On right-left symmetries of stable singularities

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Introduction

In this paper the right-left automorphism group (also called symmetry group) $Aut_{\mathcal{A}}\eta$ of stable germs $\eta : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+k}, 0)$ are studied for $k \geq 0$. The goal is, on one hand, to explicitly compute its maximal compact subgroup, on the other hand, to prove that the maximal compact subgroup carries exactly as much topological information as the whole symmetry group.

Though these problems may be of independent interest let us discuss why we are interested in them, and in the meanwhile we explain what they really mean. The starting principle is that '*local symmetry governs global topology*'. A simple example for this principle is the following: let $f : N \to P$ be a smooth map between smooth manifolds and η the most complicated singularity of f. Then a tubular neighbourhood of $\eta(f)$ (= the η-points of f in N) is diffeomorphic to the total space of a vector bundle ν with structure group $Aut_{\mathcal{A}}\eta$. In other words ν can be pulled back from a universal bundle over $B(Aut_{\mathcal{A}}\eta)$.

Following pioneering works of Szűcs (e.g. [Sz1]) in [RSz] it is shown that we can apply this simple idea simultaneously to every singularity up to a given complexity and construct a so called 'universal map'. The building blocks of the source and target spaces of this universal map are the $B(Aut_{\mathcal{A}}\eta)$'s mentioned above. It is also shown there how the analysis of the universal map can be translated to global topological properties of *all* maps with prescribed singularities. So, once we understand the $B(Aut_{\mathcal{A}}\eta)$'s well enough, we have the chance to obtain various topological results, such as e.g. computations of cobordism groups [Sz2], or results on elimination of singularities, see references in [RSz]. A recent application of the author is the computation of the cohomology classes of $\eta(f)$'s in terms of the characteristic classes of f—i.e. the so called Thom polynomials of singularities, see [R2]. A further application of the idea was the computation of multiple point formulas ([R3]) describing the cohomology class defined by double, triple, etc. points of the map f. A very simple illustration of the techniques is given below.

Before that, however, let us unearth a technical difficulty in applying the 'local symmetry governs global topology' principle as above. Namely, the group $Aut_{\mathcal{A}}\eta$ is not a finite dimensional Lie group, it does not even possess any convenient topology. So it is difficult to calculate, even to define the classifying space $B(Aut_{\mathcal{A}}\eta)$. If $Aut_{\mathcal{A}}\eta$ were finite dimensional then we could freely pass to its maximal compact subgroup, since a Lie group (with finitely many components) is always homotopy equivalent to its maximal compact subgroup (the quotient space is contractible). Jänich [J] and Wall [W] defined the maximal compact subgroup $MC\,Aut_{\mathcal{A}}\eta$ of

$Aut_{\mathcal{A}}\eta$—and proved its basic properties. Wall also suggested (and Jänich proved for right symmetries of function germs) that the quotient by the maximal compact subgroup is contractible in a generalized sense. This generalized contractibility is proved in section 2 of the present paper. Its geometric meaning is the following: in topological circumstances we meet locally trivial fibrations with vector space fibre (like ν above), which are not vector bundles, but behave like having structure group $Aut_{\mathcal{A}}\eta$. Our definition of 'generalized contractibility of $Aut_{\mathcal{A}}\eta/MC\ Aut_{\mathcal{A}}\eta$' is exactly the technical assumption needed to consider these fibrations as vector bundles with structure group $MC\ Aut_{\mathcal{A}}\eta$, e.g. these bundles can be pulled back by a map to $B(MC\ Aut_{\mathcal{A}}\eta)$. This technical addendum makes the universal map construction of the preceding paragraph work.

The application of the universal map method, however, needs one more input: the concrete knowledge of the maximal compact symmetry groups of singularities. Let us see how it works. By definition we obtain $MC\ Aut_{\mathcal{A}}\eta$ as a subgroup of $GL(n) \times GL(n+k)$, in other words, it comes with representations λ_1, λ_2 on $\mathbb{R}^n$ and $\mathbb{R}^{n+k}$ respectively. The generalized contractibility theorem implies that once we have a map $f : N \to P$, the structure group of the normal bundle of $\eta(f)$ can be reduced to $\lambda_1(MC\ Aut_{\mathcal{A}}\eta)$. Therefore if $det(\lambda_1) > 0$ then $\eta(f)$ is coorientable for any f (see more in Remark 3 below). This is an important condition turning up for example in the computation of the so called Vassilev complex of singularities, see e.g. [FR]. More delicate applications of the concrete knowledge of $MC\ Aut_{\mathcal{A}}\eta$ are given in the works cited above.

So the other result of the present paper concerns the computation of the maximal compact symmetry group of singularities. In theorem 1.4 this problem (infinite dimensional in nature) is reduced to a finite dimensional, algebraically more attackable one, namely to the computation of maximal compact symmetry groups of finite dimensional algebras. As an example we compute the maximal compact symmetry groups of Σ^1 and $\Sigma^{2,0}$ singularities, needed e.g. in the Thom polynomial computations of [R2].

We will use standard notations of singularity theory, such as: $\mathcal{E}(n,p)$ denotes the vector space of smooth germs $(\mathbb{R}^n, 0) \longrightarrow \mathbb{R}^p$. If $p = 1$ then it is an algebra and the former is a (free) module over it. The unique maximal ideal of $\mathcal{E}(n) := \mathcal{E}(n,1)$ will be denoted by $\mathfrak{m}(n)$. The group $\mathcal{A} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ acts on $\mathfrak{m}(n)\mathcal{E}(n,p)$ by the definition $(\psi, \phi)f = \phi f \psi^{-1}$. The stabiliser subgroup of η will be denoted by $Aut_{\mathcal{A}}\eta$ (which would still be an infinite dimensional group if we defined any reasonable topology on it). Germs in the same orbit are called $\mathcal{A}$-equivalent. The contact group $\mathcal{K}$ also act on $\mathcal{E}(n,p)$, the stabiliser of $\eta \in \mathcal{E}(n,p)$ is denoted by $Aut_{\mathcal{K}}\eta$. If $\eta \in \mathcal{E}(n,p)$ then θ_η will denote the vector space of germs of vector fields along η. The formula $\theta_{\mathbb{R}^n}$ stands for $\theta_{id_{\mathbb{R}^n}}$. When passing from the "manifold of germ" to its "tangent space" θ_η, it is useful to define the maps $t\eta : \theta_{\mathbb{R}^n} \longrightarrow \theta_f$ and $\omega\eta : \theta_{\mathbb{R}^p} \longrightarrow \theta_f$ by $t\eta(h) = d\eta \circ h$ and $\omega\eta(h) = h \circ \eta$.

Let $\mathcal{H}$ be one of the groups $\mathcal{A}$ or $\mathcal{K}$. Motivated by Bochner's theorem let us call a subgroup $G \leq Aut_{\mathcal{H}}\eta$ compact if it is conjugate in $\mathcal{H}$ to a compact linear group. Following Jänich [J] Wall proves the following theorem.

Theorem [W]. *If $\eta \in \mathcal{E}(n,p)$ is finitely determined then any compact subgroup of $Aut_{\mathcal{H}}\eta$ is contained in a maximal such group and any two maximal compact subgroups are conjugate in $Aut_{\mathcal{H}}\eta$.* $\square$

1. Computation of the maximal compact subgroup

In this section we reduce the "infinite dimensional" problem of computing the maximal compact subgroup of $Aut_{\mathcal{A}}F$ to a finite dimensional one, and use this reduction to determine this maximal compact subgroup in case of singularities of type Σ^1, $\Sigma^{2,0}$.

Let $F : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ $(k \geq 0)$ be an isolated stable map germ. We call a germ 'isolated' if it has the property that it is not $\mathcal{A}$-equivalent to $F : (\mathbb{R}^n, x) \longrightarrow (\mathbb{R}^{n+k}, F(x))$ for any $x \neq 0$ near $0 \in \mathbb{R}^m$. It is well known that F is the a miniversal unfolding of a germ $f \in \mathcal{E}(m, m+k)$ whose differential at 0 vanishes. Let the unfolding dimension be $r = n - m$. Recall that for two germ $f_1, f_2 \in \mathcal{E}(m, m+k)$ and for their miniversal unfoldings F_1, F_2, the following holds: $f_1 \sim_{\mathcal{K}} f_2 \Leftrightarrow F_1 \sim_{\mathcal{A}} F_2$. The following statement can be found in [W], but for completeness we repeat its proof (in this paper MC will always mean "maximal compact subgroup of").

1.1 Theorem. *MC $Aut_{\mathcal{A}}F \cong MC$ $Aut_{\mathcal{K}}f$.*

Proof. Let G be a compact subgroup of $Aut_{\mathcal{K}}f$. By definition it is conjugate in $\mathcal{K}$ to a (compact) linear group G_0, which is therefore also in $Aut_{\mathcal{A}}f_0$ for a germ $f_0 \in \mathcal{E}(m, m+k)$ $\mathcal{K}$-equivalent to f. Now recall that the miniversal unfolding of f_0 is

$$F_0 : (\mathbb{R}^m \times V, 0) \longrightarrow \mathbb{R}^{m+k} \times V$$

$$(x, \phi) \mapsto (f_0(x) + \phi(x), \phi),$$

where $V = \mathbb{R}^r$ is a complement of $tf(\theta_{\mathbb{R}^m}) + f^*\mathfrak{m}(m+k)\theta_f$ in $\mathfrak{m}(m)\theta_f$. (The expression $\phi(x)$ makes sense since the space θ_f can be identified with $\mathcal{E}(m, m+k)$.) Observe that G_0 has a natural linear action on θ_f:

$$(\alpha, \beta) \cdot \phi := \beta \circ \phi \circ \alpha^{-1}, \quad (\alpha, \beta) \in G \leq GL_m(\mathbb{R}) \times GL_{m+k}(\mathbb{R}), \phi \in \theta_f = \mathcal{E}(m, m+k).$$

Remark. The following may help to understand the G action on θ_f and the invariant subspaces of it. Think of θ_f as the tangent space at f to the infinite dimensional manifold $\mathcal{E}(m, m+k)$. Let $\mathcal{K}f$ denote the set of points in $\mathcal{E}(m, m+k)$ contact equivalent to f. This is a "submanifold" of $\mathcal{E}(m, m+k)$. As $G \leq \mathcal{K}$, and fixes f, it has an action on $\mathcal{E}(m, m+k)$ fixing f. The group G will send a germ $\mathcal{K}$-equivalent to f into another germ $\mathcal{K}$-equivalent to f (because $G \leq \mathcal{K}$). Therefore the submanifold $\mathcal{K}f$ is kept invariant. Hence the *differential* of the G-action which is again a G-action but now on the tangent space at f (i.e. on θ_f) leaves the tangent space of $\mathcal{K}f$ invariant. It is not hard to identify $tf(\theta_{\mathbb{R}^m}) + f^*\mathfrak{m}(m+k)\theta_f$ as the tangent space of $\mathcal{K}f$ in θ_f. In fact, $\mathcal{K}$ is locally a direct product of $\mathcal{R}$ and $\mathcal{C}$, and $\mathcal{R}f = tf(\theta_{\mathbb{R}^m})$, $\mathcal{C}f = f^*\mathfrak{m}(m+k)\theta_f$.

If V is chosen to be G_0-invariant (which is possible since G_0 is compact) then we can define G_0-actions on $\mathbb{R}^m \times V$ and $\mathbb{R}^{m+k} \times V$ as follows ($x \in \mathbb{R}^m$, $y \in \mathbb{R}^{m+k}$, $\phi \in V$)

$$(\alpha, \beta) \cdot (x, \phi) := (\alpha(x), (\alpha, \beta) \cdot \phi),$$

$$(\alpha, \beta) \cdot (y, \phi) := (\beta(y), (\alpha, \beta) \cdot \phi)$$

This action on the source and the target spaces of F_0 makes G a subgroup of $Aut_{\mathcal{A}}F_0$:

$$F_0((\alpha,\beta)\cdot(x,\phi)) = F_0(\alpha(x),(\alpha,\beta)\cdot\phi) = (f_0(\alpha(x)) + (\alpha,\beta)\cdot\phi(\alpha(x)),(\alpha,\beta)\cdot\phi) =$$

$$= (\beta(f_0(x)) + (\beta\phi\alpha^{-1})(\alpha(x)),(\alpha,\beta)\cdot\phi) =$$

$$= (\beta(f_0(x)+\phi(x)),(\alpha,\beta)\cdot\phi) = (\alpha,\beta)\cdot F_0(x,\phi).$$

Here we used that G is linear and that it is a subgroup of $Aut_{\mathcal{A}}f_0$. Since $F \sim_{\mathcal{A}} F_0$, we have proved that $MC\ Aut_{\mathcal{K}}f \leq MC\ Aut_{\mathcal{A}}F$.

To prove the converse let G be a compact subgroup of $Aut_{\mathcal{A}}F$. Then there is an $h \in \mathcal{A}$ such that $G_0 := hGh^{-1} \leq GL_{n+r}(\mathbb{R}) \times GL_{p+r}(\mathbb{R})$. If $F_0 = h \cdot F$ then $G_0 \leq Aut_{\mathcal{A}}F_0$ because

$$(hgh^{-1})graph\ F_0 = (hg)graph\ F = h\ graph\ F = graph\ F_0$$

for any $g \in G$. Since G_0 is linear and $graph\ F_0$ is G_0-invariant the tangent space $(dF_0)_0 = T_0(graph\ F_0) \subset \mathbb{R}^{n+r} \times \mathbb{R}^{p+r}$ is also G_0-invariant. It follows that $A := (dF_0)_0 \cap \mathbb{R}^{n+r} \times \{0\}$ and $C := pr_{\mathbb{R}^{p+r}}((dF_0)_0)$ are also G_0-invariant subspaces of dimensions n and r respectively. Choose G_0-invariant complements of A and C in $\mathbb{R}^{n+r}$ and $\mathbb{R}^{p+r}$: B and D. Therefore $A = \mathbb{R}^n$, $B = \mathbb{R}^r$, $C = \mathbb{R}^r$, $D = \mathbb{R}^p$. Denote the map $(pr_D \circ F_0)|_A : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ by f_0. It can be proved that F_0 is a miniversal unfolding of f_0. Since F_0 and F are $\mathcal{A}$-equivalent f_0 and f must be $\mathcal{K}$-equivalent.

If we project the group $G_0 \leq GL(A) \times GL(B) \times GL(C) \times GL(D)$ to $GL(A) \times GL(D)$ — let this projection be π — then the resulting group is clearly in $Aut_{\mathcal{A}}f_0$. What we want to show is that this projection is injective on G_0, that is we want to prove that the action of G_0 on A and D determines its action on B and C (both are isomorphic to $\mathbb{R}^r$). It is enough to deal with the action on C, because it (together with the actions on A and D) determines the action on B.

Let us use the notation $\mathcal{N}_\eta = \theta_\eta / t\eta(\theta_{\mathbb{R}^n}) + \eta^*(\mathfrak{m}(p))\theta_\eta$ for $\eta \in \mathcal{E}(n,p)$ and the fact that $\omega\eta$ induces isomorphism between $\theta_{\mathbb{R}^n}/\mathfrak{m}(p) \longrightarrow \mathcal{N}_\eta$ if η is an isolated stable singularity. It is also known that the natural map $\rho_{f_0,F_0} : \mathcal{N}_{f_0} \longrightarrow \mathcal{N}_{F_0}$ is an isomorphism.

Now consider the diagram

$$\theta_{\mathbb{R}^{p+r}}/\mathfrak{m}(p+r)\theta_{\mathbb{R}^{p+r}} \quad \xrightarrow{\omega F_0} \quad \mathcal{N}_{F_0} \quad \xleftarrow{\rho_{f_0,F_0}} \quad \mathcal{N}_{f_0}.$$

There is a naturally defined G_0 action on all three spaces involved here: the space $\theta_{\mathbb{R}^{p+r}}/\mathfrak{m}(p+r)\theta_{\mathbb{R}^{p+r}}$ is naturally identified with $\mathbb{R}^{p+r} \cong C \times D$, so it has a G_0 action. Since $G_0 \leq Aut_{\mathcal{A}}F_0$ it has an action on $\mathcal{N}_{F_0}$ described in the proof of the preceding theorem. The group G_0 also operates as $\mathcal{A}$-automorphism group on f_0 through π, so it has an action on $\mathcal{N}_{f_0}$, too. The discussion before the theorem says that the map ρ_{f_0,F_0} is G_0-equivariant. The other map ωF_0 is trivially G_0-equivariant. Since both maps ωF_0 and ρ_{f_0,F_0} are G_0-equivariant isomorphisms we obtain that the action of G_0 on $\mathcal{N}_{f_0}$ determines the action of G_0 on C (and D). But the action on $\mathcal{N}_{f_0}$ depends only on the π-image of G_0, so $\pi|_{G_0}$ is injective.

The end of the proof is a routine: $G_0 \leq Aut_{\mathcal{A}}f_0$ compact, so $G_0 \leq Aut_{\mathcal{K}}f_0$ compact. Since f and f_0 are $\mathcal{K}$-equivalent, there is a compact subgroup of $Aut_{\mathcal{K}}f$ isomorphic (conjugate in $\mathcal{K}$) to G_0. As $G_0 \simeq G$, the theorem is proved. $\square$

Now the problem of computing $MC\ Aut_{\mathcal{A}}F$ is reduced to the computation of $MC\ Aut_{\mathcal{K}}f$, which is still "infinite dimensional" in nature. Now our goal is to go further and embed this group in a (finite dimensional) Lie group.

Let us fix the finitely $\mathcal{K}$-determined germ $f \in \mathcal{E}(m, m+k)$ for which $df(0) = 0$. Present its local algebra Q_f (which is also the local algebra of F) as

$$\mathbb{R}[[x_1, \dots, x_m]]/(r_1, \dots, r_s),$$

where $s - m$ is minimal (m is also minimal because of the condition on df). Let us call this minimal $s - m$ the *defect* of the algebra, and denote it by $d = d(Q_f)$. The germ f has to be $\mathcal{K}$-equivalent to

$$g : (\mathbb{R}^m, 0) \longrightarrow \mathbb{R}^s \times \mathbb{R}^{k-d}$$

$$(x_1, \dots, x_n) \mapsto (r_1, \dots, r_s,\ 0, \dots, 0).$$

Let us denote by h the germ $h : (\mathbb{R}^m, 0) \longrightarrow \mathbb{R}^s$, $(x_1, \dots, x_m) \mapsto (r_1, \dots, r_s)$. The proof of the following lemma is trivial.

1.2 Lemma. *$MC\ Aut_{\mathcal{K}}g \cong MC\ Aut_{\mathcal{K}}h \times O(k-d)$.* $\square$

Now we turn to the study of $Aut_{\mathcal{K}}h$. I_h will denote the image of $h^*\mathfrak{m}(s)\mathcal{E}(m) \leq \mathcal{E}(m)$ in $\mathbb{R}[[x_1, \dots, x_m]]$.

1.3 Theorem. *Let $G \leq Aut_{\mathcal{K}}h$ be compact. Then there is a compact subgroup H of $Aut\ Q_h$ isomorphic to G.*

Proof. Without loss of generality we can suppose that G acts linearly on $\mathbb{R}^m \times \mathbb{R}^s$, because if not, then we change h by an appropriate element of $\mathcal{K}$ (which linearizes G).

Now let $(\psi, \phi) \in G \leq GL_m(\mathbb{R}) \times GL_s(\mathbb{R})$ then ψ acts on $\mathbb{R}[[x_1, \dots, x_m]]$ and it induces an action

$$\bar{\psi} : Q_h \longrightarrow Q_h$$

on the local algebra Q_h. To see this we only have to check that $\psi \cdot I_h \subset I_h$. Since $(\psi, \phi) \in G$ we have $h(x) = \phi \circ h \circ \psi^{-1}(x)$. The matrix ϕ is invertible, so $I_h = I_{\psi \circ h} = \psi I_h$, which means that $\bar{\psi}$ is well defined.

The map $\bar{\psi}$ is an automorphism for all $(\psi, \phi) \in G$. To see this, we have to check that it is an injective and surjective homomorphism. All the three of these properties are easily verified.

Now consider the homomorphism $G \longrightarrow Aut\ Q_h$ mapping (ψ, ϕ) to $\bar{\psi}$. We are going to show that it is injective. Suppose that $\bar{\psi}$ is the identity. Then every element of Q_h is mapped to an element I_h-equivalent to it. Especially $[x_i]$ and $[\sum \psi_{ij} x_j]$ are I_h equivalent. But I_h is contained in the square of the maximal ideal of $\mathbb{R}[[x_1, \dots, x_m]]$, so these linear polynomials can only be I_h equivalent if they are equal. This means that $\psi = id$.

It remains to show that ϕ is also the identity. Usually, if the action of an element of $Aut_{\mathcal{A}}h$ is given on the source space, then the action of the same element on the target space is not determined. It is determined only on $pr_{\mathbb{R}^s}(graph\ h)$ — and therefore on the linear space $W_h := span\ pr_{\mathbb{R}^s}(graph\ h)$ it generates. However we prove that in our case $W_h = \mathbb{R}^s$. Suppose that W_h is smaller than $\mathbb{R}^s$. Then h has the form (after appropriate coordinate changes):

$$(x_1, \dots, x_m) \mapsto (r'_1, \dots, r'_{s-1}, 0)$$

which means that Q_h can be presented as

$$\mathbb{R}[[x_1,\dots,x_m]]/(r'_1,\dots,r'_{s-1})$$

which contradicts to the condition that the defect of Q_h is $d=s-m$. Therefore the map $(\psi,\phi)\mapsto\bar\psi$ is injective so we have proved the theorem. $\square$

Putting all these together we have the following

1.4 Theorem. *If $F:\mathbb{R}^n\longrightarrow\mathbb{R}^{n+k}$ $(k\geq 0)$ is an isolated stable singularity then*

$$MC\ Aut_{\mathcal{A}}F\leq MC\ Aut\ Q_F\times O(k-d),$$

where d is the defect of its local algebra. $\square$

The problem of finding the maximal compact subgroup of a Lie group (such as $Aut\ Q_F$) is theoretically easy. On the other hand, computations can turn very long and difficult, since the algebras at hand are high dimensional even for the singularity types Σ^1, $\Sigma^{2,0}$. The following idea "restricts the size" of our groups.

For a commutative, finite dimensional $\mathbb{R}$-algebra Q with the maximal ideal $\mathfrak{m}$ the associated graded algebra is the vector space

$$G(Q):=Q/\mathfrak{m}\ \oplus\ \mathfrak{m}/\mathfrak{m}^2\ \oplus\ \mathfrak{m}^2/\mathfrak{m}^3\ \oplus\dots$$

with the multiplication induced by the one on Q. In addition to considering the automorphisms of $G(Q)$ we can consider its graded automorphisms, i.e. those which keep the above direct sum decomposition. Let their group be $Aut^{gr}G(Q)$.

1.5 Lemma. $MC\ Aut\ Q\leq MC\ Aut^{gr}G(Q)$.

Proof. If $\phi\in Aut\ Q$ then it induces linear automorphism on $\mathfrak{m}^l/m^{l+1}$, and therefore a linear automorphism $\bar\phi$ on $G(Q)$. It is readily checked that $\bar\phi$ is actually a graded algebra automorphism: $\bar\phi\in Aut^{gr}Q$. Its kernel consists of unipotent upper triangular elements, which can not belong to a compact subgroup (except the id), so the map $\phi\mapsto\bar\phi$ restricted to any compact subgroup is injective. $\square$

The forgetful map $Aut^{gr}G(Q)\longrightarrow GL(\mathfrak{m}/\mathfrak{m}^2)$ is clearly injective. So we have that $MC\ Aut\ Q\leq O(\mathfrak{m}/\mathfrak{m}^2)$. Let us remark that for a singularity F of Thom-Boardman type Σ^r the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is r (it is identified with $ker\ dF$).

Some other interesting statements about $Aut\ Q$ and its relation to $Aut\ G(Q)$ can be found in [GS]. They turn out to be useful in concrete computations.

Examples

The aim of this section is to compute the maximal compact subgroup of the $\mathcal{A}$-automorphism group of some stable germs. This means that we will compute $Aut_{\mathcal{A}}F$ as an abstract group and its representations λ_1 and λ_2 in the source and in the target spaces for all stable singularities $\mathbb{R}^n\longrightarrow\mathbb{R}^{n+k}(k\geq 0)$ of type Σ^1 and $\Sigma^{2,0}$.

Note that if the germ $F:(\mathbb{R}^{n+t},0)\longrightarrow\mathbb{R}^{p+t}$ has the form $(x,u)\mapsto(h(x),u)$, then $Aut_{\mathcal{A}}F\cong Aut_{\mathcal{A}}h\times O(t)$. Therefore it is enough to deal with *isolated* singu-

Notation. In what follows ρ_l will always mean the usual representation of $O(l)$ on $\mathbb{R}^l$. For the groups $O(2)$ and D_a (dihedral group) ρ_2^w will mean the two-dimensional representation which maps

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \cos w\alpha & -\sin w\alpha \\ \sin w\alpha & \cos w\alpha \end{pmatrix}$$

and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to itself. Three one-dimensional representations of the dihedral groups will also be needed. Let the dihedral group D_{2a} be presented as $< f, t | f^a = t^2 = 1, tft = f^{2a-1} >$. The following define 1-dimensional representations of D_a:

$$\lambda : f \mapsto -1, t \mapsto -1 \qquad \kappa : f \mapsto -1, t \mapsto 1 \qquad \tau : f \mapsto 1, t \mapsto -1.$$

Moreover, ε and θ will mean the non-trivial representations of $\mathbb{Z}_2$ (two letters are needed when $\mathbb{Z}_2 \times \mathbb{Z}_2$ are concerned). If one of the defined representations are written as a representation of a group $O(l) \times H$, $D_m \times H$, $\mathbb{Z}_2 \times H$ then they really mean to be a composition of that representation with the projection to $O(l), D_{2a}$ or $\mathbb{Z}_2$.

For all codimension k and for all $r \in \{1, 2, \dots\}$ there is an isolated stable singularity $F_{r,k}$ of type Σ^1. The one corresponding to r and k (the "isolated Morin singularity of type Σ^{1_r} in codimension k") has local algebra $Q_{F_{r,k}} = \mathbb{R}[[x]]/(x^{r+1})$ (defect=0), so it is the miniversal unfolding of $f_{r,k} : (\mathbb{R}, 0) \longrightarrow \mathbb{R}^{k+1}$, $x \mapsto (x^{r+1}, 0, \dots, 0)$.

The above argument shows the following theorem (which has already been proved by [Sz3] using a different, geometric approach).

1.6 Theorem. *MC* $Aut_{\mathcal{A}} F_{r,k} = O(1) \times O(k)$, $\lambda_1 = \mu_1 \oplus \mu_V$, $\lambda_2 = \mu_2 \oplus \mu_V$, *where* $\mu_1 := \rho_1, \qquad \mu_2 := \rho_1^{r+1} \oplus \rho_k, \qquad \mu_V := \left(\sum_{l=r+2}^{2r} \rho_1^{\otimes l}\right) \oplus \left(\sum_{i=1}^{r} \rho_k \otimes \rho_1^{\otimes i}\right).$ □

Now we turn to $\Sigma^{2,0}$ germs. Mather proved in [M2] that there are five infinite sequences of algebras corresponding to $\Sigma^{2,0}$ singularity types:

$I_{a,b}$	$\mathbb{R}[[x,y]] \; / \; (xy, x^a + y^b)$	$b \geq a \geq 2$
$II_{a,b}$	$\mathbb{R}[[x,y]] \; / \; (xy, x^a - y^b)$	$b \geq a \geq 2$ both even
$III_{a,b}$	$\mathbb{R}[[x,y]] \; / \; (xy, x^a, y^b)$	$3 \leq b \geq a \geq 2$
IV_a	$\mathbb{R}[[x,y]] \; / \; (x^2 + y^2, x^a)$	$a \geq 3$
V_a	$\mathbb{R}[[x,y]] \; / \; (x^2 + y^2, x^a, yx^{a-1})$	$a \geq 2$

(For convenience $III_{2,2}$ of Mather is renamed as V_2 here, since — considering symmetries — $III_{2,2}$ is closer to the V_a sequence.)

1.7 Theorem. *If F is an isolated singularity of type $\Sigma^{2,0}$ then MC $Aut_{\mathcal{A}} F$ and its representations $\lambda_1 = \mu_1 \oplus \mu_V$, $\lambda_2 = \mu_1 \oplus \mu_V$ are given by*

F		*MC Aut F*	μ_1	μ_2
$I_{a,b}, II_{a,b}$	$2 \leq a < b$ both even	$D_2 \times O(k)$	ρ_2	$\tau \oplus 1 \oplus \rho_k$
$I_{a,b}$	$2 < a$ odd, $2 \leq b$ even	$\mathbb{Z}_2 \times O(k)$	$1 \oplus \varepsilon$	$\varepsilon \oplus 1 \oplus \rho_k$
$I_{a,b}$	$2 < a < b$ both odd	$\mathbb{Z}_2 \times O(k)$	2ε	$1 \oplus \varepsilon \oplus \rho_k$
[illegible]	$2 < a$ odd [illegible]	[illegible] $\times O(k)$	[illegible]	[illegible]

$I_{a,a}$	$2 \leq a$ even	$D_4 \times O(k)$	ρ_2	$\lambda \oplus 1 \oplus \rho_k$
$II_{a,a}$	$2 < a$ even	$D_4 \times O(k)$	ρ_2	$\lambda \oplus \kappa \oplus \rho_k$
$II_{2,2}$		$O(2) \times O(k)$	ρ_2	$\rho_2^2 \oplus \rho_k$
$III_{a,a}$	$2 < a$ even	$D_4 \times O(k-1)$	ρ_2	$\lambda \oplus 1 \oplus \kappa \oplus \rho_{k-1}$
$III_{a,a}$	$2 < a$ odd	$D_4 \times O(k-1)$	ρ_2	$\lambda \oplus \rho_2 \oplus \rho_{k-1}$
$III_{a,b}$	$2 \leq a < b$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times O(k-1)$	$\varepsilon \oplus \theta$	$(\varepsilon \otimes \theta) \oplus \varepsilon^a \oplus \theta^b \oplus \rho_{k-1}$
IV_a	$3 \leq a$	$D_{2a} \times O(k)$	ρ_2	$1 \oplus \kappa \oplus \rho_k$
V_a	$2 \leq a$	$O(2) \times O(k-1)$	ρ_2	$1 \oplus \rho_2^a \oplus \rho_{k-1}$

F		μ_V
$I_{a,b}, II_{a,b}$	$2 \leq a < b$ both even	$\left[\frac{a}{2}\rho_2 \oplus \frac{b-a}{2}\lambda \oplus \frac{a+b-4}{2}1\right] \otimes (1 \oplus \rho_k) \oplus \rho_k$
$I_{a,b}$	$2 < a$ odd, $2 \leq b$ even	$\left[\frac{b}{2}\varepsilon \oplus \frac{2a+b-4}{2}1\right] \otimes (1 \oplus \rho_k) \oplus \rho_k$
$I_{a,b}$	$2 < a < b$ both odd	$\frac{a+b-2}{2}(1 \oplus \varepsilon) \otimes (1 \oplus \rho_k) \oplus (\varepsilon \otimes \rho_k)$
$I_{a,a}$	$2 < a$ odd	$\frac{a-1}{2}(\rho_2 \oplus \rho_2^2) \otimes (1 \oplus \rho_k) \oplus (\lambda \otimes \rho_k)$
$I_{a,a}$	$2 \leq a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes (1 \oplus \rho_k) \oplus (\kappa \otimes \rho_k)$
$II_{a,a}$	$2 < a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes (1 \oplus \rho_k) \oplus (\rho_k)$
$II_{2,2}$		$\rho_2^3 \oplus (\rho_2 \oplus 1) \otimes \rho_k$
$III_{a,a}$	$2 < a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes \left[1 \oplus 1 \oplus \rho_{k-1}\right]$
$III_{a,a}$	$2 < a$ odd	$\frac{a-1}{2}\left[2\rho_2 \oplus \rho_2^2 \oplus \rho_2^4 \oplus (\rho_2 \oplus 1 \oplus \kappa) \otimes \rho_{k-1}\right]$
$III_{a,b}$	$2 \leq a < b$	$\bigoplus_{j=1}^{a-1}\left[\varepsilon^{a+j} \oplus \varepsilon^j \otimes (\theta^b \oplus \rho_{k-1})\right] \oplus$ $\oplus \bigoplus_{j=1}^{b-1}\left[\theta^{b+j} \oplus \theta^j \otimes (\varepsilon^a \oplus \rho_{k-1})\right]$
IV_a	$3 \leq a$	$(\bigoplus_{j=1}^{a-1} \rho_2^j) \otimes (1 \oplus \rho_k) \oplus (\lambda \otimes \rho_k)$
V_a	$2 \leq a$	$\bigoplus_{j=1}^{a-1}(\rho_2^{a-j} \oplus \rho_2^{a+j}) \oplus (\bigoplus_{j=1}^{a-1} \rho_2^j) \otimes \rho_{k-1}$

Remark 1. It is well known that for a Thom-Boardman type Σ^r, the structure group of the normal bundle in the source (λ_1) is isomorphic to $Hom(ker, coker)$, where ker is the kernel bundle (in our notation μ_1) and $coker$ is the cokernel bundle (in our notation μ_2). For $k = 0$ type $I_{2,2}$ and $II_{2,2}$ are the simplest singularities and for $k > 0$ it is V_2. So the identity $Hom(\mu_1, \mu_2) \cong \mu_1 \oplus \mu_V$ should hold for them. Using the fact that $\rho_2 \otimes \rho_2^3 \cong \rho_2 \oplus \rho_2^2$ it can be seen from the above that they really hold.

Remark 2. The singularities $I_{a,b}$, $II_{a,b}$ are equivalent over the complex field. Therefore it seems interesting that while they usually behave similarly, their symmetries are characteristicly different for $a = b = 2$: one is discrete $(\times O(k))$ and the other is continuous $(\times O(k))$.

Remark 3. Some differential topological applications can already be drawn, without the generalized Pontrjagin-Thom construction of [RSz]. Let $\phi : M^m \longrightarrow N^{m+k}$ be a stable smooth map between smooth manifolds and let $\eta(\phi)$ be the submanifold of M consisting those points of M where the germ of ϕ is ($\mathcal{A}$-equivalent to a suspension of) η. If for an η for all $g \in MC\ Aut_{\mathcal{A}}\eta$ we have that $\det\ \lambda_1(g) > 0$ then $\eta(\phi)$ is co-orientable. (These are exactly the singularities occuring in the Vassiliev complex of right-left singularity theory if homology is considered with integer coefficients.) For example if a and k are both even then the submanifold $V_a(\phi)$ is always co-orientable. Another interesting fact is that while for $k = 0$ the submanifolds $I_{a,a}(\phi)$

and $II_{2,2}(\phi)$ are always co-orientable, their image in N is always co-orientable in case of $II_{2,2}$, but not necessarily co-orientable for $I_{2,2}$. Some more results on the orientability of singularity submanifolds in the source and in the target spaces can be found in [R1].

As an example we show the proof of the theorem for the singularity type V_a. Let us use the shorthand notations: $z = x+iy$, $Re\ z^k = x^k - \binom{k}{k-2}x^{k-2}y^2 + \dots$, $Im\ z^k = \binom{k}{1}x^{k-1}y - \binom{k}{3}x^{k-3}y^3 + \dots$, $\alpha = e^{2\pi i/a}$. Then the ideal $I = (x^2+y^2, x^a, yx^{a-1})$ is equal to $(x^2+y^2, Re\ z^a, Im\ z^a) = \mathfrak{m}^a + (x^2+y^2)$. The algebra $Q = \mathbb{R}[[x,y]]/I$ is isomorphic to $G(Q)$, it has defect 1, and the group $MC\ Aut^{gr}\ G(Q)$ is the maximal possible: $O(2)$. Therefore $G = MC\ Aut_{\mathcal{A}}F = O(2) \times O(k-1)$. The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^{k+2}$ can be chosen to have the form

$$z \mapsto (|z|^2, z^a, 0, \dots, 0),$$

and clearly $\mu_1 =$'multiplication by α', $\mu_2 = 1 \oplus$'multiplication by α^a'$\oplus \rho_k$, i.e. $\mu_1 = \rho_2$, $\mu_2 = 1 \oplus \rho_2^a \oplus \rho_{k-1}$. A G-invariant complement V of $tf(\theta_{\mathbb{R}^2}) + f^*\mathfrak{m}(k+2)\theta_f$ if $\mathfrak{m}(2)\theta_f$is generated by the following maps:

$$\left.\begin{aligned} z &\mapsto (0, Re\ z^j, Im\ z^j, 0, \dots, 0) \\ z &\mapsto (0, Re\ iz^j, Im\ iz^j, 0, \dots, 0) \\ z &\mapsto (0, Re\ \overline{z}^j, Im\ \overline{z}^j, 0, \dots, 0) \\ z &\mapsto (0, Re\ i\overline{z}^j, Im\ i\overline{z}^j, 0, \dots, 0) \end{aligned}\right\} \text{for } j = 1, \dots, a-1$$

$$\left.\begin{aligned} z &\mapsto (0,0,0, Re\ z^j, 0, \dots, 0), \dots, z \mapsto (0, \dots, 0, Re\ z^j) \\ z &\mapsto (0,0,0, Im\ z^j, 0, \dots, 0), \dots, z \mapsto (0, \dots, 0, Im\ z^j) \end{aligned}\right\} \text{for } j = 1, \dots, a-1$$

The representation of G on the first block is $\rho_2^{a-j} \oplus \rho_2^{a+j}$ and on the second it is $\rho_2^j \otimes \rho_{k-1}$. $\square$

2. The contractibility of $Aut_{\mathcal{A}}\eta/G$

In this section we prove the generalized contractibility property (explained below) of $Aut_{\mathcal{A}}F\ /\ MC\ Aut_{\mathcal{A}}F$. Here we change the notation from F to η to emphasize that results in this section are valid not only for stable germs but for finitely determined ones, too.

Let M be a differentiable manifold with boundary and let G be a subgroup of $\mathcal{A}$. Call a map $q : M \longrightarrow \mathcal{A}/G$ differentiable if M can be covered by open sets U, on which q can be represented by pairs of local diffeomorphisms ($U \times \mathbb{R}^n \longrightarrow U \times \mathbb{R}^n$ and $U \times \mathbb{R}^p \longrightarrow U \times \mathbb{R}^p$) (in fact germs at the zero section), which map all the fibres $u \times \mathbb{R}^n$ and $u \times \mathbb{R}^p$ into themselves.

2.1 Definition. *Let G be a subgroup of $Aut_{\mathcal{A}}\eta$. We call $Aut_{\mathcal{A}}\eta/G$ contractible if for every smooth manifold M with boundary, any differentiable map $q : \partial M \longrightarrow Aut_{\mathcal{A}}\eta/G$ can be extended to a differentiable map $M \longrightarrow Aut_{\mathcal{A}}\eta/G$.*

Let $Diff(M \times \mathbb{R}^n)$ denote the group of diffeomorphism germs of $M \times \mathbb{R}^n$ at the zero section.

In the sequel we will use the following characterisation of finitely determined germs (see e.g. [M1 Lemma 2, p. 149]): if $f \in \mathcal{E}(n,p)$ is finitely determined then

$$\mathfrak{m}(n)^k \mathcal{E}(n,p) \subset (tf)(\mathfrak{m}(n)^2\mathcal{E}(n)^n) + (\omega f)(\mathfrak{m}(p)^2\mathcal{E}(p)^p) \tag{1}$$

for k large enough. The following proposition is an analogue of the main lemma in [J; p. 150] (where it is stated for function germs instead of map germs) but the proof here is a bit more complicated.

Let us fix a finitely determined map germ $\eta : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$.

2.2 Proposition. *There exists an $l \in \mathbb{N}$ large enough for which the following holds. If M is an r-dimensional manifold with boundary and $f, g : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^p$ are map germs at $M \times 0$ with the following properties*

$$f \circ pr_M = pr_M \circ f, \qquad g \circ pr_M = pr_M \circ g,$$

$$f|_{\partial M \times \mathbb{R}^n} = g|_{\partial M \times \mathbb{R}^n},$$

$$j^l(f|_{u \times \mathbb{R}^n}) = j^l(g|_{u \times \mathbb{R}^n}) = j^l\eta \qquad \text{for all } u \in M,$$

then there exist $(\psi, \phi) \in Diff(M \times \mathbb{R}^n) \times Diff(M \times \mathbb{R}^p)$ such that $g = \phi \circ f \circ \psi^{-1}$ and

$$\psi|_{\partial M \times \mathbb{R}^n} = id \qquad \phi|_{\partial M \times \mathbb{R}^n} = id$$

$$j^1(\psi|_{u \times \mathbb{R}^n}) = id \qquad j^1(\phi|_{u \times \mathbb{R}^p}) = id$$

for all $u \in M$.

Proof. Instead of constructing only ψ and ϕ we prove the existence of two one-parameter families of diffeomorphisms.

Let $F : M \times \mathbb{R}^n \times \mathbb{R} \longrightarrow M \times \mathbb{R}^p \times \mathbb{R}$ be the map germ at $M \times 0 \times \mathbb{R}$ defined by

$$(u, x, t) \mapsto (u, (1-t)f(u,x) + tg(u,x), t).$$

From now on $u = (u_i)$, $x = (x_i)$, $y = (y_i)$ and t will always denote coordinates of M, $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}$ respectively; and e.g. F_y will denote the composition $pr_{\mathbb{R}^p} \circ F$.

We will construct two flows, i.e. families of curves. The first family contains curves $\gamma_{u,x} : [0,1] \longrightarrow M \times \mathbb{R}^n \times \mathbb{R}$ (for all $u \in M$, $x \in \mathbb{R}^n$) starting in $(u, x, 0)$ and ending somewhere in $M \times \mathbb{R}^n \times 1$. Suppose also that the 3rd (t) coordinate of $\gamma_{u,x}(t)$ is t. The second family contains similar curves $\delta_{u,y} : [0,1] \longrightarrow M \times \mathbb{R}^p \times \mathbb{R}$.

We want these two flows to be "compatible" with F, that is

$$F(\gamma_{u,x}(\tau)) = \delta_{f(u,x)}(\tau). \tag{2}$$

Putting $\tau = 1$ we see that the maps $\psi : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n$, $(u,x) \mapsto \gamma_{u,x}(1)$, and $\phi : M \times \mathbb{R}^p \longrightarrow M \times \mathbb{R}^p$, $(u,y) \mapsto \delta_{u,y}(1)$ satisfy $g = \phi \circ f \circ \psi^{-1}$.

We will define the two flows by their systems of differential equations. This will assure that ψ and ϕ are diffeomorphisms (at least near $M \times 0$). We will also pay attention to the other conditions on ψ and ϕ.

Suppose therefore that we are given two vector field germs: $X : M \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $Y : M \times \mathbb{R}^p \times \mathbb{R} \longrightarrow \mathbb{R}^p$ satisfying

$$\sum^n \frac{\partial F_{y_j}}{\partial x_i}(u,x,t) X_i(u,x,t) + \frac{\partial F_{y_j}}{\partial t}(u,x,t) = Y_j(F(u,x,t)) \quad j = 1, \ldots, p \tag{3}$$

$$(4) \qquad X|_{M\times 0\times\mathbb{R}} = 0 \qquad\qquad Y|_{M\times 0\times\mathbb{R}} = 0$$

$$(5) \qquad \frac{\partial X}{\partial x_i}(u,0,t) = 0 \quad i = 1,\dots,n \qquad\qquad \frac{\partial Y}{\partial y_j}(u,0,t) = 0 \quad j = 1,\dots,n$$

$$(6) \qquad X|_{\partial M\times\mathbb{R}^n\times\mathbb{R}} = 0 \qquad\qquad Y|_{\partial M\times\mathbb{R}^p\times\mathbb{R}} = 0.$$

Then let us consider the trajectories of the differential equations

$$\bar{X} : M\times\mathbb{R}^n\times\mathbb{R} \longrightarrow TM\times T\mathbb{R}^n\times T\mathbb{R}, \quad (u,x,t)\mapsto (0, X(u,x,t), 1),$$

$$\bar{Y} : M\times\mathbb{R}^p\times\mathbb{R} \longrightarrow TM\times T\mathbb{R}^p\times T\mathbb{R}, \quad (u,x,t)\mapsto (0, Y(u,x,t), 1).$$

These trajectories exist at least in a neighbourhood of $M\times 0\times\mathbb{R}$, because of (4). The equation (3) is just the condition of $\bar{X}$ being the derivative of a flow satisfying (2). The maps ψ, ϕ assigned to the trajectories as above will have the properties

$$\psi|_{\partial M\times\mathbb{R}^n} = id \qquad\qquad \phi|_{\partial M\times\mathbb{R}^p} = id$$

$$j^1(\psi|_{u\times\mathbb{R}^n}) = id \qquad\qquad j^1(\phi|_{u\times\mathbb{R}^p}) = id,$$

because of (6) and (5).

It means that we reduced the problem of finding ψ and ϕ to the existence of X and Y satisfying (3)–(6). It is enough to prove the existence of X and Y locally (near a point in $M\times[0,1]$) and to use *partition of unity* to "add up" these solutions.

(I) First we solve the local problem near a point $(u,t)\in int\ M\times[0,1]$. In this case condition (6) is vacuous and the others can be summarized in the condition:

$$\left(\frac{\partial F_{y_j}}{\partial t}(u,x,t)\right)_{j=1,\dots,p} \in \left\langle \left(\frac{\partial F_{y_1}}{\partial x_i},\dots,\frac{\partial F_{y_p}}{\partial x_i}\right) \ i=1,\dots,n \right\rangle_{\mathcal{E}(r+n+1)} \mathfrak{m}(n)^2 + \\ + F^*\left(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)\right)^p.$$

A coordinate of the left hand side is however

$$\frac{\partial F_{y_j}}{\partial t}(u,x,t) = g_{y_j}(u,x) - f_{y_j}(u,x) \in \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$$

(because f and g have the same l-jets in every fiber), so it is enough to show that

$$(7) \quad \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p \subset (tF)\left(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\right) + (\omega F)\left(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)^p\right)$$

(on the right hand side we just used the definition of tF and ωF). We know that the finite determinacy of the germ η implies a very similar equation (see (1)):

$$(8) \qquad \mathfrak{m}(n)^k\mathcal{E}(n)^p \subset (t\eta)\left(\mathfrak{m}(n)^2\mathcal{E}(n)^n\right) + (\omega\eta)\left(\mathfrak{m}(p)^2\mathcal{E}(p)^p\right)$$

The rest of the proof is showing (7) with the aid of (8). First we have to compare the corresponding terms on the right hand side of (7) and (8), that is to control the non-commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{E}(n)^n \arrow[r, "t\eta"] \arrow[d] & \mathcal{E}(n)^p \arrow[d] & \mathcal{E}(p)^p \arrow[l, "\omega\eta"'] \arrow[d] \\
\mathcal{E}(r+n+1)^n \arrow[r, "tF"] & \mathcal{E}(r+n+1)^p & \mathcal{E}(r+p+1)^p, \arrow[l, "\omega F"']
\end{tikzcd}$$

where the vertical arrows are just the natural inclusions (which we will omit from the formulas). This is done in the following two lemmas.

2.3 Lemma. *If $h \in \mathcal{E}(p)^p$ then $\omega F(h) - \omega\eta(h) \in \mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$.*

Proof. The jth coordinate of $\omega F(h) - \omega\eta(h)$ is $h_j(F_y(u,x,t)) - h_j(\eta(x))$. We will show that it is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$. Let h_j be written in the form $p(y)+q(y)$ where p is a polynomial of degree l and $q \in \mathfrak{m}(p)^{l+1}$. Then

$$h_j(F_y(u,x,t)) - h_j(\eta(x)) = \Big(p(F_y(u,x,t)) - p(\eta(x))\Big) + q(F_y(u,x,t)) - q(\eta(x)).$$

The second and third terms are in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$ so it remains to show that the first term is there too. It is no restriction to consider only the case when p is a monomial. We will use induction on the degree of p. If p is a constant, then the statement is evident. If the degree is bigger, we can write $p = y_i p'(y)$ for some i and a monomial p' with smaller degree. In this case

$$p(F_y(u,x,t)) - p(\eta(x)) = F_{y_i}(u,x,t)p'(F_y(u,x,t)) - \eta_i(x)p'(\eta(x)) =$$

$$= p'(\eta(x))\Big(F_{y_i}(u,x,t) - \eta_i(x)\Big) + F_{y_i}(u,x,t)\Big(p'(F_y(u,x,t)) - p'(\eta(x))\Big),$$

and the elements in both brackets are in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$, the first by definition and the second by the induction hypotheses. $\square$

2.4 Lemma. *If $h \in \mathcal{E}(n)^n$ then $tF(h) - t\eta(h) \in \mathfrak{m}(n)^l\mathcal{E}(r+n+1)^p$.*

Proof. The jth coordinate of $tF(h) - t\eta(h)$ is $\sum_{i=1}^n(\frac{\partial F_{y_j}}{\partial x_i}(u,x,t) - \frac{\partial \eta_j}{\partial x_i}(x))h_i(x)$. This must be in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)$ since $F_{y_j}(u,x,t) - \eta_j(x)$ is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)$. $\square$

Now let us denote by U the intersection

$$(\omega\eta)^{-1}\Big(t\eta(\mathfrak{m}(n)^2\mathcal{E}(n)^n) + \mathfrak{m}(n)^k\mathcal{E}(n)^p\Big) \;\cap\; \mathfrak{m}(p)^2\mathcal{E}(p)^p.$$

This is an $\mathcal{E}(p)$-submodule of $\mathcal{E}(p)^p$. Let V be the $\mathcal{E}(r+p+1)$-submodule of $\mathcal{E}(r+p+1)^p$ generated by the image of U under the natural inclusion $\mathcal{E}(p)^p \longrightarrow \mathcal{E}(r+p+1)^p$.

Now we claim that the following equality holds:

$$\omega F(V) + tF\Big(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\Big) + \mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p =$$

$$= tF\Big(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\Big) + \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p \tag{9}$$

To prove the $\subset$ part, it is enough to show that $\omega F(U) \subset$ the right hand side, since the right hand side is an $\mathcal{E}(r+p+1)$-submodule. Let $v \in U$ — so $v = (\omega\eta)^{-1}(t\eta(\xi)+\zeta)$ where $\xi \in \mathfrak{m}(n)^2\mathcal{E}(n)^n$ and $\zeta \in \mathfrak{m}(n)^k\mathcal{E}(n)^p$. Then

$$\omega F(v) = \Big(\omega F(v) - \omega\eta(v)\Big) + \Big(t\eta(\xi) - tF(\xi)\Big) + \zeta + tF(\xi).$$

According to the lemmas above the element in the first bracket is in $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$ and the element in the second bracket is in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)^p$. So (choosing l bigger than k) we have

$$\omega F(v) \in \mathfrak{m}(n)\mathcal{E}(r+n+1)^p + tF\left(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n\right),$$

which implies LHS $\subset$ RHS in (9).

Now it has to be shown that LHS $\supset$ RHS in (9). The only thing to be proved is that $\mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p$ is part of the LHS. Let $\rho \in \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p$, then

$$\rho(u,x,t) = x^k[h_0(x) + s_1(t,u)h_1(x) + s_2(t,u)h_2(x) + \ldots + s_{l-k}(t,u)h_{l-k}(u,x,t)],$$

where $s_i(t,u)$ is a polynomial in $t, u_1, \ldots, u_r$ of degree i, and the h_i's are smooth maps $\mathbb{R}^n \longrightarrow \mathbb{R}^p$ for $i < l-k$ and h_{l-k} is a smooth map $\mathbb{R}^{r+n+1} \longrightarrow \mathbb{R}^p$. The last term is in the LHS of (9) by definition and since the LHS of (9) is closed to the multiplication by t and u, it is enough to show that $x^k h_0(x) \in$ LHS of (9).

Because of (8) we can write $x^k h_0(x)$ in the form $t\eta(\xi) + \omega\eta(\zeta)$ for some $\xi \in \mathfrak{m}(n)^2\mathcal{E}(n)^n$ and $\zeta \in \mathfrak{m}(p)^2\mathcal{E}(p)^p$. Therefore

$$x^k h_0(x) = \Big(t\eta(\xi) - tF(\xi)\Big) + \Big(\omega\eta(\zeta) - \omega F(\zeta)\Big) + tF(\xi) + \omega F(\zeta),$$

and the elements in the two brackets are in $\mathfrak{m}(n)^l\mathcal{E}(r+n+1)$ and $\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$ respectively (see the two lemmas above), and the remaining two terms are also in the LHS of (9) by definition. So the proof of (9) is complete.

Having the formula (9) we now want to prove (7). This will be a so called Nakayama-type argument — although it needs a more sophisticated lemma than that of Nakayama, namely the following.

2.5 Lemma [M1; p. 135]. *Let $G : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be a smooth map germ. Suppose A is a finitely generated $\mathcal{E}(p)$-module; B and C are $\mathcal{E}(n)$-modules (C is finitely generated); $\beta : B \longrightarrow C$ is an $\mathcal{E}(n)$-module homomorphism and $\alpha : A \longrightarrow C$ is a homomorphism over $G^* : \mathcal{E}(p) \longrightarrow \mathcal{E}(n)$. Let a be the dimension of the vector space $A/m(p)A$ over $\mathbb{R}$. Then*

$$\alpha(A) + \beta(B) + (G^*(\mathfrak{m}(p)) + \mathfrak{m}(n)^{a+1})C = C$$

implies

$$\alpha(A) + \beta(B) = C.$$

Remark. The proof of this lemma is based on the Nakayama-lemma and the Malgrange preparation theorem.

A trivial consequence of this lemma is the following. Using the same notation as above, if $D \subset C$ satisfies

$$\alpha(A) + \beta(B) + D = C \tag{10}$$

and

$$D \subset (G^*(\mathfrak{m}(p)) + \mathfrak{m}(n)^{a+1})C, \tag{11}$$

then

$$\alpha(A) + \beta(B) = C. \tag{12}$$

We will use this lemma with the following substitutions:

$G := F$ $\qquad$ $A := V$
$B := \mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n$ $\qquad$ $D := \mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p$
$C := tF(\mathfrak{m}(n)^2\mathcal{E}(r+n+1)^n) + \mathfrak{m}(n)^k\mathcal{E}(r+n+1)^p$
$\alpha := \omega F$ $\qquad$ $\beta := tF$

We have to check (10) and (11). In fact (10) is exactly (9) which we have just proved, and (11) is

$$\mathfrak{m}(n)^k\mathfrak{m}(r+n+1)^{l-k}\mathcal{E}(r+n+1)^p \subset$$

$$\subset \Big(F^*(\mathfrak{m}(r{+}p{+}1)){+}\mathfrak{m}(r{+}n{+}1)^{a+1}\Big)\Big(tF(\mathfrak{m}(n)^2\mathcal{E}(r{+}n{+}1)^n){+}\mathfrak{m}(n)^k{+}\mathcal{E}(r{+}n{+}1)^p\Big).$$

If we chose l at least $a+k+1$ then this is clearly true (see the product of the second terms in both brackets). Therefore we have (12) in our situation, which is exactly (7) what we wanted to prove.

(II) Now we want to solve the same local problem as in (I) but near a point in $\partial M \times [0,1]$. It will turn out that this problem can be reduced to the case studied in (I).

Indeed, extend F from $\mathbb{R}^r_+ \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^r_+ \times \mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}$ still satisfying the conditions required for F in the theorem. This time we have to show that

$$\left(\frac{\partial F_{y_1}}{\partial t}, \dots, \frac{\partial F_{y_p}}{\partial t}\right) \in \left\langle \left(\frac{\partial F_{y_1}}{\partial x_i}, \dots, \frac{\partial F_{y_p}}{\partial x_i}\right) \mid i = 1, \dots, n \right\rangle_{\mathcal{E}(r+n+1)} \mathfrak{m}(n)^2\mathfrak{m}(1) +$$

$$+F^*\Big(\mathfrak{m}(p)^2\mathfrak{m}(1)\mathcal{E}(r+p+1)\Big)^p. \tag{13}$$

(Note that $\mathfrak{m}(1)$ here refers to the ideal generated by the first local coordinate u_1 of M, where the boundary ∂M is defined by the equation $u_1 = 0$.) Because of condition (6) now the left hand side is in $\mathfrak{m}(1)\mathfrak{m}(n)^{l+1}\mathcal{E}(r+n+1)^p$, so it is enough to show that this submodule is part of the right hand side of (13). If we multiply the inclusion (7) by $\mathfrak{m}(1)$ we get

$$\mathfrak{m}(1)\mathfrak{m}(n)^{l+1}\mathcal{E}(r{+}n{+}1)^p \subset \left\langle \left(\frac{\partial F_{y_1}}{\partial x_i}, \dots, \frac{\partial F_{y_p}}{\partial x_i}\right) \mid i = 1, \dots, n \right\rangle \qquad \mathfrak{m}(n)^2\mathfrak{m}(1)+$$

$$+\mathfrak{m}(1)F^*\left(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)\right)^p. \tag{14}$$

So it is enough to prove that

$$\mathfrak{m}(1)F^*\left(\mathfrak{m}(p)^2\mathcal{E}(r+p+1)\right)^p \subset F^*\left(\mathfrak{m}(p)^2\mathfrak{m}(1)\mathcal{E}(r+p+1)\right)^p,$$

which is clearly true, since a coordinate of the left hand side can be written in the form of $u_1 F_y^2(u,x,t)h(u,x,t)$ and a coordinate of the right hand side has the form $F_y^2(u,x,t)\ F_{u_1}(u,x,t)\ h(u,x,t)$ — but $F_{u_1}(u,x,t) = u_1$ and therefore these two sets are in fact equal.

The proof of the proposition is complete. $\square$

Denote by A^l the Lie group of l-jets of the elements of $\mathcal{A}$, and put

$$A^l_\eta = \{\ z = (z_1, z_2) \in A^l \mid z_2 \circ j^l\eta \circ z_1^{-1} = j^l\eta\ \}.$$

From the proof of the theorem above it is also clear that for l large enough the image of a maximal compact subgroup of $Aut_{\mathcal{A}}\eta$ under the map j^l is a maximal compact subgroup (in the classical sense) of A^l_η. In fact, if η is well chosen from its $\mathcal{A}$-equivalence class then the maximal compact subgroup G of $Aut_{\mathcal{A}}\eta$ is linear, so $j^1(G) = G$. From now on we will assume that η has this property.

2.6 Theorem. *If η is finitely determined and $G \le Aut_{\mathcal{A}}\eta$ is a maximal compact subgroup then $Aut_{\mathcal{A}}\eta/G$ is contractible.*

Proof. What we have to show is that a differentiable map $f : \partial M \longrightarrow Aut_{\mathcal{A}}\eta/G$ extends to a differentiable map $\bar{f} : M \longrightarrow Aut_{\mathcal{A}}\eta/G$. Consider the following commutative diagram

$$\begin{array}{ccc} Aut_{\mathcal{A}}\eta & \xrightarrow{j^l} & A^l_\eta \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi^l} \\ Aut_{\mathcal{A}}\eta\ /\ G & \xrightarrow{\overline{j^l}} & A^l_\eta\ /\ G. \end{array}$$

There is a section σ^l of π^l and it induces a section σ of π. It is easy to check that π and σ are differentiable — in the sense that for a differentiable map $q_1 : N \longrightarrow Aut_{\mathcal{A}}\eta$ the composition $\pi \circ q_1$ is also differentiable, and for a differentiable map $q_2 : N \longrightarrow Aut_{\mathcal{A}}\eta/G$ the composition $\sigma \circ q_2$ is also differentiable.

We want to prove that the differentiable map $k = \sigma \circ f : \partial M \longrightarrow Aut_{\mathcal{A}}\eta$ extends to a map $\bar{k} : M \longrightarrow Aut_{\mathcal{A}}\eta$. This implies the theorem since the composition of $\bar{f} = \pi \circ \bar{k}$ will extend f.

The composition $g = \overline{j^l} \circ f$ extends to $\bar{g} : M \longrightarrow A^l_\eta/G$ since G is a maximal compact subgroup (in the classical sense) of a Lie group, so the quotient is contractible. Composing g and $\bar{g}$ with σ^l we get maps h and $\bar{h}$. It is clear that $j^l \circ k = h$, our task is to construct $\bar{k}$ such that $j^l \circ \bar{k} = \bar{h}$. We will do it in two steps. First we extend k in a bigger group then $Aut_{\mathcal{A}}\eta$. Namely, let

$$\mathcal{A}_{j^l\eta} := \{\ \psi = (\psi_1, \psi_2) \in \mathcal{A} \mid j^l\psi \in A^l_\eta\ \} \supset Aut_{\mathcal{A}}\eta.$$

We will construct a map $k' : M \longrightarrow \mathcal{A}_{j^l\eta}$ which extends k and satisfies $j^l \circ k' = \bar{h}$. This is equivalent to the following problem: we seek diffeomorphism germs at $M \times 0$

$$F_1 : M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n$$

and

$$F_2 : M \times \mathbb{R}^p \longrightarrow M \times \mathbb{R}^p$$

if they are given in $\partial M \times \mathbb{R}^n$ and $\partial M \times \mathbb{R}^p$, and their l-jets are given everywhere. We show the existence of F_1, the existence of F_2 can be proved in the same way. First we show an F_1 locally and then use *partition of unity* to "add up" these solutions. The local problem near a point in *int* M is trivial (let F_1 coincide with the given l-jet), and for a coordinate function of F_1 the local problem near a point in ∂M is the following. Given a polynomial P of degree l in the variables $x_1, \dots, x_n$ with coefficients from $\mathcal{E}(r)$ and a function $\alpha_0 : \mathbb{R}^{r-1+n} \longrightarrow \mathbb{R}$ such that

$$j^l_x \alpha_0 = P(0, u_2, \dots, u_r, x_1, \dots, x_n)$$

(as usual the coordinates of $\mathbb{R}^n$ are denoted by x and the local coordinates of M are denoted by u and ∂M is given by $u_1 = 0$) a function $\alpha : \mathbb{R}^{r+n} \longrightarrow \mathbb{R}$ is needed with the properties that

$$\alpha|_{u_1=0} = \alpha_0,$$

$$j^l_x \alpha = P(u_1, \dots, u_r, x_1, \dots, x_n).$$

The following function satisfies these conditions

$$\alpha(u, x) = \alpha_0(u_2, \dots, u_r, x) - P(0, u_2, \dots, u_r, x) + P(u_1, \dots, u_r, x).$$

So we proved the existence of k'. The next step is to prove the existence of $\bar{k}$. Let the map k' be represented by the pair (F_1, F_2) of diffeomorphism germs (of $M \times \mathbb{R}^n$ and $M \times \mathbb{R}^p$ respectively). If the *fibrewise* map germs $a, b : (M \times \mathbb{R}^n, M) \longrightarrow (M \times \mathbb{R}^p, M)$ are defined by

$$pr_{\mathbb{R}^p} \circ a = \eta \circ pr_{\mathbb{R}^n}$$

$$pr_{\mathbb{R}^p} \circ F_2 \circ b = \eta \circ pr_{\mathbb{R}^n} \circ F_1,$$

then clearly $a \circ F_1 = F_2 \circ b$. Further a and b coincide over ∂M and their l-jets coincide over the whole M. So they satisfy the conditions of the proposition above, and therefore there exist diffeomorphism germs ψ and ϕ (of $M \times \mathbb{R}^n$ and $M \times \mathbb{R}^p$ respectively) such that $a \circ \psi = \phi \circ b$.

If we denote $F_1 \circ \psi$ by $\bar{\psi}$ and $F_2 \circ \phi$ by $\bar{\phi}$ then the pair $(\bar{\psi}, \bar{\phi}) : M \longrightarrow Aut_{\mathcal{A}}\eta$ represents k, because

$$a \circ \bar{\psi} = a \circ F_1 \circ \psi = F_2 \circ b \circ \psi = F_2 \circ \phi \circ b = \bar{\phi} \circ b.$$

The proof of the theorem is complete. $\square$

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