ON THE ORIENTABILITY OF SINGULARITY SUBMANIFOLDS

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Abstract. As an application of the generalized Pontrjagin-Thom construction (see [5]) here we prove some results on the orientability of singularity submanifolds. Our approach is based on the computation of symmetries of singularities and is different from the one based on the fundamental work of Boardman ([3]) which involves the high intristic derivatives. As an example we apply our method to all the Σ^{1_r} and $\Sigma^{2,0}$ singularities.

The integer $k > 0$ will be fixed throughout the paper. Let $\eta : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ be a smooth map germ. From now on, we will restrict ourselves to *simple* singularities. By a suspension of η we mean a germ $\Sigma\eta : (\mathbb{R}^{n+v}, 0) \longrightarrow (\mathbb{R}^{(n+k)+v}, 0)$ defined by $(x, u) \mapsto (\eta(x), u)$ — otherwise we will use the standard notations of singularity theory, see e.g. [2]. For a C^∞-stable map $f : N \longrightarrow P$ between smooth manifolds we define the singularity submanifold

$$\eta(f) = \{\ y \in P \mid \quad \begin{array}{l} f^{-1}(y) \text{ has only one element and the germ of } f \\ \text{at } f^{-1}(y) \text{ is } \mathcal{A}\text{-equivalent to a suspension of } \eta\ \}. \end{array}$$

We may think of $\eta(f)$ either as an abstract manifold (not necessarily closed), or a submanifold of P, or a submanifold $f^{-1}(\eta(f))$ of N.

Now let η be a C^∞-stable germ, and suppose that it is not $\mathcal{A}$-equivalent to the suspension of any other germ — germs having this property will be called "isolated". Another description of an isolated stable germ η is that $d_e(\eta, \mathcal{K}) =$ its target dimension (see [2; p. 166] for the definition of $d_e(\ ,\mathcal{K})$). According to Mather's classification theorem: $\mathcal{A}$-equivalence classes of isolated stable singularities are in one-to-one correspondence with finite dimensional local $\mathbb{R}$-algebras. In [5] the maximal compact subgroup G of η's automorphism group

$$Aut_{\mathcal{A}}\eta = \{\ (\varphi, \phi) \in Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^{n+k}, 0) \mid \phi \circ \eta \circ \varphi^{-1} = \eta\ \}$$

is considered. We can assume it is linear, and its representation on the source and target spaces will be denoted by λ_1 and λ_2. The vector bundle associated to the universal principal G-bundle using the representation λ_i will be denoted by $E\lambda_i \longrightarrow BG$.

The following two theorems are byproducts of the mail lemma in [5]. The letters N and P will always denote closed smooth manifolds, and the letter ν will refer to normal bundles.

1991 *Mathematics Subject Classification.* 57R45.

The author received financial support from OTKA 014906. The author thanks the referee the very useful comments.

Theorem 1. *For any stable smooth map $f : N^m \longrightarrow P^{m+k}$ there exists a continuous map $g : \eta(f) \longrightarrow BG$, such that*

$$\nu(\eta(f) \subset N) = g^* E\lambda_1, \qquad \nu(\eta(f) \subset P) = g^* E\lambda_2.$$

Theorem 2. *For any closed manifold K and any continuous map $g : K \longrightarrow BG$ there exist smooth manifolds N^m, P^{m+k} and a stable smooth map $f : N \longrightarrow P$, such that K is a component of $\eta(f)$ and*

$$\nu(K \subset N) = g^* E\lambda_1, \qquad \nu(K \subset P) = g^* E\lambda_2.$$

□

Remark. Observe that these two theorems together completely describe the normal bundles of the singularity submanifolds $\eta(f)$ in the source and in the target manifolds. Namely, these normal bundles can be any pull-back bundles (so any finite dimensional approximations) of $E\lambda_1$ and $E\lambda_2$ using the same map into BG.

The smooth map $f : N \longrightarrow P$ will be called k-codimensional if $\dim P - \dim N = k$. The following two statements are easy conseqences.

Theorem 3. *The following two conditions are equivalent:*
1. *for every k-codimensional map $f : N \longrightarrow P$, where P is orientable, the manifold $\eta(f)$ is orientable;*
2. $\det \lambda_2(g) > 0$ *for all $g \in G$.*

Theorem 4. *The following two conditions are equivalent:*

(3) *for every k-codimensional map $f : N \longrightarrow P$, where N is orientable, the manifold $\eta(f)$ is orientable;*

(4) $\det \lambda_1(g) > 0$ *for all $g \in G$.*

Proof. Condition (1) is equivalent to the following: for every k-codimensional map $f : N \longrightarrow P$, where P is orientable, $\nu(\eta(f) \subset P)$ is an orientable bundle. Because of Theorem 2 it implies that for all K and $g : K \longrightarrow BG$ the bundle $g^* E\lambda_2$ is orientable. Then it follows that $E\lambda_2$ is orientable, which is equivalent to condition (2).

Conversely, if $E\lambda_2$ is orientable, then (using Theorem 1) for any $f : N^m \longrightarrow P^{m+k}$ the bundle $\nu(\eta(f) \subset P)$ is orientable. If P is orientable, then this implies that $\eta(f)$ is also orientable.

The proof of Theorem 4 goes the same way. □

Now we turn to the investigation of conditions (2) and (4). First we recall from [5] some results about the maximal compact automorphism group of $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$. If η is a miniversal unfolding of $\zeta : \mathbb{R}^a \longrightarrow \mathbb{R}^{a+k}$, where $d\zeta(0) = 0$, and V is a complement of the subspace $t\zeta(\theta_a) + \zeta^* \mathfrak{m}(a+k)\theta_\zeta$ in the vector space θ_ζ, then η is $\mathcal{A}$-equivalent to

$$\mathbb{R}^a \times V \longrightarrow \mathbb{R}^{a+k} \times V$$

$$(x, \phi) \mapsto (x + \phi(x), \phi).$$

Now let G be the maximal compact subgroup of the $\mathcal{K}$-equivalence group $Aut_{\mathcal{K}}\zeta$. If ζ is well chosen from its $\mathcal{K}$-equvivalence class then we can suppose G acting linearly on $\mathbb{R}^a \times \mathbb{R}^{a+k}$. Then, in particular, G acts as an $\mathcal{A}$ automorphism group, so it has representations α and β on $\mathbb{R}^a$ and $\mathbb{R}^{a+k}$ respectively. The group G also acts on θ_ζ by $(\alpha, \beta) \cdot \phi = \beta \circ \phi \circ \alpha^{-1}$ — leaving $t\zeta(\theta_a) + \zeta^*\mathfrak{m}(a+k)\theta_\zeta$ invariant. If V is chosen to be G-invariant (G compact, so it is possible) then G also acts on V. Let this action be γ. A theorem in [5] proves that the maximal compact subgroup of $Aut_{\mathcal{A}}\eta$ is G with the representations $\lambda_1 := \alpha \oplus \gamma$, $\lambda_2 := \beta \oplus \gamma$ on the source $(\mathbb{R}^a \times V)$ and target $(\mathbb{R}^{a+k} \times V)$ spaces, respectively. This reduces the problem of finding $MC\ Aut_{\mathcal{A}}\eta$ to finding $MC\ Aut_{\mathcal{K}}\zeta$ and the representations α, β. (MC stands for 'maximal compact subgroup of'.) This latter problem is also essentially solved (reduced to a finite dimensional one) in [5], we will come back to these results in concrete examples.

Notation. In what follows ρ_r will always mean the standard r-dimensional representation of $O(r)$. If ρ_r is written as a representation of $O(r) \times H$ then it really means $\rho_r \circ pr_{O(r)}$.

Example 1. For all $r \geq 0$ there is a unique isolated germ η^{1_r} (in codimension k) corresponding to the local algebra $\mathbb{R}[[t]]/(t^{r+1})$. This germ is called the isolated Morin singularity type of Σ^{1_r}. This is the miniversal unfolding of $\zeta^{1_r} : \mathbb{R} \longrightarrow \mathbb{R}^{k+1}$, $x \mapsto (x^{r+1}, 0, \ldots, 0)$. It is clear that $MC\ Aut_{\mathcal{K}}\zeta^{1_r} = O(1) \times O(k)$, and the representations α, β are as follows:

$$\alpha = \rho_1, \qquad \beta = \rho_1^{r+1} \oplus \rho_k.$$

Indeed, $O(1) \times O(k) \leq MC\ Aut_{\mathcal{A}}\zeta^{1_r} \leq MC\ Aut_{\mathcal{K}}\zeta^{1_r}$, as the representations α and β show. On the other hand — by Theorem 7 in [5] — $MC\ Aut_{\mathcal{K}}\zeta^{1_r} \leq MC\ Aut\ Q_{\zeta^{1_r}} \times O(k-d) = O(1) \times O(k)$ where $Q_{\zeta^{1_r}}$ is the local algebra of ζ^{1_r} and d is its defect.

The space V can be chosen to be spanned by the vectors

$$\begin{array}{lcc} x \mapsto & (x^i, 0, \ldots, 0) & i = 1, \ldots, r-1 \\ x \mapsto & (0, 0, \ldots, 0, x^j, 0, \ldots, 0) & j = 1, \ldots, r \\ & & \text{the coordinate of } x^j \text{ is from } 2, \ldots, k+1 \end{array}$$

and (using the definition of γ above) the action of $O(1) \times O(k)$ on V can be computed:

$$\gamma = \left\lceil \frac{r-1}{2} \right\rceil 1 \oplus \left\lfloor \frac{r-1}{2} \right\rfloor \rho_1 \oplus \left\lfloor \frac{r}{2} \right\rfloor \rho_k \oplus \left\lceil \frac{r}{2} \right\rceil \rho_1 \otimes \rho_k.$$

As an application of this example and Theorem 3 and 4 we can prove the following two theorems about the orientability of the Morin-singularity submanifolds.

Theorem 5. *Let η^{1_r} be as in the example above. Then the following two conditions are equivalent:*

(5) *for every k-codimensional map $f : N \longrightarrow P$, where P is orientable, the manifold $\eta^{1_r}(f)$ is orientable;*
(6) *k is even and $r \equiv 1 \mod 4$.*

Theorem 6. *Let η^{1_r} be as in the example above. Then the following two conditions are equivalent:*

(7) *for every k-codimensional map $f : N \longrightarrow P$, where N is orientable, the manifold $\eta^{1_r}(f)$ is orientable;*
(8) *either k is odd and r is even, or k is even and $r \equiv 0$ mod 4.*

Proofs. According to the Theorems 3 and 4 we only have to analyse the sign of the determinants of $\lambda_1(g)$, $\lambda_2(g)$. Since explicit formulae are given for $\lambda_1 = \alpha \oplus \gamma$ and $\lambda_2 = \beta \oplus \gamma$, easy computation gives the proofs. $\square$

Example 2. Let η_r be the miniversal unfolding of $\zeta_r : \mathbb{R}^r \longrightarrow \mathbb{R}^{r+k}$,

$$(x_1, \dots, x_r) \mapsto (x_1^2, \dots, x_r^2, x_1x_2, x_1x_3, \dots, x_{r-1}x_r, 0, \dots, 0),$$

(where there are $t = k - \binom{r}{2}$ 0's at the end). This η_r is the "simplest" singularity of Thom-Boardman type $\Sigma^{r,0}$. The group $MC\ Aut_{\mathcal{K}}\zeta_r$ is $O(r) \times O(t)$. Indeed, $O(r) \times O(t)$ clearly acts as a $\mathcal{K}$-symmetry (in fact as an $\mathcal{A}$-symmetry) group of ζ_r, so $O(r) \times O(t) \leq MC\ Aut_{\mathcal{K}}\zeta_r$. On the other hand $MC\ Aut_{\mathcal{K}}\zeta_r \leq MC\ Aut\ Q_{\zeta_r} \times O(k-d) = O(r) \times O(t)$, where Q_{ζ_r} is ζ_r's local algebra, and d is its defect. The representation $\alpha = \rho_r$, but we will not need to determine β explicitly.

If we choose V to be spanned by

$$(x_1, \dots, x_r) \mapsto \quad (0, \dots, 0, x_i, 0, \dots, 0) \qquad \begin{array}{c} i = 1, \dots, r \\ \text{the coordinate is } j = 1, \dots, r,\ i \neq j \end{array}$$

$$(x_1, \dots, x_r) \mapsto \quad (0, \dots, 0, x_i, 0, \dots, 0) \qquad \begin{array}{c} i = 1, \dots, r \\ \text{the coordinate is } j = r+1, \dots, r+k, \end{array}$$

then V will be $O(r) \times O(t)$-invariant.

Although we have not written up explicit formulae for β and γ, we will need some information on the sign of $\det \beta(g)$, $\det \gamma(g)$ ($g \in MC\ Aut_{\mathcal{A}}\eta_r$). Let $g_1, g_2 \in O(r) \times O(t)$ be given by

$$g_1 := \left(\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, I_{t\times t} \right), \qquad g_2 := \left(I_{r\times r}, \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right).$$

Easy computation shows that

$$\det \alpha(g_1) = -1, \qquad \det \beta(g_1) = (-1)^{r-1}, \qquad \det \gamma(g_1) = (-1)^{k-r+1},$$

$$\det \alpha(g_2) = 1, \qquad \det \beta(g_2) = -1, \qquad \det \gamma(g_2) = (-1)^r.$$

Theorem 7. *Let η_r be as in the example above. Then the following two conditions are equivalent:*
(9) *for every k-codimensional map $f : N \longrightarrow P$, where P is orientable, the manifold $\eta_r(f)$ is orientable;*
(10) *k is even and r is odd.*

THEOREM 8. *Let η_r be as in the example above. Then the following two conditions are equivalent:*
(11) for every k-codimensional map $f : N \longrightarrow P$, where N is orientable, the manifold $\eta_r(f)$ is orientable;
(12) k is even and r is even.

PROOF. According to Theorem 3 condition (9) is equivalent to $\det\beta(g)\cdot\det\gamma(g) > 0$ for every $g \in O(r) \times O(t)$. This latter is equivalent to $\det\beta(g_1)\cdot\det\gamma(g_1) > 0$ and $\det\beta(g_2)\cdot\det\gamma(g_2) > 0$, that is (using the computation above): $r-1+k-r+1 \equiv 0 \mod 2$ and $1+r \equiv 0 \mod 2$. This is exactly condition (10). The proof of Theorem 8 is similar. □

For a stable map $f : N \longrightarrow P$ we can define the submanifold

$$\Sigma^r(f) = \{\ x \in N \mid \text{the germ of } f \text{ at } x \text{ is of Thom-Boardman type } \Sigma^r.\ \}.$$

Clearly $f^{-1}(\eta_r(f)) \subset \Sigma^r(f)$, and the difference is a union of submanifolds all of codimension $\geq k$. Since a manifold of codimension ≥ 2 can not alter orientability, we have the following two corollaries.

COROLLARY 9. *Let $k, r > 1$. The condition*
(13) for every k-codimensional map $f : N \longrightarrow P$, where P is orientable, the manifold $\Sigma^r(f)$ is orientable
is equivalent to condition (9) (and therefore to condition (10)). □

COROLLARY 10. *Let $k, r > 1$. The condition*
(14) for every k-codimensional map $f : N \longrightarrow P$, where N is orientable, the manifold $\Sigma^r(f)$ is orientable
is equivalent to condition (11) (and therefore to condition (12)). □

EXAMPLE 3. We turn to $\Sigma^{2,0}$ germs, which is — according to the authors knowledge — the last Thom-Boardman type for which the $\mathcal{A}$-classification is complete. Mather proved that there are five infinite sequences of algebras corresponding to $\Sigma^{2,0}$ singularity types:

$I_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a + y^b)$	$2 \leq a, b$
$II_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a - y^b)$	$2 \leq a, b$ both even
$III_{a,b}$	$\mathbb{R}[[x,y]]\ /\ (xy, x^a, y^b)$	$2 \leq a, 3 \leq b$
IV_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a)$	$3 \leq a$
V_a	$\mathbb{R}[[x,y]]\ /\ (x^2 + y^2, x^a, yx^{a-1})$	$2 \leq a$

Here the only coincidenses are $I_{a,b} = I_{b,a}$, etc. (For convenience $III_{2,2}$ of Mather is renamed as V_2 here, since — considering symmetries — $III_{2,2}$ is closer to the V_a sequence.) Using the method descibed above we can compute their maximal compact symmetry groups (some more details in [4]), in which we will use the following notations: for the groups $O(2)$ and D_n (dihedral group) ρ_2^w means the two-dimensional representation which maps

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \cos w\alpha & -\sin w\alpha \\ \sin w\alpha & \cos w\alpha \end{pmatrix}$$

and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to itself. Three one-dimensional representations of the dihedral groups will also be used. Let the dihedral group D_{2a} (of order $4a$) be presented as $< f, t | f^{2a} = t^2 = 1, tft = f^{2a-1} >$. The following define 1-dimensional representations of D_{2a}:

$$\lambda : f \mapsto -1, t \mapsto -1 \qquad \kappa : f \mapsto -1, t \mapsto 1 \qquad \tau : f \mapsto 1, t \mapsto -1.$$

Moreover, ε and θ will mean the non-trivial 1-dimensional representation of $\mathbb{Z}_2$ (two letters are needed when $\mathbb{Z}_2 \times \mathbb{Z}_2$ is concerned). If one of the defined representations are written as a representation of a group $O(l) \times H$, $D_{2a} \times H$, $\mathbb{Z}_2 \times H$ then they really mean to be a composition of that representation with the projection to $O(l), D_{2a}$ or $\mathbb{Z}_2$.

Theorem 11. *If η is an isolated singularity of type $\Sigma^{2,0}$ then $MC\ Aut_{\mathcal{A}}\eta$ and its representations $\lambda_1 = \alpha \oplus \gamma$, $\lambda_2 = \beta \oplus \gamma$ are given by*

η		$MC\ Aut\ \eta$	α	β
$I_{a,b}, II_{a,b}$	$2 \le a < b$ both even	$D_2 \times O(k)$	ρ_2	$\tau \oplus 1 \oplus \rho_k$
$I_{a,b}$	$2 < a$ odd, $2 \le b$ even	$\mathbb{Z}_2 \times O(k)$	$1 \oplus \varepsilon$	$\varepsilon \oplus 1 \oplus \rho_k$
$I_{a,b}$	$2 < a < b$ both odd	$\mathbb{Z}_2 \times O(k)$	2ε	$1 \oplus \varepsilon \oplus \rho_k$
$I_{a,a}$	$2 < a$ odd	$D_2 \times O(k)$	ρ_2	$1 \oplus \kappa \oplus \rho_k$
$I_{a,a}$	$2 \le a$ even	$D_4 \times O(k)$	ρ_2	$\lambda \oplus 1 \oplus \rho_k$
$II_{a,a}$	$2 < a$ even	$D_4 \times O(k)$	ρ_2	$\lambda \oplus \kappa \oplus \rho_k$
$II_{2,2}$		$O(2) \times O(k)$	ρ_2	$\rho_2^2 \oplus \rho_k$
$III_{a,a}$	$2 < a$ even	$D_4 \times O(k-1)$	ρ_2	$\lambda \oplus 1 \oplus \kappa \oplus \rho_{k-1}$
$III_{a,a}$	$2 < a$ odd	$D_4 \times O(k-1)$	ρ_2	$\lambda \oplus \rho_2 \oplus \rho_{k-1}$
$III_{a,b}$	$2 \le a < b$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times O(k-1)$	$\varepsilon \oplus \theta$	$(\varepsilon \otimes \theta) \oplus \varepsilon^{\otimes a} \oplus \theta^{\otimes b} \oplus \rho_{k-1}$
IV_a	$3 \le a$	$D_{2a} \times O(k)$	ρ_2	$1 \oplus \lambda \oplus \rho_k$
V_a	$2 \le a$	$O(2) \times O(k-1)$	ρ_2	$1 \oplus \rho_2^a \oplus \rho_{k-1}$

η		γ
$I_{a,b}, II_{a,b}$	$2 \le a < b$ both even	$\left[\frac{a}{2}\rho_2 \oplus \frac{b-a}{2}\lambda \oplus \frac{a+b-4}{2}1\right] \otimes (1 \oplus \rho_k) \oplus \rho_k$
$I_{a,b}$	$2 < a$ odd, $2 \le b$ even	$\left[\frac{b}{2}\varepsilon \oplus \frac{2a+b-4}{2}1\right] \otimes (1 \oplus \rho_k) \oplus \rho_k$
$I_{a,b}$	$2 < a < b$ both odd	$\frac{a+b-2}{2}(1 \oplus \varepsilon) \otimes (1 \oplus \rho_k) \oplus (\varepsilon \otimes \rho_k)$
$I_{a,a}$	$2 < a$ odd	$\frac{a-1}{2}(\rho_2 \oplus \rho_2^2) \otimes (1 \oplus \rho_k) \oplus (\lambda \otimes \rho_k)$
$I_{a,a}$	$2 \le a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes (1 \oplus \rho_k) \oplus (\kappa \otimes \rho_k)$
$II_{a,a}$	$2 < a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes (1 \oplus \rho_k) \oplus (\rho_k)$
$II_{2,2}$		$\rho_2^3 \oplus (\rho_2 \oplus 1) \otimes \rho_k$
$III_{a,a}$	$2 < a$ even	$\left[\frac{a}{2}\rho_2 \oplus \frac{a-2}{2}(1 \oplus \kappa)\right] \otimes [1 \oplus 1 \oplus \rho_{k-1}]$
$III_{a,a}$	$2 < a$ odd	$\frac{a-1}{2}\left[2\rho_2 \oplus \rho_2^2 \oplus \rho_2^4 \oplus (\rho_2 \oplus 1 \oplus \kappa) \otimes \rho_{k-1}\right]$
$III_{a,b}$	$2 \le a < b$	$\bigoplus_{j=1}^{a-1}\left[\varepsilon^{\otimes a+j} \oplus \varepsilon^{\otimes j} \otimes (\theta^{\otimes b} \oplus \rho_{k-1})\right] \oplus$ $\oplus \bigoplus_{j=1}^{b-1}\left[\theta^{\otimes b+j} \oplus \theta^{\otimes j} \otimes (\varepsilon^{\otimes a} \oplus \rho_{k-1})\right]$
IV_a	$3 \le a$	$(\bigoplus_{j=1}^{a-1}\rho_2^j) \otimes (1 \oplus \rho_k) \oplus (\lambda \otimes \rho_k)$
V_a	$2 \le a$	$\bigoplus_{j=1}^{a-1}(\rho_2^{a-j} \oplus \rho_2^{a+j}) \oplus (\bigoplus_{j=1}^{a-1}\rho_2^j) \otimes \rho_{k-1}$ $\square$

As an application of this theorem and Theorems 3 and 4 we have the following characterization of the orientability of $\Sigma^{2,0}$-singularity submanifolds.

THEOREM 12. *In the next table — for some singularities of Thom-Boardman type $\Sigma^{2,0}$ — equivalent conditions are given to condition (2) (and therefore to condition (1)):*

$\eta = I_{a,b}$:	$a \equiv b \equiv 2 \mod 4$	$a \neq b$	k is even
$\eta = II_{a,b}$:	$a \equiv b \equiv 2 \mod 4$		k is even
$\eta = I_{a,b}$:	$a \equiv b \equiv 1 \mod 2$	$a \neq b$	k is odd
$\eta = I_{a,b}, I_{b,a}$:	$a \equiv 1,\, b \equiv 3 \mod 4$		
$\eta = III_{a,b}, III_{b,a}$:	$a \equiv 2,\, b \equiv 3 \mod 4$		k is even
$\eta = IV_a$:			k is odd.

Moreover, for the $\Sigma^{2,0}$ singularities above these are the only values of a, b, k for which condition (2) holds. $\square$

THEOREM 13. *In the next table — for some singularities of Thom-Boardman type $\Sigma^{2,0}$ — equivalent conditions are given to condition (4) (and therefore to condition (3)):*

$\eta = I_{a,b}, I_{b,a}$:	$a \equiv 1 \mod 2,\, b \equiv 2 \mod 4$		k is even
$\eta = III_{a,b}$:	$a \equiv b \equiv 3 \mod 4$	$a \neq b$	k is even
$\eta = III_{a,b}$:	$a \equiv b \equiv 2 \mod 4$		k is even
$\eta = V_a$:	$a \equiv 0 \mod 2$		k is even.

Moreover, for the $\Sigma^{2,0}$ singularities above these are the only values of a, b, k for which condition (4) holds. $\square$

Besides Theorems 3 and 4 there is a third type of results we can prove about the orientability of singularity submanifolds $\eta(f)$, this time in case both the source N and the target P manifolds are oriented. The arguments follows the same line with the only difference that now we must start with the "universal singular map" $Y^{SO}\tau \xrightarrow{f_\tau} X^{SO}\tau$ in [5]. The only change in this case is that we have to replace the group $G = MC\ Aut_{\mathcal{A}}\eta$ to

$$G^+ = G\, \cap\, \big(Diff^+(\mathbb{R}^n,0)\times Diff^+(\mathbb{R}^{n+k},0)\cup Diff^-(\mathbb{R}^n,0)\times Diff^+(\mathbb{R}^{n+k},0)\,\big)$$

for any map germ $\eta : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^{n+k},0)$. Otherwise all the proofs goes the same way, so we will restrict ourselves to only stating the results.

THEOREM 14. *The following conditions are equivalent:*

(15) for every $f : N \longrightarrow P$, where N and P are orientable, the manifold $\eta(f)$ is orientable;
(16) $\det \lambda_1(G^+) > 0$;
(17) $\det \lambda_1(G^-) > 0$;
(18) there is no $g \in G$ such that $\det \lambda_1(g) < 0$ *and* $\det \lambda_2(g) < 0$;
(19) either $\det \lambda_1(g) > 0$ *for all $g \in G$ or* $\det \lambda_2(g) > 0$ *for all $g \in G$ ($\Leftrightarrow$ (2) or (4)).*

$\square$

In case $\eta = \eta^{1_r}$ of Example 1 condition (19) is equivalent to the condition: (6) or (8). In case $\eta = \eta_r$ of Example 2 condition (19) reads: (10) or (12), that is

(20) k is even.

Just like above, we can use that if $k > 0$ then the orientability of η_r is equivalent to the orientability of $\Sigma^r(f)$ therefore we have the following corollary.

Corollary 15. *Let $k > 1$. The condition*
(21) for every k-codimensional map $f : N \longrightarrow P$, where N and P are orientable, the manifold $\Sigma^r(f)$ is orientable;
is equivalent to condition (20). $\square$

This last corollary can also be derived from a result of Ando [1, Proposition 4.1].

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