LACES: A GENERALISATION OF BRAIDS

ROGER FENN, GYO TAEK JIN AND RICHÁRD RIMÁNYI

1. Introduction

The subject of this paper is a previously little studied object which we call a *lace*. A lace of n components is represented by a disjoint union of n arcs in the plane which join n fixed points to n other fixed points. We take as the initial points of the arcs the points $(1,1),(2,1),\dots,(n,1)$ and the final points of the arcs are $(1,0),(2,0),\dots,(n,0)$ in some order.

There are several notions of equivalence of laces. Apart from the obvious notion of isotopy in the plane there is a notion of 3-isotopy in which the interiors of the arcs are allowed to move in the upper half space. There is also a notion of cobordism and to each of the previous equivalences can be added a similar equivalence where the arcs are allowed to lie in the extended Riemannian plane or sphere. Clearly isotopy implies cobordism and, because the 3-isotopy has one extra dimensional freedom, it is weaker than the cobordism.

Lemma 8. *Cobordant laces are 3-isotopic.*

Laces are a very natural generalisation of braids. Given an n-lace we can construct an n-braid as follows: Consider the xy-plane in which the lace lies as being the horizontal plane $z = 0$ in space with the

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z-axis vertical[1]. In the upper half space add vertical half lines with end points $(i, 1, 0)$ and in the lower half space add vertical half lines with end points $(i, 0, 0)$. After a small isotopy the result is a braid with n strings. Conversely given an n-braid we can construct an n-lace. Consider the braid as usual with the strings running from top to bottom with the z coordinate monotonic. Rotate the braid about the x-axis so that the z-axis becomes the y-axis (and the y-axis becomes the $(-z)$-axis). A projection gives a picture of a braid lying in the (x, y) plane. Starting from (one of) the bottom crossings push the over crossing string down the undercrossing string to the bottom. Now repeat till all the crossings are eliminated. The result is a special type of lace called a *lower lace*[2], that is none of the arcs venture into the region $y > 1$. There is a bijection between braids and lower laces consisting of the two above constructions.

Our initial interest in laces was due to the following group action. Let C_n be the group of automorphisms of the free group F_n with n free generators $x_1, \dots, x_n$, mapping each x_i to a conjugate $w_i^{-1} x_i w_i$. The group C_n has the following finite presentation:

$$\langle \alpha_{ij},\ i \neq j,\ 1 \leq i, j \leq n \mid \alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij},\ \alpha_{ij}\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{kj}\alpha_{ij},$$
$$\alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij}, \text{for distinct } i, j, k, l \rangle$$

where α_{ij} is defined by $\alpha_{ij}(x_i) = x_j^{-1} x_i x_j$ and $\alpha_{ij}(x_k) = x_k$, for $k \neq i$. Elements of C_n are called *basis-conjugating automorphisms* [4].

The action of $\alpha_{ij}^{\pm 1}$ on an n-lace ℓ is to replace the j-th component ℓ_j by a connected sum of ℓ_j with the boundary of a regular neighborhood of the i-th component ℓ_i along a path not intersecting ℓ in its interior.

One can easily see that this action respects the relations in the presentation of C_n, provided the paths are well chosen. This action is well

[1]Although a lace will normally lie in $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$, we allow laces at other levels, typically in $\mathbb{R}^2 \times \{1\}$ when considering cobordism of laces.

[2]Equivalent to the *normal dissections* of [1, p. 143].

defined only up to the choice of the paths. But the path choices are related by a cobordism involving two saddle points. Therefore C_n acts on cobordism classes of laces in the plane and on the sphere. There are also actions of the braid group and of the framed braid group on pure laces.

Like braids, laces have closures which result in knots or links. The standard closure is to adjoin to a lace, semi-circular arcs in the lower half space from $(i, 0)$ to $(i, 1)$. The result is a link with n bridges[3]. The plat closure, or simply the plat, of a $2n$-lace also results in a link with n bridges. Take a $2n$-lace and adjoin n semi-circular arcs in the upper half space from $(2i-1, 1)$ to $(2i, 1)$ and n semi-circular arcs in the lower half space from $(2i-1, 0)$ to $(2i, 0)$. The link so obtained is the same as the plat of the $2n$-braid obtained by raising the endpoints $(j, 1), j = 1, 2, \ldots, 2n$ vertically.

Theorem 7. *A pure n-lace has trivial closure if and only if it is in the orbit of the trivial n-lace under the representative actions of C_n.*

Finally we prove the following theorem which is a variation of results due to Otal [5, 6].

Theorem 16. *Let ℓ be an n-lace whose closure $\hat{\ell}$ is a trivial n-component link. There is an isotopy deforming $\hat{\ell}$ to the plat of the trivial $2n$-braid such that the number of bridges is unchanged during the isotopy.*

2. Definition of laces and their various equivalences

2.1. **Laces and their closures.** A *lace* ℓ is an n-tuple $(\ell_1, \ldots, \ell_n)$ of disjoint simple arcs in the xy-plane $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ such that $\partial \ell_i = \{(i, 1), (\pi_\ell(i), 0)\}$ for $i = 1, \ldots, n$, where π_ℓ is a permutation of the set $\{1, \ldots, n\}$, for some positive integer n. Each arc ℓ_i will be

[3] A 5-lace and its closure in shown in [1, Fig.2.11b]

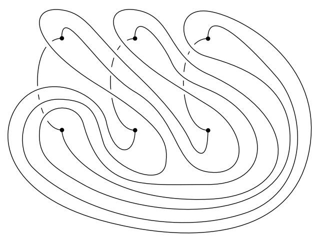


FIGURE 1

called a *component* or, more precisely, the *i-th component* of ℓ. A lace with n components will be called an *n-lace*. The permutation π_ℓ will be called the *permutation type* of ℓ. A lace whose permutation type is the identity permutation will be called a *pure* lace. A lace which lies in the region $y \le 1$ will be called a *lower lace*. An *oriented lace* is a lace with each of its components oriented. The *preferred orientation* on ℓ is the one which directs from $(i,1)$ to $(\pi_\ell(i),0)$ for each component. The *i-th spine* of a lace is the segment $\{i\} \times [0,1]$ and the *spine* of an n-lace is the union of the i-th spine for $i = 1, \ldots, n$. The *closure* of an n-lace ℓ, denoted $\hat{\ell}$, is the union of simple closed curves in space obtained from ℓ by adding the n semi-circles

$$\{(i,y,z) \mid y^2 + z^2 = 1, z \le 0\},\ \ i = 1, \ldots, n.$$

Figure 1 shows a pure 3-lace. The broken curves indicate the three semi-circles added for the closure. The *plat closures*, or simply *plats* of a $2n$-lace ℓ and a $2n$-braid b will be denoted $\bar{\ell}$ and $\bar{b}$, respectively.

2.2. **Equivalences.**

2.2.1. *Isotopy and $*$-isotopy.* Two n-laces are *isotopic* if they are isotopic in $\mathbb{R}^2$ relative to the endpoints. A *trivial lace* is a lace which is isotopic to its spine. For a *$*$-isotopy*, we allow arcs to pass through the point at infinity. This is the same as considering laces on S^2, the one-point compactification of the xy-plane. We will denote the set of isotopy classes and the the set of $*$-isotopy classes of n-laces as L_n and

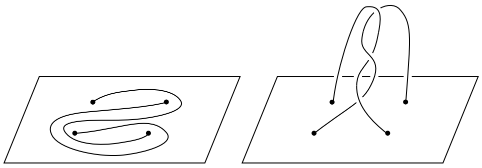


FIGURE 2

L_n^*, respectively. Provided no confusion can result we will also call laces the elements of L_n and L_n^* which they represent. We denote the subset of L_n whose elements are represented by lower laces by LL_n.

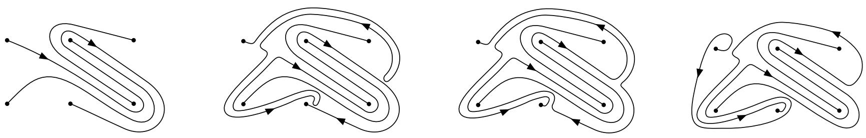
2.2.2. *3-isotopy.* Two n-laces are *3-isotopic* if they are isotopic in $\mathbb{R}^3_+ = \mathbb{R}^2 \times [0, \infty)$ relative to their endpoints. We will denote the set of 3-isotopy classes of n-laces as L_n^3. Pushing the interior of each component of a lace ℓ off the xy-plane into $\mathbb{R}^3_+$, we can represent the 3-isotopy class of ℓ by a "string link", again denoted by $\ell = (\ell_1, \dots, \ell_n)$, satisfying the following conditions:

(1) $\partial \ell_i = \{(i, 1, 0), (\pi_\ell(i), 0, 0)\}$, for $i = 1, \dots, n$.
(2) The z coordinate of ℓ_i has only one critical point, a local maximum.

2.2.3. *Cobordism and $*$-cobordism.* Two n-laces ℓ and m of the same permutation type are *cobordant* if there exists a surface V whose connected components are n disjointly embedded locally-flat orientable surfaces $V_1, \dots, V_n$ in $\mathbb{R}^2 \times [0, 1]$ such that, for each i,

(1) $\ell_i \times \{0\} = V_i \cap \mathbb{R}^2 \times \{0\}$,
(2) $m_i \times \{1\} = V_i \cap \mathbb{R}^2 \times \{1\}$,
(3) $\partial V_i = \ell_i \times \{0\} \cup m_i \times \{1\} \cup \{(i, 1), (\pi_\ell(i), 0)\} \times [0, 1]$.

The word *cobordism* will be used either to indicate the manifold V, or the equivalence relation "is cobordant to". We will denote the set of cobordism classes of n-laces as L_n^C. Allowing passages over the point at infinity, as in the $*$-isotopy, we define *$*$-cobordism.* We can interpret



(a) (b) (c) (d)

Figure 3

$*$-cobordism as cobordism of laces in S^2. The set of $*$-cobordism classes of n-laces is denoted by L_n^{C*}.

Example. In Figure 3, (a)$\leftrightarrow$(b) is an isotopy, (b)$\leftrightarrow$(c) a cobordism, (c)$\leftrightarrow$(d) a $*$-isotopy and (a)$\leftrightarrow$(d) a $*$-cobordism.

Theorem 1. *The obvious natural maps in the diagram below are all well defined and surjective and the diagram is commutative.*

$$\begin{array}{ccccc} L_n & \rightarrow & L_n^* & \rightarrow & L_n^3 \\ \downarrow & & \downarrow & \nearrow & \\ L_n^C & \rightarrow & L_n^{C*} & & \end{array}$$

The proof of Theorem 1 will be delayed until after Lemma 8. Notice that laces which are equivalent under any of the above relations have the same permutation type. Therefore we can define the sets PL_n, PL_n^*, PL_n^3, PL_n^C, PL_n^{C*} of equivalence classes of pure n-laces and hence the same statement of Theorem 1 holds for the following diagram.

$$\begin{array}{ccccc} PL_n & \rightarrow & PL_n^* & \rightarrow & PL_n^3 \\ \downarrow & & \downarrow & \nearrow & \\ PL_n^C & \rightarrow & PL_n^{C*} & & \end{array}$$

3. Laces and braids

We regard braids as placed vertically with respect to the xy-plane. More precisely, an n-braid b is properly embedded in $\mathbb{R}^2 \times [0,1]$ with $\partial b = \{(i,0,0),(i,0,1) \mid i = 1,\ldots,n\}$. Let B_n denote the group of

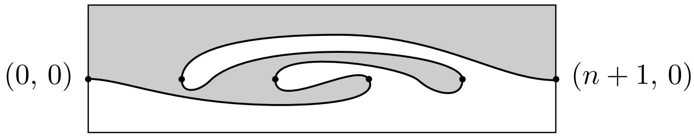


Figure 4

n-braids in which the multiplication ab is defined by stacking a over b with respect to the z-coordinate and scaling the result to be fitted in $\mathbb{R}^2 \times [0,1]$. Given an n-lace ℓ, we construct an n-braid $\beta(\ell)$ by attaching to ℓ_i the line segment joining $(i,1,0)$ to $(i,0,1)$ and pushing the interior of the new arc into the half space $\mathbb{R}^3_+$, for $i = 1,\ldots,n$. Note that $\beta(\ell)$ is uniquely determined by ℓ. This is essentially the same as adding vertical half lines as described in Section 1. The braid $\beta(\ell)$ will be called the braid type of ℓ. This function $\beta : L_n \to B_n$ has a right inverse $\iota : B_n \to L_n$ whose image is LL_n. We will describe ι using *path braids*. A *path n-braid* is defined to be a simple path $\mathfrak{b} : [0,n+1] \to [0,n+1] \times (-1,1)$ such that $\mathfrak{b}(0) = (0,0)$, $\mathfrak{b}(n+1) = (n+1,0)$ and $\mathfrak{b}(i) = (\pi(i),0)$ for some permutation $\pi = \pi_{\mathfrak{b}}$ of the set $\{1,\ldots,n\}$. See Figure 4 for an example when $n = 4$. The two path n-braids are said to be equivalent if they are isotopic relative to the points, $(i,0)$, $i = 0,\ldots,n+1$. The *trivial* path n-braid $\mathfrak{t}$ is defined by $\mathfrak{t}(t) = (t,0)$ for $t \in [0,n+1]$. Let Π_n denote the set of equivalence classes of path n-braids. First we establish a bijection $\lambda : \Pi_n \to LL_n$. Given a path n-braid $\mathfrak{b}$, let $d_{\mathfrak{b}}$ be the disc with boundary $\{0\}\times[0,1]\cup[0,n+1]\times\{1\}\cup\{n+1\}\times[0,1]\cup\mathfrak{b}([0,n+1])$. For an example see the shaded region in Figure 4. Given a path n-braid we can define a lower lace $\lambda(\mathfrak{b})$ by joining the points $(i,1)$ to $\mathfrak{b}(i)$ by a collection of disjoint simple paths properly embedded in $d_{\mathfrak{b}}$, see Figure 5(a). The construction of $\lambda^{-1} : LL_n \to \Pi_n$ should be clear from Figure 5.

Now we construct a bijection $\mu : \Pi_n \to B_n$. Given a path n-braid $\mathfrak{b}$, there is an isotopy $h_t : [0,n+1] \times \mathbb{R} \to [0,n+1] \times \mathbb{R}$ such that

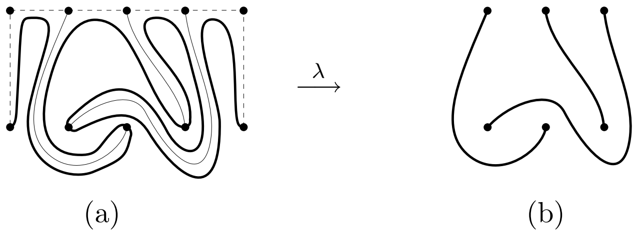


FIGURE 5

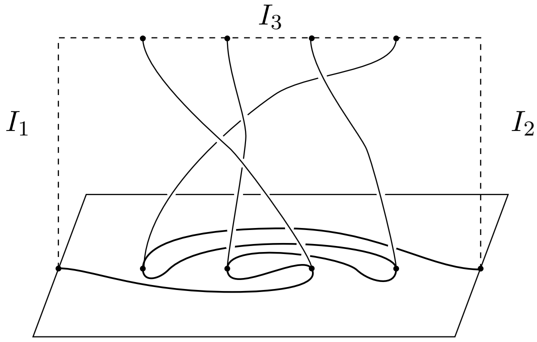


FIGURE 6

$h_t(0,0) = (0,0)$, $h_t(n+1,0) = (n+1,0)$, for all $0 \le t \le 1$, and $h_1(\mathfrak{b}) = \mathfrak{t}$ where $\mathfrak{t}$ is the trivial path braid. Let $\mu(\mathfrak{b})$ be the braid represented by $\{(h_t(i,0),t) \mid i = 1,\dots,n,\ 0 \le t \le 1\}$. Conversely, suppose an n-braid b is given. We may assume that b is contained in $(0,n+1)\times\mathbb{R}\times[0,1]$. Then there is a 2-disc D which is properly embedded in $[0,n+1]\times\mathbb{R}\times[0,1]$, which contains b and whose boundary is the union of $I_1 = \{0\}\times\{0\}\times[0,1]$, $I_2 = \{n+1\}\times\{0\}\times[0,1]$, $I_3 = [0,1]\times\{0\}\times\{1\}$ and a curve in the xy-plane. This curve represents the path braid $\mu^{-1}(b)$. Figure 6 shows a path 4-braid together with the corresponding 4-braid.

We define ι to be the composition $\lambda\,\mu^{-1}$. Given a braid b, $\iota(b)$ can be obtained by the process described in Section 1. See Figure 6.

The function μ can also be seen by the action of the braid generators σ_i, $i = 1,\dots,n-1$, as an isotopy of the plane described below. Let $\bar{\sigma}_i$

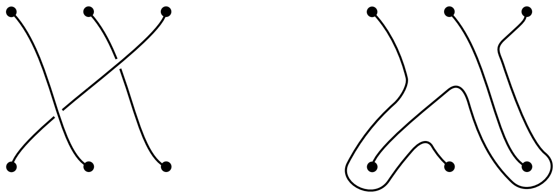


FIGURE 7

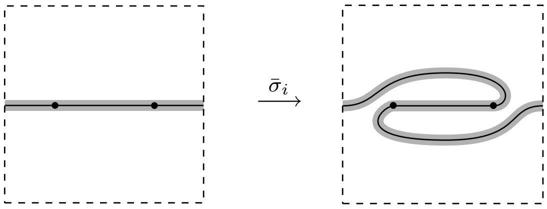


FIGURE 8

be a homeomorphism of $\mathbb{R}^2$ which is fixed outside the open rectangle $(i-\frac{1}{2}, i+\frac{3}{2}) \times (-1,1)$ and is, as indicated in Figure 8, the result of rotating the segment $[i, i+1] \times \{0\}$ through $180°$ clockwise. We define right actions of σ_i^ϵ, on a path braid $\mathfrak{b}$ and a lace ℓ by the formulae

$$\mathfrak{b} \cdot \sigma_i^\epsilon = \bar{\sigma}_i^\epsilon(\mathfrak{b})$$
$$\ell \cdot \sigma_i^\epsilon = \bar{\sigma}_i^\epsilon(\ell)$$

for $i = 1, \dots, n-1$, $\epsilon = \pm 1$. These actions extend to an action of the braid group B_n. Let $b = \sigma_{i_1}^{\epsilon_{i_1}} \sigma_{i_2}^{\epsilon_{i_2}} \cdots \sigma_{i_k}^{\epsilon_{i_k}}$. Then

$$\mu^{-1}(b) = \mathfrak{t} \cdot b = \mathfrak{t} \cdot \sigma_{i_1}^{\epsilon_{i_1}} \cdot \sigma_{i_2}^{\epsilon_{i_2}} \cdots \sigma_{i_k}^{\epsilon_{i_k}} = \bar{\sigma}_{i_k}^{\epsilon_{i_k}} \cdots \bar{\sigma}_{i_1}^{\epsilon_{i_1}}(\mathfrak{t})$$
$$\iota(b) = 1 \cdot b = 1 \cdot \sigma_{i_1}^{\epsilon_{i_1}} \cdot \sigma_{i_2}^{\epsilon_{i_2}} \dots \sigma_{i_k}^{\epsilon_{i_k}} = \bar{\sigma}_{i_k}^{\epsilon_{i_k}} \cdots \bar{\sigma}_{i_1}^{\epsilon_{i_1}}(1)$$

where $\mathfrak{t}$ is the trivial path braid and 1 the trivial n-lace.

Now we define a binary operation on laces by $(\ell, m) \mapsto \ell m := \ell \cdot \beta(m)$, which makes the following diagram commutative.

$$\begin{array}{ccccc} & L_n & \times & L_n & \longrightarrow & L_n & \\ \beta\times\beta & \downarrow & & & & \downarrow & \beta \\ & B_n & \times & B_n & \longrightarrow & B_n & \end{array}$$

Then L_n together with the trivial n-lace becomes a monoid containing B_n as a direct product factor by the following theorem. Let K_n denote the subset $\beta^{-1}(1)$ of L_n. Elements of K_n have a trivial associated braid and we call them *pseudo trivial* laces.

Theorem 2. *The sequence*

$$1 \to K_n \hookrightarrow L_n \xrightarrow{\beta}\!\!\to B_n \to 1$$

is a split exact sequence of monoids.

Proof. The result follows from the existence of a right inverse ι of β as described above. □

Corollary 3. $L_n = K_n \times LL_n \cong K_n \times B_n$ *where the product is*

$$(k_1, b_1)(k_2, b_2) = (k_1, b_1 b_2).$$

If k_1 and k_2 are pseudo trivial n-laces, their product is $k_1k_2 = k_1$. So the monoid structure collapses the pseudo trivial laces by a projection. Finding more about K_n seems an interesting problem.

4. Actions on laces by automorphism groups of free groups

C_n is the group of basis-conjugating automorphisms of the free group F_n with n free generators $x_1, \dots, x_n$, mapping each x_i to a conjugate $w_i^{-1}x_iw_i$. C_n has the following finite presentation:

$$\langle \alpha_{ij},\ i \neq j,\ 1 \leq i, j \leq n \mid \alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij},\ \alpha_{ij}\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{kj}\alpha_{ij},$$
$$\alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij}, \text{for distinct } i, j, k, l\rangle$$

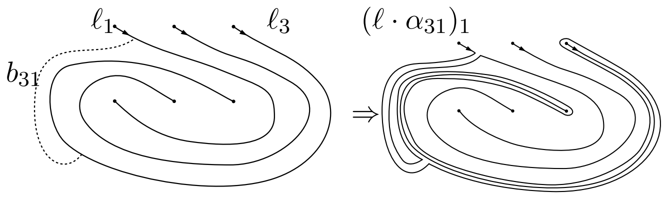


FIGURE 9. $\ell \cdot \alpha_{31}$

where α_{ij} is defined by $\alpha_{ij}(x_i) = x_j^{-1} x_i x_j$ and $\alpha_{ij}(x_k) = x_k$, for $k \neq i$. For a proof see [4].

The action of $\alpha_{ij}^{\pm 1}$ on an n-lace ℓ is to replace ℓ_j by a connected sum of the boundary of a regular neighborhood of ℓ_i and ℓ_j along a simple path. To be more precise, we start by choosing a path b_{ij} which satisfies:

(1) b_{ij} is a smooth simple path from an interior point of the component ℓ_i to an interior point of the component ℓ_j not intersecting any other part of ℓ.
(2) b_{ij} is attached to the right hand side of ℓ_j with respect to the preferred orientation.

Also choose a regular neighbourhood N of ℓ_i containing the interior of b_{ij} in its interior such that $\delta = N \cap \ell_j = \partial N \cap \ell_j$ is a connected arc. Define

$$(\ell \cdot \alpha_{ij})_k = \begin{cases} \overline{\partial N \cup \ell_j \setminus \delta} & \text{if } k = j \\ \ell_k & \text{if } k \neq j \end{cases}$$

The condition (2) makes the preferred orientation of $(\ell \cdot \alpha_{ij})_j$ coherent with the anticlockwise orientation of ∂N. To define $\ell \cdot \alpha_{ij}^{-1}$, we replace the condition (2) by

(3) b_{ij} is attached to the left hand side of ℓ_j with respect to the preferred orientation,

and use the same formula for $(\ell \cdot \alpha_{ij}^{-1})_k$'s. The condition (3) makes the preferred orientation of $(\ell \cdot \alpha_{ij}^{-1})_k$ opposite to the anticlockwise orientation of ∂N.

One can easily see that this action respects the relations in the presentation of C_n, provided the paths are well chosen. This action is well defined only up the choice of the paths b_{ij} but the path choices are related by a cobordism involving two saddle points. Therefore C_n acts on L_n^C, and hence on L_n^{C*}. This fact is an interesting geometric interpretation of the Yang-Baxter equation — because the relations of C_n are (not taking into consideration the evident commutativity relations) the Yang-Baxter equations. The induced action of C_n is trivial on L_n^3, because one can lift the interior of the disc N off the xy-plane into $\mathbb{R}^3_+$ and shrink the arc $\overline{\partial N \setminus \delta}$ to δ in the disc. Later we will need to distinguish the C_n actions on cobordism classes of laces from the repetitions of the construction above along a specific choice of the paths b_{ij}. We will call the latter a *representative action* of C_n.

5. Lace links

A link in $\mathbb{R}^3$ or in S^3 will be called a *lace link* if it is the closure of a pure lace. In fact, a lace link is a link whose bridge index is equal to the number of components.

Let ℓ be a pure n-lace. Suppose that the i-th spine intersects ℓ with the components $\ell_{i_1}, \dots, \ell_{i_k}$, in this order from $(i, 0)$ to $(i, 1)$ with signs $\epsilon_{i_1}, \dots, \epsilon_{i_k}$, determined as indicated in Figure 10 where the arrows indicate the preferred orientation. Let λ_i denote the unreduced word $a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_k}^{\epsilon_{i_k}}$ in the letters $a_1, \dots, a_n$. Then the link group $G_\ell = \pi_1(\mathbb{R}^3 \setminus \hat{\ell})$ has Wirtinger presentation

$$\langle a_1, \dots, a_n \mid a_i = \lambda_i^{-1} a_i \lambda_i,\ i = 1, \dots, n \rangle$$

where each a_i corresponds to the meridian of ℓ_i.

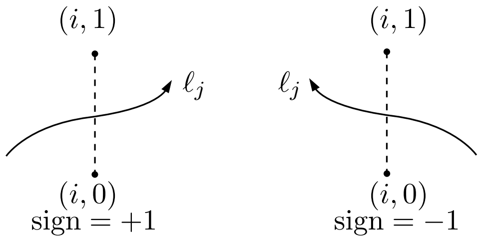


Figure 10

$*$-isotopies may change the unreduced words λ_i, but the presentation is unchanged because the only possible changes in λ_i by $*$-isotopies are insertions of $a_j a_j^{-1}$ or $a_j^{-1} a_j$ and left or right multiplications by a_i or a_i^{-1}. Notice that λ_i is a longitude of the component ℓ_i.

Proposition 4. *A lace link which is a homology boundary link is a trivial link.*

Proof. Let ℓ be a pure n-lace whose closure $\hat{\ell}$ is a homology boundary link. Then there is an epimorphism of G_ℓ onto a rank n free group F_n [8]. If $\hat{\ell}$ is not a trivial link, then the group G_ℓ is a proper factor group of F_n. Since a free group of finite rank is Hopfian, this is impossible. □

Corollary 5. *Let V be a cobordism or a $*$-cobordism between two laces. Then ∂V is a trivial link.*

Proof. Notice that by an isotopy ∂V can be made into a link with one maximum for each component and so is a lace link and also a boundary link. By Proposition 4, it is a trivial link. □

Corollary 6. *The closure of a (pure) lace which is cobordant or $*$-cobordant to a trivial lace is the trivial link.*

Proof. The boundary of a cobordism between a pure lace ℓ and a trivial lace is the same as the closure $\hat{\ell}$. Therefore by Corollary 5, $\hat{\ell}$ is trivial. □

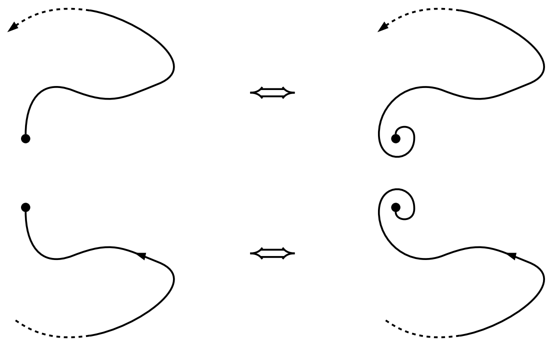


Figure 11

Theorem 7. *A pure n-lace has trivial closure if and only if it is in the orbit of the trivial n-lace under the representative actions of C_n.*

Proof. The sufficiency is obvious since the representative action does not change the link type of the closures.

Let ℓ be a pure n-lace whose closure is the trivial link. By isotopies and by representative actions of C_n, we will deform ℓ to its spine, component by component.

Suppose ℓ_i is the first component which is not deformed to its spine. Since the group G_ℓ must be a free group, the word λ_i is constructed from a power of a_i by inserting the words $a_j a_j^{-1}$ or $a_j^{-1} a_j$, for some $j = 1, \ldots, n$, repeatedly. For each subword of the form $a_j a_j^{-1}$ or $a_j^{-1} a_j$, there corresponds a pair of intersections of the i-th spine and ℓ_j. Let δ be the segment of the i-th spine cut by the pair. Then there are no other intersections on δ. Let δ' be the arc on ℓ_j which is cut by the same points. Then $\delta \cup \delta'$ is a simple closed curve bounding a disc D. Let ℓ' be obtained by replacing ℓ_j by $\ell'_j = (\ell_j \setminus \delta') \cup \delta$ which is pushed off the i-th spine. If the disc D contains no component of ℓ', then the change $\ell \mapsto \ell'$ is an isotopy. If $\ell_{j_1}, \ldots, \ell_{j_k}$ are the components of ℓ contained in D, then $\ell' = \ell \cdot (\alpha_{j_1 j} \cdots \alpha_{j_k j})^\epsilon$ where ϵ is -1 if the anticlockwise orientation on $\delta \cup \delta'$ is coherent with the preferred orientation of ℓ_j

and 1 otherwise. This representative action can be performed along any disjoint paths inside D. Now we replace ℓ by ℓ' and continue this process until the word λ_i becomes a word in a_i and a_i^{-1}. By an isotopy as indicated in Figure 11, we can make the sum of the powers of a_i in λ_i into zero. By similar processes as above, we can deform ℓ so that λ_i becomes the empty word. If the disc D' bounded by ℓ_i and the i-th spine contains no components of ℓ then we use isotopy to deform ℓ into its spine and proceed to the $(i+1)$-st component. If there are any components inside D', we can perform a representative action of C_n as above to replace ℓ_i by its spine and proceed to the next component. □

Lemma 8. *Cobordant laces are 3-isotopic.*

Proof. Let V be a cobordism between two n-laces ℓ and m. There is an orientation preserving homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\ell_i)$ is the i-th spine and $\phi(i,1) = (i,1)$, for $i = 1, \ldots, n$. Then $\phi(m)$ is a lace which is cobordant to the trivial lace $\phi(\ell)$ with cobordism $\phi \times id_{[0,1]}(V)$. According to Corollary 6, $\phi(m)$ has the trivial closure. As in the proof of Theorem 7, we can perform a series of representative actions of C_n on $\phi(m)$ with a suitable choice of paths to deform it into the trivial lace. Performing the same sequence of these representative actions on m, using the preimages of the paths under ϕ, we get a lace which is isotopic to ℓ. □

To see that the converse of Lemma 8 is false, we consider the 2-laces ℓ and ℓ' of Figure 12. Suppose there is a cobordism V_1 of the first components of ℓ and ℓ'. Then ∂V_1 has linking number 1 with the relative 1-cell $\{(1,0)\} \times [0,1]$ which is a part of any cobordism V_2 of the second components of ℓ and ℓ'. Therefore V_1 and V_2 cannot be disjoint and hence ℓ and ℓ' are not cobordant.

Proof of Theorem 1. The only difficult fact to check is the commutativity of the triangle. By Lemma 8, we know that there is a well-defined

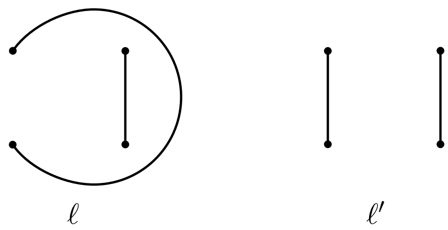


Figure 12

surjection $L_n^C \to L_n^3$. Since passages of arcs through the point at infinity can be realized by 3-isotopies, the above map factors though L_n^{C*}. □

Definition. Let ℓ and m be two n-laces. We define $\ell - m$ as the link obtained by taking the union of m and the string link obtained by pushing the interior of ℓ into $\mathbb{R}^3_+$. So for example $\ell - 1$ is $\hat{\ell}$, the closure of ℓ.

Theorem 9. *Let ℓ and m be two n-laces with the same permutation type. The following are equivalent:*

(1) *ℓ and m are 3-isotopic.*
(2) *$\ell - m$ is the trivial n-component link.*
(3) *ℓ and m are in the same orbit of the C_n representative actions.*

Proof. (1)⇒(2) Since ℓ and m are 3-isotopic, the links $\ell - m$ and $m - m$ are equivalent. Obviously the latter is the n-component trivial link.
(2)⇒(3) Let ϕ be a homeomorphism of $\mathbb{R}^2$ which is isotopic to the identity map and which maps the i-th component of m into the i-th spine. Then $\ell - m$, $\phi(\ell) - \phi(m)$ and $\widehat{\phi(\ell)}$ are all equivalent[4] links. Since $\ell - m$ is the trivial n-component link, $\phi(\ell)$ is in the orbit of the trivial lace $\phi(m)$ under C_n representative actions, according to Theorem 7. If we apply to ℓ the same representative actions to make $\phi(\ell)$ into the trivial lace along the preimages of the arcs used for $\phi(\ell)$, under ϕ, then

[4] In the strong sense, i.e., as ordered oriented links. This is true without the condition (2).

we must get the lace m.

(3)$\Rightarrow$(1) This is clear. $\square$

Corollary 10. *$L_n^C/C_n \cong L_n^3$ and $PL_n^C/C_n \cong PL_n^3$.*

Corollary 11. *A pure n-lace is 3-isotopic to the trivial n-lace if and only if its closure is the trivial n-component link.*

Theorem 12. *A strongly slice[5] lace link is a trivial link.*

Proof. Let ℓ be a pure n-lace whose closure is concordant to the trivial link. Since the lower central series quotients are invariants of link concordance, $G_\ell/(G_\ell)_q$ is isomorphic to $F_n/(F_n)_q$, for all q [9]. Having n generators, G_ℓ is isomorphic to F_n [3, p. 346]. Since only trivial links have fundamental group of their complement free, ℓ is trivial. $\square$

6. Braid actions and plats

6.1. **More group actions.** We introduce two more group actions on pure laces. These actions permute and rotate the spines. Now consider the following example.

EXAMPLE. The two 3-laces given in Figure 13 as string links have the same closures. But they are not 3-isotopic. For each case, consider the loop obtained by the third component together with the third spine in the complement of the first and second components. In the case (b), the loop is not contractible. In fact, this loop is the commutator of the two meridians, which are free generators of the fundamental group of the exterior of the first and second components.

One can deform the lace in Figure 13(a) to the one in (b) by these actions. These actions give one more step toward the classification of lace links by laces.

[5]concordant to a trivial link.

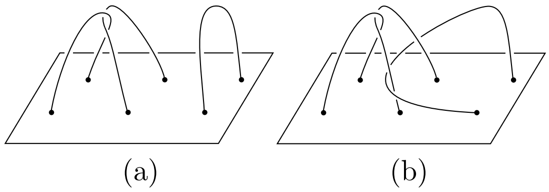


FIGURE 13

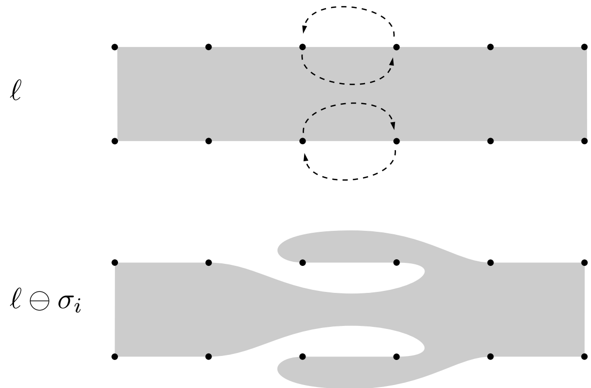


FIGURE 14

6.1.1. *The action of the group B_n of n-braids.* This action permutes the initial and end points of the lace components. Let σ_i be one of the standard generators of B_n. Then $\ell \ominus \sigma_i$ is the result of the isotopy of $\mathbb{R}^2$ which rotates the segments $[i, i+1] \times \{0, 1\}$ through $180°$ and which is fixed outside a regular neighborhood of the segments. The rotation is anticlockwise on $[i, i+1] \times \{1\}$ and clockwise on $[i, i+1] \times \{0\}$. So the action is a combination of the lower action $\ell \cdot \sigma_i$ considered in section 3 and a similar but opposite action at the top. In Figure 14, the change on the region between the horizontal lines $y = 0, y = 1$ is shown together with the endpoints of laces.

6.1.2. *The action of the group FB_n of the framed n-braids.* The group FB_n of the framed n-braids is the semi-direct product $B_n \ltimes \mathbb{Z}^n$ with the action of B_n on $\mathbb{Z}^n$ given by $b \cdot (m_1, \ldots, m_n) = (m_{\pi_b(1)}, \ldots, m_{\pi_b(n)})$

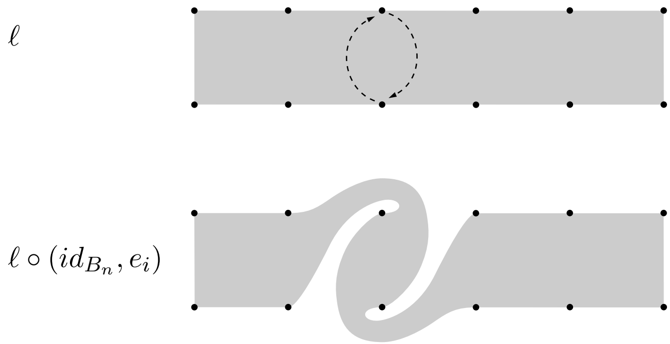


FIGURE 15

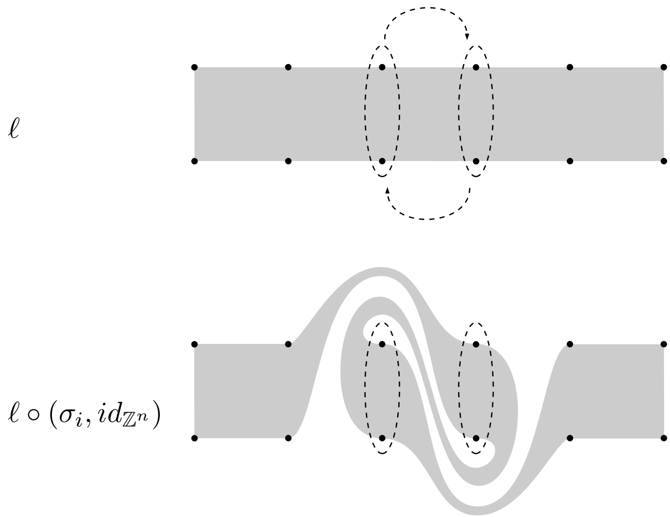


FIGURE 16

where π_b is the permutation of $\{1, \dots, n\}$ determined by b in the usual way.

Now we look at the action of FB_n on laces. The action of the standard basis element $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ of $\mathbb{Z}^n$ is the result of an isotopy of $\mathbb{R}^2$ rotating the i-th spine through $180°$ clockwise with the outside of a regular neighborhood of the i-th spine fixed. This action results in reversing the preferred orientation on the i-th component.

The action of the framed braid generator σ_i permutes the components of the laces in a way slightly different from the action of the braid group described above. This action is the result of an isotopy of $\mathbb{R}^2$ which switches around the i-th and $(i+1)$-st spines as in Figure 16 keeping them upright throughout the isotopy and fixing the outside of an open disc containing the two spines.

6.2. **Plats of trivial braids.** Otal's results in [5, 6] imply that given any two n bridge presentations of a trivial knot or a rational link, one can be isotoped to the other without changing the number of bridges during the isotopy. We show the same for trivial links.

Lemma 13. *Let b be a $2n$-braid whose plat $\bar{b}$ is the trivial n-component link. Then there exist n embedded discs D_i, $i = 1, \dots, n$, with the following properties:*

(1) $\cup \partial D_i = \bar{b}$.
(2) *Each D_i has no critical points with respect to the z-coordinate function.*
(3) *For $i \neq j$, $D_i \cap D_j$ consists of finitely many disjoint ribbon intersections which are located in distinct horizontal levels of the z-coordinate function.*

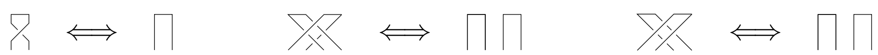
Proof. Let $\mathfrak{b}_b$ be the path $2n$-braid in $\mathbb{R}^2 \times \{0\}$ corresponding to b. Consider the 2-disc D which gives the correspondence between b and $\mathfrak{b}_b$ in Section 3. See Figure 6. Inside D, there exist n 2-discs d_i, $i = 1, \dots, n$, disjointly embedded in $R = [0, 2n+1] \times \mathbb{R} \times [0, 1]$ with

$$\partial d_i = b_{2i-1} \cup b_{2i} \cup \mathfrak{b}_b^i \cup [2i-1, 2i] \times \{(0, 1)\}$$

where b_j is the j-th string of b and $\mathfrak{b}_b^i$ is the subarc of $\mathfrak{b}_b$ joining the bottom end points of b_{2i-1} and b_{2i}. We may choose D so that it does not have critical points in the z-coordinate. Then the d_i's also do not have critical points in the z-coordinate.

Let $\phi = \phi' \times id_{[0,1]}$ where ϕ' is a homeomorphism of $\mathbb{R}^2$ isotopic to $id_{\mathbb{R}^2}$ mapping $[2i-1, 2i] \times \{0\}$ into the i-th spine and outside a regular neighborhood of $\bigcup_{i=1}^{n}[2i-1,2i]\times\{0\}$ by an affine map. By hypothesis the lace $\phi(b\cup\bigcup_{i=1}^{n}([2i-1,2i]\times\{(0,1)\}))$ is a string link representing the 3-isotopy class of the n-lace $\phi(\cup\mathfrak{b}_b^i)$. It must be noticed that the lace $\phi(\cup\mathfrak{b}_b^i)$ has a trivial closure. Therefore by Theorem 7, there is a sequence of n-laces $\ell^0,\dots,\ell^p$ from $\phi(\cup\mathfrak{b}_b^i)$ to a trivial lace such that ℓ^k is obtained from ℓ^{k-1} by a representative action of $a_{i_kj_k}^{\pm1}$ along a path $b_{i_kj_k}$. To perform a representative action of $a_{i_1j_1}^{\pm1}$, we need to take a regular neighbourhood N_1 of $\ell_{i_1}^0$ containing the interior of the path $b_{i_1j_1}$ in its interior such that $\delta_1 = N_1\cap\ell_{j_1}^0 = \partial N_1\cap\ell_{j_1}^0$ is a connected arc. Then $\ell_{j_1}^1 = \overline{\partial N_1\cup\ell_{j_1}^0\setminus\delta_1}$ and $\ell_j^1=\ell_j^0$ for $j\neq j_1$. We will replace d_{j_1} by the disc, again denoted d_{j_1}, obtained from $d_{j_1}\cup N_1$ by pushing $int(N_1)\cup int(\delta_1)$ into $\mathbb{R}^3_+$. We may choose the new d_{j_1} so that it has no critical points in the z-coordinate and such that $d_{i_1}\cap d_{j_1}$ is a ribbon intersection contained in the level $z=\epsilon_1$ for some $0<\epsilon_1<1$. We repeat this process so that the j-th ribbon intersection lies in the level $z=\epsilon_j$ with $0<\epsilon_j<\epsilon_{j-1}$ for $j=2,\dots,p$. Then we obtain discs d_i, $i=1,\dots,n$ satisfyng conditions similar to (1)–(3). Now let $D_i=\phi^{-1}(d_i)$. Then we are done. $\square$

Theorem 14. *A $2n$-plat presentation of an n-component trivial link can be made into the plat of the trivial $2n$-braid by the following three local moves together with the $2n$-braid relations.*

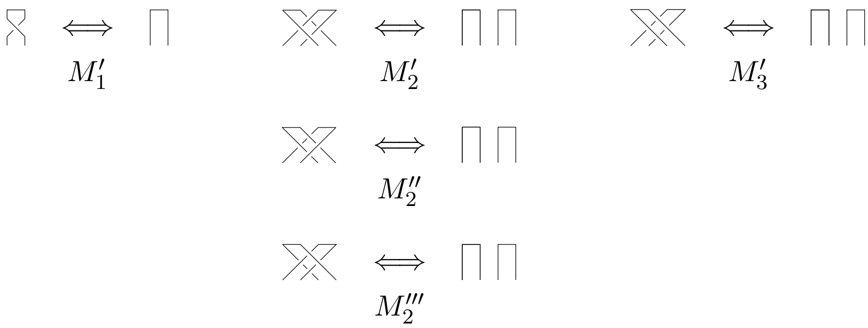


M_1 $\qquad$ M_2 $\qquad$ M_3

Proof. Let b be a $2n$-braid whose plat $\bar{b}$ is a trivial n-component link. Then there exist discs D_i, $i=1,\dots,n$, satisfying (1)–(3) in Lemma 13. Suppose that the ribbon intersections lie in the levels $z=\epsilon_i$, where $0<\epsilon_1<\cdots<\epsilon_p<1$. Then $\cup D_i$ is isotopic in $\mathbb{R}^2\times[0,1]$ relative to $\mathbb{R}^2\times[0,1]$ to a braid of n bands with horizontal ribbon intersections.

Then the moves M_1, M_2 and M_3 can be applied on the top of the braid to obtain a trivial braid of n bands. □

REMARKS. 1. Consider the moves M_1', M_2', M_3', M_2'' and M_2''' shown below. We observe that the moves M_1 and M_1' are equivalent up to braid relations. Likewise M_2 and M_2', M_2'' and M_2''', M_3 and M_3' are pairwise equivalent. But the moves M_2 and M_2'' are not equivalent unless the move M_1 is allowed.



2. The moves M_1 and M_3 on trivial $2n$-braids give rise to braids of n bands, which are the same as framed n-braids. Therefore these moves correspond to framed braid actions on laces. On the other hand the M_2-moves corresponds to the braid group actions on laces.

Looking at the changes on braids corresponding the moves M_1, M_2 and M_3, we easily have

Corollary 15. *The plat of a $2n$-braid represents the trivial n-component link if and only if the $2n$-braid is an element of the subgroup of B_{2n} generated by the following elements.*

(1) $\sigma_{2i-1},\ i = 1, \dots , n$

(2) $\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{\varepsilon}\sigma_{2i}^{\varepsilon},\ i = 1, \dots , n-1,\ \varepsilon = \pm 1$

The subgroup of B_{2n} mentioned in Corollary 15 can be seen as the braid group of n-bands with horizontal ribbon intersections as described in the proof of Theorem 14.

Theorem 16. *Let ℓ be an n-lace whose closure $\hat{\ell}$ is a trivial n-component link. There is an isotopy deforming $\hat{\ell}$ to the plat of the trivial $2n$-braid such that the number of bridges is unchanged during the isotopy.*

Proof. Let ℓ be a pure n-lace whose closure is a trivial link. Then we can choose a string link $\ell' = \cup \ell'_i$ representing the 3-isotopy class of ℓ such that ℓ'_i is the union of two arcs with no critical points in the z-coordinate joining $(i, \pm 1, 0)$ to $(i, \pm 1, 1)$ and the segment $\{i\} \times [-1, 1] \times \{1\}$. Let ϕ be as in the proof of Lemma 13. Then $\phi^{-1}(\ell')$ is a $2n$-plat with the bottom segments missing. As in the proof of Theorem 14, we can find an n-braid of bands with horizontal ribbon intersections bounding this $2n$-plat. To make the n-braid of bands into n straight bands, we consider the upside-down version of the moves M_1, M_2 and M_3 on its bottom. Applying ϕ, these moves correspond to the B_n actions and the FB_n actions. The number of bridges is unchanged under these moves. □

Corollary 17. *A pure n-lace whose closure is the trivial link can be transformed into the trivial n-lace by the B_n actions and the FB_n actions.*

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Roger Fenn
University of Sussex
Falmer, Brighton BN1 9QH
England
rogerf@central.susx.ac.uk

Gyo Taek Jin
Korea Advanced Institute of
Science and Technology
Taejon, 305-701
Korea
trefoil@math.kaist.ac.kr

Richárd Rimányi
Eötvös Loránd University
Budapest, Rákóczi út 5. 1088
Hungary
rimanyi@cs.elte.hu

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