{s_j=0}^{e_j} \frac{(-1)^{s_j}}{s_j!(e_j-s_j)!} \cdot \frac{1}{i \cdot Q_{e_i}(\,(je_ju - js_j(u-v)\,)/i)}.$$

For example, assume that $\lambda = i^{e_i} j$, i.e. $e_j = 1$. Then $s_j = 0$ or 1, hence

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{i(v-u)} \cdot \Big(\frac{1}{Q_{e_i}(ju/i)} - \frac{1}{Q_{e_i}(jv/i)} \Big).$$

It is convenient to express this in the language of *divided difference*: If $P(u,v)$ is a polynomial in two variables (u,v), we denote by $\partial(P)$ the polynomial $(P(u,v) - P(v,u))/(u-v)$. Then

$$\mathrm{Tp}_{(i^{e_i}j)} = \frac{1}{i} \cdot \partial \Big(\frac{C_{d+1}(S^d)}{Q_{e_i}(jv/i)} \Big) = i^{e_i} \cdot \partial \Big(\prod \big((d-k)v + ku \big) \Big),$$

where the product is over k with $0 \le k \le d$, but $k \neq is$ with $0 \le s \le e_i$. In particular,

$$\mathrm{Tp}_{(1^{e_1}j)} = \partial \Big(\prod_{l=0}^{j-1} \big(lv + (e_1 + j - l)u \big) \Big) \quad (\text{for } j \ge 2), \tag{1}$$

which is equivalent with Kirwan's formula [Kir92, page 902].

Example 3.8. Assume that $d = 2h$ is even, $h > 2$ and $\lambda = (1^{h-j}, j, h)$ for some $1 < j < h$. By a similar argument as in 3.7 and by a computation, one has

$$\mathrm{Tp}_\lambda = \frac{C_{d+1}(S^d)}{(u-v)^2} \cdot \Big(\frac{1}{Q_{h-j}(hu+ju)} - \frac{1}{Q_{h-j}(hu+jv)} - \frac{1}{Q_{h-j}(hv+ju)} + \frac{1}{Q_{h-j}(hv+jv)} \Big)$$

$$= \partial \Big[\frac{C_{d+1}(S^d)}{u-v} \cdot \Big(\frac{1}{Q_{h-j}(hv+jv)} - \frac{1}{Q_{h-j}(hv+ju)} \Big) \Big] = \partial \Big[D_j \cdot \prod_{l=0}^{h-1} \big(lv + (d-l)u \big) \Big],$$

where

$$D_j := \frac{1}{u-v} \cdot \Big[\prod_{l=h-j+1}^{h} (\, lv + (d-l)u\,) - \prod_{l=0}^{j-1} (\, lv + (d-l)u\,) \Big].$$

E.g., if $j = 2$, then

$$\mathrm{Tp}_\lambda = h(h-1) \cdot \partial \Big[(u+3v) \prod_{l=0}^{h-1} \big(lv + (d-l)u \big) \Big].$$

Remarks 3.9. (1) The Thom polynomials are connected by many interesting polynomial relations. E.g., the next section presents two situations when the ideal generated by natural families of Thom polynomials is generated only by two of them. Some of these relations can be verified easily. E.g., assume $d = 2h$ as in 3.8, consider the partitions $\lambda_0' = (1^{h-2}, 2, h)$, $\lambda_0 = (1^h, h)$, $\lambda_1 = (1^{h-1}, h+1)$ and $\lambda_2 = (1^{h-2}, h+2)$. Then from 3.7(1) and 3.8, one gets $\mathrm{Tp}_{\lambda_1} = hc_1 \cdot \mathrm{Tp}_{\lambda_0}$ and

$$(h-1) \cdot \mathrm{Tp}_{\lambda_2} = (h-1)(h-2)c_1 \cdot \mathrm{Tp}_{\lambda_1} + c_1 \, \mathrm{Tp}_{\lambda_0'} \, .$$

(2) Using [FNR], one can determine $\deg(\mathbb{P}X_\lambda)$ by the substitution $u = v = 1/d$ in $\mathrm{Tp}_\lambda \in \mathbb{Z}[u, v]$. The interested reader is invited to verify the compatibility of Hilbert's result (cf. introduction) with this section.
(3) In the sequel we will use many times the following divided difference formula. For any polynomial $A \in \mathbb{Q}[u, v]$ write $A^*(u, v) := A(v, u)$. Then

$$\partial(AB) = B^* \cdot \partial(A) + A \cdot \partial(B).$$

4. Thom polynomial description of the cohomology ring of the moduli space

In this section we apply the coincident root loci formulas in the study of the cohomology ring of the moduli space of the representation $S^d\mathbb{C}^2$ (in the Geometric Invariant Theory sense). We calculate the rational cohomology rings $H_G^*(X^{ss})$, $H^*(X^{ss}//G)$ and $H_G^*(X^s) \cong H^*(X^s//G)$ in terms of generators and relations. If d is odd then all these rings coincide, but for the even case they are different.

There is an extensive literature on these cohomology rings, both from combinatorial-algebraic (see e.g. [Bri91], [Mar99]) and from geometric point of view (the Atiyah-Bott-Kirwan theory [Kir84]). Our approach (in the odd d case) is closest to that of Kirwan. The advantage of our approach is that we treat the odd and even cases in a uniform language, and that we provide for the above cohomology rings a very transparent structure: we obtain explicit presentations of them in terms of generators and relations with clear geometric meanings.

Let us sketch our approach in the odd case first (for details see below). In this case the Kirwan stratification of $S^d\mathbb{C}^2$ is G-perfect since the normal (equivariant) Euler classes of the strata are not zero-divisors. It implies that the spectral sequence of the corresponding filtration degenerates. It is not difficult to calculate all but the 0th column of the E_1-table, so by subtraction we can calculate the ranks of the 0th column: the Betti numbers of $H_G^*(X^{ss})$. Also by G-perfectness the natural map

$$\kappa : H_G^*(S^d\mathbb{C}^2) \cong \mathbb{Q}[c_1, c_2] \to H_G^*(X^{ss})$$

is surjective so we need to find relations in terms of c_1 and c_2, ie. we have to find generators of $\mathrm{Ker}(\kappa)$. If $Y \cap X^{ss} = \emptyset$ for an invariant subvariety Y then clearly $[Y] \in \mathrm{Ker}(\kappa)$. (This idea was studied in [FR03]). So all the higher Kirwan strata provide relations. But the Kirwan strata are coincident root loci for specific partitions and we can calculate their equivariant Poincaré dual using the first part of the paper. Our main point is that the first two Kirwan strata are enough to generate $\mathrm{Ker}(\kappa)$ which can be checked by a simple Betti number calculation.

In the d even case we will refine the Kirwan stratification (Discussion 4.6). The main difficulty is that for one of the strata in this refined stratification the normal (equivariant) Euler class is a zero-divisor. To prove G-perfectness we use the results of the first part of the paper.

Namely we show that certain elements in the E_1-table can be represented by the Poincaré dual of coincident root loci (these are not Kirwan strata!) and they survive to E_∞, hence they could not be hit by a differential. After G-perfectness is proven the process is the same as in the odd case. We can find coincident root loci in the null cone such that their Poincaré dual generate $\mathrm{Ker}(\kappa)$. Here we also need two coincident root loci but one of them is not a Kirwan stratum.

In this section all cohomologies are meant with rational coefficients.

Let us consider the Kirwan-stratification (see [Kir92] and [Kir84]) of the vector space V_d:

- $X^{ss} = \{B \mid B \text{ has no root of multiplicity} > d/2\}$,
- $X_i = \{B \mid B \text{ has a root of multiplicity } i \text{ but no with multiplicity } i+1\}$ $(d/2 < i \leq d)$,
- $X_0 = \{0\}$.

The strata are smooth open submanifolds, the complex codimensions are $0,\ i-1,\ d+1$ in the three cases. By $F_i = \cup$ strata of complex codimension $\leq i$ we get a filtration of V_d:

$$\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \ldots \subset F_{d+1} = S^d\mathbb{C}^2.$$

Let $E_*^{*,*}$ be the associated spectral sequence in G-equivariant cohomology with $\mathbb{Q}$-coefficients.

Proposition 4.1.

(1) $E_1^{0,*} = H_G^*(X^{ss};\mathbb{Q})$
(2) $E_1^{2p,*} = H^*(BU(1);\mathbb{Q})$ *for* $p = [d/2], \ldots, d-1$*;*
(3) $E_1^{2(d+1),*} = H^*(BG;\mathbb{Q})$*;*
(4) $E_1^{*,*} = 0$ *for all cases not covered by (1), (2), (3);*
(5) The spectral sequence converges to $H^*(BG;\mathbb{Q})$*;*
(6) The spectral sequence degenerates at $E_1^{*,*}$ *(in particular,* $H_G^{odd}(X^{ss},\mathbb{Q}) = 0$*).*

Proof. By definition we have $E_1^{2p,*} = H_G^{2p+*}(F_p, F_{p-1})$ which is by Thom isomorphism $H_G^*(F_p \setminus F_{p-1})$. This proves (1) and (4). For $p = d+1$ we have $E_1^{2(d+1),*} = H_G^*(\{0\}) = H^*(BG)$ which proves (3). For $d/2 < i \leq d$ we define $Y_i = \{B \in X_i : x^i | B \text{ and } coeff(x^i y^{d-i}) = 1\}$. Let H be the stabilizer subgroup of Y_i, i.e. the group of matrices of the form $\begin{pmatrix} \alpha_1 & \beta \\ 0 & \alpha_2 \end{pmatrix}$ with $\alpha_1^i \alpha_2^{d-i} = 1$. Since Y_i is contractible, and $X_i = G \times_H Y_i$, part (2) follows from

$$H_G^*(X_i) \cong H_G^*(G \times_H Y_i) \cong H_H^*(Y_i) \cong H^*(BH) \cong H^*(BU(1)) \text{ (over } \mathbb{Q}).$$

The degeneracy of the spectral sequence—called G-perfectness by Atiyah-Bott in [AB83]—follows from usual arguments, as follows. Let us build up V_d by gluing the strata one by one together in order of increasing codimension. Then at one step we have U and glue a new stratum X of complex codimension c to it. We need to prove that the first map in the diagram

$$H_G^{n-2c}(X) \cong H_G^n(U \cup X, U) \to H_G^n(U \cup X) \to H_G^n(X)$$

is injective. However, the whole composition is the multiplication with the equivariant Euler class of the stratum X. This is an injective map being a multiplication by a non-zero element in a polynomial ring. (For a computation of an equivariant Euler class see the proof of 4.7.) □

Since $E_\infty = E_1$, the sum of the ranks of the groups in diagonal (i.e. $p+q=r$) entries must be the rank of the appropriate cohomology group of $H^*(BG;\mathbb{Q})$. Thus we have the following

Corollary 4.2. *Let $h := [d/2]$. The Poincaré series of the ring $H^*_G(X^{ss};\mathbb{Q})$ is*

$$\frac{1}{(1-t)(1-t^2)}(1-t^{d+1}) - \frac{1}{1-t}(t^h + \ldots + t^{d-1}) = \frac{1-t^h-t^{h+1}+t^d}{(1-t)(1-t^2)} \quad (\deg(t)=2).$$

□

What we obtained so far is basically equivalent to the Atiyah-Bott-Kirwan theory applied to our representation, see [Kir84, 16.2].

What can also be seen from the spectral sequence is that $H^*_G(X^{ss}) = H^*(BG)/I$ where the ideal comes from the $p > 0$ columns of the spectral sequence. Thus among the elements of I we have the ones that are the images of the generators of $E_1^{2p,0}$ under the edge-homomorphism. For $[d/2] \le p \le d-1$, these are exactly the Thom polynomials corresponding to the strata X_i, $i = p+1$. We have $\mathrm{Tp}(X_i) = \mathrm{Tp}_\lambda$ with $\lambda = (1^{d-i}, i)$, since the closures of X_i and X_λ are the same. The above Betti number computation can be used to test if a few of these Thom polynomials are enough to generate I.

Theorem 4.3. *Set $\lambda_1 = (1^{d-h-1}, h+1)$ and $\lambda_2 = (1^{d-h-2}, h+2)$, where $h = [d/2]$. Then I is generated by Tp_{λ_1} and Tp_{λ_2}. In particular,*

$$H^*_G(X^{ss};\mathbb{Q}) = \mathbb{Q}[c_1,c_2]\Big/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}).$$

Proof. We already observed that the given two Tp's are in I. Now we prove that the ring on the right hand side has the same Poincaré series as the one given in Corollary 4.2.

We claim that the ideal $J := (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$ has the following R–resolution: $0 \leftarrow J \leftarrow (\mathrm{Tp}_{\lambda_1}) \oplus (\mathrm{Tp}_{\lambda_2}) \leftarrow U \leftarrow 0$, where U is a principal ideal generated by a $\deg d$ polynomial in $R = \mathbb{Q}[c_1,c_2]$. If $d = 2h+1$ then for this we only need to prove that Tp_{λ_1} and Tp_{λ_2} have no nontrivial common divisor D. We know that $\mathrm{Tp}_{\lambda_1} = \partial(\Pi)$, $\mathrm{Tp}_{\lambda_2} = \partial(\Pi L)$, where $\Pi(u,v) = \prod_{l=0}^{h}(lv + (d-l)u)$ and $L(u,v) = (h+1)v + hu$. By 3.9(3), if $D | gcd(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$, then $D|\Pi$, hence $D | gcd(\Pi, \partial(\Pi))$ as well. But $gcd(\Pi, \Pi^*) = 1$, which ends the proof of the claim.

So we get the Poincaré series of R/J as $(1-t^h-t^{h+1}+t^{2h+1})/(1-t)(1-t^2)$, which is the same as the Poincaré series of $H^*_G(X^{ss};\mathbb{Q})$. For d even the proof is similar. □

Discussion 4.4. The cohomology ring of $X^{ss}//G$. Observe that if d is odd then $X^{ss} = X^s$, and all stabilizers of polynomials in X^{ss} are finite. Therefore, we have the ring isomorphism $H^*_G(X^{ss};\mathbb{Q}) = H^*(X^{ss}//G;\mathbb{Q})$ with Poincaré polynomial $(1-t^h)(1-t^{h+1})/(1-t)(1-t^2)$.

If $d = 2h$ is even, then $X^{ss}//G = X^s//G \cup \{p^{ss}\}$, where p^{ss} is the unique "semisimple point" of $X^{ss}//G$. The Poincaré series of $H^*_G(X^{ss})$ is infinite; it is:

$$\frac{1}{1-t^2} + t\cdot P(t), \text{ where } P(t) \text{ is the polynomial } \frac{(1-t^{h-1})(1-t^h)}{(1-t)(1-t^2)} \quad (\deg(t)=2). \tag{1}$$

All the stabilizers of the stable part are finite, and there is only one orbit in the strict semistable part with infinite stabilizer H^{ss}, namely the orbit of the partition (h,h). H^{ss} can be described explicitly, and one has an exact sequence $1 \to U(1)\times\mathbb{Z}_h \to H^{ss} \to \mathbb{Z}_2 \to 1$. Hence BH^{ss} is a double covering of $BU(1)\times B\mathbb{Z}_h$ with rational cohomology $H^*(BH^{ss}) = H^*(BU(1))^{\mathbb{Z}_2} = \mathbb{Q}[t]^{\mathbb{Z}_2}$ ($\deg_t = 2$). Here the $\mathbb{Z}_2$-action is $t \mapsto \pm t$, hence the invariant part is $\mathbb{Q}[t^2]$ with an infinite Poincaré series $1/(1-t^2)$. This is exactly the "infinite contribution" in the above Poincaré series of $H^*_G(X^{ss})$.

In fact, the map $r : H^*(BG) \to H^*(BH^{ss})$ (induced by the inclusion) is the following. At the level of roots, it is given by $u \mapsto \pm t$ and $v \mapsto \mp t$, hence it is the epimorphism $r : \mathbb{Q}[c_1, c_2] \to \mathbb{Q}[t^2]$ given by $c_1 \mapsto 0$ and $c_2 \mapsto -t^2$.

As usual, for any connected space Z, let $\tilde{H}^*(Z)$ be the kernel of $H^*(Z) \to H^*(\text{point})$, as an ideal (or subring without unit) in $H^*(Z)$. The ring $H^*(Z)$ can be reconstructed from $\tilde{H}^*(Z)$ by adding the unit: $H^*(Z) = \mathbb{Q}\langle 1\rangle \oplus \tilde{H}^*(Z)$ (with the natural multiplication).

Let o be the orbit corresponding to the partition (h, h) and consider the natural inclusion $j : o \times_G EG \to X^{ss} \times_G EG$. Obviously, $o \times_G EG$ can be identified with BH^{ss}. Moreover, $j^* : H^*_G(X^{ss}) \to H^*(BH^{ss})$ induced by j can be identified with the epimorphism $\mathbb{Q}[c_1, c_2]/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) \to \mathbb{Q}[t^2]$, $c_1 \mapsto 0$ and $c_2 \mapsto -t^2$ induced by r above. In fact, Tp_{λ_1} and Tp_{λ_2} are both divisible by c_1 (cf. 3.9(1)), hence r sends the ideal generated by them to zero.

Finally, notice that $H^*(X^{ss} \times_G EG, BH^{ss}) = \tilde{H}^*(X^{ss} \times_G EG/BH^{ss})$, and the natural map $r : X^{ss} \times_G EG/BH^{ss} \to X^{ss}//G$ induces an isomorphism at the level of rational cohomology rings. In particular, the long exact cohomology sequence of the pair $(X^{ss} \times_G EG, BH^{ss})$ transforms into the short exact sequences:

$$0 \to \tilde{H}^*(X^{ss}//G) \to H^*_G(X^{ss}) \xrightarrow{j^*} H^*(BH^{ss}) \to 0. \tag{2}$$

Analyzing the kernel of j^*, we get:

Corollary 4.5. *With the notations of 4.3, one has the following ring isomorphisms:*

$$H^*(X^{ss}//G; \mathbb{Q}) = \mathbb{Q}[c_1, c_2]\Big/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) \quad \textit{if } d \textit{ is odd;}$$

$$H^*(X^{ss}//G; \mathbb{Q}) = \mathbb{Q}\langle 1\rangle \oplus (c_1\mathbb{Q}[c_1, c_2])\Big/(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) \quad \textit{if } d \textit{ is even.}$$

Notice that the Poincaré series formula 4.4(1) is compatible with the 4.4(2) and 4.5. In particular, if $d = 2h$, the Poincaré polynomial of $H^*(X^{ss}//G)$ is $1 + tP(t)$.

Discussion 4.6. The cohomology ring of $X^s//G$. Next, for the case $d = 2h$, we wish to determine the cohomology ring of the geometric quotient $X^s//G$. In the notations below it is convenient to assume $h > 2$ (if $h = 2$, then $X^{ss}//G = \mathbb{P}^1$, and $X^s//G = \mathbb{C}$).

We consider a similar spectral sequence, but now associated with the following 'refined' stratification

- $X^s = \{B \mid B \text{ has no root of multipicity} \geq h\}$,
- $X_i = \{B \mid B \text{ has exactly one root of multiplicity } i \text{ but no roots of multiplicity } i+1\}$ $(h \leq i \leq d)$,
- $o = \{\text{the orbit associated with the partition } (h, h)\}$,
- $X_0 = \{0\}$.

In lemma 4.1, $E_1^{0,*}$ will be replaced by $H^*_G(X^s)$. For $i > h$, the stratum X_i is the same as in the previous case. But there are two new strata, namely X_h and o. Since o is an orbit with stabilizer H^{ss}, $H^*_G(o) = H^*(BH^{ss})$. The complex codimension of o in V_d is $d - 2$, hence this will provide an additional direct sum contribution in $E_1^{2(d-2),*}$. Hence, $E_1^{2p,*} = H^*(BU(1))$ if $h \leq p \leq d-1$, but $p \neq d-2$; and $E_1^{2(d-2),*} = H^*(BU(1)) \oplus H^*(BH^{ss})$. Finally, we compute $E_1^{2(h-1),*} = H^*_G(X_h)$. Set

$$Y_h = \{B \in X_h \,:\, B = x^h \cdot B' = x^h(y^h + a_2x^2y^{h-2} + \cdots + a_hx^h);\ \text{and } B' \text{ is not an } h\text{-power}\}.$$

The stabilizer subgroup H of Y_h is the group of diagonal matrices of the form $diag(\alpha_1, \alpha_2)$ with $\alpha_1^h \alpha_2^h = 1$. One can verify that $X_h = G \times_H Y_h$. Moreover, B' is not a h-power if and only if $(a_2, \dots, a_h) \neq (0, \dots, 0)$. Hence Y_h is $\mathbb{C}^{h-1} \setminus \{0\}$ and the action of H is a diagonal torus action (modulo a finite group). In particular, $E_1^{2(h-1),*} = H_G^*(X_h)$ equals the cohomology ring of a weighted projective space of dimension $h-2$, which is $\mathbb{Q}[t]/(t^{h-1})$ ($\deg(t) = 2$).

Proposition 4.7. *The spectral sequence converges to $H^*(BG; \mathbb{Q})$ and it degenerates at $E_1^{*,*}$.*

Proof. The Euler classes of the strata are not zero-divisors except for X_h. So we need the following local version of the Atiyah-Bott argument:

Lemma 4.8. *Suppose that $\{X_i\}$ is a G-equivariant stratification of V and the equivariant normal Euler class of X_i is not a zero-divisor if* $\operatorname{codim}(X_i) > c$. *Then all differentials of the corresponding spectral sequence $E_r^{p,q}$ starting or landing in the region $p > c$ are zero.*

Proof of Lemma. Let X be the union of X_i with $\operatorname{codim}(X_i) > c$. Then the Lemma is equivalent with the statement that $H_G^*(V, V \setminus X) \to H_G^*(V)$ is injective, since $\{E_r^{p,q} : p > c\}$ converges to $H_G^*(V, V \setminus X)$. Injectivity can be proved by adding the X_i's one by one, and noticing that the composition

$$H_G^{n-2c}(X) \cong H_G^n(U \cup X_i, U) \to H_G^n(U \cup X_i) \to H_G^n(X_i)$$

is multiplication with the equivariant normal Euler class of the stratum X_i (where U is an open subset of V in which X_i is closed). □

For the convenience of the reader we show how one determines the equivariant Euler class of o. Fix an element, say $x^h y^h$ on o, let H^{ss} be its stabilizer, consider an H^{ss} invariant normal slice N at $x^h y^h$. In fact, for N one can take the vector space spanned by $x^i y^{d-i}$, where $0 \leq i \leq d$, but $i \notin \{h-1, h, h+1\}$. H^{ss} acts on N, and our goal is the computation of the Euler class $e^{ss} \in H^*(BH^{ss})$ of $EH^{ss} \times_{H^{ss}} N \to BH^{ss}$. Consider now the subgroup $U(1)$ of H^{ss} (see 4.4). The Euler class $e \in H^*(BU(1)) = \mathbb{Q}[t]$ of $EH^{ss} \times_{U(1)} N \to BU(1)$ can be computed as follows. The eigenvalues of $diag(\alpha, \bar{\alpha}) \in U(1)$ on N are $(\alpha^d, \alpha^{d-2}, \dots, \alpha^4, \alpha^{-4}, \dots, \alpha^{-d})$, hence $e = (dt)((d-2)t) \cdots (4t)(-4t) \cdots (-dt) = mt^{d-2}$ for some $m \neq 0$. Since d is even, this is in the invariants part $H^*(BH^{ss}) = \mathbb{Q}[t^2]$ and can be identified in this ring by e^{ss}. Hence $e^{ss} \neq 0$.

This type of argument is not working for the stratum X_h (since the stabilizer of its points are finite, and also $H_G^*(X_h)$ has zero divisors).

In order to show that the differentials $d_{2h-2}^{0,q}$ (q odd and $2h-3 \leq q \leq 4h-7$) of the spectral sequence are trivial, we consider another spectral sequence associated with only two strata, namely with X^s and X_h. The differential $d_{2h-2}^{0,q}$ in the two spectral sequences coincides. If we compare them by the natural maps, then we get the exact sequence

$$0 \to I' \to H_G^*(V_d) \xrightarrow{\tau} H_G^*(X^s \cup X_h)$$

where the ideal I' is generated by all the columns $E_1^{>2h-2,*}$. In $E_\infty^{2(h-1),2(j-1)}$ we can find special elements, those represented by the Thom polynomials $\mathrm{Tp}_j \in H_G^*(V_d)$ associated with the partitions $(1^{h-j}, j, h)$, where $0 < j < h$. Hence, $d_{2h-2}^{0,2j+2h-5} = 0$ if $\tau(\mathrm{Tp}_j) \neq 0$, or equivalently, if $\mathrm{Tp}_j \notin I'$. Notice that the graded ideal I' and the graded ideal I considered in 4.2 and 4.3 are the same in the relevant degrees, hence it is enough to verify that $\mathrm{Tp}_j \notin I$ for any j. But in 4.3 we verified that $I = (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})$. Hence, we need to prove:

$$\mathrm{Tp}_j \notin (\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}). \tag{1}$$

Set

$$\Pi := \prod_{l=0}^{h-1}(\, lv + (d-l)u\,) \quad \text{and} \quad L = (h+1)v + (h-1)u.$$

From 3.7(1) one gets $\mathrm{Tp}_{\lambda_1} = hc_1 \cdot \partial(\Pi)$ and $\mathrm{Tp}_{\lambda_2} = hc_1 \cdot \partial(\Pi L)$. In particular, by 3.9(3), $\mathrm{Tp}_{\lambda_2} = hL^* c_1 \cdot \partial(\Pi) - 2hc_1\Pi$, hence

$$(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2}) = (c_1 \cdot \partial(\Pi), c_1 \cdot \Pi) \tag{2}$$

in $\mathbb{Q}[u,v]$. Assume that (1) is not true and we have $\mathrm{Tp}_j = Ac_1 \cdot \partial(\Pi) + Bc_1\Pi$. Since the degrees of Tp_j and Π are $h+j-2$ and h respectively, the degree of Ac_1 is $j-1$. From 3.8 and 3.9(3), $\mathrm{Tp}_j = \partial(\Pi \cdot D_j) = D_j^* \cdot \partial(\Pi) + \Pi \cdot \partial(D_j)$. This means that

$$\Pi(\partial(D_j) - Bc_1) = \partial(\Pi)(Ac_1 - D_j^*). \tag{3}$$

But it is easy to verify that $gcd(\Pi, \partial(\Pi)) = 1$. Indeed, if $F | gcd(\Pi, \partial(\Pi))$, then also $F|(u-v)\partial(\Pi) = \Pi - \Pi^*$, hence $F|\Pi^*$ as well. But $gcd(\Pi, \Pi^*) = 1$.

This fact together with (3) show that $\Pi | Ac_1 - D_j^*$, but $\deg(Ac_1 - D_j^*) = j-1 < \deg \Pi$, hence $Ac_1 = D_j^*$. In particular, $c_1 | D_j^*$, or $u+v | D_j$. But this leads to a contradiction. Indeed, analyzing in 3.8 the expression of $(u-v)D_j$, one sees that the first product is divisible by $u+v$ (take $l = h$) but the second is not. Hence, (1) is true. □

By similar argument as in the case of $H_G^*(X^{ss})$, for $H_G^*(X^s) = H^*(X^s//G)$ one gets:

Corollary 4.9. *$H^{odd}(X^s//G, \mathbb{Q}) = 0$, and the Poincaré series of $H^*(X^s//G)$ is the polynomial $P(t)$ introduced in 4.4(1).*

Let I'' be the ideal in $H^*(BG) = \mathbb{Q}[c_1, c_2]$ generated by the columns $E_1^{>0,*}$. Then one has the ring isomorphism $H^*(X^s//G) = \mathbb{Q}[c_1, c_2]/I''$. Now we will consider two special elements of I'', namely the Thom polynomials Tp_{λ_0} and $\mathrm{Tp}_{\lambda_0'}$, where $\lambda_0 = (1^h, h)$ and $\lambda_0' = (1^{h-2}, 2, h)$. Their degrees are $h-1$ an h respectively. We will verify now that they are relative prime. Indeed, using the above notations, $\mathrm{Tp}_{\lambda_0} = \partial(\Pi)$ (from 3.7). Moreover, by 3.8 and 3.9(3) one has $\mathrm{Tp}_{\lambda_0'} = h(h-1)\partial((u+3v)\Pi) = h(h-1)[(v+3u)\partial\Pi - 2\Pi]$. In particular, in $\mathbb{Q}[u,v]$ we have $gcd(\mathrm{Tp}_{\lambda_0}, \mathrm{Tp}_{\lambda_0'}) = gcd(\Pi, \partial\Pi)$ which is 1 by the proof of 4.7. Then the usual Poincaré polynomial argument shows $I'' = (\mathrm{Tp}_{\lambda_0}, \mathrm{Tp}_{\lambda_0'})$.

Hence we proved:

Theorem 4.10. *For $d = 2h$ we have*

$$H^*(X^s//G; \mathbb{Q}) = \mathbb{Q}[c_1, c_2]/(\mathrm{Tp}_{\lambda_0}, \mathrm{Tp}_{\lambda_0'}),$$

$$H^*(X^{ss}//G; \mathbb{Q}) = \mathbb{Q}\langle 1 \rangle \oplus \frac{c_1\mathbb{Q}[c_1, c_2]}{(\mathrm{Tp}_{\lambda_1}, \mathrm{Tp}_{\lambda_2})}$$

and the restriction map $H^*(X^{ss}//G) \to H^*(X^s//G)$ is induced by the identity of $\mathbb{Q}[c_1, c_2]$.

Remark 4.11. Since the quotient $X^{ss}//G$ has a unique singular point ($d = 2h$), its intersection cohomology can be computed from the cohomology description of $X^s//G$ and $X^{ss}//G$, using

(the intersection cohomology version of) Theorem 3.5.1 of [Kir88]. Namely, in our case we obtain that $IH^{odd}(X^{ss}//G) = 0$ and for i even we have

$$IH^i(X^{ss}//G) = \begin{cases} H^i(X^s//G) & \text{if } i < d-3 \\ H^i(X^{ss}//G) & \text{if } i > d-3. \end{cases} \tag{1}$$

Since the Poincaré polynomials of $X^s//G$ and $X^{ss}//G$ are $P(t)$ and $1 + tP(t)$, respectively (see 4.4(1)), we obtain that the Poincaré polynomial of $IH^*(X^{ss}//G)$ is

$$\frac{(1-t^{2\lfloor \frac{d}{4} \rfloor})(1-t^{2\lceil \frac{d}{4} \rceil})}{(1-t)(1-t^2)} \qquad (\deg t = 2), \tag{2}$$

where $\lfloor\ \rfloor$ is the (usual, "lower") integer part function, and $\lceil\ \rceil$ is the upper integer part function. The same intersection cohomology Poincaré polynomial (in a different disguise) can be computed using the method of [Kir86], as is presented e.g. in [Kie] as

$$\frac{1+t+\ldots+t^d}{1-t^2} - \sum_{\frac{d}{2}<r\leq d} \frac{t^{r-1}}{1-t} - \frac{t^{2\lfloor \frac{d}{4} \rfloor}}{1-t^2}.$$

Discussion 4.12. The cohomology ring of the link. Denote by L^{ss} the link of the unique semisimple point p^{ss} in $X^{ss}//G$ (i.e. $L^{ss} = \rho^{-1}(\epsilon)$, where $\rho : X^{ss}//G \to [0,\infty)$ is a real analytic map with $\rho^{-1}(0) = \{p^{ss}\}$ and ϵ is sufficiently small). Write CL^{ss} for the real cone over it (i.e. $CL^{ss} = [0,1] \times L^{ss}/\{0\} \times L^{ss}$). Then $H^*(CL^{ss}, L^{ss}) = H^*(X^{ss}//G, X^s//G)$. Hence $H^*(L^{ss})$ is completely determined by the restriction morphism from 4.10. In fact, L^{ss} is a rational homological manifold of real dimension $4h-7$ (with Poincaré duality). [This can also be proved as follows: The geometric quotient of the set of ordered d-points of $\mathbb{P}^1$ is smooth, and one has only finitely many ordered semisimple points. Hence, L^{ss} is the quotient by a finite permutation group of a smooth $(4h-7)$–dimensional link]. 4.10, this duality and a computation give:

Theorem 4.13. *$H^*(L^{ss}, \mathbb{Q})$ can be generated by two elements, c_2 of degree 4 and g (the Poincaré dual of $c_2^{[h/2]-1}$) of degree $4h - 4[h/2] - 3$ with relations $c_2^{[h/2]} = 0$ and $g^2 = 0$. (Notice that all the Betti numbers are 0 or 1.)*

Remark 4.14. 4.10 implies the following: the cohomology ring of the quasi-projective variety $X^s//G$ of (complex) dimension $d-3$ shares the Poincaré duality properties of a smooth projective variety of dimension $d-4$. In fact, cohomologically (over $\mathbb{Q}$), $X^s//G$ behaves like a line bundle $\mathcal{L}$ with Chern class c_1 over a smooth projective variety M with cohomology $\mathbb{Q}[c_1, c_2]/(\partial\Pi, \Pi)$; and $X^{ss}//G$ behaves like the Thom space of this line bundle (or equivalently, the complex cone over M associated with $\mathcal{L}$). In particular, L^{ss} has the cohomology of the S^1-bundle of $\mathcal{L}$.

Remark 4.15. Assume that $d = 2h+1$ is odd. It is tempting to compare the moduli space $X^s//G$ with the (possibly weighted) Grassmanian $Gr_2\mathbb{C}^{h+1}$ because the presentation of their cohomology rings have the same structure $\mathbb{Q}[c_1, c_2]/(\partial p_1, \partial p_2)$ (where $\deg p_1 = h+1$ and $\deg p_2 = h+2$), and they share the same Betti numbers. Indeed for the Grassmanian we can take $p_1 = u^{h+1}, p_2 = u^{h+2}$. In fact, this analogy can be continued: in both cases the set of relations are

guided by some nice generating function, as follows. Set $\Pi_0 := 1$ and $\Pi_j := \prod_{l=0}^{j-1}(lv+(d-l)u)$, and consider the generating function

$$\mathcal{G}(q) = \sum_{j\geq 0} \mathcal{G}_j q^j := \sum_{j\geq 0} \Pi_j q^j/j! = [1+(u-v)q]^{du/(u-v)} \ \in \mathbb{Q}[u,v][[q]].$$

Then $H^*(X^s//G) = \mathbb{Q}[c_1, c_2]/I$, where I is generated by $\partial\mathcal{G}_j$, $j > h$.

In the Grassmanian case the same fact is true with $\mathcal{G}(q) = 1 + uq + u^2q^2 + \cdots = 1/(1-uq)$. However, easy computation shows that, as graded rings, these cohomology rings are *not* isomorphic (unless for small d's).

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