TRIGONOMETRIC WEIGHT FUNCTIONS AS K-THEORETIC STABLE ENVELOPE MAPS FOR THE COTANGENT BUNDLE OF A FLAG VARIETY

R. RIMÁNYI⋆, V. TARASOV∘, A. VARCHENKO⋄

⋆⋄*Department of Mathematics, University of North Carolina at Chapel Hill*
Chapel Hill, NC 27599-3250, USA

∘*Department of Mathematical Sciences, Indiana University – Purdue University Indianapolis*
402 North Blackford St, Indianapolis, IN 46202-3216, USA

∘*St. Petersburg Branch of Steklov Mathematical Institute*
Fontanka 27, St. Petersburg, 191023, Russia

Abstract. We consider the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of a GL_n partial flag variety, $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$, $|\boldsymbol{\lambda}|=\sum_i\lambda_i=n$, and the torus $T=(\mathbb{C}^\times)^{n+1}$ equivariant K-theory algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We introduce K-theoretic stable envelope maps $\mathrm{Stab}_\sigma:\bigoplus_{|\boldsymbol{\lambda}|=n}K_T((T^*\mathcal{F}_{\boldsymbol{\lambda}})^T)\to\bigoplus_{|\boldsymbol{\lambda}|=n}K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, where $\sigma\in S_n$. Using these maps we define a quantum loop algebra action on $\bigoplus_{|\boldsymbol{\lambda}|=n}K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We describe the associated Bethe algebra $\mathcal{B}^q(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$ by generators and relations in terms of a discrete Wronski map. We prove that the limiting Bethe algebra $\mathcal{B}^\infty(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$, called the Gelfand-Zetlin algebra, coincides with the algebra of multiplication operators of the algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We conjecture that the Bethe algebra $\mathcal{B}^q(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$ coincides with the algebra of quantum multiplication on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ introduced by Givental and Lee [G, GL].

The stable envelope maps are defined with the help of Newton polygons of Laurent polynomials representing elements of $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ and with the help of the trigonometric weight functions introduced in [TV1, TV3] to construct q-hypergeometric solutions of trigonometric qKZ equations.

The paper has five appendices. In particular, in Appendix 5 we describe the Bethe algebra of the XXZ model by generators and relations.

Contents

⋆*E-mail*: rimanyi@email.unc.edu, supported in part by NSF grant DMS-1200685
∘*E-mail*: vt@math.iupui.edu, vt@pdmi.ras.ru
⋄*E-mail*: anv@email.unc.edu, supported in part by NSF grant DMS-1362924

1. Introduction

In [MO], Maulik and Okounkov study the classical and quantum equivariant cohomology of Nakajima quiver varieties for a quiver Q. They construct a Hopf algebra Y_Q, the Yangian of Q, acting on the cohomology of these varieties, and show that the associated Bethe algebra $\mathcal{B}^q$ acting on the cohomology of these varieties coincides with the algebra of quantum multiplication. The construction of the Yangian and the Yangian action is based on the notion of the stable envelope maps introduced in [MO]. In this paper we construct the analog of the stable envelope maps for the equivariant K-theory algebras of the cotangent bundles of the GL_n partial flag varieties.

Let $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ be the cotangent bundle of a GL_n partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, $|\boldsymbol{\lambda}| = \sum_i \lambda_i = n$. We consider the torus $T = (\mathbb{C}^\times)^{n+1}$ equivariant K-theory algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ and introduce K-theoretic stable envelope maps $\mathrm{Stab}_\sigma : \bigoplus_{|\boldsymbol{\lambda}|=n} K_T((T^*\mathcal{F}_{\boldsymbol{\lambda}})^T) \to \bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, where $(T^*\mathcal{F}_{\boldsymbol{\lambda}})^T \subset T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is the torus fixed point set and σ is an element of the symmetric group S_n. We describe the composition maps $\mathrm{Stab}^{-1}_{\sigma'} \circ \mathrm{Stab}_\sigma$ in terms of the standard $\mathfrak{gl}_N$ trigonometric R-matrix. Using these maps we define a quantum loop algebra action on $\bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We describe the associated Bethe algebra $\mathcal{B}^q(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$ by generators and relations in terms of a discrete Wronski map. We prove that the limiting Bethe algebra $\mathcal{B}^\infty(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$, called the Gelfand-Zetlin algebra, coincides with the algebra of multiplication operators of the algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. We conjecture that the Bethe algebra $\mathcal{B}^q(K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}))$ is isomorphic to the algebra of quantum multiplication on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ introduced by Givental and Lee [G, GL]. That conjecture is the K-theoretic analog of the main theorem in [MO] that describes the quantum multiplication.

In [MO], the stable envelope maps for equivariant cohomology of Nakajima varieties are defined axiomatically and then the uniqueness and existence are proved. Our definition of K-theoretic stable envelope maps for the cotangent bundles of partial flag varieties is also axiomatic and then we also prove the uniqueness and existence. The difference with axioms in [MO] is that we do not consider the supports of the stable envelope maps and we replace the notion of the degree of a polynomial with the notion of the Newton polygon of a Laurent polynomial. Another difference with [MO] is that we prove the existence by giving an explicit formula for the stable envelope maps. The formula for the stable envelope maps is given in terms of the trigonometric weight functions introduced in [TV1, TV3] to construct q-hypergeometric solutions of the trigonometric qKZ equations. The arguments of the weight functions in [TV1, TV3] are h, $z_1, \dots, z_n$, $t_{i,j}$, where h is the parameter of the quantum loop algebra, $z_1, \dots, z_n$ are positions of sites in the associated XXZ model and

$t_{i,j}$ are the integration variables in the q-hypergeometric integrals. Another interpretation of the variables $t_{i,j}$ in [TV1, TV3] is that they are variables in the Bethe ansatz equations associated with the XXZ model. In this paper, the arguments $h, z_1, \dots, z_n$ are interpreted as the equivariant parameters corresponding to the torus T and the arguments $t_{i,j}$ are interpreted as the Chern roots of the associated bundles over $\mathcal{F}_{\boldsymbol{\lambda}}$. This correspondence between the variable in the Bethe ansatz equations and the Chern roots is the indication of a K-theoretic Landau-Ginzburg mirror correspondence.

The paper is organized as follows. In Section 2, we introduce our geometric objects: cotangent bundles of partial flag varieties, the torus action, the equivariant K-theory algebras. In Section 3, we formulate axioms defining certain classes $\kappa_{\sigma,I} \in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ and formulate Theorem 3.1 that such classes exist and are unique. The classes $\kappa_{\sigma,I}$ are building blocks for the K-theoretic stable envelope maps and are the main novelty objects of our paper. Theorem 3.1 is our first main result. In Section 3.2, we prove the uniqueness of the classes $\kappa_{\sigma,I}$. In Section 4, we introduce the trigonometric weight functions and in Section 5 describe useful combinatorial presentation of the weight functions as sums of 'elementary' rational functions assigned to certain 'filled tables'. In Section 6, we describe properties of the weight functions and prove the existence of the classes $\kappa_{\sigma,I}$. In Section 7, using the classes $\kappa_{\sigma,I}$, we define the stable envelope maps and describe the compositions $\mathrm{Stab}_{\sigma'}^{-1} \circ \mathrm{Stab}_{\sigma}$ in terms of the standard trigonometric R-matrix. In Section 8, we describe the inverse map to the stable envelope map $\mathrm{Stab}_{\mathrm{id}}$.

In Section 9, we consider the space $(\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(z_1, \dots, z_n, h)$ with an S_n-action. We introduce the important subspace $\frac{1}{D}\mathcal{V}^- \subset (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(z_1, \dots, z_n, h)$ invariant with respect to the S_n-action. In Section 10, we define the quantum loop algebra and its commutative Bethe subalgebra $\mathcal{B}^q$ depending on parameters $q = (q_1, \dots, q_N)$. In Section 11, we describe a quantum loop algebra action on $\frac{1}{D}\mathcal{V}^- \otimes \mathbb{C}(h)$.

In Section 12, we describe a quantum loop algebra action on $\bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(h)$. This is done through the isomorphism $\nu : \bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(h) \to \frac{1}{D}\mathcal{V}^- \otimes \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \otimes \mathbb{C}(h)$ defined with the help of the map $\mathrm{Stab}_{\mathrm{id}}^{-1}$. We describe the close relations between our quantum loop algebra action on $\oplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(h)$ and the quantum loop algebra action studied by Ginzburg and Vasserot in [GV, Vas1, Vas2]. In Theorem 11.12, we identify the action on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ of the Gelfand-Zetlin algebra $\mathcal{B}^\infty$ and the action on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ of its own elements by multiplication. This is our second main result.

In Section 13, we introduce the discrete Wronski map $\mathrm{Wr}^q_{\boldsymbol{\lambda}}$ and define the associated algebra $\mathcal{K}^q_{\boldsymbol{\lambda}}$ by generators and relations. We describe a construction that identifies the Bethe algebra $\mathcal{B}^q$ action on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ and the regular representation of the algebra $\mathcal{K}^q_{\boldsymbol{\lambda}}$. This statement is formulated in Corollary 13.13.

In Section 13.4, we introduce a new commutative associative multiplication $*^q$ on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, depending on the parameters $q_1, \dots, q_N$. In Section 13.5, we formulate Conjecture 13.15 that the new multiplication $*^q$ on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ coincides with the quantum multiplication introduced by Givental and Lee in [G, GL]. By taking the $h \to 0$ limit of this conjectural statement, we formulate in Section 13.6 a conjectural description of the equivariant quantum K-theory algebra of the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ by generators and relations.

In Appendix 1, we show how the weight functions specialize to Grothendieck polynomials introduced by Lascoux and Schutzenberger in [LS]. In Appendix 2, we give an interpolation definition of the K-theory classes of Schubert varieties in the equivariant K-theory algebra $K_{(\mathbb{C}^\times)^n}(\mathcal{F}_{\boldsymbol{\lambda}})$ of a partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$. This definition is the $h \to 0$ limit of the definition of the classes $\kappa_{\sigma,I} \in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. In Appendix 3, we describe presentations and structure constants of the K-theory algebras associated with the projective line. Some of them follows from the conjecture in Section 13.5 and others are known. In Appendix 4, we give formulae for multiplication of the classes $\kappa_{\mathrm{id},I}$. These formulae can be viewed as a beginning of Schubert calculus on $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$.

In Appendix 5 we describe the Bethe algebra of the XXZ model by generators and relations. In particular we show that the Bethe algebra of the XXZ model on $(\mathbb{C}^N)^{\otimes n}$ is a maximal commutative subalgebra of $\mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big)$.

2. Preliminaries from Geometry

2.1. **Partial flag varieties.** Fix natural numbers N, n. Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = \lambda_1 + \ldots + \lambda_N = n$. Consider the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \ldots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \ldots, N$. Denote by $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ the cotangent bundle of $\mathcal{F}_{\boldsymbol{\lambda}}$, and let $\pi : T^*\mathcal{F}_{\boldsymbol{\lambda}} \to \mathcal{F}_{\boldsymbol{\lambda}}$ be the projection of the bundle. Denote

$$\mathcal{X}_n = \coprod_{|\boldsymbol{\lambda}|=n} T^*\mathcal{F}_{\boldsymbol{\lambda}}\,.$$

Example. If $n = 1$, then $\boldsymbol{\lambda} = (0, \ldots, 0, 1_i, 0, \ldots, 0)$, $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is a point and $\mathcal{X}_1$ is the union of N points.

If $n = 2$ then $\boldsymbol{\lambda} = (0, \ldots, 0, 1_i, 0, \ldots, 0, 1_j, 0, \ldots, 0)$ or $\boldsymbol{\lambda} = (0, \ldots, 0, 2_i, 0, \ldots, 0)$. In the first case $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is the cotangent bundle of projective line, in the second case $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is a point. Thus $\mathcal{X}_2$ is the union of N points and $N(N-1)/2$ copies of the cotangent bundle of projective line.

Let $I = (I_1, \ldots, I_N)$ be a partition of $\{1, \ldots, n\}$ into disjoint subsets $I_1, \ldots, I_N$. Denote $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all partitions I with $|I_j| = \lambda_j$, $j = 1, \ldots, N$.

Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of $\mathbb{C}^n$. For any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, let $x_I \in \mathcal{F}_{\boldsymbol{\lambda}}$ be the point corresponding to the coordinate flag $F_1 \subset \ldots \subset F_N$, where F_i is the span of the standard basis vectors $\epsilon_j \in \mathbb{C}^n$ with $j \in I_1 \cup \ldots \cup I_i$. We embed $\mathcal{F}_{\boldsymbol{\lambda}}$ in $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ as the zero section and consider the points x_I as points of $T^*\mathcal{F}_{\boldsymbol{\lambda}}$.

2.2. **Schubert cells, conormal bundles.** For any $\sigma \in S_n$, we consider the coordinate flag in $\mathbb{C}^n$,

$$V^\sigma \;:\; 0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n$$

where V_i is the span of $\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(i)}$. For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ we define the *Schubert cell*

$$\Omega_{\sigma,I} = \{F \in \mathcal{F}_{\boldsymbol{\lambda}} \mid \dim(F_p \cap V^\sigma_q) = \#\{i \in I_1 \cup \ldots \cup I_p \mid \sigma^{-1}(i) \leqslant q\} \;\forall p \leqslant N, \forall q \leqslant n\}.$$

The Schubert cell $\Omega_{\sigma,I}$ is an affine space of dimension

$$\ell_{\sigma,I} = \#\{(i,j) \in \{1, \ldots, n\}^2 \mid \sigma(i) \in I_a,\ \sigma(j) \in I_b,\ a < b,\ i > j\}.$$

For a fixed σ, the flag manifold is the disjoint union of the cells $\Omega_{\sigma,I}$. We have $x_I \in \Omega_{\sigma,I}$, see e.g. [FP, Sect.2.2].

For $\sigma \in S_n$, we define the *geometric* partial ordering on the set $\mathcal{I}_{\boldsymbol{\lambda}}$. For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, we say that $J \leqslant_g I$ if x_J lies in the closure of $\Omega_{\sigma,I}$.

We also define the *combinatorial* partial ordering. For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, let

$$\sigma^{-1}\Big(\bigcup_{\ell=1}^{k} I_\ell\Big) \;=\; \{a_1^k < \ldots < a_{\lambda^{(k)}}^k\}, \qquad \sigma^{-1}\Big(\bigcup_{\ell=1}^{k} J_\ell\Big) \;=\; \{b_1^k < \ldots < b_{\lambda^{(k)}}^k\}$$

for $k = 1, \ldots, N-1$, where $\lambda^{(k)} = \lambda_1 + \ldots + \lambda_k$. We say that $J \leqslant_c I$ if $b_i^k \leqslant a_i^k$ for $k = 1, \ldots, N-1$, $i = 1, \ldots, \lambda^{(k)}$.

Lemma 2.1. *The geometric and combinatorial partial orderings are the same.*

Proof. This is the so-called "Tableau Criterion" for the Bruhat (i.e. geometric) order, see e.g. [BB, Theorem 2.6.3]. □

In what follows we will denote both partial orderings by $\leqslant_\sigma$.

The Schubert cell $\Omega_{\sigma,I}$ is a smooth submanifold of $\mathcal{F}_{\boldsymbol{\lambda}}$, hence we can consider its conormal space

$$C\Omega_{\sigma,I} = \{\alpha \in \pi^{-1}(\Omega_{\sigma,I}) \mid \alpha(T_{\pi(\alpha)}\Omega_{\sigma,I}) = 0\} \subset T^*\mathcal{F}_{\boldsymbol{\lambda}}\,.$$

The conormal space $C\Omega_{\sigma,I}$ is the total space of a vector subbundle of $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ over $\Omega_{\sigma,I}$. The rank of this subbundle is $\dim \mathcal{F}_{\boldsymbol{\lambda}} - \dim \Omega_{\sigma,I}$. Hence, as a manifold $C\Omega_{\sigma,I}$ is an affine cell of dimension $\dim \mathcal{F}_{\boldsymbol{\lambda}}$. In particular, the dimension is independent of σ, I. Define

$$\mathrm{Slope}_{\sigma,I} \;=\; \bigcup_{J \leqslant_\sigma I} C\Omega_{\sigma,J}. \tag{2.1}$$

2.3. **Equivariant K-theory.** The diagonal action of the torus $(\mathbb{C}^\times)^n$ on $\mathbb{C}^n$ induces an action on $\mathcal{F}_{\boldsymbol{\lambda}}$, and hence on the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$.

Remark. One may use this action to give an equivalent ("unstable submanifold") definition of the spaces $C\Omega_{\sigma,I}$ of the last section. Namely $x \in C\Omega_{\sigma,I}$ if and only if

$$\lim_{z\to 0}(z^{-\sigma(1)}, z^{-\sigma(2)}, \ldots, z^{-\sigma(n)}) \cdot x = x_I,$$

cf. [MO, Section 3.2.2].

We extend this $(\mathbb{C}^\times)^n$-action to the action of $T = (\mathbb{C}^\times)^n \times \mathbb{C}^\times$ so that the extra $\mathbb{C}^\times$ acts on the fibers of $T^*\mathcal{F}_{\boldsymbol{\lambda}} \to \mathcal{F}_{\boldsymbol{\lambda}}$ by multiplication.

We consider the equivariant K-theory algebras $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ and

$$K_T(\mathcal{X}_n) \;=\; \bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}). \tag{2.2}$$

Our general reference for equivariant K-theory is [ChG, Ch.5].

Denote $S_{\boldsymbol{\lambda}} = S_{\lambda_1}\times\ldots\times S_{\lambda_N}$ the product of symmetric groups. Consider variables $\Gamma_i = \{\gamma_{i,1}, \ldots, \gamma_{i,\lambda_i}\}$, $i = 1, \ldots, N$. Let $\boldsymbol{\Gamma} = (\Gamma_1; \ldots; \Gamma_N)$. The group $S_{\boldsymbol{\lambda}}$ acts on the set $\boldsymbol{\Gamma}$ by permuting the variables with the same first index. Let $\mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]$ be the algebra of Laurent polynomials in variables $\gamma_{i,j}$ and $\mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]^{S_{\boldsymbol{\lambda}}}$ the subalgebra of invariants with respect to the $S_{\boldsymbol{\lambda}}$-action.

Consider variables $\boldsymbol{z}=\{z_1,\dots,z_n\}$ and h. The group S_n acts on the set $\boldsymbol{z}$ by permutations. Let $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$ be the algebra of Laurent polynomials in variables $\boldsymbol{z},h$ and $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_n}$ the subalgebra of invariants with respect to the S_n-action. We have

$$K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \,=\, \mathbb{C}[\boldsymbol{\Gamma}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]\,\big/\big\langle f(\boldsymbol{\Gamma})=f(\boldsymbol{z}) \quad \text{for any } f\in\mathbb{C}[\boldsymbol{z}^{\pm1}]^{S_n}\big\rangle\,. \tag{2.3}$$

Here $\gamma_{i,j}$ correspond to (virtual) line bundles also denoted by $\gamma_{i,j}$ with

$$\bigoplus_{j=1}^{\lambda_i}\gamma_{i,j} \,=\, F_i/F_{i-1}$$

while z_a and h correspond to the factors of $T=(\mathbb{C}^\times)^n\times\mathbb{C}^\times$.

The algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ is a module over $K_T(pt;\mathbb{C})=\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$.

Example. If $n=1$, then

$$K_T(\mathcal{X}_1) \,=\, \bigoplus_{i=1}^{N} K_T(T^*\mathcal{F}_{(0,\dots,0,1_i,0,\dots,0)})$$

is naturally isomorphic to $\mathbb{C}^N\otimes\mathbb{C}[z_1^{\pm1},h^{\pm1}]$ with the basis $v_i=(0,\dots,0,1_i,0,\dots,0)$, $i=1,\dots,N$.

2.4. **Fixed point sets.** The set $(T^*\mathcal{F}_{\boldsymbol{\lambda}})^T$ of fixed points of the torus T action is $(x_I)_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$. We have

$$(\mathcal{X}_n)^T=\mathcal{X}_1\times\ldots\times\mathcal{X}_1.$$

The algebra $K_T\big((\mathcal{X}_n)^T\big)$ is naturally isomorphic to $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$. This isomorphism sends the identity element $1_I\in K_T(x_I)$ to the vector

$$v_I \,=\, v_{i_1}\otimes\ldots\otimes v_{i_n}\,, \tag{2.4}$$

where $i_j=k$ if $j\in I_k$. We denote by $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ the span of $\{v_I\mid I\in\mathcal{I}_{\boldsymbol{\lambda}}\}$.

2.5. **Equivariant localization.** Consider the *equivariant localization* map

$$\mathrm{Loc}\,:\,K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\,\to\,K_T((T^*\mathcal{F}_{\boldsymbol{\lambda}})^T)\,=\,\bigoplus_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}K_T(x_I) \tag{2.5}$$

whose components are the restrictions to the fixed points x_I. Namely, the I-component Loc_I of this map is the substitution

$$\{\gamma_{k,1},\dots,\gamma_{k,\lambda_k}\}\mapsto\{z_a\mid a\in I_k\}\qquad\text{for all } k=1,\dots,N. \tag{2.6}$$

Equivariant localization theory (see e.g. [ChG, Ch.5], [RoKu, Appendix]) asserts that Loc is an injection of algebras. Moreover, an element of the right-hand side is in the image of Loc if the difference of the I-th and $s_{i,j}(I)$-th components is divisible by $1-z_i/z_j$ in $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$ for all $I\in\mathcal{I}_{\boldsymbol{\lambda}}$ and $i,j\in\{1,\dots,n\}$. Here $s_{i,j}(I)$ is the partition obtained from I by switching the numbers i and j.

Let $\mathbb{C}(\boldsymbol{z},h)$ be the algebra of rational functions in $z_1,\dots,z_n,h$. The map

$$\mathrm{Loc}\;:\;K_T(\mathcal{F}_{\boldsymbol{\lambda}})\otimes\mathbb{C}(\boldsymbol{z},h)\xrightarrow{\ \cong\ }\oplus_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}K_T(x_I)\otimes\mathbb{C}(\boldsymbol{z},h) \tag{2.7}$$

is an isomorphism.

2.6. **K-theory fundamental class of $\Omega_{\sigma,I}$ at x_I.** Define the following classes

$$e^{hor}_{\sigma,I,+}=\prod_{k<l}\prod_{\sigma(a)\in I_k}\prod_{\substack{\sigma(b)\in I_l\\ b<a}}\left(1-z_{\sigma(a)}/z_{\sigma(b)}\right),\qquad e^{hor}_{\sigma,I,-}=\prod_{k<l}\prod_{\sigma(a)\in I_k}\prod_{\substack{\sigma(b)\in I_l\\ b>a}}\left(1-z_{\sigma(a)}/z_{\sigma(b)}\right),\tag{2.8}$$

$$e^{vert}_{\sigma,I,+}=\prod_{k<l}\prod_{\sigma(a)\in I_k}\prod_{\substack{\sigma(b)\in I_l\\ b>a}}\left(1-hz_{\sigma(b)}/z_{\sigma(a)}\right),\qquad e^{vert}_{\sigma,I,-}=\prod_{k<l}\prod_{\sigma(a)\in I_k}\prod_{\substack{\sigma(b)\in I_l\\ b<a}}\left(1-hz_{\sigma(b)}/z_{\sigma(a)}\right),\tag{2.9}$$

in $K_T(x_I)=\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$. We also set $e_{\sigma,I}=e^{hor}_{\sigma,I,-}\,e^{vert}_{\sigma,I,-}$.

Recall that if $\mathbb{C}^\times$ acts on a line $\mathbb{C}$ by $\alpha\cdot x=\alpha^r x$, then the $\mathbb{C}^\times$-equivariant Euler class of the line bundle $\mathbb{C}\to\{0\}$ is $e(\mathbb{C}\to\{0\})=1-z^r\in K_{\mathbb{C}^\times}(\text{point})=\mathbb{C}[z^{\pm1}]$. Thus standard knowledge on the tangent bundle of flag manifolds imply that

$$e(T\Omega_{\sigma,I}|_{x_I})\,=\,e^{hor}_{\sigma,I,+}\,,\qquad e(\nu(\Omega_{\sigma,I}\subset\mathcal{F}_{\boldsymbol{\lambda}})|_{x_I})\,=\,e^{hor}_{\sigma,I,-}\,,\tag{2.10}$$

where $\nu(A\subset B)$ means the normal bundle of a submanifold A in the ambient manifold B, and $\xi|_x$ means the restriction of the bundle ξ over the point x in the base space. Therefore we also have

$$e(C\Omega_{\sigma,I}|_{x_I})\,=\,e^{vert}_{\sigma,I,+}\,,\qquad e((\pi^{-1}(\Omega_{\sigma,I})-C\Omega_{\sigma,I})|_{x_I})\,=\,e^{vert}_{\sigma,I,-}\,,$$

where $C\Omega_{\sigma,I}$ and $\pi^{-1}(\Omega_{\sigma,I})$ are considered bundles over $\Omega_{\sigma,I}$. Now consider $C\Omega_{\sigma,I}$ as a(n open) submanifold of $T^*\mathcal{F}_{\boldsymbol{\lambda}}$. Then we obtain

$$e(\nu(C\Omega_{\sigma,I}\subset T^*\mathcal{F}_{\boldsymbol{\lambda}})|_{x_I})\,=\,e^{hor}_{\sigma,I,-}\,e^{vert}_{\sigma,I,-}\,=\,e_{\sigma,I}\,.\tag{2.11}$$

3. Axiomatic definition of the $\kappa_{\sigma,I}$ classes

3.1. **Main result.** In this section we phrase a theorem that axiomatically defines some special classes in $K_T(\mathcal{F}_{\boldsymbol{\lambda}})$.

Define the *polarization* of $\sigma\in S_n$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, to be

$$P_{\sigma,I}\,=\prod_{k<l}\prod_{\sigma(a)\in I_k}\prod_{\substack{\sigma(b)\in I_l\\ b>a}}\left(-z_{\sigma(b)}/z_{\sigma(a)}\right).\tag{3.1}$$

Observe that $P_{\sigma,I}$ is an invertible element of $K_T(x_I)$. It is the inverse of the "top" term of $e^{hor}_{\sigma,I,-}$. In particular, the number of $-z_{\sigma(b)}/z_{\sigma(a)}$ factors is the codimension of $\Omega_{\sigma,I}$ in $\mathcal{F}_{\boldsymbol{\lambda}}$, see (2.10). The quantity

$$P_{\sigma,I}\,e_{\sigma,I}\,=\prod_{k<l}\prod_{\sigma(a)\in I_k}\Bigg(\prod_{\substack{\sigma(b)\in I_l\\ b<a}}\left(1-hz_{\sigma(b)}/z_{\sigma(a)}\right)\prod_{\substack{\sigma(b)\in I_l\\ b>a}}\left(1-z_{\sigma(b)}/z_{\sigma(a)}\right)\Bigg)\tag{3.2}$$

will play a role below.

For a Laurent polynomial $f\in\mathbb{C}[\boldsymbol{z}^{\pm1}]$ let $N(f)\subset\mathbb{R}^n$ be the Newton polygon of f, that is the convex hull of the points $m\in\mathbb{Z}^n\subset\mathbb{R}^n$ such that the coefficient of $\prod z_a^{m_a}$ in f is not 0. For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, consider the linear map $\varphi_I:\mathbb{R}^n\to\mathbb{R}$ defined by

$$\varphi_I(\epsilon_a)=k\text{ if }a\in I_k.$$

We will study the convex set (closed interval) $\varphi_I(N(f)) \subset \mathbb{R}$ for certain f's. For example $\varphi_{(\{1\},\{2\})}(N(1-z_2/z_1)) = [0,1]$. Observe that

$$\varphi_I(N(P_{\sigma,I}\, e_{\sigma,I})) \;=\; \Big[0\,,\ \sum_{1\leqslant k<l\leqslant N} \lambda_k\lambda_l\,(l-k)\Big] \tag{3.3}$$

is independent of σ and I; it only depends on $\boldsymbol{\lambda}$. Define an element $f \in K_T(x_I)$ to be I-*small* if

$$\varphi_I(N(f)) \subset \Big[0\,,\ \sum_{1\leqslant k<l\leqslant N} \lambda_k\lambda_l\,(l-k) - 1\Big]\,. \tag{3.4}$$

Note that in our Newton polygon considerations $h \in K_T(x_I)$ is considered a constant, not a variable.

Theorem 3.1. *Let $n, N, \boldsymbol{\lambda}$ be as above, and $\sigma \in S_n$. Then for any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, there exists a unique element $\kappa_{\sigma,I} \in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ satisfying the following conditions:*

(I) *$\kappa_{\sigma,I}|_{x_J}$ is divisible by $e^{vert}_{\sigma,J,-}$ for all J;*
(II) *$\kappa_{\sigma,I}|_{x_I} = P_{\sigma,I}\, e_{\sigma,I}$;*
(III) *$\kappa_{\sigma,I}|_{x_J}$ is J-small if $J \neq I$.*

Remark. From the proof of Theorem 3.1 it will turn out that the condition

(0) $\kappa_{\sigma,I}|_{x_J} = 0$ if $J \not\leqslant_\sigma I$

also holds. Condition (0) together with property (I) is a "localized" version of the statement

(I$'$) $\kappa_{\sigma,I}$ is supported on $\mathrm{Slope}_{\sigma,I}$.

Indeed, if a class is supported on $\mathrm{Slope}_{\sigma,I}$ then (0) and (I) follow, see Section 2.6. Condition (I$'$) appeared in [MO] and [RTV].

First we prove uniqueness in Section 3.2 following the arguments of [MO] in which we replace the property of smallness of the degrees of restrictions $\kappa_{\sigma,I}|_{x_J}$ by the smallness of their Newton polygons $N(\kappa_{\sigma,I}|_{x_J})$.

One novelty of our treatment is that our axioms (I)-(III) are all *local* properties of $\kappa_{\sigma,I}$ (unlike (I$'$)). That is, they are properties of fix point restrictions. Another key novelty of our treatment is Section 6.3 where we will show existence by giving a *formula* for $\kappa_{\sigma,I}$. Moreover, this formula for $\kappa_{\sigma,I}$ is a version of the trigonometric weight functions that had appeared in [TV1] – [TV3] in hypergeometric solutions of qKZ equations.

3.2. **Proof of uniqueness in Theorem 3.1.** Suppose $\kappa_{\sigma,I}$ and $\kappa'_{\sigma,I}$ both satisfy the conditions. Let ω be their difference.

Refine the partial order $\leqslant_\sigma$ to a total order $\preceq$ on $\mathcal{I}_{\boldsymbol{\lambda}}$.

Assume $\omega \neq 0$. Since the Loc map of Section 2.5 is injective there is at least one $J \in \mathcal{I}_{\boldsymbol{\lambda}}$ such that the restriction $\omega|_{x_J} \neq 0$. Let J be the $\preceq$-largest such element of $\mathcal{I}_{\boldsymbol{\lambda}}$.

We claim that $\omega|_{x_J}$ is

- J-small,
- divisible by $e^{vert}_{\sigma,J,-}$,
- divisible by $e^{hor}_{\sigma,J,-}$.

The first two properties follow explicitly from conditions (I), (II), (III).

To prove the third property choose a pair $a<b$ such that $\sigma(a)\in J_k,\sigma(b)\in J_l$ and $k<l$. Let $U=s_{\sigma(a),\sigma(b)}(J)\in\mathcal{I}_{\boldsymbol\lambda}$. From the definition of $\leqslant_\sigma$ it follows that $J<_\sigma U$ and hence $J\prec U$. The choice of J implies that $\omega|_{x_U}=0$. Differences of components of the Loc map satisfy divisibility conditions, see Section 2.5, hence we obtain that $\omega|_{x_J}$ is divisible by $1-z_{\sigma(a)}/z_{\sigma(b)}$. This argument holds for all a,b with $\sigma(a)\in J_k,\sigma(b)\in J_l$, $k<l$ hence we proved the third claim.

A Laurent polynomial divisible by $e^{vert}_{\sigma,J,-}$ and $e^{hor}_{\sigma,J,-}$ must be divisible by their product too. Comparing (3.3) and (3.4) we see that a J-small class divisible by $e_{\sigma,J}=e^{hor}_{\sigma,J,-}e^{vert}_{\sigma,J,-}$ must be 0. Hence $\omega|_{x_J}$ is 0. This is a contradiction, which proves that $\omega=0$.

4. Trigonometric weight functions

4.1. **Definition.** Let $n,N\in\mathbb{Z}_{\geqslant0}$ and let $\boldsymbol\lambda\in\mathbb{Z}^N_{\geqslant0}$ be such that $\sum_{k=1}^N\lambda_k=n$. Set $\lambda^{(k)}=\lambda_1+\ldots+\lambda_k$ for $k=0,\ldots,N$ and $\lambda^{\{1\}}=\lambda^{(1)}+\ldots+\lambda^{(N-1)}$.

Recall that the set of partitions $I=(I_1,\ldots,I_N)$ of $\{1,\ldots,n\}$ with $|I_k|=\lambda_k$ is denoted by $\mathcal{I}_{\boldsymbol\lambda}$. For $I\in\mathcal{I}_{\boldsymbol\lambda}$ we will use the notation $\bigcup_{a=1}^k I_a=\{i^{(k)}_1<\ldots<i^{(k)}_{\lambda^{(k)}}\}$.

Consider variables $t^{(k)}_a$ for $k=1,\ldots,N$, $a=1,\ldots,\lambda^{(k)}$, where $t^{(N)}_a=z_a, a=1,\ldots,n$. Denote $t^{(j)}=(t^{(j)}_k)_{k\leqslant\lambda^{(j)}}$ and $\boldsymbol t=(t^{(1)},\ldots,t^{(N-1)})$.

For $I\in\mathcal{I}_{\boldsymbol\lambda}$, define the *trigonometric weight function*

$$W_I(\boldsymbol t,\boldsymbol z,h)\;=\;(1-h)^{\lambda^{\{1\}}}\,\mathrm{Sym}_{t^{(1)}}\ldots\mathrm{Sym}_{t^{(N-1)}}U_I\,, \tag{4.1}$$

where

$$U_I\;=\;\prod_{k=1}^{N-1}\prod_{a=1}^{\lambda^{(k)}}\Bigl(\prod_{\substack{c=1\\ i^{(k+1)}_c<i^{(k)}_a}}^{\lambda^{(k)}}\bigl(1-ht^{(k+1)}_c/t^{(k)}_a\bigr)\prod_{\substack{c=1\\ i^{(k+1)}_c>i^{(k)}_a}}^{\lambda^{(k)}}\bigl(1-t^{(k+1)}_c/t^{(k)}_a\bigr)\prod_{b=a+1}^{\lambda^{(k)}}\frac{1-ht^{(k)}_b/t^{(k)}_a}{1-t^{(k)}_b/t^{(k)}_a}\Bigr), \tag{4.2}$$

and $\mathrm{Sym}_{t^{(k)}}$ is the symmetrization with respect to the variables $t^{(k)}_1,\ldots,t^{(k)}_{\lambda^{(k)}}$,

$$\mathrm{Sym}_{t^{(k)}}f\bigl(t^{(k)}_1,\ldots,t^{(k)}_{\lambda^{(k)}}\bigr)\;=\;\sum_{\sigma\in S_{\lambda^{(k)}}}f\bigl(t^{(k)}_{\sigma(1)},\ldots,t^{(k)}_{\sigma(\lambda^{(k)})}\bigr).$$

The trigonometric weight functions are Laurent polynomials in the $\boldsymbol t,\boldsymbol z,h$ variables since the factors $1-t^{(k)}_b/t^{(k)}_a$ in the denominator cancel out in the symmetrization.

For $\sigma\in S_n$ and $I\in\mathcal{I}_{\boldsymbol\lambda}$, define the *trigonometric weight function*

$$W_{\sigma,I}(\boldsymbol t,\boldsymbol z,h)\;=\;W_{\sigma^{-1}(I)}(\boldsymbol t,z_{\sigma(1)},\ldots,z_{\sigma(n)},h)\,, \tag{4.3}$$

where $\sigma^{-1}(I)=\bigl(\sigma^{-1}(I_1),\ldots,\sigma^{-1}(I_N)\bigr)$. Hence, $W_I=W_{\mathrm{id},I}$.

Remark. The weight functions were described in [TV1] to solve the *qKZ* equations and describe eigenvectors of the Hamiltonians of the *XXZ*-type integrable models. Namely, the

$(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of the qKZ equations have the form

$$I(\boldsymbol{z}) \,=\, \int \Phi(\boldsymbol{t},\boldsymbol{z},h) \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} W_{\mathrm{id},I}(\boldsymbol{t},\boldsymbol{z},h)\, v_I\, d\boldsymbol{t}$$

where $\Phi(\boldsymbol{t},\boldsymbol{z},h)$ is a suitable scalar (master) function. If $\boldsymbol{t}$ satisfies Bethe ansatz equations, the vector $\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} W_{\mathrm{id},I}(t,z,h)\, v_I$ becomes an eigenvector of the Hamiltonians of the XXZ-type model. Convenient formulae for the weight functions were suggested in [TV3]. The weight functions $W_{\mathrm{id},I}$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, in this paper are Laurent polynomials. They differ from the corresponding weight functions in [TV3], that are rational functions, by a common factor independent of I.

4.2. **Remarks.** In this subsection we first give an alternative formula for the weight function, and then define two modified versions. Strictly speaking none of these are necessary in the rest of the paper, they just help the reader to get familiar with weight functions.

Consider variables $u^{(k)}_a$ for $k=1,\dots,N$, $a=1,\dots,\lambda^{(k)}$. Define

$$U'_I \,=\, \prod_{k=1}^{N-1}\prod_{a,b=1}^{\lambda^{(k)}} \big(1-t^{(k)}_a/u^{(k)}_b\big)^{-1} \times$$
$$\times \prod_{k=1}^{N-1}\prod_{a=1}^{\lambda^{(k)}} \Bigg(\prod_{\substack{c=1\\ i^{(k+1)}_c<i^{(k)}_a}}^{\lambda^{(k)}} (1-hu^{(k+1)}_c/u^{(k)}_a) \prod_{\substack{c=1\\ i^{(k+1)}_c>i^{(k)}_a}}^{\lambda^{(k)}} (1-u^{(k+1)}_c/u^{(k)}_a) \prod_{b=a+1}^{\lambda^{(k)}} \frac{1-hu^{(k)}_b/u^{(k)}_a}{1-u^{(k)}_a/u^{(k)}_b}\Bigg).$$

Replace in U' each variable $u^{(N)}_a$ with z_a, $a=1,\dots,n$. The obtained function U'' depends on the variables $\boldsymbol{t},\boldsymbol{z},h$ and $u^{(k)}_a$, $k=1,\dots,N-1$, $a=1,\dots,\lambda^{(k)}$.

Theorem 4.1. *We have*

$$W_I(\boldsymbol{t},\boldsymbol{z},h) = (h-1)^{\lambda^{\{1\}}}\, \mathrm{Res}\Big(U''_I\cdot \prod_{k=1}^{N-1}\prod_{a=1}^{\lambda^{(k)}} du^{(k)}_a/u^{(k)}_a\Big), \tag{4.4}$$

where Res *is the iterated application of the* $\mathrm{Res}_{u^{(k)}_a=0}+\mathrm{Res}_{u^{(k)}_a=\infty}$ *operations for all* $k=1,\dots,N-1$, $a=1,\dots,\lambda^{(k)}$ *(in arbitrary order).*

Proof. Consider the right-hand side of (4.4). Applying the Residue Theorem for each $u^{(k)}_a$, we obtain that this is equal to

$$(h-1)^{\lambda^{\{1\}}}(-1)^{\lambda^{\{1\}}} \sum_{\sigma_1,\dots,\sigma_{N-1}} \mathrm{Res}_{u^{(k)}_a=t^{(k)}_{\sigma(a)}} \Big(U''_I\cdot \prod_{k=1}^{N-1}\prod_{a=1}^{\lambda^{(k)}} du^{(k)}_a/u^{(k)}_a\Big),$$

where σ_k is a map $\{1,\dots,\lambda^{(k)}\}\to\{1,\dots,\lambda^{(k)}\}$ for each k. First observe that the term corresponding to $\sigma_1,\dots,\sigma_{N-1}$ is zero unless each of the σ_k's are permutations — this is essentially due to the factors $(1-u^{(k)}_a/u^{(k)}_b)$ in the numerator of U'_I. The term corresponding to permutations σ_k are exactly the analogous terms in the definition of weight functions. $\square$

The following modified version of weight functions are sometimes useful.

Definition. For $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$ define $q(\boldsymbol{\lambda})$ to be the greatest q with $\lambda_q > 0$. Let $M_1 W_I$ be obtained from W_I by substituting $t^{(k)}_i = z_i$ for all $k \geqslant q(\boldsymbol{\lambda})$ and all i.

Definition. Consider variables $\gamma_{k,a}$ for $k = 1, \dots, q(\boldsymbol{\lambda}) - 1$, $a = 1, \dots, \lambda_k$. Let $M_2 W_I$ be obtained from $M_1 W_I$ by carrying out the substitution

$$(t^{(k)}_1, \dots, t^{(k)}_{\lambda^{(k)}}) \quad\mapsto\quad (\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}, \gamma_{2,1}, \dots, \gamma_{2,\lambda_2}, \dots, \gamma_{k,1}, \dots, \gamma_{k,\lambda_k})$$

for all $k < q(\boldsymbol{\lambda})$.

Here are some examples of weight functions.

- For $N = 2$, $n = 2$, $\boldsymbol{\lambda} = (1, 1)$, we have

$$W_{(\{1\},\{2\})} = (1-h)(1 - z_2/t^{(1)}_1), \qquad W_{(\{2\},\{1\})} = (1-h)(1 - hz_1/t^{(1)}_1).$$

  The residue formula (4.4) gives

$$\begin{aligned} W_{(\{1\},\{2\})} &= (h-1)\Big(\mathrm{Res}_{u^{(1)}_1=0} + \mathrm{Res}_{u^{(1)}_1=\infty}\Big)\Big(\frac{1 - z_2/u^{(1)}_1}{1 - t^{(1)}_1/u^{(1)}_1}\frac{du^{(1)}_1}{u^{(1)}_1}\Big), \\ W_{(\{2\},\{1\})} &= (h-1)\Big(\mathrm{Res}_{u^{(1)}_1=0} + \mathrm{Res}_{u^{(1)}_1=\infty}\Big)\Big(\frac{1 - hz_1/u^{(1)}_1}{1 - t^{(1)}_1/u^{(1)}_1}\frac{du^{(1)}_1}{u^{(1)}_1}\Big). \end{aligned}$$

- For $N = 2$, $\boldsymbol{\lambda} = (1, n-1)$, we have

$$W_{(\{i\},\{1,\dots,n\}-\{i\})} = (1-h)\prod_{j=1}^{i-1}(1 - hz_j/t^{(1)}_1)\prod_{j=i+1}^{n}(1 - z_j/t^{(1)}_1).$$

- Let $\boldsymbol{\lambda} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^N_{\geqslant 0}$, where the nonzero coordinates are at positions i and j, $i < j$. The set $\mathcal{I}_{\boldsymbol{\lambda}}$ consists of two elements

$$\begin{aligned} I &= (\{\}, \dots, \{\}, \{1\}, \{\}, \dots, \{\}, \{2\}, \{\}, \dots, \{\}), \\ J &= (\{\}, \dots, \{\}, \{2\}, \{\}, \dots, \{\}, \{1\}, \{\}, \dots, \{\}). \end{aligned}$$

  We have

$$\begin{aligned} M_1 W_I &= (1-h)^{2N-i-j}(1 - hz_2/z_1)^{N-j}(1 - hz_1/z_2)^{N-j}(1 - z_2/t^{(j-1)}_1), \\ M_1 W_J &= (1-h)^{2N-i-j}(1 - hz_2/z_1)^{N-j}(1 - hz_1/z_2)^{N-j}(1 - hz_1/t^{(j-1)}_1), \\ M_2 W_I &= (1-h)^{2N-i-j}(1 - hz_2/z_1)^{N-j}(1 - hz_1/z_2)^{N-j}(1 - z_2/\gamma_{i,1}), \\ M_2 W_J &= (1-h)^{2N-i-j}(1 - hz_2/z_1)^{N-j}(1 - hz_1/z_2)^{N-j}(1 - hz_1/\gamma_{i,1}). \end{aligned}$$

- We have

$$\begin{aligned} M_2 W_{(\{1\},\{2\},\{3\})} = {} & (1-h)^3 (1 - z_2/\gamma_{1,1})(1 - z_3/\gamma_{1,1}) \times \\ & \times (1 - z_3/\gamma_{2,1})(1 - hz_1/\gamma_{2,1})(1 - h\gamma_{2,1}/\gamma_{1,1}), \end{aligned}$$

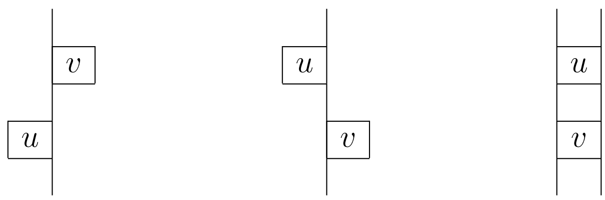
  although $W_{(\{1\},\{2\},\{3\})}$ does not factor into analogous simple factors.

5. Combinatorics of terms of the weight function

In this section we show a diagrammatic interpretation of the rich combinatorics encoded in the weight function. Let $I \in \mathcal{I}_{\boldsymbol{\lambda}}$. Consider a table with n rows and N columns. Number the rows from top to bottom and number the columns from left to right. Certain boxes of this table will be distinguished, as follows. In the first column distinguish boxes in the i'th row if $i \in I_1$, in the second column distinguish boxes in the i'th row if $i \in I_1 \cup I_2$, etc. This way all the boxes in the last column will be distinguished since $I_1 \cup \ldots \cup I_N = \{1, \ldots, n\}$.

Now we will define fillings of the tables by putting various variables in the distinguished boxes. First, put the variables $z_1, \ldots, z_n$ into the last column from top to bottom. Now choose permutations $\sigma_1 \in S_{\lambda^{(1)}}$, $\sigma_2 \in S_{\lambda^{(2)}}$, ..., $\sigma_{N-1} \in S_{\lambda^{(N-1)}}$. Put the variables $t^{(k)}_{\sigma_k(1)}, \ldots, t^{(k)}_{\sigma_k(\lambda^{(k)})}$ in the k'th column from top to bottom.

Each such filled table will define a rational function as follows. Let u be a variable in the filled table in one of the columns $1, \ldots, N-1$. If v is a variable in the next column, but above the position of u then consider the factor $1 - hv/u$ ('type-1 factor'). If v is a variable in the next column, but below the position of u then consider the factor $1 - v/u$ ('type-2 factor'). If v is a variable in the same column, but below the position of u then consider the factor $(1 - hv/u)/(1 - v/u)$ ('type-3 factor'). The rule is illustrated in the following figure.



$$(1 - hv/u) \qquad\qquad (1 - v/u) \qquad\qquad \frac{(1 - hv/u)}{(1 - v/u)}$$

type-1 type-2 type-3

For each variable u in the table consider all these factors and multiply them together. This is "the term associated with the filled table".

One sees that W_I is the sum of terms associated with the filled tables corresponding to all choices $\sigma_1, \ldots, \sigma_{N-1}$. For example, $W_{\{2\},\{1\},\{3\}}$ is the sum of two terms associated with the filled tables

	$t^{(2)}_1$	z_1
$t^{(1)}_1$	$t^{(2)}_2$	z_2
		z_3

,

	$t^{(2)}_2$	z_1
$t^{(1)}_1$	$t^{(2)}_1$	z_2
		z_3

.

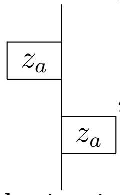
The term corresponding to the first filled table is hence

$$\underbrace{(1 - ht^{(2)}_1/t^{(1)}_1)(1 - hz_1/t^{(2)}_2)}_{type-1}\ \underbrace{(1 - z_2/t^{(2)}_1)(1 - z_3/t^{(2)}_1)(1 - z_3/t^{(2)}_2)}_{type-2}\ \underbrace{\frac{(1 - ht^{(2)}_2)}{(1 - t^{(2)}_1)}}_{type-3},$$

and the term corresponding to the second filled table is

$$\underbrace{(1-ht_2^{(2)}/t_1^{(1)})\,(1-hz_1/t_1^{(2)})}_{type-1}\ \underbrace{(1-z_2/t_2^{(2)})\,(1-z_3/t_2^{(2)})\,(1-z_3/t_1^{(2)})}_{type-2}\ \underbrace{\frac{(1-ht_1^{(2)})}{(1-t_2^{(2)})}}_{type-3}\ .$$

Note that variables in consecutive columns but in the same row do not produce a factor.

In the next section we will substitute z_i's into the $t_a^{(k)}$ variables according to some rules. Thus we obtain terms corresponding to tables filled with only z_a variables (no $t_a^{(k)}$'s). If in such a substitution we have a filled table containing , then the term corresponding to that table is 0. This phenomenon is behind the substitution lemmas of the next section.

6. Properties of weight functions

6.1. **Substitutions.** Recall that for $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ we use the notation $\cup_{a=1}^{k} I_a = \{i_1^{(k)} < \ldots < i_{\lambda^{(k)}}^{(k)}\}$. For a function $f(\boldsymbol{t},\boldsymbol{z},h)$, we denote $f(\boldsymbol{z}_I,\boldsymbol{z},h)$ the substitution under

$$t_a^{(k)} = z_{i_a^{(k)}} \qquad \text{for} \qquad k=1,\ldots,N, \quad a=1,\ldots,\lambda^{(k)}.$$

For a function $f(\boldsymbol{\Gamma},\boldsymbol{z},h)$, we denote $f(\boldsymbol{z}_I,\boldsymbol{z},h)$ the substitution under

$$\{\gamma_{k,a} \mid a=1,\ldots,\lambda_k\} \mapsto \{z_a \mid a \in I_k\},$$

cf. equivariant localization in Section 2.5.

Observe that the various substitutions are set up in such a way that

$$W_I(\boldsymbol{z}_J,\boldsymbol{z},h) = M_1 W_I(\boldsymbol{z}_J,\boldsymbol{z},h) = M_2 W_I(\boldsymbol{z}_J,\boldsymbol{z},h).$$

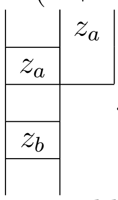
Define

$$E(\boldsymbol{t},h) \;=\; \prod_{k=1}^{N-1}\prod_{a=1}^{\lambda^{(k)}}\prod_{b=1}^{\lambda^{(k)}} \bigl(1-ht_b^{(k)}/t_a^{(k)}\bigr)\,. \tag{6.1}$$

Lemma 6.1. *For any $\sigma \in S_n$ and $I,J \in \mathcal{I}_{\boldsymbol{\lambda}}$, the function $W_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ is divisible by $E(\boldsymbol{z}_J,h)$ in the algebra of Laurent polynomials $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$.*

Proof. For notational simplicity we consider the case $\sigma = \mathrm{id}$. As explained in Section 5, $W_I(\boldsymbol{z}_J,\boldsymbol{z},h)$ is the sum of terms corresponding to certain tables filled with the variables z_a. Consider such a term, and the corresponding filled table. Let us fix $k \leqslant N-1$ and $a \neq b \in J_1 \cup \ldots \cup J_k$. In the next paragraph we will specify some positions in the filled table that are responsible for the appearance of the factors $(1-hz_a/z_b)(1-hz_b/z_a)$ in this term. These positions will be different for different triples (k,a,b).

Suppose z_a is above z_b in the k-th column. Then the type-3 factor $(1-hz_b/z_a)$ is the factor of this term, because both z_a and z_b are in the k-th column. Also, $z_a \in J_1 \cup \ldots \cup J_{k+1}$ and, hence, z_a is in the $(k+1)$-st column as well. If our term is nonzero, then the position of z_a in the $(k+1)$-st column is *weakly above* the position of z_a in the k-th column.

Hence the position of z_a in the $(k+1)$-st column is *strictly above* the position of z_b in the k-th column, as in the picture . Then the z_b variable in the k-th column and the z_a variable in the $(k+1)$-st column yield the type-1 factor $(1-hz_a/z_b)$.

The function $E(\boldsymbol{z}_J,h)$ is a product of factors $(1-hz_a/z_b)(1-hz_b/z_a)$ for certain triples $(k,a\neq b)$, and the factor $(1-h)^{\lambda^{\{1\}}}$. The argument above shows that $W_I(\boldsymbol{z}_J,\boldsymbol{z},h)$ is divisible by the desired product of factors $(1-hz_a/z_b)(1-hz_b/z_a)$. The factor $(1-h)^{\lambda^{\{1\}}}$ is explicit in the definition of the weight functions. $\square$

Define

$$\widetilde{W}_{\sigma,I}(\boldsymbol{t},\boldsymbol{z},h) \;=\; \frac{W_{\sigma,I}(\boldsymbol{t},\boldsymbol{z},h)}{E(\boldsymbol{t},h)}\,. \tag{6.2}$$

The function $\widetilde{W}_{\sigma,I}$ is not a Laurent *polynomial* in the $\boldsymbol{t}$ variables any more, but Lemma 6.1 asserts that all its $\boldsymbol{z}_J$-substitutions are Laurent polynomials in the $\boldsymbol{z},h$ variables.

Lemma 6.2. *We have* $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)=0$ *unless* $J\leqslant_\sigma I$.

Proof. If the condition $J\leqslant_\sigma I$ is not satisfied, then the table of every term in $W_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)=0$ contains a part described in the last paragraph of Section 5. Hence every term is 0, yielding $W_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)=0$ and $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)=0$. $\square$

Lemma 6.3. *For all* $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, *the function* $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ *is divisible by* $e^{vert}_{\sigma,J,-}$.

Proof. This proof is a continuation of the proof of Lemma 6.1, so, in particular, we focus on the special case $\sigma=\mathrm{id}$. Let us chose $a\in J_k$ and $b\in J_l$ with $k<l$, $a>b$. Consider again a term in $W_I(\boldsymbol{z}_J,\boldsymbol{z},h)$ and its filled table. Our goal is to specify a pair of variables in the table that produces the $1-hz_b/z_a$ factor.

From the variables z_a and z_b only z_a appears in the k-th column and both of them appear in the l-th column. We will study two cases.

Assume first that in the l-th column z_a is below z_b. Then, in the $(l-1)$-st column z_a is further below the position of z_b in the l-th column (otherwise the term equals 0). This pair, z_a in the $(l-1)$-st column and z_b in the l-th column is the desired pair — they produce the type-1 factor $1-hz_b/z_a$. This factor was not indicated and specified in the proof of Lemma 6.1.

Assume now that in the l-th column z_a is above z_b. Since $a>b$, their position is reversed in the N-th column. Hence there must exist a number s such that in the s-th column z_a is above z_b, and in the $(s+1)$-st column z_a is below z_b. Since z_a in the s-th column is below z_a in the $s+1$-th column (otherwise the term equals 0), we have that z_a in the s-th column and z_b in the $(s+1)$-st column is the desired pair — they produce the type-1 factor $1-hz_b/z_a$.

In both cases above we found positions in the filled table which are different from positions already "used" in the proof of Lemma 6.1. Hence we proved that $1-hz_b/z_a$ divides not only every non-zero term of $W_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$, but also $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$. $\square$

Lemma 6.4. *For* $I \in \mathcal{I}_{\boldsymbol{\lambda}}$*, we have* $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_I, \boldsymbol{z}, h) = P_{\sigma,I}\, e_{\sigma,I}$ *.*

Proof. In this case only one term of the symmetrization in (4.1) is nonzero, see Section 5. This term equals the right-hand side of the formula of the lemma. □

Recall the notion of $f(\boldsymbol{z}, h)$ being J-small from Section 3.

Lemma 6.5. *For all* $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$*,* $I \neq J$*, the function* $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h)$ *is* J*-small.*

Proof. By definition we have

$$W_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h) = (1-h)^{\lambda^{\{1\}}} \sum_{\boldsymbol{\pi}} \Big(\prod_{(a,b)} (1 - hz_a/z_b) \prod_{(c,d)} (1 - z_c/z_d) \prod_{(e,f)} \frac{1 - hz_e/z_f}{1 - z_e/z_f} \Big),$$

where the products are for certain pairs (a,b), (c,d), and (e,f), and the summation is for an $(N-1)$-tuple of permutations $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{N-1})$ with $\pi_k \in S_{\lambda^{(k)}}$. The last product can be rewritten as

$$\prod_{(e,f)} \frac{1 - hz_e/z_f}{1 - z_e/z_f} = \prod_{(e,f)} \frac{z_f - hz_e}{z_f - z_e}.$$

Denote

$$A_{I,J,\boldsymbol{\pi}} = \prod_{(e,f)} (z_f - hz_e)\,, \quad B_{I,J,\boldsymbol{\pi}} = \prod_{(e,f)} (z_f - z_e)\,, \quad C_{I,J,\boldsymbol{\pi}} = \prod_{(a,b)} (1 - hz_a/z_b) \prod_{(c,d)} (1 - z_c/z_d)\,.$$

Observe that $A_{I,J,\boldsymbol{\pi}}$ and $B_{I,J,\boldsymbol{\pi}}$ do not depend on I and for different $\boldsymbol{\pi}$'s the products $B_{I,J,\boldsymbol{\pi}}$ only differ possibly by a sign. Denote $A_{J,\boldsymbol{\pi}} = A_{I,J,\boldsymbol{\pi}}$ and $B_J = B_{I,J,\mathbf{id}}$. Then

$$B_J W_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h) = (1-h)^{\lambda^{\{1\}}} \sum_{\boldsymbol{\pi}} \pm C_{I,J,\boldsymbol{\pi}} A_{J,\boldsymbol{\pi}}. \tag{6.3}$$

If I were equal to J, then only one term of the summation is nonzero and

$$B_J W_{\sigma,J}(\boldsymbol{z}_J, \boldsymbol{z}, h) = (1-h)^{\lambda^{\{1\}}} C_{J,J,\mathbf{id}} A_{J,\mathbf{id}}. \tag{6.4}$$

That equation leads to the statement of Lemma 6.4. For $I \neq J$, we reason as follows.

Let $U_1 = [m_1, M_1]$ and $U_2 = [m_2, M_2]$ be closed intervals with $m_1, m_2, M_1, M_2 \in \mathbb{Z}$. For the purpose of this proof let $U_1 \ll U_2$ mean that $U_1 \subset [m_2, M_2 - 1]$. Also for Laurent polynomials f, g, let $f \ll_J g$ mean $\varphi_J(N(f)) \ll \varphi_J(N(g))$.

We claim that $C_{I,J,\boldsymbol{\pi}} \ll_J C_{J,J,\mathbf{id}}$ for all $\boldsymbol{\pi}$. Indeed, let

$$V = \{(k,a,b) \mid k = 1, \dots, N-1\,,\ \ a \in J_1 \cup \ldots \cup J_k\,,\ \ b \in J_1 \cup \ldots \cup J_{k+1}\}\,,$$

$$V_1 = \{(k,a,b) \mid k = 1, \dots, N-1\,,\ \ a \in J_1 \cup \ldots \cup J_k\,,\ \ b \in J_1 \cup \ldots \cup J_{k+1}\,,\ \ a = b\}\,,$$

Then

$$C_{J,J,\mathbf{id}} = \prod_{(k,a,b) \in V - V_1} (1 - h_{k,a,b}\, z_b/z_a)\,, \tag{6.5}$$

where the factors $h_{k,a,b}$ equal either 1 or h depending on the subscript (k,a,b). In other words, all factors of the product (6.5) are either $1 - z_b/z_a$ or $1 - hz_b/z_a$. We also have

$$C_{I,J,\boldsymbol{\pi}} = \prod_{(k,a,b) \in V - V_2} (1 - h_{k,a,b}\, z_b/z_a) \tag{6.6}$$

with the same meaning of $h_{k,a,b}$. Here $V_2 \subset V$, $V_1 \neq V_2$, and $|V_1| = |V_2|$. Therefore $C_{I,J,\boldsymbol{\pi}}$ either contains either a factor $(1 - z_a/z_a) = 0$ or a factor $(1 - hz_a/z_a) = 1 - h$.[1]

If f, g are Laurent polynomials, then $\varphi_J(N(fg)) = \varphi_J(N(f)) + \varphi_J(N(g))$, where $+$ is the Minkowski sum. Thus $\varphi_J(N(C_{I,J,\pi}))$ is the Minkowski sum of the intervals labeled by $(k, a, b) \in V - V_2$, with the vertices of the corresponding interval at 0 and $\varphi_J(N(z_b/z_a))$.

First notice that $V_2 - V_2 \cap V_1$ is not empty if $J \neq I$ and hence there are elements (k, a, b) which are present in the product (6.5) but not in the product (6.6). We claim that there is an element $(k, a, b) \in V_2 - V_2 \cap V_1$ with $\varphi_J(b) > \varphi_J(a)$. Indeed, choose k with $I_1 \cup \ldots \cup I_{k+1} = J_1 \cup \ldots \cup J_{k+1}$, $I_1 \cup \ldots \cup I_k \neq J_1 \cup \ldots \cup J_k$, and any $b \in I_1 \cup \ldots \cup I_k - J_1 \cup \ldots \cup J_k$; then there is an a with $(k, a, b) \in V_2 - V_2 \cap V_1$. The appearance of a factor $(1 - (h)z_b/z_a)$ with $\varphi_J(b) > \varphi_J(a)$ in (6.5) but not in (6.6) proves that $C_{I,J,\boldsymbol{\pi}} \ll_J C_{J,J,\mathbf{id}}$.

Consider the Laurent polynomials $A_{J,\boldsymbol{\pi}}$ in equations (6.3) and (6.4). Clearly the Newton polygon of $A_{J,\boldsymbol{\pi}}$ does not depend on $\boldsymbol{\pi}$. Therefore

$$(1-h)^{\lambda^{\{1\}}} \sum_{\boldsymbol{\pi}} \pm C_{I,J,\boldsymbol{\pi}} A_{J,\boldsymbol{\pi}} \ \ll_J \ (1-h)^{\lambda^{\{1\}}} C_{J,J,\mathbf{id}} A_{J,\mathbf{id}}\,.$$

Consequently, we have

$$W_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h) \ \ll_J \ W_{\sigma,J}(\boldsymbol{z}_J, \boldsymbol{z}, h).$$

The Laurent polynomials on both sides of this relation are divisible by the same Laurent polynomial $E(\boldsymbol{z}_J, h)$, see Lemma 6.3. Hence the same relation holds for the quotients,

$$\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h) \ll_J \widetilde{W}_{\sigma,J}(\boldsymbol{z}_J, \boldsymbol{z}, h).$$

By Lemma 6.4 this means that $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J, \boldsymbol{z}, h)$ is J-small. $\square$

6.2. **Orthogonality.** The number of inversions in an ordered sequence $j_1, \ldots, j_n$ is the number of pairs (a, b) with $a < b$, $j_a > j_b$. Let $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ where $I_k = \{i_1^{(k)} < \ldots < i_{\lambda_k}^{(k)}\}$ as before. Let $p(I)$ denote the number of inversions in the ordered sequence

$$I_N, I_{N-1}, \ldots, I_1 \ = \ i_1^{(N)}, \ldots, i_{\lambda_N}^{(N)}, i_1^{(N-1)}, \ldots, i_{\lambda_{N-1}}^{(N-1)}, \ldots, i_1^{(1)}, \ldots, i_{\lambda_1}^{(1)}\,.$$

We saw in Section 2.2 that $p(I)$ is the codimension of $\Omega_{\mathrm{id},I}$ in $\mathcal{F}_{\boldsymbol{\lambda}}$.

Theorem 6.6. *Let σ_0 be the longest permutation in S_n. For $J, K \in \mathcal{I}_{\boldsymbol{\lambda}}$, we have*

$$\sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} h^{p(K)}\, P(\boldsymbol{z}_I)\, \frac{\widetilde{W}_{\mathrm{id},J}(\boldsymbol{z}_I, \boldsymbol{z}, h)\, \widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1}, \boldsymbol{z}^{-1}, h^{-1})}{R(\boldsymbol{z}_I)\, Q(\boldsymbol{z}_I, h)} \ = \ \delta_{J,K}\,, \tag{6.7}$$

where

$$P(\boldsymbol{z}_I) \ = \ P_{\mathrm{id},I}\, P_{\sigma_0,I} \ = \ \prod_{k<l} \prod_{a \in I_k} \prod_{b \in I_l} (-z_b/z_a)\,, \tag{6.8}$$

$$R(\boldsymbol{z}_I) \ = \ P(\boldsymbol{z}_I)\, e^{hor}_{\mathrm{id},I,+}\, e^{hor}_{\mathrm{id},I,-} \ = \ \prod_{k<l} \prod_{a \in I_k} \prod_{b \in I_l} (1 - z_b/z_a)\,, \tag{6.9}$$

$$Q(\boldsymbol{z}_I, h) \ = \ e^{vert}_{\mathrm{id},I,+}\, e^{vert}_{\mathrm{id},I,-} \ = \ \prod_{k<l} \prod_{a \in I_k} \prod_{b \in I_l} (1 - hz_b/z_a)\,, \tag{6.10}$$

[1] A note to the experts: this observation is enough for the argument in cohomology, where J-smallness is measured by the smallness of z-degree. In K-theory the argument of the next few sentences is needed.

and $\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1},\boldsymbol{z}^{-1},h^{-1})$ *is the function obtained from* $\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I,\boldsymbol{z},h)$ *by the substitution* $\{z_1,\dots,z_n,h\}\mapsto\{z_1^{-1},\dots,z_n^{-1},h^{-1}\}$,

The proof is given in Section 7.5.

6.3. **Proof of existence in Theorem 3.1.** We show the existence of $\kappa_{\sigma,I}$ by giving an explicit formula for it.

Theorem 6.7. *For any* $\sigma\in S_n$ *and* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, *there exists a unique element* $[\widetilde{W}_{\sigma,I}]\in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ *such that for any* $J\in\mathcal{I}_{\boldsymbol{\lambda}}$ *we have* $\mathrm{Loc}_J[\widetilde{W}_{\sigma,I}]=\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$. *Moreover, the classes*

$$\kappa_{\sigma,I}=[\widetilde{W}_{\sigma,I}]\ \in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \tag{6.11}$$

satisfy conditions (I–III) *of Theorem 3.1.*

Proof. According to Lemma 6.1, $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ is a Laurent polynomial for all J, and hence $(\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h))_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}$ is an element of the right hand side of (2.5). We claim that it is in the image of Loc. Consider $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ and $\widetilde{W}_{\sigma,I}(\boldsymbol{z}_{s_{i,j}(J)},\boldsymbol{z},h)$. The $z_i=z_j$ substitution makes these two Laurent polynomials equal. Hence their difference is divisible by $1-z_i/z_j$. Therefore the element $[\widetilde{W}_{\sigma,I}]\in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ with $\mathrm{Loc}_J[\widetilde{W}_{\sigma,I}]=\widetilde{W}_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ exists.

Properties (I) – (III) are all about restrictions of $[\widetilde{W}_{\sigma,I}]$ to fixed points x_J. Hence they are computed by various $\boldsymbol{t}=\boldsymbol{z}_J$ substitutions in $\widetilde{W}_{\sigma,I}$. Therefore, Lemmas 6.2, 6.3, 6.4 prove properties (I), (II), (III) respectively. □

Observe that a byproduct of our proof of Theorem 3.1 and Lemma 6.2 is that conditions (I) – (III) imply $\kappa_{\sigma,I}|_{x_J}=0$ if $J\not\leqslant_\sigma I$.

Like in the proof above, we observe that for any σ and I, there is a unique element $[W_{\sigma,I}]\in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ with $\mathrm{Loc}_J[W_{\sigma,I}]=W_{\sigma,I}(\boldsymbol{z}_J,\boldsymbol{z},h)$ for all J.

Theorem 6.8. *For a fixed* $\sigma\in S_n$,

- *The set* $\{[W_{\sigma,I}]\}_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ *is a basis of the* $\mathbb{C}(\boldsymbol{z},h)$*-module* $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\otimes\mathbb{C}(\boldsymbol{z},h)$.
- *The set* $\{\kappa_{\sigma,I}\}_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ *is a basis of the* $\mathbb{C}(\boldsymbol{z},h)$*-module* $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\otimes\mathbb{C}(\boldsymbol{z},h)$.

Proof. As we claimed in Section 2.5 the map (2.7) is an isomorphism. Hence the statements follow from the triangularity properties

$$\mathrm{Loc}_J[W_{\sigma,I}]=\begin{cases}0 & \text{if } J\not\leqslant_\sigma I\\ \neq 0 & \text{if } J=I,\end{cases}\qquad\qquad \mathrm{Loc}_J[\kappa_{\sigma,I}]=\begin{cases}0 & \text{if } J\not\leqslant_\sigma I\\ \neq 0 & \text{if } J=I,\end{cases}$$

see Lemmas 6.2, 6.4. □

6.4. **Recursive properties.** Let $\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}|=n$. Define an action of the symmetric group S_n on the set $\mathcal{I}_{\boldsymbol{\lambda}}$. Let $I=(I_1,\dots,I_N)\in\mathcal{I}_{\boldsymbol{\lambda}}$, where $I_j=\{i_1,\dots,i_{\lambda_j}\}\subset\{1,\dots,n\}$. For $\sigma\in S_n$, recall $\sigma(I)=(\sigma(I_1),\dots,\sigma(I_N))$.

Let

$$\beta(x_1,x_2,y_1,y_2)\,=\,\mathrm{Sym}_{x_1,x_2}\,(1-hy_1/x_2)\,(1-y_2/x_1)\,\frac{1-hx_2/x_1}{1-x_2/x_1}\ . \tag{6.12}$$

It is straightforward to see that

$$\beta(x_1,x_2,y_1,y_2) \,=\, \mathrm{Sym}_{y_1,y_2}\,(1-hy_1/x_2)\,(1-y_2/x_1)\,\frac{1-hy_2/y_1}{1-y_2/y_1}\,. \tag{6.13}$$

Lemma 6.9. *$\beta(x_1,x_2,y_1,y_2)$ is symmetric in x_1,x_2 and in y_1,y_2.* □

Let $s_{a,b}\in S_n$ denote the transposition of a and b.

Theorem 6.10. *Let $\sigma\in S_n$ be such that $\sigma(a)\in I_k$ and $\sigma(a+1)\in I_l$. Then*

$$W_{\sigma s_{a,a+1},I} \,=\, W_{\sigma,I} \tag{6.14}$$

for $k=l$,

$$W_{\sigma s_{a,a+1},I} \,=\, h\,\frac{1-z_{\sigma(a)}/z_{\sigma(a+1)}}{1-hz_{\sigma(a)}/z_{\sigma(a+1)}}\,W_{\sigma,I} + \frac{1-h}{1-hz_{\sigma(a)}/z_{\sigma(a+1)}}\,W_{\sigma,\,s_{\sigma(a),\sigma(a+1)}(I)} \tag{6.15}$$

for $k<l$, and

$$W_{\sigma s_{a,a+1},I} \,=\, \frac{1-z_{\sigma(a)}/z_{\sigma(a+1)}}{1-hz_{\sigma(a)}/z_{\sigma(a+1)}}\,W_{\sigma,I} + (1-h)\,\frac{z_{\sigma(a)}/z_{\sigma(a+1)}}{1-hz_{\sigma(a)}/z_{\sigma(a+1)}}\,W_{\sigma,\,s_{\sigma(a),\sigma(a+1)}(I)} \tag{6.16}$$

for $k>l$.

Proof. By formula (4.3), it suffices to prove the statement when σ is the identity permutation. Next, formula (4.2) implies that it is enough to consider only the case $n=2$. To simplify the notation, we write $W_I=W_{\mathrm{id},I}$ and $s=s_{1,2}$.

Let $k=l$, $I=(\varnothing,\dots,\varnothing,\{1,2\},\varnothing,\dots,\varnothing)$, the set $\{1,2\}$ being at the k-th place. We compute W_I starting symmetrization in formula (4.1) from $t^{(k)}_1,t^{(k)}_2$, and using formula (6.12) and Corollary 6.9:

$$W_I(\boldsymbol t,z_1,z_2) \,=\, (1-h)^{2\delta_{k,1}}\beta(t^{(N-1)}_1,t^{(N-1)}_2,z_1,z_2)\prod_{p=k}^{N-2}\beta(t^{(p)}_1,t^{(p)}_2,t^{(p+1)}_1,t^{(p+1)}_2)\,.$$

Hence by Corollary 6.9, we have $W_{s,I}(\boldsymbol t,z_1,z_2)=W_I(\boldsymbol t,z_2,z_1)=W_I(\boldsymbol t,z_1,z_2)$.

Let $k<l$, $I=(\varnothing,\dots,\varnothing,\{1\},\varnothing,\dots,\varnothing,\{2\},\varnothing,\dots,\varnothing)$, the sets $\{1\}$ and $\{2\}$ being at the k-th and l-th places, respectively, and $s(I)=(\varnothing,\dots,\varnothing,\{2\},\varnothing,\dots,\varnothing,\{1\},\varnothing,\dots,\varnothing)$. Formula (6.15) is equivalent to the equality

$$hW_I(\boldsymbol t,z_1,z_2)+W_{s(I)}(\boldsymbol t,z_1,z_2) \,=\, \mathrm{Sym}_{z_1,z_2}\,W_{s(I)}(\boldsymbol t,z_1,z_2)\,\frac{1-hz_2/z_1}{1-z_2/z_1}\,. \tag{6.17}$$

We compute the left-hand side of (6.17) starting symmetrization in formula (4.1) from $t^{(l)}_1,t^{(l)}_2$, and using formula (6.12) and Corollary 6.9. The result of calculation is

$$(1-h)^{\delta_{k,1}}(1+h-ht^{(l)}_1/t^{(l-1)}_1-ht^{(l)}_1/t^{(l-1)}_1)\,\times \\ \times\,\beta(t^{(N-1)}_1,t^{(N-1)}_2,z_1,z_2)\prod_{p=l}^{N-2}\beta(t^{(p)}_1,t^{(p)}_2,t^{(p+1)}_1,t^{(p+1)}_2)\,. \tag{6.18}$$

We compute the right-hand side of (6.17) starting symmetrization from z_1,z_2, and using formula (6.12) and Corollary 6.9, and get the same answer (6.18). Formula (6.15) is proved.

The proof of formula (6.16) is similar. □

Theorem 6.10 implies recursions for weight functions. Set

$$s_{a,a+1}(\boldsymbol{z}) = (z_1, \dots, z_{a-1}, z_{a+1}, z_a, z_{a+2}, \dots, z_n).$$

Corollary 6.11. *Suppose* $a \in I_k$ *and* $a+1 \in I_l$. *Then*

$$W_I\big(\boldsymbol{t}, s_{a,a+1}(\boldsymbol{z})\big) = W_I(\boldsymbol{t}, \boldsymbol{z}) \tag{6.19}$$

for $k = l$,

$$W_{s_{a,a+1}(I)}(\boldsymbol{t}, \boldsymbol{z}) = \frac{1 - hz_a/z_{a+1}}{1 - z_a/z_{a+1}} W_I\big(\boldsymbol{t}, s_{a,a+1}(\boldsymbol{z})\big) + (h-1)\frac{z_a/z_{a+1}}{1 - z_a/z_{a+1}} W_I(\boldsymbol{t}, \boldsymbol{z}) \tag{6.20}$$

for $k < l$, *and*

$$W_{s_{a,a+1}(I)}(\boldsymbol{t}, \boldsymbol{z}) = \frac{1 - h^{-1}z_{a+1}/z_a}{1 - z_{a+1}/z_a} W_I\big(\boldsymbol{t}, s_{a,a+1}(\boldsymbol{z})\big) + (h^{-1}-1)\frac{z_{a+1}/z_a}{1 - z_{a+1}/z_a} W_I(\boldsymbol{t}, \boldsymbol{z}) \tag{6.21}$$

for $k > l$.

Proof. We take $\sigma = \mathrm{id}$ in Theorem 6.10 and apply formula (4.3). Then formulae (6.19), (6.20), (6.21) are respective counterparts of formulae (6.14), (6.16), (6.15). □

Remark. For a function $f(x, y)$, define

$$\partial_{x,y} f(x,y) = \frac{f(x,y) - f(y,x)}{x - y}, \qquad \pi_{x,y} f(x,y) = \partial_{x,y}\big(xf(x,y)\big). \tag{6.22}$$

We call $\partial_{x,y}$ and $\pi_{x,y}$ the rational and trigonometric divided difference operators, respectively. Formulae (6.20) and (6.21) respectively read

$$\begin{aligned} W_{s_{a,a+1}(I)} &= \pi_{z_a, z_{a+1}} W_I - hz_a \cdot \partial_{z_a, z_{a+1}} W_I, \\ W_{s_{a+1,a}(I)} &= \pi_{z_{a+1}, z_a} W_I - h^{-1}z_{a+1} \cdot \partial_{z_{a+1}, z_a} W_I. \end{aligned} \tag{6.23}$$

7. Stable envelope maps and R-matrices

7.1. **Definition.** For $\sigma \in S_n$, we define the *stable envelope map*

$$\mathrm{Stab}_\sigma : K_T\big((\mathcal{X}_n)^T\big) \to K_T(\mathcal{X}_n), \qquad 1_I \mapsto \kappa_{\sigma,I}, \tag{7.1}$$

where $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda} \in \mathbb{Z}^n_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$.

The maps Stab_σ become isomorphisms after tensoring the K-theory algebras with $\mathbb{C}(\boldsymbol{z}, h)$, see Theorem 6.8. For $\sigma', \sigma \in S_n$, we define the *geometric* R*-matrix*

$$R_{\sigma',\sigma} = \mathrm{Stab}_{\sigma'}^{-1} \circ \mathrm{Stab}_\sigma \in \mathrm{End}(K_T\big((\mathcal{X}_n)^T\big)) \otimes \mathbb{C}(\boldsymbol{z}, h) = \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}(\boldsymbol{z}, h). \tag{7.2}$$

7.2. **Trigonometric R-matrix.** Let h, z be parameters. Define the *trigonometric* R*-matrix*, an element $\mathcal{R}(z, h) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}(z, h)$, by the conditions:

- For $i = 1, \dots, N$,

$$\mathcal{R}(z, h) : v_i \otimes v_i \mapsto v_i \otimes v_i, \tag{7.3}$$

- For $1 \leqslant i < j \leqslant N$, on the two-dimensional subspace with ordered basis $v_i \otimes v_j$, $v_j \otimes v_i$, the trigonometric R-matrix is given by the matrix

$$\begin{pmatrix} \dfrac{1-z}{1-hz} & \dfrac{1-h}{1-hz} \\ \dfrac{(1-h)z}{1-hz} & \dfrac{h(1-z)}{1-hz} \end{pmatrix}. \tag{7.4}$$

The trigonometric R-matrix depends on two parameters z, h. We often omit in the notation the dependence on h.

The 2×2-matrix in (7.4), satisfies the following relation

$$\begin{pmatrix} \dfrac{1-z}{1-hz} & \dfrac{(1-h)z}{1-hz} \\ \dfrac{1-h}{1-hz} & \dfrac{h(1-z)}{1-hz} \end{pmatrix} = \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1-z^{-1}}{1-h^{-1}z^{-1}} & \dfrac{1-h^{-1}}{1-h^{-1}z^{-1}} \\ \dfrac{(1-h^{-1})z^{-1}}{1-h^{-1}z^{-1}} & \dfrac{h^{-1}(1-z^{-1})}{1-h^{-1}z^{-1}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}. \tag{7.5}$$

The trigonometric R-matrix satisfies the Yang-Baxter equation

$$\mathcal{R}^{(1,2)}(z_2/z_1)\,\mathcal{R}^{(1,3)}(z_3/z_1)\,\mathcal{R}^{(2,3)}(z_3/z_2) \,=\, \mathcal{R}^{(2,3)}(z_3/z_2)\,\mathcal{R}^{(1,3)}(z_3/z_1)\,\mathcal{R}^{(1,2)}(z_2/z_1)\,. \tag{7.6}$$

This is an identity in $\mathrm{End}((\mathbb{C}^N)^{\otimes 3})$ and $\mathcal{R}^{(i,j)}(z_j/z_i)$ is the R-matrix $\mathcal{R}(z_j/z_i)$ acting on the i-th and j-th factors of $(\mathbb{C}^N)^{\otimes 3}$.

The trigonometric R-matrix satisfies the inversion relation

$$\mathcal{R}^{(1,2)}(z_2/z_1)\,\mathcal{R}^{(2,1)}(z_1/z_2) \,=\, 1\,. \tag{7.7}$$

7.3. **Geometric R-matrix for $n = 2$.** The group S_2 has two elements: the identity id and the transposition s. After the identification $K_T\big((\mathcal{X}_n)^T\big) \otimes \mathbb{C}(\boldsymbol{z}, h) = (\mathbb{C}^N)^{\otimes 2} \otimes \mathbb{C}(\boldsymbol{z}, h)$, we calculate the geometric R-matrix $R_{s,\mathrm{id}}$ as follows.

For $\boldsymbol{\lambda} = (0, \dots, 0, 2, 0, \dots, 0)$ with the coordinate 2 being at i-th position, both maps $\mathrm{Stab}_{\mathrm{id}}$ and Stab_s send the vector $v_i \otimes v_i$ to $1 \in K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$. Hence, $R_{s,\mathrm{id}}\,(v_i \otimes v_i) = v_i \otimes v_i$.

For $\boldsymbol{\lambda} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0 \dots, 0)$ with the nonzero coordinates 1 being at i-th and j-th positions, $i < j$, the set $\mathcal{I}_{\boldsymbol{\lambda}}$ consists of two elements: $I = (\varnothing, \dots, \varnothing, \{1\}, \varnothing, \dots, \varnothing, \{2\}, \varnothing, \dots, \varnothing)$ and $J = (\varnothing, \dots, \varnothing, \{2\}, \varnothing, \dots, \varnothing, \{1\}, \varnothing, \dots, \varnothing)$.

By formulae for $M_2 W_I$ and $M_2 W_J$ from Section 4.2 and the equality

$$E(\boldsymbol{z}_I, h) \,=\, (1-h)^{2N-i-j}(1-hz_2/z_1)^{N-j}(1-hz_1/z_2)^{N-j}\,,$$

see (6.1), we have

$$\mathrm{Stab}_{\mathrm{id}}(v_i \otimes v_j) \,=\, 1 - z_2/\gamma_{i,1}\,, \qquad \mathrm{Stab}_{\mathrm{id}}(v_j \otimes v_i) \,=\, 1 - hz_1/\gamma_{i,1}\,.$$

Similarly,

$$\mathrm{Stab}_s(v_i \otimes v_j) \,=\, 1 - hz_2/\gamma_{i,1}\,, \qquad \mathrm{Stab}_s(v_j \otimes v_i) \,=\, 1 - z_1/\gamma_{i,1}\,.$$

Thus,

$$R_{s,\mathrm{id}}\,(v_i\otimes v_j)\;=\;\frac{1-z_2/z_1}{1-hz_2/z_1}\,v_i\otimes v_j+\frac{(1-h)\,z_2/z_1}{1-hz_2/z_1}\,v_j\otimes v_i\,,$$

$$R_{s,\mathrm{id}}\,(v_j\otimes v_i)\;=\;\frac{1-h}{1-hz_2/z_1}\,v_i\otimes v_j+\frac{h\,(1-z_2/z_1)}{1-hz_2/z_1}\,v_j\otimes v_i\,.$$

Therefore, $R_{s,\mathrm{id}}=\mathcal{R}(z_2/z_1,h)$.

7.4. **Geometric R-matrices for arbitrary n.** Since for any permutations σ,σ',σ'',

$$R_{\sigma'',\sigma}\;=\;R_{\sigma'',\sigma'}\,R_{\sigma',\sigma}\,,\tag{7.8}$$

it is enough to describe the geometric R-matrices $R_{\sigma s_{a,a+1},\sigma}$ that correspond to permutations $\sigma s_{a,a+1}$, $\sigma\in S_n$.

Theorem 7.1. *We have*

$$R_{\sigma s_{a,a+1},\sigma}\;=\;\mathcal{R}^{(\sigma(a),\sigma(a+1))}(z_{\sigma(a+1)}/z_{\sigma(a)})\,\in\,\mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big)\otimes\mathbb{C}(\boldsymbol{z},h)\,,$$

where $\mathcal{R}^{(\sigma(a),\sigma(a+1))}$ is the trigonometric R-matrix (7.3), (7.4) *acting in the $\sigma(a)$-th and $\sigma(a+1)$-th tensor factors.*

Proof. The geometric R-matrix is defined by formulae (7.2), (7.1), (6.11), (6.2). Now the statement follows from Theorem 6.10 and formulae (7.3), (7.4). □

7.5. **Proof of Theorem 6.6.** For $\sigma\in S_n$, introduce a matrix

$$\widehat{W}_\sigma(\boldsymbol{z},h)=(\widetilde{W}_{\sigma,J}(\boldsymbol{z}_I,\boldsymbol{z},h))_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}}\tag{7.9}$$

where the subscripts I,J label rows and columns, respectively. Consider the matrix

$$\widehat{R}(\boldsymbol{z},h)\;=\;\widehat{W}^{-1}_{\sigma_0}(\boldsymbol{z},h)\,\widehat{W}_{\mathrm{id}}(\boldsymbol{z},h)\,.\tag{7.10}$$

This is the matrix of the restriction of the geometric R-matrix $R_{\sigma_0,\mathrm{id}}$, see (7.2), on the span of $\{v_I\mid I\in\mathcal{I}_{\boldsymbol{\lambda}}\}$. By Theorem 7.1 and formulae (7.8), (7.5), we have

$$(\widehat{R}(\boldsymbol{z},h))^t\;=\;M\,\widehat{R}(\boldsymbol{z}^{-1},h^{-1})\,\widetilde{M}\,,\tag{7.11}$$

where the superscript t denotes transposition and $M,\widetilde{M}$ are diagonal matrices. The entries of M are $M_{I,I}=h^{-p(I)}$, and an explicit formula for the entries of $\widetilde{M}$ will not be used. Formulae (7.10), (7.11) yield

$$\big(\widehat{W}_{\mathrm{id}}(\boldsymbol{z},h)\big)^t\,\big(\widehat{W}^{-1}_{\sigma_0}(\boldsymbol{z},h)\big)^t\;=\;M\,\widehat{W}^{-1}_{\sigma_0}(\boldsymbol{z}^{-1},h^{-1})\,\widehat{W}_{\mathrm{id}}(\boldsymbol{z}^{-1},h^{-1})\,\widetilde{M}\,.$$

Hence,

$$\widehat{W}_{\sigma_0}(\boldsymbol{z}^{-1},h^{-1})\,M^{-1}\,\big(\widehat{W}_{\mathrm{id}}(\boldsymbol{z},h)\big)^t\;=\;\widehat{W}_{\mathrm{id}}(\boldsymbol{z}^{-1},h^{-1})\,\widetilde{M}\big(\widehat{W}_{\sigma_0}(\boldsymbol{z},h)\big)^t\,.\tag{7.12}$$

By Lemma 6.2, $\widetilde{W}_{\mathrm{id},J}(\boldsymbol{z}_I,\boldsymbol{z},h)=0$ if $I>_{\mathrm{id}}J$ and $\widetilde{W}_{\sigma_0,J}(\boldsymbol{z}_I,\boldsymbol{z},h)=0$ if $I<_{\mathrm{id}}J$. That is, the matrices $\big(\widehat{W}_{\mathrm{id}}(\boldsymbol{z},h)\big)^t$ and $\widehat{W}_{\sigma_0}(\boldsymbol{z}^{-1},h^{-1})$ are lower triangular, and so is the left-hand side of (7.12). Similarly, the matrices $\widehat{W}_{\mathrm{id}}(\boldsymbol{z}^{-1},h^{-1})$ and $\big(\widehat{W}_{\sigma_0}(\boldsymbol{z},h)\big)^t$ are upper triangular, and so is the right-hand side of (7.12). Therefore,

$$\widehat{W}_{\sigma_0}(\boldsymbol{z}^{-1},h^{-1})\,M^{-1}\,\big(\widehat{W}_{\mathrm{id}}(\boldsymbol{z},h)\big)^t\;=\;S\,,$$

where S is a diagonal matrix with entries

$$S_{I,I} = h^{p(I)}\,\widetilde W_{\mathrm{id},I}(\boldsymbol z_I,\boldsymbol z,h)\,\widetilde W_{\sigma_0,I}(\boldsymbol z_I^{-1},\boldsymbol z^{-1},h^{-1}) \,=\, \frac{R(\boldsymbol z_I)\,Q(\boldsymbol z_I,h)}{P(\boldsymbol z_I)}\,. \tag{7.13}$$

Here the second equality follows from Lemma 6.4 and formula (3.2). Hence,

$$\bigl(\widehat W_{\mathrm{id}}(\boldsymbol z,h)\bigr)^t\,S^{-1}\,\widehat W_{\sigma_0}(\boldsymbol z^{-1},h^{-1})\,M^{-1} \,=\, 1\,, \tag{7.14}$$

which is the matrix form of formula (6.7).

8. Inverse of the map $\mathrm{Stab}_{\mathrm{id}}$

8.1. **S_n-action on functions.** Let $P^{(i,j)}$ be the permutation of the i-th and j-th factors of $(\mathbb C^N)^{\otimes n}$. Let

$$K_i : f(z_1,\dots,z_n)\mapsto f(z_1,\dots,z_{i-1},z_{i+1},z_i,z_{i+2},\dots,z_n) \tag{8.1}$$

be the operator interchanging the variables z_i and z_{i+1}.

Define an action of the symmetric group S_n on $(\mathbb C^N)^{\otimes n}$-valued functions of $z_1,\dots,z_n,h$. Let the i-th elementary transposition $s_i\in S_n$ act by the formula

$$\tilde s_i \,=\, P^{(i,i+1)}\,\mathcal R^{(i,i+1)}(z_i/z_{i+1})\,K_i\,, \tag{8.2}$$

where $\mathcal R$ is the trigonometric R-matrix (7.3), (7.4).

Lemma 8.1. *The S_n-action (8.2) is well-defined, that is,*

$$(\tilde s_i)^2 = 1,\qquad \tilde s_i\tilde s_{i+1}\tilde s_i \,=\, \tilde s_{i+1}\tilde s_i\tilde s_{i+1}\,,\qquad \tilde s_i\tilde s_j = \tilde s_j\tilde s_i\quad \text{if}\ \ |i-j|>1\,.$$

Moreover, $\tilde s_i z_i\tilde s_i = z_{i+1}$ and $\tilde s_i z_j = z_j\tilde s_i$ if $j\neq i,i+1$, where $z_1,\dots,z_n$ are considered as the scalar operators on $(\mathbb C^N)^{\otimes n}$ of multiplication by the respective variable.

Proof. The S_n-action is well-defined due to the inversion relation (7.7) and the Yang-Baxter equation (7.6). The rest of the statement is clear. □

8.2. **Vectors ξ_I.** Recall the partial ordering $\leqslant_\sigma$ on $\mathcal I_{\boldsymbol\lambda}$ defined in Section 2.2. Set

$$I^{\min} \,=\, \bigl(\{1,\dots,\lambda_1\},\{\lambda_1+1,\dots,\lambda_1+\lambda_2\},\ \dots\ ,\{n-\lambda_N+1,\dots,n\}\bigr)\in\mathcal I_{\boldsymbol\lambda}\,, \tag{8.3}$$

$$I^{\max} \,=\, \bigl(\{n-\lambda_1+1,\dots,n\},\{n-\lambda_1-\lambda_2+1,\dots,n-\lambda_1\},\ \dots\ ,\{1,\dots,\lambda_N\}\bigr)\in\mathcal I_{\boldsymbol\lambda}\,.$$

Clearly, $I^{\min}\leqslant_{\mathrm{id}} I\leqslant_{\mathrm{id}} I^{\max}$ for any $I\in\mathcal I_{\boldsymbol\lambda}$.

Let

$$D \,=\, \prod_{1\leqslant b<a\leqslant n}(1-hz_b/z_a)\,. \tag{8.4}$$

Theorem 8.2. *There exist unique elements $\{\xi_I\in(\mathbb C^N)^{\otimes n}\otimes\mathbb C[\boldsymbol z^{\pm1},h^{\pm1},D^{-1}]\ \mid\ I\in\mathcal I_{\boldsymbol\lambda}\}$ such that $\xi_{I^{\min}}=v_{I^{\min}}$ and*

$$\xi_{s_i(I)} \,=\, \tilde s_i\,\xi_I \tag{8.5}$$

for every $I\in\mathcal I_{\boldsymbol\lambda}$ and $i=1,\dots,n-1$. Moreover,

$$\xi_I \,=\, \sum_{J\leqslant_{\mathrm{id}} I} X_{I,J}\,v_J\,, \tag{8.6}$$

where $X_{I,J} \in \mathbb{C}[\boldsymbol{z}^{\pm 1}, h^{\pm 1}, D^{-1}]$ *and*

$$X_{I,I} = \prod_{k<l} \prod_{a \in I_k} \prod_{\substack{b \in I_l \\ b<a}} \frac{1 - z_b/z_a}{1 - h z_b/z_a} .$$

In particular,

$$X_{I^{\max}, I^{\max}} = \frac{R(\boldsymbol{z}_{I^{\max}})}{Q(\boldsymbol{z}_{I^{\max}}, h)} . \tag{8.7}$$

Furthermore,

$$X_{I,J} = h^{p(J)} \, \widetilde{W}_{\sigma_0, J}(\boldsymbol{z}_I^{-1}, \boldsymbol{z}^{-1}, h^{-1}) \, \frac{P(\boldsymbol{z}_I)}{Q(\boldsymbol{z}_I, h)} . \tag{8.8}$$

Here $\sigma_0 \in S_n$ *is the longest permutation, and* $p(J)$, $P(\boldsymbol{z}_I)$, $R(\boldsymbol{z}_I)$, $Q(\boldsymbol{z}_I, h)$ *are defined in Section 6.2, see formulae* (6.8)–(6.10)*.*

Notice that

$$X_{I,I} = \frac{P_{\sigma_0, I} \, e^{hor}_{\mathrm{id}, I, +}}{e^{vert}_{\mathrm{id}, I, -}} ,$$

where $e^{hor}_{\mathrm{id}, I, +}$, $e^{vert}_{\mathrm{id}, I, -}$, and $P_{\sigma_0, I}$ are given by formulae (2.8), (2.9), and (3.1), respectively.

Proof. The properties $\xi_{I^{\min}} = v_{I^{\min}}$ and property (8.5) imply that $\xi_{\sigma(I^{\min})} = \tilde{\sigma} \, v_{I^{\min}}$ for any $\sigma \in S_n$, which proves uniqueness.

To show existence, define the elements ξ_I by the rule: $\xi_I = \tilde{\sigma} \, v_{I^{\min}}$ provided $I = \sigma(I^{\min})$. By Lemma 8.1 and the property $\tilde{\sigma} \, v_{I^{\min}} = v_{I^{\min}}$ for any $\sigma \in S_n$ such that $\sigma(I^{\min}) = I^{\min}$, these elements ξ_I are well-defined and satisfy (8.5), and $\xi_{I^{\min}} = v_{I^{\min}}$.

Let σ be the shortest permutation such that $\sigma(I^{\min}) = I$, and $\sigma = s_{i_1} \dots s_{i_k}$ be a reduced presentation. Then the equality $\xi_I = \tilde{s}_{i_1} \dots \tilde{s}_{i_k} v_{I^{\min}}$ and formulae (7.3), (7.4) for the R-matrix $\mathcal{R}(z, h)$ yield formula (8.6) with some coefficients $X_{I,J}$, as well as the explicit formula for $X_{I,I}$.

To get formula (8.8) for $X_{I,J}$, we denote by $Y_{I,J}$ the right-hand side of formula (8.8) and will show that the elements

$$\eta_I = \sum_{J \in \mathcal{I}_{\boldsymbol{\lambda}}} Y_{I,J} \, v_J \tag{8.9}$$

satisfy the defining properties of the elements ξ_I. The property $\eta_{I^{\min}} = v_{I^{\min}}$ is immediate from Lemmas 6.2, 6.4, that imply $Y_{I^{\min}, J} = \delta_{I^{\min}, J}$.

Let $\widetilde{Y}_{I,J} = Y_{I,J} / R(\boldsymbol{z}_I)$ and $\tilde{\eta} = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} \widetilde{Y}_{IJ} \, v_J$. The properties $\eta_{s_i(I)} = \tilde{s}_i \, \eta_I$ and $\tilde{\eta}_{s_i(I)} = \tilde{s}_i \, \tilde{\eta}_I$ are equivalent, and we will check the second one. Define a matrix $\widehat{Y} = (\widetilde{Y}_{I,J})_{J, I \in \mathcal{I}_{\boldsymbol{\lambda}}}$, where the subscripts J, I label rows and columns, respectively, Consider $\widehat{Y}$ as a linear operator on the space $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ with basis $\{ v_I \mid I \in \mathcal{I}_{\boldsymbol{\lambda}} \}$, see Section 2.4. Then $\widehat{Y} : v_I \mapsto \tilde{\eta}_I$.

The linear maps $\mathcal{R}^{(i,i+1)}$ and $P^{(i,i+1)}$ preserve the space $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. The relations $\tilde{\eta}_{s_i(I)} = \tilde{s}_i \, \tilde{\eta}_I$ are equivalent to

$$P^{(i,i+1)} \, \mathcal{R}^{(i,i+1)}(z_i / z_{i+1}) \, K_i \, \widehat{Y} \, K_i \, P^{(i,i+1)} = \widehat{Y} . \tag{8.10}$$

Recall the matrices $\widehat W_\sigma$, S given by (7.9), (7.13), respectively, and M with entries $M_{I,J} = h^{-p(I)}\delta_{I,J}$. We have

$$\widehat Y \,=\, M^{-1}\bigl(\widehat W_{\sigma_0}(\boldsymbol z^{-1},h^{-1})\bigr)^t S^{-1} \,=\, \bigl(\widehat W_{\mathrm{id}}(\boldsymbol z,h)\bigr)^{-1},$$

the second equality following from formula (7.14). Thus formula (8.10) transforms to

$$P^{(i,i+1)}\,\mathcal R^{(i,i+1)}(z_i/z_{i+1})\,K_i\,\bigl(\widehat W_{\mathrm{id}}(\boldsymbol z,h)\bigr)^{-1}K_i\,P^{(i,i+1)} \,=\, \bigl(\widehat W_{\mathrm{id}}(\boldsymbol z,h)\bigr)^{-1}$$

and then to

$$\mathcal R^{(i,i+1)}(z_{i+1}/z_i) \,=\, K_i\,P^{(i,i+1)}\bigl(\widehat W_{\mathrm{id}}(\boldsymbol z,h)\bigr)^{-1}P^{(i,i+1)}K_i\,\widehat W_{\mathrm{id}}(\boldsymbol z,h)\,. \tag{8.11}$$

The right-hand side of (8.11) equals the geometric R-matrix $R_{s_i,\mathrm{id}}$, see (7.2). Thus formula (8.11) follows from Theorem 7.1, which proves the desired relation $\tilde\eta_{s_i(I)} = \tilde s_i\,\tilde\eta_I$. ☐

Example. Let $N=n=2$ and $\boldsymbol\lambda=(1,1)$. Then $\xi_{(\{1\},\{2\})}(z_1,z_2,h)=v_{(\{1\},\{2\})}$ and

$$\xi_{(\{2\},\{1\})}(z_1,z_2,h) \,=\, (1-h)\,\frac{z_1/z_2}{1-hz_1/z_2}\,v_{(\{1\},\{2\})} + \frac{1-z_1/z_2}{1-hz_1/z_2}\,v_{(\{2\},\{1\})}\,.$$

Corollary 8.3. *The set of vectors ξ_I, $I\in\mathcal I_{\boldsymbol\lambda}$, is a $\mathbb C(\boldsymbol z,h)$-basis of the space $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes \mathbb C(\boldsymbol z,h)$.* ☐

Let $f_I(\boldsymbol z,h)$, $I\in\mathcal I_{\boldsymbol\lambda}$, be a collection of scalar functions.

Lemma 8.4. *The function $\sum_{I\in\mathcal I_{\boldsymbol\lambda}} f_I(\boldsymbol z,h)\,\xi_I$ is invariant under the S_n-action* (8.2) *if and only if $f_{\sigma(I)}(\boldsymbol z,h) \,=\, f_I(z_{\sigma(1)},\dots,z_{\sigma(n)},h)$ for any $I\in\mathcal I_{\boldsymbol\lambda}$ and any $\sigma\in S_n$.* ☐

8.3. **Inverse of** $\mathrm{Stab}_{\mathrm{id}}$**.** Recall the stable envelope map

$$\mathrm{Stab}_{\mathrm{id}}\,:\,(\mathbb C^N)^{\otimes n}\otimes\mathbb C(\boldsymbol z,h)\,\to\,K_T(\mathcal X_n)\otimes\mathbb C(\boldsymbol z,h)\,,\qquad v_I\,\mapsto\,\kappa_{\mathrm{id},I}\,, \tag{8.12}$$

where $I\in\mathcal I_{\boldsymbol\lambda}$ and $\boldsymbol\lambda\in\mathbb Z^n_{\geqslant 0}$, $|\boldsymbol\lambda|=n$, cf. (7.1).

Define the homomorphism $\nu: K_T(\mathcal X_n)\otimes\mathbb C(\boldsymbol z,h)\,\to\,(\mathbb C^N)^{\otimes n}\otimes\mathbb C(\boldsymbol z,h)$ of $\mathbb C(\boldsymbol z,h)$-modules by the rule

$$\nu\,:\,[f(\boldsymbol\Gamma,\boldsymbol z,h)]\,\mapsto\,\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{f(\boldsymbol z_I,\boldsymbol z,h)}{R(\boldsymbol z_I)}\,\xi_I \tag{8.13}$$

for any $f\in\mathbb C[\boldsymbol\Gamma^{\pm1}]^{S_{\boldsymbol\lambda}}\otimes\mathbb C(\boldsymbol z,h)$. Here $f(\boldsymbol z_I,\boldsymbol z,h)$ is obtained from $f(\boldsymbol\Gamma,\boldsymbol z,h)$ by the substitution $\{\gamma_{k,1},\dots,\gamma_{k,\lambda_k}\}\mapsto\{z_a\,|\,a\in I_k\}$ for all $k=1,\dots,N$. This substitution was denoted Loc_I, see (2.6).

Theorem 8.5. *The maps* $\mathrm{Stab}_{\mathrm{id}}$ *and* ν *are the inverse isomorphisms.*

Proof. The statement follows from the orthogonality relation (6.7) and the formulae (8.6), (8.8) for vectors ξ_I. ☐

Theorem 8.5 is a K-theoretic analog of [RTV, Lemma 6.7].

Remark. The group S_n acts on $K_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z},h)$ by permutations of $z_1,\dots,z_n$ in the second factor, $s_i:[f]\mapsto[K_i f]$ for $i=1,\dots,n-1$ and $f\in\mathbb{C}[\boldsymbol{\Gamma}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(\boldsymbol{z},h)$. Under the isomorphism $\mathrm{Stab}_{\mathrm{id}}$, this S_n-action is identified with the S_n-action (8.2) on $(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}(\boldsymbol{z},h)$. This corollary of Theorem 7.1 could be considered as a motivation for the S_n-action (8.2).

9. Space $\frac{1}{D}\mathcal{V}^-$

9.1. **Invariant functions.** Define the operators $\hat s_1,\dots,\hat s_{n-1}$ acting on functions of $z_1,\dots,z_n,h$ as follows:

$$\hat s_i\,=\,\frac{1-hz_{i+1}/z_i}{1-z_{i+1}/z_i}\,K_i+\frac{h-1}{1-z_{i+1}/z_i}\,. \tag{9.1}$$

We consider $z_1,\dots,z_n$ as operators of multiplication by the respective variable.

Lemma 9.1. *The operators $\hat s_1,\dots,\hat s_{n-1}$, $z_1,\dots,z_n$ satisfy the relations*

$$(\hat s_i+1)(\hat s_i-h)=0\,,\qquad \hat s_i\,\hat s_{i+1}\hat s_i=\hat s_{i+1}\hat s_i\,\hat s_{i+1}\,,\qquad \hat s_i\hat s_j=\hat s_j\hat s_i\quad\text{if}\ \ |i-j|>1\,, \tag{9.2}$$
$$\hat s_i z_{i+1}\hat s_i=hz_i\,,\qquad \hat s_i z_j=z_j\hat s_i\,,\quad\text{if}\ \ j\neq i,i+1\,.$$

Proof. The statement follows from formula (9.1) by direct verification. $\square$

Remark. Set $h=q^{-2}$, $t_i=q^{-1}\hat s_i^{-1}$. Then $t_1,\dots,t_{n-1}$, $z_1,\dots,z_n$ satisfy the relations

$$(t_i-q)(t_i+q^{-1})=0\,,\qquad t_i\,t_{i+1}t_i=t_{i+1}t_i\,t_{i+1}\,,\qquad t_it_j=t_jt_i\quad\text{if}\ \ |i-j|>1\,, \tag{9.3}$$
$$t_iz_it_i=z_{i+1}\,,\qquad t_iz_j=z_jt_i\quad\text{if}\ \ j\neq i,i+1\,.$$

The algebra generated by $t_1,\dots,t_{n-1}$, $z_1,\dots,z_n$ subject to relations (9.3) is the affine Hecke algebra of type A_{n-1}.

Let $f_I(\boldsymbol{z},h)$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, be a collection of scalar functions and

$$f(\boldsymbol{z},h)\,=\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}f_I(\boldsymbol{z},h)\,v_I\,. \tag{9.4}$$

Recall the S_n-action (8.2) on $(\mathbb{C}^N)^{\otimes n}$-valued functions defined in Section 8.1.

Lemma 9.2. *Given j, we have $\tilde s_j f=f$ if and only if for any $I=(I_1,\dots,I_N)\in\mathcal{I}_{\boldsymbol{\lambda}}$ the following three conditions are satisfied:*
(i) $f_I=K_jf_I$, *if* $j,j+1\in I_a$ *for some* a;
(ii) $f_{s_j(I)}=\hat s_j^{-1}f_I$, *if* $j\in I_a$, $j+1\in I_b$, *and* $a<b$;
(iii) $f_{s_j(I)}=\hat s_jf_I$, *if* $j\in I_a$, $j+1\in I_b$, *and* $a>b$.

Proof. Denote by $\tilde f_I$ the I-th coordinate of $\tilde s_jf$. Then

$$\tilde f_I\,=\,K_jf_I \tag{9.5}$$

if $j,j+1\in I_a$ for some a,

$$\tilde f_I\,=\,h\,\frac{1-z_j/z_{j+1}}{1-hz_j/z_{j+1}}\,K_jf_{s_j(I)}+(1-h)\,\frac{z_j/z_{j+1}}{1-hz_j/z_{j+1}}\,K_jf_I \tag{9.6}$$

if $j \in I_a$, $j+1 \in I_b$, $a < b$, and

$$\tilde{f}_I = \frac{1 - z_j/z_{j+1}}{1 - hz_j/z_{j+1}} K_j f_{s_j(I)} + \frac{1-h}{1 - hz_j/z_{j+1}} K_j f_I \tag{9.7}$$

if $j \in I_a$, $j+1 \in I_b$, $a > b$. If $\tilde{s}_j f = f$, we have $\tilde{f}_I = f_I$ for any I, and formulae (9.5) – (9.7) are equivalent to conditions (i) – (iii), respectively. □

Remark. Lemma 9.2 could be considered as a motivation for the operators $\hat{s}_1, \dots, \hat{s}_{n-1}$.

For $\sigma \in S_n$, let $\sigma = s_{j_1} \dots s_{j_l}$ be a reduced presentation. Define $\hat{\sigma} = \hat{s}_{j_1} \dots \hat{s}_{j_l}$. By Lemma 9.1, the operator $\hat{\sigma}$ does not depend on the choice of the reduced presentation.

Let $\boldsymbol{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$. Denote by $S^{\max}_{\boldsymbol{\lambda}} \subset S_n$ the isotropy subgroup of $I^{\max}$, see (8.3). For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, let $\sigma_I \in S_n$ be the shortest permutation such that $I = \sigma_I(I^{\max})$.

Proposition 9.3. *The function $f(\boldsymbol{z}, h)$, see* (9.4), *is invariant under the S_n-action* (8.2) *if and only if $f_{I^{\max}}(\boldsymbol{z}, h) = f_{I^{\max}}(\boldsymbol{z}_\sigma, h)$ for any $\sigma \in S^{\max}_{\boldsymbol{\lambda}}$, and $f_I = \hat{\sigma}_I(f_{I^{\max}})$ for any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$. Moreover,*

$$f(\boldsymbol{z}, h) = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} f_{I^{\max}}(\boldsymbol{z}_{\sigma_0(I)}, h) \frac{Q(\boldsymbol{z}_I, h)}{R(\boldsymbol{z}_I)} \xi_I , \tag{9.8}$$

where $\sigma_0 \in S_n$ is the longest permutation.

Proof. The first part of the proposition follows from Lemmas 9.1 and 9.2. To prove formula (9.8), observe that the function in the right-hand side of (9.8) is invariant under the S_n-action (8.2) by Lemma 8.4, and has the $I^{\max}$-th coordinate $f_{I^{\max}}(\boldsymbol{z}, h)$ in the basis $\{v_I \mid I \in \mathcal{I}_{\boldsymbol{\lambda}}\}$ by Theorem 8.2. □

Example. By Theorem 6.10, the collection $f_I(\boldsymbol{z}, h) = h^{p(I)} \widetilde{W}_{\sigma_0, I}(\boldsymbol{t}^{-1}, \boldsymbol{z}^{-1}, h^{-1})$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, satisfies the assumption of Lemma 9.2. Hence,

$$\widetilde{W}_{\bullet}(\boldsymbol{t}, \boldsymbol{z}, h) = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} h^{p(I)} \widetilde{W}_{\sigma_0, I}(\boldsymbol{t}^{-1}, \boldsymbol{z}^{-1}, h^{-1}) \, v_I \tag{9.9}$$

as a function of $\boldsymbol{z}, h$ is invariant under the S_n-action (8.2), and formula (9.8) yields

$$\widetilde{W}_{\bullet}(\boldsymbol{t}, \boldsymbol{z}, h) = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} \widetilde{W}_{\mathrm{id}, I^{\min}}(\boldsymbol{t}^{-1}, \boldsymbol{z}_I^{-1}, h^{-1}) \frac{Q(\boldsymbol{z}_I, h)}{R(\boldsymbol{z}_I)} \xi_I , \tag{9.10}$$

since $\widetilde{W}_{\sigma_0, I^{\max}}(\boldsymbol{t}^{-1}, \boldsymbol{z}^{-1}_{\sigma_0(I)}, h^{-1}) = \widetilde{W}_{\mathrm{id}, I^{\min}}(\boldsymbol{t}^{-1}, \boldsymbol{z}_I^{-1}, h^{-1})$ by formula (4.3). Notice also that

$$\xi_I = \widetilde{W}_{\bullet}(\boldsymbol{z}_I, \boldsymbol{z}, h) \frac{P(\boldsymbol{z}_I)}{Q(\boldsymbol{z}_I, h)} , \tag{9.11}$$

either by formulae (9.9) and (8.8), or by formulae (9.10), (6.8), (6.10), and Lemmas 6.2, 6.4. See also a remark on the function $\widetilde{W}_{\bullet}(\boldsymbol{z}_I, \boldsymbol{z}, h)$ at the end of Section 11.2.

9.2. **Space $\frac{1}{D}\mathcal{V}^-$.** Let $\frac{1}{D}\mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$ be the space of functions of the form $\frac{1}{D}f$, where $f \in \mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$ and D is given by formula (8.4). Set $\frac{1}{D}\mathcal{V} = (\mathbb{C}^N)^{\otimes n} \otimes \frac{1}{D}\mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$ and $\frac{1}{D}\mathcal{V}_{\boldsymbol{\lambda}} = (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes \frac{1}{D}\mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$.

Lemma 9.4. *The operators $\hat{s}_1, \dots, \hat{s}_{n-1}$ preserve the space $\frac{1}{D}\mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$.*

Proof. Let $f \in \mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$. Then

$$D \cdot \hat{s}_i\bigl(\tfrac{1}{D}f\bigr) \;=\; \frac{(1 - hz_i/z_{i+1})\, K_i\, f + (h-1)\, f}{1 - z_{i+1}/z_i} \in \mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$$

since the numerator of the right-hand side vanishes if $z_i = z_{i+1}$. □

Define the operators $\check{s}_1, \dots, \check{s}_{n-1}$ as follows:

$$\check{s}_i \;=\; \frac{1 - hz_{i+1}/z_i}{1 - z_{i+1}/z_i}\, \tilde{s}_i + (h-1)\, \frac{z_{i+1}/z_i}{1 - z_{i+1}/z_i}\,, \tag{9.12}$$

where $\tilde{s}_i$ is given by formula (8.2).

Lemma 9.5. *The operators $\check{s}_1, \dots, \check{s}_{n-1},\ z_1, \dots, z_n$ satisfy the relations*

$$(\check{s}_i - 1)(\check{s}_i + h) = 0\,, \qquad \check{s}_i\,\check{s}_{i+1}\check{s}_i = \check{s}_{i+1}\check{s}_i\check{s}_{i+1}\,, \qquad \check{s}_i\check{s}_j = \check{s}_j\check{s}_i \quad \textit{if}\ \ |i-j| > 1\,, \tag{9.13}$$
$$\check{s}_i z_i \check{s}_i = hz_{i+1}\,, \qquad \check{s}_i z_j = z_j \check{s}_i\,, \quad \textit{if}\ \ j \neq i, i+1\,.$$

Proof. The statement follows from Lemma 8.1 and formula (9.12) by direct verification. □

Lemma 9.6. *The operators $\check{s}_1, \dots, \check{s}_{n-1}$ preserve the spaces $\frac{1}{D}\mathcal{V}$ and $\frac{1}{D}\mathcal{V}_{\boldsymbol{\lambda}}$.*

Proof. Let $g \in (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes \mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$. Then by formulae (9.12), (8.2), (8.4),

$$D \cdot \check{s}_i\bigl(\tfrac{1}{D}g\bigr) \;=\; \frac{(1 - hz_i/z_{i+1})\, P^{(i,i+1)}\, \mathcal{R}^{(i,i+1)}(z_i/z_{i+1})\, K_i\, g + (h-1)\,(z_{i+1}/z_i)\, g}{1 - z_{i+1}/z_i}\,.$$

The numerator of the right-hand side belongs to $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes \mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]$ by formulae (7.3), (7.4), and vanishes if $z_i = z_{i+1}$. This proves the statement for $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$.

Since $\frac{1}{D}\mathcal{V}^- = \bigoplus_{|\boldsymbol{\lambda}|=n} \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, the lemma follows. □

Lemma 9.7. *Let $f(z_1, \dots, z_n, h)$ be a $(\mathbb{C}^N)^{\otimes n}$-valued function. Then for any $i = 1, \dots, n-1$, $\tilde{s}_i f = f$ if and only if $\check{s}_i f = f$.*

Proof. The statement follows from formula (9.12). □

Remark. Lemmas 9.6, 9.7 could be considered as a motivation for the operators $\check{s}_1, \dots, \check{s}_{n-1}$.

Denote by $\frac{1}{D}\mathcal{V}^- \subset \frac{1}{D}\mathcal{V}$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} \subset \frac{1}{D}\mathcal{V}_{\boldsymbol{\lambda}}$ the subspaces of invariants of the operators $\check{s}_1, \dots, \check{s}_{n-1}$. By Lemma 9.7, $\frac{1}{D}\mathcal{V}^- \subset \frac{1}{D}\mathcal{V}$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} \subset \frac{1}{D}\mathcal{V}_{\boldsymbol{\lambda}}$ are also the subspaces of functions invariant under the S_n-action (8.2). All four spaces $\frac{1}{D}\mathcal{V}$, $\frac{1}{D}\mathcal{V}_{\boldsymbol{\lambda}}$, $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$, and $\frac{1}{D}\mathcal{V}^-$ are $\mathbb{C}[\boldsymbol{z}^{\pm1}, h^{\pm1}]^{S_n}$-modules.

For a function $f(\boldsymbol{z},h)$, let $\check f(\boldsymbol{z},h)=f(z_n,\dots,z_1,h)$. Denote by $S^{\min}_{\boldsymbol{\lambda}}\subset S_n$ the isotropy subgroup of $I^{\min}$, see (8.3). Let $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ be the algebra of Laurent polynomials in z_1, $\dots,z_n,h$ such that $f(\boldsymbol{z},h)=f(\boldsymbol{z}_\sigma,h)$ for any $\sigma\in S^{\min}_{\boldsymbol{\lambda}}$. Set $\check Q(\boldsymbol{z},h)=Q(\boldsymbol{z}_{I^{\max}},h)$.

Lemma 9.8. *The homomorphism* $\vartheta_{\boldsymbol{\lambda}}:\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_{\boldsymbol{\lambda}}}\to\frac1D\mathcal{V}^-_{\boldsymbol{\lambda}}$,

$$\vartheta_{\boldsymbol{\lambda}}:f(\boldsymbol{z},h)\mapsto\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\hat\sigma_I\Big(\frac{\check f(\boldsymbol{z},h)}{\check Q(\boldsymbol{z},h)}\Big)\,v_I \tag{9.14}$$

is an isomorphism of $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_n}$*-modules.*

Proof. Since $\check Q(\boldsymbol{z},h)$ divides D in $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$, the right-hand side of formula (9.14) belongs to $\frac1D\mathcal{V}_{\boldsymbol{\lambda}}$ by Lemma 9.4. Moreover, it is invariant under the S_n-action (8.2) by Proposition 9.3 and is preserved by the operators $\check s_1,\dots,\check s_{n-1}$ by Lemma 9.7. Hence, $\vartheta_{\boldsymbol{\lambda}}(f)\in\frac1D\mathcal{V}^-_{\boldsymbol{\lambda}}$.

The map $\vartheta_{\boldsymbol{\lambda}}$ is clearly injective. To prove surjectivity, let $\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}g_I(\boldsymbol{z},h)\,v_I\in\frac1D\mathcal{V}^-_{\boldsymbol{\lambda}}$. By Proposition 9.3, $g_I=\hat\sigma_I(g_{I^{\max}})$ for any $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, and $g_{I^{\max}}(\boldsymbol{z},h)=g_{I^{\max}}(\boldsymbol{z}_\sigma,h)$ for any $\sigma\in S^{\max}_{\boldsymbol{\lambda}}$. Therefore, the function $g_{I^{\max}}$ cannot have poles at the hyperplanes $z_i=hz_j$ if $i,j\in I^{\max}_a$ for some a. Hence, $g_{I^{\max}}=\check f/\check Q$ for some $f\in\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_{\boldsymbol{\lambda}}}$. $\square$

Corollary 9.8 yields another formula for the map $\vartheta_{\boldsymbol{\lambda}}$:

$$\vartheta_{\boldsymbol{\lambda}}:f(\boldsymbol{z},h)\mapsto\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z}_I,h)}{R(\boldsymbol{z}_I)}\,\xi_I\,. \tag{9.15}$$

Corollary 9.9. *The homomorphism* $\nu:K_T(\mathcal{X}_n)\to\frac1D\mathcal{V}^-\otimes\mathbb{C}[\boldsymbol{z}^{\pm1}]$

$$\nu:[f(\boldsymbol{\Gamma},\boldsymbol{z},h)]\mapsto\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z}_I,\boldsymbol{z},h)}{R(\boldsymbol{z}_I)}\,\xi_I \tag{9.16}$$

for any $f\in\mathbb{C}[\boldsymbol{\Gamma}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$, *is an isomorphism of* $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$*-modules.* $\square$

Abusing notation, we use here the same letter for the homomorphism ν as in Section 8.3.

Corollary 9.10. *The canonical embeddings*

$$\iota:\tfrac1D\mathcal{V}^-\otimes\mathbb{C}(\boldsymbol{z},h)\to(\mathbb{C}^N)^{\otimes n}\otimes\mathbb{C}(\boldsymbol{z},h)\,,\qquad \iota_{\boldsymbol{\lambda}}:\tfrac1D\mathcal{V}^-_{\boldsymbol{\lambda}}\otimes\mathbb{C}(\boldsymbol{z},h)\to(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes\mathbb{C}(\boldsymbol{z},h)$$

are isomorphisms of $\mathbb{C}(\boldsymbol{z},h)$*-modules.* $\square$

9.3. Subspace $\frac1D\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$.

Let $\frac1{\widetilde D}\mathbb{C}[\boldsymbol{z},h]$ be the space of functions of the form $\frac1{\widetilde D}f$, where $f\in\mathbb{C}[\boldsymbol{z},h]$ and $\widetilde D=\prod_{1\leqslant i<j\leqslant n}(z_j-hz_i)$, cf. (8.4).

Lemma 9.11. *The operators* $\hat s_1,\dots,\hat s_{n-1}$ *preserve the space* $\frac1{\widetilde D}\mathbb{C}[\boldsymbol{z},h]$.

Proof. Let $f\in\mathbb{C}[\boldsymbol{z},h]$. Then

$$\widetilde D\cdot\hat s_i\big(\tfrac1{\widetilde D}f\big)\;=\;\frac{(z_{i+1}-hz_i)\,K_i\,f+(h-1)\,z_i f}{z_i-z_{i+1}}\in\mathbb{C}[\boldsymbol{z},h]$$

since the numerator of the right-hand side vanishes if $z_i=z_{i+1}$. $\square$

Let $\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}$ be the algebra of polynomials in $z_1,\dots,z_n,\,h$ such that $f(\boldsymbol{z})=f(\boldsymbol{z}_\sigma)$ for any $\sigma\in S^{\min}_{\boldsymbol{\lambda}}$. Recall $\lambda^{(a)}=\lambda_1+\ldots+\lambda_a\,,\ a=1,\dots,N$. For any $f\in\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}$, set

$$\widetilde{\vartheta}_{\boldsymbol{\lambda}}(f)=\vartheta_{\boldsymbol{\lambda}}\Bigl(f(\boldsymbol{z},h)\prod_{a=1}^{N}\prod_{i\in I^{\min}_a} z_i^{\lambda^{(a)}-n}\Bigr)\,. \tag{9.17}$$

Denote $\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}=\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}\cap\bigl((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes\frac{1}{D}\mathbb{C}[\boldsymbol{z},h]\bigr)\,.$

Lemma 9.12. *The homomorphism* $\widetilde{\vartheta}_{\boldsymbol{\lambda}}:\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}\to\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$, *is an isomorphism of* $\mathbb{C}[\boldsymbol{z}]^{S_n}\otimes\mathbb{C}[h]$*-modules.*

Proof. Let $f\in\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}$ and $g=f\cdot\prod_{a=1}^N\prod_{i\in I^{\min}_a} z_i^{\lambda^{(a)}-n}$. Then $\widetilde{\vartheta}_{\boldsymbol{\lambda}}(f)=\vartheta_{\boldsymbol{\lambda}}(g)\in\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ by Lemma 9.8. Also, $\check g/\check Q\in\frac{1}{D}\mathbb{C}[\boldsymbol{z},h]$, so $\widetilde{\vartheta}_{\boldsymbol{\lambda}}(f)\in(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes\frac{1}{D}\mathbb{C}[\boldsymbol{z},h]$ by Lemma 9.11. Thus $\widetilde{\vartheta}_{\boldsymbol{\lambda}}(f)\in\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$, and the map $\widetilde{\vartheta}_{\boldsymbol{\lambda}}$ is injective.

To prove surjectivity of $\widetilde{\vartheta}_{\boldsymbol{\lambda}}$, let $\tilde g(\boldsymbol{z},h)=\frac{1}{D}\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}g_I(\boldsymbol{z},h)\,v_I\in\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$. By Proposition 9.8, $g_{I^{\max}}=\widetilde{D}\check f/\check Q$ for some $f\in\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_{\boldsymbol{\lambda}}}$. Hence, the function

$$\tilde f(\boldsymbol{z},h)\,=\,f(\boldsymbol{z},h)\prod_{a=1}^N\prod_{i\in I^{\min}_a} z_i^{n-\lambda^{(a)}}\,=\,\check g_{I^{\max}}(\boldsymbol{z},h)\prod_{a=1}^N\prod_{\substack{i,j\in I^{\min}_a\\ i<j}}(z_i-hz_j)^{-1}$$

is regular at $z_k=0$ for all $k=1,\dots,n$, and at $h=0$. Therefore, $\tilde f\in\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}$ and $\tilde g=\widetilde{\vartheta}_{\boldsymbol{\lambda}}(\tilde f)$. $\square$

9.4. **Grading.** Introduce the degrees of the variables $z_1,\dots,z_n$ and h by the rule

$$\deg z_1=\ldots=\deg z_n=1\,,\qquad \deg h=0\,. \tag{9.18}$$

This defines the grading on the space $\mathbb{C}[\boldsymbol{z}^{\pm1}]\otimes\mathbb{C}(h)$. This grading induces gradings on tensor products of $\mathbb{C}[\boldsymbol{z}^{\pm1}]\otimes\mathbb{C}(h)$ with other vector spaces and subspaces of those tensor products. In particular, $\frac{1}{D}\mathcal{V}^-$ and $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ are graded $\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]$-modules, and $\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$ is a graded $\mathbb{C}[\boldsymbol{z},h]$-module.

Lemma 9.13. *The maps* $\vartheta_{\boldsymbol{\lambda}}:\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]^{S_{\boldsymbol{\lambda}}}\to\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ *and* $\widetilde{\vartheta}_{\boldsymbol{\lambda}}:\mathbb{C}[\boldsymbol{z},h]^{S_{\boldsymbol{\lambda}}}\to\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}$ *are isomorphisms of graded spaces.* $\square$

For the variables $\gamma_{i,j}$, set $\deg\gamma_{i,j}=1$ for all i,j. This defines the grading on the algebra $K_T(\mathcal{X}_n)$, see (2.2), (2.3), making it into a graded algebra.

Lemma 9.14. *The map* $\nu:K_T(\mathcal{X}_n)\to\frac{1}{D}\mathcal{V}^-\otimes\mathbb{C}[\boldsymbol{z}^{\pm1}]$ *is an isomorphism of graded spaces.* $\square$

10. Quantum loop algebra

10.1. **Quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_N})$.** Let $q,\ u$ be parameters. Let $\mathbb{C}(u,q)$ be the algebra of rational functions of $q,\ u$. The *Cherednik-Drinfeld-Jimbo R-matrix* $R(u,q)\in\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)\otimes\mathbb{C}(u,q)$ is defined by the conditions:

- For $i = 1, \dots, N$,

$$R(u,q) : v_i \otimes v_i \mapsto (qu - q^{-1})\, v_i \otimes v_i; \tag{10.1}$$

- For $1 \leqslant i < j \leqslant N$, on the two-dimensional subspace with ordered basis $v_i \otimes v_j$, $v_j \otimes v_i$, the R-matrix $R(u,q)$ is given by the matrix

$$\begin{pmatrix} u-1 & u(q-q^{-1}) \\ q-q^{-1} & u-1 \end{pmatrix}, \tag{10.2}$$

cf. (7.3), (7.4).

The quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_N})$ is a unital associative algebra over $\mathbb{C}(q)$ with generators $\widehat{L}_{i,j,+0}, \widehat{L}_{j,i,-0}$, $1 \leqslant j \leqslant i \leqslant N$, and $\widehat{L}_{i,j,\pm s}$, $i,j = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, subject to relations (10.3), (10.4), see [RS], [DF]. For convenience, set $\widehat{L}_{i,j,+0} = \widehat{L}_{j,i,-0} = 0$ for $1 \leqslant i < j \leqslant N$.

Introduce the generating series $\widehat{L}_{i,j,\pm}(u) = \widehat{L}_{i,j,\pm 0} + \sum_{s=1}^{\infty} \widehat{L}_{i,j,\pm s}\, u^{\pm s}$, and consider them as entries of $N \times N$ matrices $\widehat{L}_{\pm}(u) = (\widehat{L}_{i,j,\pm}(u))_{i,j=1}^{N}$. The relations in $U_q(\widetilde{\mathfrak{gl}_N})$ have the form

$$\widehat{L}_{i,i,+0}\, \widehat{L}_{i,i,-0} \,=\, \widehat{L}_{i,i,-0}\, \widehat{L}_{i,i,+0} \,=\, 0\,, \qquad i = 1, \dots, N, \tag{10.3}$$

$$R^{(1,2)}(u/v,q)\, \widehat{L}^{(1)}_{\alpha}(u)\, \widehat{L}^{(2)}_{\beta}(v) \,=\, \widehat{L}^{(2)}_{\beta}(v)\, \widehat{L}^{(1)}_{\alpha}(u)\, R^{(1,2)}(u/v,q)\,, \tag{10.4}$$

where $(\alpha, \beta) = (+,+), (+,-), (-,-)$.

10.2. **Algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$.** Let $h = q^{-2}$. Let $\mathbb{C}(h)$ be the algebra of rational functions of h. The algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ is the unital associative algebra over $\mathbb{C}(h)$ generated by the following elements of $U_q(\widetilde{\mathfrak{gl}_N})$:

$$\widetilde{L}_{i,j,s} \,=\, \widehat{L}_{1,1,+0} \dots \widehat{L}_{i-1,i-1,+0}\ \widehat{L}_{i,j,s}\ \widehat{L}_{1,1,+0} \dots \widehat{L}_{j,j,+0} \tag{10.5}$$

and

$$\widetilde{L}^{-1}_{i,i,+0} = \bigl(\widehat{L}_{1,1,-0} \dots \widehat{L}_{i,i,-0}\bigr)^2 \tag{10.6}$$

for all possible i,j,s. Notice that $\widetilde{L}_{i,j,+0} = \widetilde{L}_{j,i,-0} = 0$ for $1 \leqslant i < j \leqslant N$. Consider the generating series

$$\widetilde{L}_{i,j,\pm}(u) \,=\, \widetilde{L}_{i,j,\pm 0} + \sum_{s=1}^{\infty} \widetilde{L}_{i,j,\pm s}\, u^{\pm s}$$

and the matrices $\widetilde{L}_{\pm}(u) = \bigl(\widetilde{L}_{i,j,\pm}(u)\bigr)_{i,j=1}^{N}$. The relations in $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ have the form

$$\widetilde{L}_{i,i,+0}\, \widetilde{L}^{-1}_{i,i,+0} \,=\, \widetilde{L}^{-1}_{i,i,+0}\, \widetilde{L}_{i,i,+0} \,=\, 1\,, \qquad i = 1, \dots, N, \tag{10.7}$$

$$\widetilde{L}_{1,1,-0} \,=\, 1, \qquad \widetilde{L}_{i,i,+0} \,=\, \widetilde{L}_{i+1,i+1,-0}\,, \quad i = 1, \dots, N-1\,, \tag{10.8}$$

$$\mathcal{R}^{(1,2)}(v/u,h)\, \widetilde{L}^{(1)}_{\alpha}(u)\, \widetilde{L}^{(2)}_{\beta}(v) \,=\, \widetilde{L}^{(2)}_{\beta}(v)\, \widetilde{L}^{(1)}_{\alpha}(u)\, \mathcal{R}^{(1,2)}(v/u,h)\,, \tag{10.9}$$

where $(\alpha, \beta) = (+,+), (+,-), (-,-)$, and $\mathcal{R}(z,h)$ is defined by (7.3), (7.4).

Denote by $\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\widetilde{\mathfrak{gl}_N})$ the subalgebra generated over $\mathbb{C}(h)$ by the elements

$$\widetilde{L}_{1,1,+0}\,, \dots, \widetilde{L}_{N,N,+0}\,,\ \widetilde{L}^{-1}_{1,1,+0}\,, \dots, \widetilde{L}^{-1}_{N,N,+0}\,. \tag{10.10}$$

The subalgebra $\mathcal{U}(\mathfrak{h})$ is commutative. The elements $\widetilde{L}_{N,N,+0}$ and $\widetilde{L}^{-1}_{N,N,+0}$ are central.

The algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ is a Hopf algebra with the coproduct $\Delta : \mathcal{U}(\widetilde{\mathfrak{gl}_N}) \to \mathcal{U}(\widetilde{\mathfrak{gl}_N}) \otimes \mathcal{U}(\widetilde{\mathfrak{gl}_N})$ given by

$$\Delta : \widetilde{L}_{i,j,\pm}(u) \mapsto \sum_{k=1}^{N} \widetilde{L}_{k,j,\pm}(u) \otimes \widetilde{L}_{i,k,\pm}(u)\,, \qquad i,j = 1,\dots,N. \tag{10.11}$$

The algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ is graded by the rule: $\deg \widetilde{L}_{i,j,\pm s} = \pm s$ for all $i,j = 1,\dots,N$ and $s \in \mathbb{Z}_{\geqslant 0}$, and $\deg h = 0$.

10.3. **Quantum minors.** For $p = 1,\dots,N$, $\boldsymbol{i} = \{1 \leqslant i_1 < \dots < i_p \leqslant N\}$, $\boldsymbol{j} = \{1 \leqslant j_1 < \dots < j_p \leqslant N\}$, define quantum minors

$$M_{\boldsymbol{i},\boldsymbol{j},\pm}(u) \,=\, \sum_{\sigma\in S_p} (-1)^\sigma\, \widetilde{L}_{i_1,j_{\sigma(1)},\pm}(u)\, \widetilde{L}_{i_2,j_{\sigma(2)},\pm}(uh) \,\dots\, \widetilde{L}_{i_p,j_{\sigma(p)},\pm}(uh^{p-1})\,.$$

Lemma 10.1. *For any permutation $\pi \in S_p$, we have*

$$M_{\boldsymbol{i},\boldsymbol{j},\pm}(u) \,=\, (-1)^\pi \sum_{\sigma\in S_p} (-1)^\sigma\, \widetilde{L}_{i_{\pi(1)},j_{\sigma(1)},\pm}(u)\, \widetilde{L}_{i_{\pi(2)},j_{\sigma(2)},\pm}(uh) \,\dots\, \widetilde{L}_{i_{\pi(p)},j_{\sigma(p)},\pm}(uh^{p-1})\,.$$

Proof. The statement follows from commutation relations (10.9), see, for example, [MTV1, formulae (4.9), (4.10)] □

Remark. Though [MTV1] deals with the case of the Yangian $Y(\mathfrak{gl}_N)$, the proofs given there rely upon general properties of R-matrices and can be easily tuned for the case of the algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ under consideration.

We also have

$$\Delta : M_{\boldsymbol{i},\boldsymbol{j},\pm}(u) \mapsto \sum_{\boldsymbol{k}=\{1\leqslant k_1<\dots<k_p\leqslant N\}} M_{\boldsymbol{i},\boldsymbol{k},\pm}(u) \otimes M_{\boldsymbol{k},\boldsymbol{j},\pm}(u)\,, \tag{10.12}$$

see, for example, [NT1, Proposition 1.11] or [MTV1, Lemma 4.3].

Introduce the series $A_{1,\pm}(u),\dots,A_{N,\pm}(u)$, $E_{1,\pm}(u),\dots,E_{N-1,\pm}(u)$, $F_{1,\pm}(u),\dots,F_{N-1,\pm}(u)$ as follows: given p, take $\boldsymbol{i} = \{1,\dots,p\}$, $\boldsymbol{j} = \{1,\dots,p-1,p+1\}$, and set

$$A_{p,\pm}(u) \,=\, M_{\boldsymbol{i},\boldsymbol{i},\pm}(u)\, \widetilde{L}^{-1}_{1,1,\pm 0} \dots \widetilde{L}^{-1}_{p,p,\pm 0} \,=\, 1 + \sum_{s=1}^{\infty} B^{\infty}_{p,\pm s}\, u^{\pm s}, \tag{10.13}$$

$$E_{p,\pm}(u) \,=\, (1-h)^{-1}\, M_{\boldsymbol{j},\boldsymbol{i},\pm}(u) \big(M_{\boldsymbol{i},\boldsymbol{i},\pm}(u)\big)^{-1}, \tag{10.14}$$

$$F_{p,\pm}(u) \,=\, (1-h)^{-1} \big(M_{\boldsymbol{i},\boldsymbol{i},\pm}(u)\big)^{-1} M_{\boldsymbol{i},\boldsymbol{j},\pm}(u)\,. \tag{10.15}$$

The coefficients of these series together with $\mathcal{U}(\mathfrak{h})$ generate the algebra $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$. In what follows we will describe the action of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ by using the series $A_{p,\pm}(u)$, $E_{p,\pm}(u)$, $F_{p,\pm}(u)$.

By formula (10.12), the series $A_{N,\pm}(u)$ are group-like:

$$\Delta : A_{N,\pm}(u) \mapsto A_{N,\pm}(u) \otimes A_{N,\pm}(u)\,. \tag{10.16}$$

Let $\mathcal{B}^\infty \subset \mathcal{U}(\widetilde{\mathfrak{gl}_N})$ be the unital subalgebra generated by $\mathcal{U}(\mathfrak{h})$ and the elements $B^\infty_{p,\pm s}$ for $p = 1, \ldots, N$, $s \in \mathbb{Z}_{>0}$, see (10.13). The subalgebra $\mathcal{B}^\infty$ is called the *Gelfand-Zetlin subalgebra* of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$. For any $\mathcal{B}^\infty$-module V we denote by $\mathcal{B}^\infty(V)$ the image of $\mathcal{B}^\infty$ in $\mathrm{End}(V)$ and call $\mathcal{B}^\infty(V)$ the *Gelfand-Zetlin algebra of* V.

Theorem 10.2 ([KS]). *The subalgebra $\mathcal{B}^\infty$ is commutative. The elements $B^\infty_{N,\pm s}$, $s \in \mathbb{Z}_{>0}$, are central.* □

10.4. **Bethe algebra $\mathcal{B}^q$.** For $q = (q_1, \ldots, q_N) \in (\mathbb{C}^\times)^N$ and $p = 1, \ldots, N$, define

$$B^q_{p,\pm}(u) \,=\, \sum_{\boldsymbol{i}=\{1\leqslant i_1<\ldots<i_p\leqslant N\}} q_{i_1}\ldots q_{i_p}\, M_{\boldsymbol{i},\boldsymbol{i},\pm}(u) \prod_{r=1}^{p} \widetilde{L}^{-1}_{i_r,i_r,-0} \,=\, \sum_{s=0}^{\infty} B^q_{p,\pm s}\, u^{\pm s}\,. \tag{10.17}$$

In particular,

$$B^q_{p,+0} \,=\, e_p(\tilde q_1, \ldots, \tilde q_N)\,, \qquad B^q_{p,-0} \,=\, e_p(q_1, \ldots, q_N)\,,$$

where e_p is the p-th elementary symmetric function, and $\tilde q_{i,\pm} = q_i\, \widetilde{L}_{i,i,+0} \widetilde{L}^{-1}_{i,i,-0}$.

Let $\mathcal{B}^q \subset \mathcal{U}(\widetilde{\mathfrak{gl}_N})$ be the unital subalgebra generated by $\mathcal{U}(\mathfrak{h})$ and the elements $B^q_{p,\pm s}$, $p = 1, \ldots, N$, $s \in \mathbb{Z}_{>0}$. It is easy to see that the subalgebra $\mathcal{B}^q$ does not change if all q_1, $\ldots, q_N$ are multiplied simultaneously by the same number. The algebra $\mathcal{B}^q$ is called the *Bethe subalgebra* of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$. For any $\mathcal{B}^q$-module V we denote by $\mathcal{B}^q(V)$ the image of $\mathcal{B}^q$ in $\mathrm{End}(V)$ and call $\mathcal{B}^q(V)$ the *Bethe algebra of* V.

Theorem 10.3 ([KS]). *The subalgebra $\mathcal{B}^q$ is commutative.* □

The elements $B^q_{p,\pm s}$ depend polynomially on $q_1, \ldots, q_N$. Suppose $q_1 = 1$ and $q_{i+1}/q_i \to 0$ for all $i = 1, \ldots, N-1$. In this limit,

$$B^q_{p,+}(u) \,=\, q_1 \ldots q_p \bigl(A_{p,+}(u)\, \widetilde{L}_{p,p,+0} + o(1)\bigr)\,, \quad B^q_{p,-}(u) \,=\, q_1 \ldots q_p \bigl(A_{p,-}(u) + o(1)\bigr)\,. \tag{10.18}$$

10.5. **Difference operators.** Let τ be the multiplicative shift operator acting on functions of u as follows: $\tau f(u) = f(hu)$. For $r = (r_1, \ldots, r_N) \in (\mathbb{C}^\times)^N$ and $i, j = 1, \ldots, N$, set $X_{i,j,\pm} = \delta_{i,j} - r_i\, \widetilde{L}_{i,j,\pm}(u)\, \tau$. Define the difference operators

$$\mathcal{D}_\pm \,=\, \sum_{\sigma \in S_N} (-1)^\sigma X_{1,\sigma(1),\pm} X_{2,\sigma(2),\pm} \ldots X_{N,\sigma(N),\pm}\,.$$

Then

$$\mathcal{D}_\pm \,=\, 1 + \sum_{p=1}^{N} \sum_{\boldsymbol{i}=\{1\leqslant i_1<\ldots<i_p\leqslant N\}} r_{i_1}\ldots r_{i_p}\, M_{\boldsymbol{i},\boldsymbol{i},\pm}(u)\, (-\tau)^p\,, \tag{10.19}$$

cf. formula (10.17)

10.6. **More subalgebras of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$.** Let $\mathcal{U}_\pm$, $\mathcal{B}^\infty_\pm$, $\mathcal{B}^q_\pm$ be the subalgebras of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ generated by $\mathcal{U}(\mathfrak{h})$ and the following elements, respectively:

$\mathcal{U}_\pm$: $\widetilde{L}_{i,j,\pm s}$ for $i, j = 1, \ldots, N$, $s \in \mathbb{Z}_{>0}$;
$\mathcal{B}^\infty_\pm$: $B^\infty_{p,\pm s}$ for $p = 1, \ldots, N$, $s \in \mathbb{Z}_{>0}$;
$\mathcal{B}^q_\pm$: $B^q_{p,\pm s}$ for $p = 1, \ldots, N$, $s \in \mathbb{Z}_{>0}$;

Let $\mathcal{Z}_\pm$ be the subalgebras of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ generated over $\mathbb{C}(h)$ by $\widetilde{L}_{N,N,+0}$, $\widetilde{L}^{-1}_{N,N,+0}$ and the respective elements $B^\infty_{N,\pm s}$, $s \in \mathbb{Z}_{>0}$. Let $\mathcal{Z} \subset \mathcal{U}(\widetilde{\mathfrak{gl}_N})$ be the subalgebra generated by $\mathcal{Z}_+$ and $\mathcal{Z}_-$. Recall that $\mathcal{Z}$ lies in the center of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$.

Denote by $\mathcal{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak{h}}$ the commutant of $\mathcal{U}(\mathfrak{h})$ in $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$. Recall that $\mathcal{B}^\infty, \mathcal{B}^q \subset \mathcal{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak{h}}$.

11. Space $\frac{1}{D}\mathcal{V}^- \otimes \mathbb{C}(h)$ as a module over $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$

11.1. **Action of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$.** Recall the R-matrix $\mathcal{R}(u)$ defined in Section 7.2. Set

$$L(u) \,=\, \mathcal{R}^{(0,n)}(z_n/u)\dots\mathcal{R}^{(0,1)}(z_1/u)\,\prod_{i=1}^n \frac{1-hz_i/u}{1-z_i/u}\,, \tag{11.1}$$

where the factors of $(\mathbb{C}^N)^{\otimes(n+1)} = \mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes n}$ are labeled by $0,1,\dots,n$ from left to right. We think of $L(u)$ as an $N\times N$-matrix with $\mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big)$-valued entries $L_{i,j}(u)$, depending on $u, z_1, \dots, z_n, h$.

Expand $L_{i,j}(u)$ into Laurent series at $u=0$ and $u=\infty$:

$$L_{i,j}(u) \,=\, L_{i,j,+0} + \sum_{s=1}^\infty L_{i,j,s}\,u^s\,, \qquad L_{i,j}(u) \,=\, L_{i,j,-0} + \sum_{s=1}^\infty L_{i,j,-s}\,u^{-s}\,. \tag{11.2}$$

Then $L_{i,j,+s} \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{-1}, h]$ and $L_{i,j,-s} \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}, h]$ for $s \in \mathbb{Z}_{\geqslant 0}$, and the degree in h of each $L_{i,j,s}$ is at most n.

Proposition 11.1. *The assignment $\phi : \widetilde{L}_{i,j,s} \mapsto L_{i,j,s}$ for all i,j,s, defines a homomorphism of graded algebras $\mathcal{U}(\widetilde{\mathfrak{gl}_N}) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$.*

Proof. The claim follows from formulae (7.3), (7.4) and the Yang-Baxter equation (7.6). □

Notice that $\phi(\mathcal{U}_+) \subset \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{-1}] \otimes \mathbb{C}(h)$ and $\phi(\mathcal{U}_-) \subset \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}(h)$.

We will indicate for a while the dependence of the homomorphism ϕ on n by denoting it ϕ_n. Notice that ϕ_n is the composition of a tensor power of ϕ_1 with the iteration of the coproduct (10.11):

$$\phi_n \,=\, \phi_1^{\otimes n} \circ \Delta^{(n)}\,. \tag{11.3}$$

This observation will be explored in several proofs in this section.

Lemma 11.2. *All operators $\phi(B^\infty_{N,s})$ are scalar, and*

$$1 + \sum_{s=1}^\infty \phi(B^\infty_{N,s})\,u^s \,=\, \prod_{i=1}^n \frac{1-h^{-1}u/z_i}{1-u/z_i}\,, \tag{11.4}$$

$$1 + \sum_{s=1}^\infty \phi(B^\infty_{N,-s})\,u^{-s} \,=\, \prod_{i=1}^n \frac{1-hz_i/u}{1-z_i/u}\,, \tag{11.5}$$

where the products in the right-hand sides are expanded at $u=0$ and $u=\infty$, respectively.

Proof. We will prove formula (11.5). The proof of formula (11.4) is similar.

Let $n=1$. Let $v_1,\ldots,v_N$ be the standard basis of $\mathbb{C}^N$. To show that

$$\phi\big(A_{N,-}(u)\big)\,v_i\,=\,\frac{1-hz_1/u}{1-z_1/u}\;v_i\,,$$

we compute the quantum minor in $A_{N,-}(u)$ using Lemma 10.1 and taking the permutation π such that $\pi(1)=i$.

Formula (11.5) for general n follows from formulae (11.3) and (10.16), and the result for $n=1$. □

Corollary 11.3. *We have* $\phi(\mathcal{Z}_+)=\mathbb{C}[\boldsymbol{z}^{-1}]^{S_n}\otimes\mathbb{C}(h)$ *and* $\phi(\mathcal{Z}_-)=\mathbb{C}[\boldsymbol{z}]^{S_n}\otimes\mathbb{C}(h)$.

Proof. The images $\phi(\mathcal{Z}_\mp)$ contain the power sums of $z_1,\ldots,z_n$ or $z_1^{-1},\ldots,z_n^{-1}$, respectively. For instance, formula (11.5) yields

$$\log\Big(1+\sum_{s=1}^{\infty}\phi(B^{\infty}_{p,-s})\,u^{-s}\Big)\,=\,\sum_{s=1}^{\infty}\frac{h^s-1}{s}\,(z_1^s+\ldots+z_n^s)\,u^{-s}\,.$$

□

Denote $y=z_1\ldots z_n$. Notice that $y\in\phi(\mathcal{Z}_-)$ and $y^{-1}\in\phi(\mathcal{Z}_+)$. Corollary 11.3 implies

$$\phi(\mathcal{Z})\,=\,\phi(\mathcal{Z}_+)\otimes\mathbb{C}[y]\,=\,\phi(\mathcal{Z}_-)\otimes\mathbb{C}[y^{-1}]\,. \tag{11.6}$$

Corollary 11.4. *We have*

$$\begin{aligned}\phi\big(\mathcal{U}(\widetilde{\mathfrak{gl}_N})\big)\,&=\,\phi(\mathcal{U}_+)\otimes\mathbb{C}[y]\,=\,\phi(\mathcal{U}_+)\otimes\phi(\mathcal{Z}_-)\,,\\&=\,\phi(\mathcal{U}_-)\otimes\mathbb{C}[y^{-1}]\,=\,\phi(\mathcal{U}_-)\otimes\phi(\mathcal{Z}_+)\,,\end{aligned} \tag{11.7}$$

$$\begin{aligned}\phi(\mathcal{B}^{\infty})\,&=\,\phi(\mathcal{B}^{\infty}_+)\otimes\mathbb{C}[y]\,=\,\phi(\mathcal{B}^{\infty}_+)\otimes\phi(\mathcal{Z}_-)\,,\\&=\,\phi(\mathcal{B}^{\infty}_-)\otimes\mathbb{C}[y^{-1}]\,=\,\phi(\mathcal{B}^{\infty}_-)\otimes\phi(\mathcal{Z}_+)\,,\end{aligned} \tag{11.8}$$

$$\begin{aligned}\phi(\mathcal{B}^{q})\,&=\,\phi(\mathcal{B}^{q}_+)\otimes\mathbb{C}[y]\,=\,\phi(\mathcal{B}^{q}_+)\otimes\phi(\mathcal{Z}_-)\,,\\&=\,\phi(\mathcal{B}^{q}_-)\otimes\mathbb{C}[y^{-1}]\,=\,\phi(\mathcal{B}^{q}_-)\otimes\phi(\mathcal{Z}_+)\,.\end{aligned} \tag{11.9}$$

Proof. The product $T_+(u)=L(u)\,\prod_{i=1}^n(1-u/z_i)$ is a polynomial in u. By Corollary 11.3, the coefficients of $T_+(u)$ belong to $\phi(\mathcal{U}_+)$ and together with $\phi(\mathcal{Z}_+)$ generate $\phi(\mathcal{U}_+)$. Similarly, $T_-(u)=L(u)\,\prod_{i=1}^n(1-z_i/u)$ is a polynomial in u^{-1} and the coefficients of $T_-(u)$ together with $\phi(\mathcal{Z}_-)$ generate $\phi(\mathcal{U}_-)$. Since $(-u)^nT_-(u)=y\,T_+(u)$ and taking into account (11.6), the first equalities in relation (11.7) follow. The second equalities in (11.7) hold because $y\in\phi(\mathcal{Z}_-)$ and $y^{-1}\in\phi(\mathcal{Z}_+)$.

Relations (11.8) and (11.9) follow by the same reasoning from the definition of the subalgebras involved. □

The homomorphism $\phi : \mathfrak{U}(\widetilde{\mathfrak{gl}_N}) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$ defines an action of the algebra $\mathfrak{U}(\widetilde{\mathfrak{gl}_N})$ on $(\mathbb{C}^N)^{\otimes n}$-valued functions of $z_1, \dots, z_n$ and h. In what follows when acting by $X \in \mathfrak{U}(\widetilde{\mathfrak{gl}_N})$ on a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(\boldsymbol{z}, h)$, we will write Xf instead of $\phi(X)\, f$. Clearly, for any $i = 1, \dots, N$ and $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ we have

$$\widetilde{L}_{i,i,+0}\, v_I \,=\, h^{\lambda_1 + \ldots + \lambda_i} v_I \,, \qquad \widetilde{L}_{i,i,-0}\, v_I \,=\, h^{\lambda_1 + \ldots + \lambda_{i-1}} v_I \,. \tag{11.10}$$

Therefore, for any $X \in \mathfrak{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak{h}}$ the operator $\phi(X)$ preserves the subspaces $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. We denote by

$$\phi_{\boldsymbol{\lambda}} : \mathfrak{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak{h}} \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h) \tag{11.11}$$

the corresponding homomorphism.

Lemma 11.5. *The $\mathfrak{U}(\widetilde{\mathfrak{gl}_N})$-action commutes with the S_n-action defined by* (8.2)*, that is, $\tilde{s}_i\, \phi(X) = \phi(X)\, \tilde{s}_i$ for any $i = 1, \dots, n-1$ and $X \in \mathfrak{U}(\widetilde{\mathfrak{gl}_N})$.*

Proof. The statement follows from the Yang-Baxter equation (7.6). □

Corollary 11.6. *The $\mathfrak{U}(\widetilde{\mathfrak{gl}_N})$-action commutes with the operators $\check{s}_1, \dots, \check{s}_{n-1}$ given by* (9.12)*.* □

Corollary 11.7. *The homomorphism $\phi : \mathfrak{U}(\widetilde{\mathfrak{gl}_N}) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$ makes the spaces $\frac{1}{D}\mathcal{V}^- \otimes \mathbb{C}(h)$ and $\frac{1}{D}\mathcal{V}^- \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$ into graded $\mathfrak{U}(\widetilde{\mathfrak{gl}_N})$-modules.* □

11.2. **Action of $\mathcal{B}^\infty$ on the vectors ξ_I.** By formulae (11.10), for any $i = 1, \dots, N$ and $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ we have

$$\widetilde{L}_{i,i,+0}\, \xi_I \,=\, h^{\lambda_1 + \ldots + \lambda_i} \xi_I \,, \qquad \widetilde{L}_{i,i,-0}\, \xi_I \,=\, h^{\lambda_1 + \ldots + \lambda_{i-1}} \xi_I \,. \tag{11.12}$$

Theorem 11.8. *We have*

$$A_{p,+}(u)\, \xi_I \,=\, \xi_I \prod_{a=1}^{p} \prod_{i \in I_a} \frac{1 - h^{-1} u / z_i}{1 - u/z_i} \,, \qquad A_{p,-}(u)\, \xi_I \,=\, \xi_I \prod_{a=1}^{p} \prod_{i \in I_a} \frac{1 - h z_i / u}{1 - z_i/u} \tag{11.13}$$

$$E_{p,+}(u)\, \xi_I \,=\, -\sum_{i \in I_{p+1}} \frac{\xi_{I^{i\prime}}}{1 - u/z_i} \prod_{\substack{j \in I_{p+1} \\ j \neq i}} \frac{1 - h z_j / z_i}{1 - z_j / z_i} \,, \tag{11.14}$$

$$E_{p,-}(u)\, \xi_I \,=\, \sum_{i \in I_{p+1}} \xi_{I^{i\prime}} \frac{z_i/u}{1 - z_i/u} \prod_{\substack{j \in I_{p+1} \\ j \neq i}} \frac{1 - h z_j / z_i}{1 - z_j / z_i} \,, \tag{11.15}$$

$$F_{p,+}(u)\, \xi_I \,=\, -\sum_{i \in I_p} \xi_{I^{\prime i}} \frac{u/z_i}{1 - u/z_i} \prod_{\substack{j \in I_p \\ j \neq i}} \frac{1 - h z_i / z_j}{1 - z_i / z_j} \,, \tag{11.16}$$

$$F_{p,-}(u)\, \xi_I \,=\sum_{i \in I_p} \frac{\xi_{I^{\prime i}}}{1 - z_i/u} \prod_{\substack{j \in I_p \\ j \neq i}} \frac{1 - h z_i / z_j}{1 - z_i / z_j} \,, \tag{11.17}$$

where the sequences $I^{i\prime}$ *and* $I^{\prime i}$ *are defined as follows:* $I^{i\prime}_a = I^{\prime i}_a = I_a$ *for* $a \neq p, p+1$, *and* $I^{i\prime}_p = I_p \cup \{i\}$, $I^{i\prime}_{p+1} = I_{p+1} - \{i\}$, $I^{\prime i}_p = I_p - \{i\}$, $I^{\prime i}_{p+1} = I_{p+1} \cup \{i\}$.

Proof. First observe that by formula (8.5) and Lemma 11.5, it suffices to prove formulae (11.13) – (11.17) only for $I = I^{\min}$. In this case, formula (11.13) for $n > 1$ follows from the coproduct formula (10.12) and the $n = 1$ case of (11.13). The proof of (11.13) for $n = 1$ is straightforward by using Lemma 10.1.

The proofs of formulae (11.14) – (11.17) are similar to each other. As an example, we prove formula (11.17). To verify (11.17) for $I = I^{\min}$, observe that by formulae (8.6), (10.12), (10.15), (11.13), we have $F_{p,-}(u)\,\xi_{I^{\min}} = \sum_{i \in I^{\min}_p} c_i\, \xi_{I^{\min,\prime i}}$. The smallest element of $I^{\min}_p$ equals $i_{\min} = \lambda_1 + \ldots + \lambda_{p-1} + 1$. The coefficient $c_{i_{\min}}$ can be calculated due to the triangularity property (8.6), and does have the required form. The coefficient c_i for other $i \in I^{\min}_p$ can be obtained from $c_{i_{\min}}$ by permuting z_i and $z_{i_{\min}}$ because $I^{\min}$ is invariant under the transposition of i and $i_{\min}$. Thus all the coefficients c_i are as required, which proves formula (11.17). □

Remark. Notice that the right-hand sides of formulae (11.14) and (11.15) coincide as rational functions. This function is expanded at $u = 0$ in (11.14) and at $u = \infty$ in (11.15). Similarly, the right-hand sides of (11.16) and (11.17) are the same rational function that is expanded at $u = 0$ in (11.16) and at $u = \infty$ in (11.17).

Remark. The $(\mathbb{C}^N)^{\otimes n}$-valued function $\widetilde{W}_{\bullet}(\boldsymbol{t}, \boldsymbol{z}, h)$ in formula (9.9) is known as the off-shell Bethe vector. The values of that function at $\boldsymbol{t} = \boldsymbol{z}_I$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, give the eigenvectors ξ_I of the Bethe algebra $\mathcal{B}^{\infty}$, see formula (9.11).

11.3. **Isomorphism** $\psi_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h) \to \phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty})$**.** Let $\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}}$ be the algebra of Laurent polynomials such that $f(\boldsymbol{z}) = f(\boldsymbol{z}_{\sigma})$ for any $\sigma \in S^{\min}_{\boldsymbol{\lambda}}$. For $g \in \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)$, define $\psi_{\boldsymbol{\lambda}}(g) \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big) \otimes \mathbb{C}(\boldsymbol{z}, h)$ by the rule

$$\psi_{\boldsymbol{\lambda}}(g) : \xi_I \mapsto g(\boldsymbol{z}_I, h)\, \xi_I, \qquad I \in \mathcal{I}_{\boldsymbol{\lambda}}, \tag{11.18}$$

see Lemma 8.3. The map

$$\psi_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big) \otimes \mathbb{C}(\boldsymbol{z}, h). \tag{11.19}$$

is clearly a monomorphism of graded algebras.

Lemma 11.9. *For any* $f \in \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_n} \otimes \mathbb{C}(h)$, *we have* $\psi_{\boldsymbol{\lambda}}(f) = \mathrm{id} \otimes f$. □

Theorem 11.10. *We have* $\psi_{\boldsymbol{\lambda}}\big(\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)\big) = \phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty})$.

Proof. By formulae (11.13), $\phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty})$ is generated over $\mathbb{C}(h)$ by the images of the power sums $z_1^s + \ldots + z^s_{\lambda^{(p)}}$, where $\lambda^{(p)} = \lambda_1 + \ldots + \lambda_p$, for all $p = 1, \ldots, N$ and $s \in \mathbb{Z}$, cf. the proof of Corollary 11.3. □

Corollary 11.11. *We have* $\psi_{\boldsymbol{\lambda}}\big(\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)\big) \subset \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$. □

For any $g \in \mathbb{C}[\boldsymbol{z}^{\pm 1}, h^{\pm 1}]^{S_{\boldsymbol{\lambda}}}$, the operator $\psi_{\boldsymbol{\lambda}}(g)$ preserves the space $\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}}$, see Lemma 9.8 and formula (9.15). The restriction of $\psi_{\boldsymbol{\lambda}}(g)$ to $\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}}$ can be also presented as follows:

$$\psi_{\boldsymbol{\lambda}}(g) = \vartheta_{\boldsymbol{\lambda}} \circ m(g) \circ \vartheta^{-1}_{\boldsymbol{\lambda}}, \tag{11.20}$$

where $m(g)$ is the operator of multiplication by g on $\mathbb{C}[\boldsymbol{z}^{\pm 1}, h^{\pm 1}]^{S_{\boldsymbol{\lambda}}}$, see (9.14).

Let $\mathcal{A}$ be a commutative algebra. The algebra $\mathcal{A}$ considered as an $\mathcal{A}$-module with any element of $\mathcal{A}$ acting by multiplication on itself is called the *regular representation* of $\mathcal{A}$.

Theorem 11.12. *The* $\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_n} \otimes \mathbb{C}(h)$*-module isomorphism*

$$\vartheta_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h) \to \tfrac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}} \otimes \mathbb{C}(h) \tag{11.21}$$

and the $\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_n} \otimes \mathbb{C}(h)$*-algebra isomorphism* $\psi_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h) \to \phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty})$ *identify the* $\phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty})$*-module* $\frac{1}{D}\mathcal{V}^{-}_{\boldsymbol{\lambda}} \otimes \mathbb{C}(h)$ *with the regular representation of* $\mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)$.

The isomorphism $\vartheta_{\boldsymbol{\lambda}}$ in (11.21) is the natural extension of the isomorphism (9.14) denoted by the same letter.

12. Space $K_T(\mathcal{X}_n) \otimes \mathbb{C}(h)$ as a module over $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$

12.1. **Action of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ on $K_T(\mathcal{X}_n) \otimes \mathbb{C}(h)$.** Recall

$$K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) = \mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}, h^{\pm 1}] \big/ \big\langle f(\boldsymbol{\Gamma}) = f(\boldsymbol{z}),\ f \in \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_n} \big\rangle$$

and

$$K_T(\mathcal{X}_n) = \bigoplus_{|\boldsymbol{\lambda}|=n} K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}).$$

Define the graded algebra monomorphism

$$\mu_{\boldsymbol{\lambda}} : K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(h) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$$

by the rule

$$\mu_{\boldsymbol{\lambda}}\big([f(\boldsymbol{\Gamma}, \boldsymbol{z}, h)]\big) : \xi_I \mapsto f(\boldsymbol{z}_I, \boldsymbol{z}, h)\, \xi_I, \qquad I \in \mathcal{I}_{\boldsymbol{\lambda}},$$

for any $f \in \mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$, cf. (11.18). Let

$$\mu : K_T(\mathcal{X}_n) \otimes \mathbb{C}(h) \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$$

be the direct sum of the monomorphisms $\mu_{\boldsymbol{\lambda}}$.

Lemma 12.1. *For any* $f \in \mathbb{C}[\boldsymbol{z}^{\pm 1}]^{S_n} \otimes \mathbb{C}(h)$, *we have* $\mu\big([1 \otimes f]\big) = \mathrm{id} \otimes f$. □

Theorem 12.2. *We have the induced isomorphisms of graded algebras*

$$\mu_{\boldsymbol{\lambda}} : K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(h) \to \phi_{\boldsymbol{\lambda}}(\mathcal{B}^{\infty}) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}],$$
$$\mu : K_T(\mathcal{X}_n) \otimes \mathbb{C}(h) \to \phi(\mathcal{B}^{\infty}) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}],$$

where the homomorphisms ϕ and $\phi_{\boldsymbol{\lambda}}$ are defined by Lemma 11.1 and formula (11.11).

Proof. The statement follows from Theorem 11.10. □

Consider the graded $\mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$-module isomorphism

$$\nu : K_T(\mathcal{X}_n) \otimes \mathbb{C}(h) \to \tfrac{1}{D}\mathcal{V}^{-} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h), \tag{12.1}$$

that is the natural extension of the isomorphism (9.16) denoted by the same letter.

Theorem 12.3. *The* $\mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$*-module isomorphism* $\nu : K_T(\mathcal{X}_n) \otimes \mathbb{C}(h) \to \frac{1}{D}\mathcal{V}^{-} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$ *and the* $\mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$*-algebra isomorphism* $\mu : K_T(\mathcal{X}_n) \to \phi(\mathcal{B}^{\infty}) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}]$ *identify the* $\phi(\mathcal{B}^{\infty}) \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}]$*-module* $\frac{1}{D}\mathcal{V}^{-} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \otimes \mathbb{C}(h)$ *with the regular representation of* $K_T(\mathcal{X}_n) \otimes \mathbb{C}(h)$. □

Recall that the space $\frac{1}{D}\mathcal{V}^-\otimes\mathbb{C}[\boldsymbol{z}^{\pm1}]\otimes\mathbb{C}(h)$ is a $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-module, see Corollary 11.7. Thus the isomorphism (12.1) makes $K_T(\mathcal{X}_n)\otimes\mathbb{C}(h)$ into a $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-module. We denote the $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-module structure on $K_T(\mathcal{X}_n)\otimes\mathbb{C}(h)$ by ρ.

Let $K_{\boldsymbol{\lambda}}=K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\otimes\mathbb{C}(h)$. For any $i=1,\dots,N$ and $f\in K_{\boldsymbol{\lambda}}$,

$$\widetilde{L}_{i,i,+0}\,f \,=\, h^{\lambda_1+\ldots+\lambda_i}\,f\,,\qquad \widetilde{L}_{i,i,-0}\,f \,=\, h^{\lambda_1+\ldots+\lambda_{i-1}}f\,,\tag{12.2}$$

see (11.12). Thus $K_{\boldsymbol{\lambda}}$ are eigenspaces for the action of the subalgebra $\mathcal{U}(\mathfrak{h})\subset\mathcal{U}(\widetilde{\mathfrak{gl}_N})$.

The operators $\rho\bigl(A_{p,\pm}(u)\bigr)$, $p=1,\dots,N$, preserve the subspaces $K_{\boldsymbol{\lambda}}$ and act as follows:

$$\rho\bigl(A_{p,+}(u)\bigr)\,:\,[f(\boldsymbol{\Gamma},\boldsymbol{z},h)]\,\mapsto\,\Bigl[f(\boldsymbol{\Gamma},\boldsymbol{z},h)\prod_{a=1}^{p}\prod_{i=1}^{\lambda_a}\frac{1-h^{-1}u/\gamma_{a,i}}{1-u/\gamma_{a,i}}\Bigr],\tag{12.3}$$

$$\rho\bigl(A_{p,-}(u)\bigr)\,:\,[f(\boldsymbol{\Gamma},\boldsymbol{z},h)]\,\mapsto\,\Bigl[f(\boldsymbol{\Gamma},\boldsymbol{z},h)\prod_{a=1}^{p}\prod_{i=1}^{\lambda_a}\frac{1-h\gamma_{a,i}/u}{1-\gamma_{a,i}/u}\Bigr],$$

for any $f\in\mathbb{C}[\boldsymbol{\Gamma}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}^{\pm1}]\otimes\mathbb{C}(h)$, see (11.13) and (9.16).

For $p=1,\dots,n-1$, let $\alpha_p=(0,\dots,0,1,-1,0,\dots,0)$, with $p-1$ first zeros.

Theorem 12.4. *We have*

$$\rho\bigl(E_{p,\pm}(u)\bigr)\,:\,K_{\boldsymbol{\lambda}-\alpha_p}\,\mapsto\,K_{\boldsymbol{\lambda}}\,,$$

$$\rho\bigl(E_{p,+}(u)\bigr)\,:\,[f]\,\mapsto\,\Bigl[-\sum_{i=1}^{\lambda_p}\frac{f(\boldsymbol{\Gamma}'^{i},\boldsymbol{z},h)}{1-u/\gamma_{p,i}}\prod_{\substack{j=1\\ j\neq i}}^{\lambda_p}\frac{1}{1-\gamma_{p,i}/\gamma_{p,j}}\prod_{k=1}^{\lambda_{p+1}}(1-h\gamma_{p+1,k}/\gamma_{p,i})\Bigr],$$

$$\rho\bigl(E_{p,-}(u)\bigr)\,:\,[f]\,\mapsto\,\Bigl[\sum_{i=1}^{\lambda_p}f(\boldsymbol{\Gamma}'^{i},\boldsymbol{z},h)\,\frac{\gamma_{p,i}/u}{1-\gamma_{p,i}/u}\prod_{\substack{j=1\\ j\neq i}}^{\lambda_p}\frac{1}{1-\gamma_{p,i}/\gamma_{p,j}}\prod_{k=1}^{\lambda_{p+1}}(1-h\gamma_{p+1,k}/\gamma_{p,i})\Bigr],$$

$$\rho\bigl(F_{p,\pm}(u)\bigr)\,:\,K_{\boldsymbol{\lambda}+\alpha_p}\,\mapsto\,K_{\boldsymbol{\lambda}}\,,$$

$$\begin{aligned}\rho\bigl(F_{p,+}(u)\bigr)\,:\,[f]\,\mapsto\,\Bigl[-\sum_{i=1}^{\lambda_{p+1}}f(\boldsymbol{\Gamma}^{i\prime},\boldsymbol{z},h)\,\frac{u/\gamma_{p+1,i}}{1-u/\gamma_{p+1,i}}\,\times\\ \times\prod_{\substack{j=1\\ j\neq i}}^{\lambda_{p+1}}\frac{1}{1-\gamma_{p+1,j}/\gamma_{p+1,i}}\prod_{k=1}^{\lambda_p}(1-h\gamma_{p+1,i}/\gamma_{p,k})\Bigr],\end{aligned}$$

$$\rho\bigl(F_{p,-}(u)\bigr)\,:\,[f]\,\mapsto\,\Bigl[\sum_{i=1}^{\lambda_{p+1}}\frac{f(\boldsymbol{\Gamma}^{i\prime},\boldsymbol{z},h)}{1-\gamma_{p+1,i}/u}\prod_{\substack{j=1\\ j\neq i}}^{\lambda_{p+1}}\frac{1}{1-\gamma_{p+1,j}/\gamma_{p+1,i}}\prod_{k=1}^{\lambda_p}(1-h\gamma_{p+1,i}/\gamma_{p,k})\Bigr],$$

where

$$\boldsymbol{\Gamma}'^{i} = (\Gamma_1;\ldots;\Gamma_{p-1};\Gamma_p-\{\gamma_{p,i}\};\Gamma_{p+1}\cup\{\gamma_{p,i}\};\Gamma_{p+2};\ldots;\Gamma_N)\,,$$

$$\boldsymbol{\Gamma}^{i\prime} = (\Gamma_1;\ldots;\Gamma_{p-1};\Gamma_p\cup\{\gamma_{p+1,i}\};\Gamma_{p+1}-\{\gamma_{p+1,i}\};\Gamma_{p+2};\ldots;\Gamma_N)\,.$$

Proof. The statement follows from Corollary 9.9 and Theorem 11.8. □

Remark. Notice that the expressions for $\rho\bigl(E_{p,\pm}(u)\bigr)$ coincide as rational functions, and the same is true for $\rho\bigl(F_{p,\pm}(u)\bigr)$. These functions are expanded at $u=0$ for $\rho\bigl(E_{p,+}(u)\bigr)$, $\rho\bigl(F_{p,+}(u)\bigr)$, and at $u=\infty$ for $\rho\bigl(E_{p,-}(u)\bigr)$, $\rho\bigl(F_{p,-}(u)\bigr)$.

Remark. The $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-action ρ on $K_T(\mathcal{X}_n)$ defined via the isomorphism (12.1), that is, with the help of the map $\mathrm{Stab}_{\mathrm{id}}$ introduced in Section 7 is related to the $U_q(\widetilde{\mathfrak{gl}_N})$-action introduced in [GV, Vas1, Vas2] in terms of the convolution operators acting on $K_T(\mathcal{X}_n)$.

More precisely, let

$$Q_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma},h) = \prod_{1\leqslant a<b\leqslant n}\,\prod_{i=1}^{\lambda_a}\prod_{j=1}^{\lambda_b}(1-h\gamma_{b,j}/\gamma_{a,i})\,.$$

Denote by $\widetilde{K}_{\boldsymbol{\lambda}}$ the ideal in $K_{\boldsymbol{\lambda}}$ generated by $[Q_{\boldsymbol{\lambda}}]$. Since $[Q_{\boldsymbol{\lambda}}]$ is not a zero divisor, the map $\chi_{\boldsymbol{\lambda}}: K_{\boldsymbol{\lambda}}\to\widetilde{K}_{\boldsymbol{\lambda}}$, $[f]\mapsto[Q_{\boldsymbol{\lambda}}f]$ is an isomorphism of vector spaces.

Let $\widetilde{K}=\bigoplus_{|\boldsymbol{\lambda}|=n}\widetilde{K}_{\boldsymbol{\lambda}}$, and let $\chi: K_T(\mathcal{X}_n)\to\widetilde{K}$ be the direct sum of isomorphisms $\chi_{\boldsymbol{\lambda}}$. It is straightforward to verify that $\widetilde{K}$ is a $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-submodule of $K_T(\mathcal{X}_n)$. This defines a new $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-action ρ^+ on $K_T(\mathcal{X}_n)$ by the rule: $\rho^+(X)=\chi^{-1}\rho(X)\,\chi$ for any $X\in\mathcal{U}(\widetilde{\mathfrak{gl}_N})$. The $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-actions ρ and ρ^+ on $K_T(\mathcal{X}_n)$ are conjecturally not isomorphic, but become isomorphic as actions on $K_T(\mathcal{X}_n)\otimes\mathbb{C}(\boldsymbol{z},h)$ since $[Q_{\boldsymbol{\lambda}}]$ is invertible in $K_{\boldsymbol{\lambda}}\otimes\mathbb{C}(\boldsymbol{z},h)$.

Now the $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-action ρ^+ essentially coincides with the $U_q(\widetilde{\mathfrak{gl}_N})$-action introduced in [GV, Vas1, Vas2] in terms of the convolution operators acting on $K_T(\mathcal{X}_n)$, cf. formulae in Theorem 12.4 and in [Vas2, page 287].

12.2. **Topological interpretation of the actions in Theorem 12.4.** Consider the vectors $\boldsymbol{\mu}'=(\lambda_1,\ldots,\lambda_p-1,\,1,\,\lambda_{p+1},\ldots,,\lambda_N)$ (if $\lambda_p>0$), and $\boldsymbol{\mu}''=(\lambda_1,\ldots,\lambda_p,\,1,\,\lambda_{p+1}-1,\ldots,\lambda_N)$ (if $\lambda_{p+1}>0$). There are natural forgetful maps

$$\mathcal{F}_{\boldsymbol{\lambda}-\alpha_p}\xleftarrow{\pi_1'}\mathcal{F}_{\boldsymbol{\mu}'}\xrightarrow{\pi_2'}\mathcal{F}_{\boldsymbol{\lambda}}\,,\qquad\qquad \mathcal{F}_{\boldsymbol{\lambda}+\alpha_p}\xleftarrow{\pi_1''}\mathcal{F}_{\boldsymbol{\mu}''}\xrightarrow{\pi_2''}\mathcal{F}_{\boldsymbol{\lambda}}\,. \tag{12.4}$$

The rank λ_p-1, 1, λ_{p+1} bundles over $\mathcal{F}_{\boldsymbol{\mu}'}$ with fibers F_p/F_{p-1}, F_{p+1}/F_p, F_{p+2}/F_{p+1} will be respectively denoted by A', B', C'. The rank λ_p, 1, $\lambda_{p+1}-1$ bundles over $\mathcal{F}_{\boldsymbol{\mu}''}$ with fibers F_p/F_{p-1}, F_{p+1}/F_p, F_{p+2}/F_{p+1} will be respectively denoted by A'', B'', C''.

For a $(\mathbb{C}^\times)^n$-equivariant bundle ξ, let $e(\xi)$ be its equivariant K-theoretic Euler class. We can make the extra $\mathbb{C}^\times$ (whose Chern root is h) act on any bundle by fiberwise multiplication. The equivariant Euler class with this extra action will be denoted by $e_h(\xi)$.

Recall that an equivariant proper map $f: X\to Y$ induces the pullback $f^*: K_T(Y)\to K_T(X)$ and push-forward (also known as Gysin) $f_*: K_T(X)\to K_T(Y)$ maps.

Theorem 12.5. *The operators* $\rho\big(E_{p,\pm}(u)\big)$, $\rho\big(F_{p,\pm}(u)\big)$, *are equal to the following topological operations*

$$\rho\big(E_{p,+}(u)\big)\,:\,x\,\mapsto\,(-1)^{\lambda_p}\pi'_{2*}\bigg(\pi_1'^{\,*}(x)\cdot\frac{1}{1-u/[B']}\,\frac{e_h\big(\mathrm{Hom}(C',B')\big)\,[\Lambda^{\mathrm{top}}A']}{[B']^{\lambda_p-1}}\bigg)\,,$$

$$\rho\big(E_{p,-}(u)\big)\,:\,x\,\mapsto\,(-1)^{\lambda_p-1}\pi'_{2*}\bigg(\pi_1'^{\,*}(x)\cdot\frac{[B']/u}{1-[B']/u}\,\frac{e_h\big(\mathrm{Hom}(C',B')\big)\,[\Lambda^{\mathrm{top}}A']}{[B']^{\lambda_p-1}}\bigg)\,,$$

$$\rho\big(F_{p,+}(u)\big)\,:\,x\,\mapsto\,(-1)^{\lambda_{p+1}}\pi''_{2*}\bigg(\pi_1''^{\,*}(x)\cdot\frac{u/[B'']}{1-u/[B'']}\,\frac{e_h\big(\mathrm{Hom}(B'',A'')\big)\,[B'']^{\lambda_{p+1}-1}}{[\Lambda^{\mathrm{top}}C'']}\bigg)\,,$$

$$\rho\big(F_{p,-}(u)\big)\,:\,x\,\mapsto\,(-1)^{\lambda_{p+1}-1}\pi''_{2*}\bigg(\pi_1''^{\,*}(x)\cdot\frac{1}{1-[B'']/u}\,\frac{e_h\big(\mathrm{Hom}(B'',A'')\big)\,[B'']^{\lambda_{p+1}-1}}{[\Lambda^{\mathrm{top}}C'']}\bigg)\,,$$

Proof. If we write down equivariant localization formulae for the given topological operations we obtain the formulae of Theorem 12.4. □

13. Bethe algebra $\mathcal{B}^q$ and discrete Wronskian

13.1. **Wronski map.** Throughout this section we use the following grading of functions in the variables $\gamma_{i,j}$, z_i, and h: $\deg\gamma_{i,j}=\deg z_i=1$ for all $i=1,\dots,N$, $j=1,\dots,\lambda_i$, and $\deg h=0$, cf. Section 9.4.

Let $q_1,\dots,q_N$ be distinct nonzero complex numbers. Set

$$W^q(u)\,=\,\det\bigg(q_i^{\,i-j}\prod_{k=1}^{\lambda_i}(1-h^{i-j}\gamma_{i,k}/u)\bigg)_{i,j=1}^{N}.\tag{13.1}$$

The function $W^q(u)$ is essentially a discrete Wronskian (multiplicative Casorati determinant) of functions

$$g_i(u)\,=\,q_i^{-\log u/\log h}\prod_{k=1}^{\lambda_i}\big(1-h^{i-1}\gamma_{i,k}/u\big)\,.\qquad i=1,\dots,N,\tag{13.2}$$

namely,

$$W^q(u)\,=\,\det\big(g_i(uh^{j-1})\big)_{i,j=1}^{N}\,\prod_{i=1}^{N}q_i^{\,i-1+\log u/\log h}\,.$$

Let $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$ be the algebra of polynomials in the variables $\gamma_{i,j}$ invariant under the permutations of the variables with the same first subscript. Define the elements $e^q_0(\boldsymbol{\Gamma},h),\dots,$ $e^q_N(\boldsymbol{\Gamma},h)\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[h^{\pm1}]$ via the coefficients of $W^q(u)$:

$$W^q(u)\,=\,\sum_{p=0}^{N}(-1)^p\,e^q_p(\boldsymbol{\Gamma})\,u^{-p}\prod_{1\leqslant i<j\leqslant N}(1-q_j/q_i)\,.$$

In particular,

$$e^q_0(\boldsymbol{\Gamma},h)\,=\,1\,,\qquad e^q_N(\boldsymbol{\Gamma},h)\,=\,\gamma_{1,1}\dots\gamma_{N,\lambda_N}\,.$$

Define the *Wronski map*

$$\mathrm{Wr}^q_{\boldsymbol{\lambda}}:\mathbb{C}[\boldsymbol{z}]^{S_n}\to\,\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[h^{\pm1}]\tag{13.3}$$

to be the algebra homomorphism such that

$$e_p(\boldsymbol{z}) \mapsto e^q_p(\boldsymbol{\Gamma}, h)\,, \qquad p = 1, \dots, N\,,$$

where $e_p(\boldsymbol{z})$ is the p-th elementary symmetric polynomial of $z_1, \dots, z_n$. The map $\mathrm{Wr}^q_{\boldsymbol{\lambda}}$ makes the space $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[h^{\pm 1}]$ into the graded $\mathbb{C}[\boldsymbol{z}]^{S_n}$-module.

Lemma 13.1. *The map* $\mathrm{Wr}^q_{\boldsymbol{\lambda}}$ *is injective.*

Proof. Observe that $e^q_p(\boldsymbol{\Gamma}, 1) = e_p(\gamma_{1,1}, \dots, \gamma_{N,\lambda_N})$ for all $p = 0, \dots, N$. The statement follows. □

The Wronski map (13.3) induces the monomorphism $\mathbb{C}[\boldsymbol{z}]^{S_n} \otimes \mathbb{C}(h) \to \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)$ that will be also denoted $\mathrm{Wr}^q_{\boldsymbol{\lambda}}$.

13.2. **Polynomial isomorphisms.** Recall the multiplicative shift operator τ acting on functions of u: $\tau f(u) = f(hu)$. Let

$$\mathcal{D} \,=\, 1 + \sum_{p=1}^{N} b^q_p(u)\,(-\tau)^p\,. \tag{13.4}$$

be the N-th order linear difference operator annihilating the functions $g_1(u/h), \dots, g_N(u/h)$. Its coefficients $b^q_1(u), \dots, b^q_N(u)$ are described below, see formula (13.6). We set $b^q_0(u) = 1$.

Let x be a complex variable. Set

$$\widehat{W}^q(u, x) \,=\, \det\left(q_i^{\,i-j} \prod_{k=1}^{\lambda_i} (1 - h^{i-j}\gamma_{i,k}/u) \right)_{i,j=0}^{N}, \tag{13.5}$$

where $q_0 = x^{-1}$ and $\lambda_0 = 0$. Then

$$\frac{\widehat{W}^q(u, x)}{W^q(u)} \,=\, 1 + \sum_{p=1}^{N} b^q_p(u)\,(-x)^p\,, \tag{13.6}$$

and the coefficients $b^q_p(u)$ have the following expansion at $u = \infty$:

$$b^q_p(u) \,=\, e_p(q_1, \dots, q_N) + \sum_{s=1}^{\infty} b^q_{p,-s} u^{-s}, \tag{13.7}$$

where $e_p(q_1, \dots, q_N)$ is the p-th elementary symmetric polynomial. Notice that

$$b^q_N(u) \,=\, q_1 \dots q_N\, \frac{W^q(u/h)}{W^q(u)}\,. \tag{13.8}$$

Proposition 13.2. *The elements* $b^q_{p,-s}$, $p = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, *together with* $\mathbb{C}(h)$ *generate the algebra* $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}(h)$.

Proof. Recall the functions $g_1, \dots, g_N$, see (13.2), and the difference operator (13.4). The equation $\mathcal{D} g_i(u/h) = 0$ yields

$$\sum_{p=0}^{N} (-q_i)^{-p}\, b^q_p(u) \prod_{k=1}^{\lambda_i} (1 - h^{i-p}\gamma_{i,k}/u) \,=\, 0\,. \tag{13.9}$$

Let $e_j(\Gamma_i)$ be the j-th elementary symmetric polynomial of $\gamma_{i,1},\dots,\gamma_{i,\lambda_i}$. Collecting the coefficient of u^{-j} in (13.9), we get that $e_j(\Gamma_i) = C_j$, where C_j is expressed via the elements $b^q_{k,s}$ for $k \leqslant j$, integer powers of h, and $e_m(\Gamma_i)$ for $m < j$. The proposition follows. □

Recall the subalgebras $\mathcal{B}^q_-$ and $\mathcal{Z}_-$ defined in Section 10.6, and the homomorphism $\phi_{\boldsymbol\lambda}$ given by (11.11). Corollary 11.3 yields

$$\phi_{\boldsymbol\lambda}(\mathcal{Z}_-) = \mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h) \subset \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}\big) \otimes \mathbb{C}[\boldsymbol z] \otimes \mathbb{C}(h)\,.$$

Theorem 13.3. *The assignment* $b^q_{p,-s} \mapsto \phi_{\boldsymbol\lambda}(B^q_{p,-s})$, $p = 1,\dots,N$, $s \in \mathbb{Z}_{>0}$, *defines an isomorphism*

$$\psi^q_{\boldsymbol\lambda} : \mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h) \to \phi_{\boldsymbol\lambda}(\mathcal{B}^q_-) \tag{13.10}$$

of graded $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$*-algebras. Here the algebra* $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$ *acts on* $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)$ *via the Wronski map* $\mathrm{Wr}^q_{\boldsymbol\lambda}$.

Proof. The proof of Theorem 13.3 is similar to the analogous proofs of [MTV2, Theorem 6.3] and [MTV3, Theorem 5.2, item (i)]. The proof is based on the Bethe ansatz technique. The details will be published elsewhere. □

Recall the spaces $\frac{1}{D}\mathcal{V}_{\boldsymbol\lambda}$, $\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda}$ defined in Sections 9.2, 9.3, and the isomorphism $\widetilde\vartheta_{\boldsymbol\lambda}$, see Lemma 9.12. Keeping the same notation, we will think of $\widetilde\vartheta_{\boldsymbol\lambda}$ as the isomorphism $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h) \to \frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda} \otimes \mathbb{C}(h)$, where we identified $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}$ with $\mathbb{C}[\boldsymbol z]^{S_{\boldsymbol\lambda}}$ by the rule $f(\boldsymbol\Gamma) \mapsto f(\boldsymbol z_{I^{\min}})$ for any $f \in \mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}$, and replaced $\mathbb{C}[h^{\pm1}]$ by $\mathbb{C}(h)$.

Define the homomorphism

$$\widetilde\vartheta^q_{\boldsymbol\lambda} : \mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h) \to \tfrac{1}{D}\mathcal{V}_{\boldsymbol\lambda} \otimes \mathbb{C}(h)\,, \tag{13.11}$$

$$f \mapsto \sum_{I \in \mathcal{I}_{\boldsymbol\lambda}} \frac{1}{R(\boldsymbol z_I)} \prod_{a=1}^{N} \prod_{i \in I_a} z_i^{\lambda^{(a)}-n}\, \psi^q_{\boldsymbol\lambda}(f)\, \xi_I\,,$$

cf. (9.15) and (9.17). The map $\widetilde\vartheta^q_{\boldsymbol\lambda}$ is graded.

Lemma 13.4. *We have* $\widetilde\vartheta^q_{\boldsymbol\lambda}\big(\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)\big) \subset \frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda} \otimes \mathbb{C}(h)$.

Proof. Let $f \in \mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)$. Then $\widetilde\vartheta^q_{\boldsymbol\lambda}(f) = \psi^q_{\boldsymbol\lambda}(f)\,\widetilde\vartheta_{\boldsymbol\lambda}(1)$. By Theorem 13.3 and Lemma 11.5, $\psi^q_{\boldsymbol\lambda}(f) \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}\big) \otimes \mathbb{C}[\boldsymbol z] \otimes \mathbb{C}(h)$, and $\psi^q_{\boldsymbol\lambda}(f)$ commutes with the S_n-action (8.2). Since $\widetilde\vartheta_{\boldsymbol\lambda}(1) \in \frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda}$, the statement follows. □

Theorem 13.5. *The map* $\widetilde\vartheta^q_{\boldsymbol\lambda} : \mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h) \to \frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda} \otimes \mathbb{C}(h)$ *is an isomorphism of graded* $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$*-modules, where* $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$ *acts on* $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)$ *via the Wronski map* $\mathrm{Wr}^q_{\boldsymbol\lambda}$.

Proof. Notice that the graded components of the spaces $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$, $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)$, $\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda} \otimes \mathbb{C}(h)$ are finite-dimensional $\mathbb{C}(h)$-modules.

By Theorem 13.3, the map $\widetilde\vartheta^q_{\boldsymbol\lambda}$ is a homomorphism of graded $\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)$-modules. Its kernel is an ideal in $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}} \otimes \mathbb{C}(h)$ that has zero intersection with $\mathrm{Wr}^q_{\boldsymbol\lambda}\big(\mathbb{C}[\boldsymbol z]^{S_n} \otimes \mathbb{C}(h)\big)$, hence, it is the zero ideal. Thus the composition $(\widetilde{\vartheta}_{\boldsymbol{\lambda}})^{-1}\widetilde{\vartheta}^q_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(h) \to \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes \mathbb{C}(h)$ is an injective graded $\mathbb{C}(h)$-endomorphism, that sends 1 to 1, since $\widetilde{\vartheta}^q_{\boldsymbol{\lambda}}(1) = \widetilde{\vartheta}_{\boldsymbol{\lambda}}(1)$. Therefore, both $(\widetilde{\vartheta}_{\boldsymbol{\lambda}})^{-1}\widetilde{\vartheta}^q_{\boldsymbol{\lambda}}$ and $\widetilde{\vartheta}^q_{\boldsymbol{\lambda}}$ are bijections. □

Corollary 13.6. *The $\mathbb{C}[\boldsymbol{z}]^{S_n}\otimes\mathbb{C}(h)$-module isomorphism $\widetilde{\vartheta}^q_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(h) \to \frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}\otimes \mathbb{C}(h)$ and the $\mathbb{C}[\boldsymbol{z}]^{S_n}\otimes\mathbb{C}(h)$-algebra isomorphism $\psi^q_{\boldsymbol{\lambda}} : \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(h) \to \phi_{\boldsymbol{\lambda}}(\mathcal{B}^q_-)$ identify the $\phi_{\boldsymbol{\lambda}}(\mathcal{B}^q_-)$-module $\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol{\lambda}}\otimes\mathbb{C}(h)$ with the regular representation of $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(h)$. Here $\mathbb{C}[\boldsymbol{z}]^{S_n}\otimes\mathbb{C}(h)$ acts on $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}(h)$ via the Wronski map $\mathrm{Wr}^q_{\boldsymbol{\lambda}}$.* □

13.3. **K-theoretic isomorphisms.** Let $\Delta(q_1,\dots,q_N) = \prod_{1\leqslant i<j\leqslant N}(1-q_j/q_i)$. Define the algebra

$$\mathcal{K}^q_{\boldsymbol{\lambda}} \,=\, \mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}]\Big/\Big\langle W^q(u) \,=\, \Delta(q_1,\dots,q_N)\prod_{a=1}^n(1-z_a/u)\Big\rangle\,. \tag{13.12}$$

Let $y = z_1\dots z_n$. Since in the algebra $\mathcal{K}^q_{\boldsymbol{\lambda}}$,

$$y \,=\, \frac{\Delta(q_1h^{\lambda_1},\dots,q_Nh^{\lambda_N})}{\Delta(q_1,\dots,q_N)}\,\prod_{i=1}^N\prod_{j=1}^{\lambda_i}\gamma_{i,j}\,,$$

we have

$$\mathcal{K}^q_{\boldsymbol{\lambda}}\otimes\mathbb{C}(h) \,=\, \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z},y^{-1}]\otimes\mathbb{C}(h)\Big/\Big\langle W^q(u) \,=\, \Delta(q_1,\dots,q_N)\prod_{a=1}^n(1-z_a/u)\Big\rangle\,. \tag{13.13}$$

Example. Let $N=n=2$ and $\boldsymbol{\lambda}=(1,1)$. Then

$$W^q(u) \,=\, \det\begin{pmatrix} 1-\gamma_{1,1}/u & q_1^{-1}(1-h^{-1}\gamma_{1,1}/u) \\ q_2(1-h\gamma_{2,1}/u) & 1-\gamma_{2,1}/u\end{pmatrix},$$

and the relations in $\mathcal{K}^q_{\boldsymbol{\lambda}}$ are

$$\gamma_{1,1}\frac{q_1-h^{-1}q_2}{q_1-q_2}+\gamma_{2,1}\frac{q_1-hq_2}{q_1-q_2} \,=\, z_1+z_2\,, \qquad \gamma_{1,1}\gamma_{2,1} \,=\, z_1z_2\,. \tag{13.14}$$

It is easy to see that the algebra $\mathcal{K}^q_{\boldsymbol{\lambda}}$ does not change if all $q_1,\dots,q_N$ are multiplied simultaneously by the same number. Notice that in the limit $q_{i+1}/q_i \to 0$ for all $i=1,\dots,N-1$, the relations in $\mathcal{K}^q_{\boldsymbol{\lambda}}$ turn into the relations in $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})$, see (2.3).

Theorem 13.7. *The isomorphism* (13.10) *induces the isomorphism*

$$\mu^q_{\boldsymbol{\lambda}} : \mathcal{K}^q_{\boldsymbol{\lambda}}\otimes\mathbb{C}(h) \,\to\, \phi_{\boldsymbol{\lambda}}(\mathcal{B}^q)\otimes\mathbb{C}[\boldsymbol{z}^{\pm1}] \tag{13.15}$$

of graded $\mathbb{C}[\boldsymbol{z}^{\pm1}]\otimes\mathbb{C}(h)$-algebras.

Proof. The statement follows from formulae (13.13) and (11.9). □

Expand the coefficients $b^q_1(u),\dots,b^q_N(u)$ of the difference operator $\mathcal{D}$, see (13.4), at $u=0$:

$$b^q_p(u) \,=\, \Big(e_p(q_1h^{\lambda_1},\dots,q_Nh^{\lambda_N}) + \sum_{s=1}^{\infty} b^q_{p,s}u^s\Big)(-u)^{-n}\prod_{i=1}^N\prod_{j=1}^{\lambda_i}\gamma_{i,j}\,, \tag{13.16}$$

where $e_p(q_1h^{\lambda_1},\dots,q_Nh^{\lambda_N})$ is the p-th elementary symmetric polynomial.

Corollary 13.8. *We have* $\mu^q_{\boldsymbol\lambda}\bigl([b^q_{p,\pm s}]\bigr) = \phi_{\boldsymbol\lambda}(B^q_{p,\pm s})$ *for all* $p=1,\dots,N,\ s\in\mathbb{Z}_{>0}$. □

Lemma 13.9. *The elements* $[b^q_{p,\pm s}]$, $p=1,\dots,N,\ s\in\mathbb{Z}_{>0}$, *together with* $\mathbb{C}(h)$ *generate the algebra* $\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$.

Proof. By formula (13.8), $[b^q_N(u)] = q_1\dots q_N h^n\bigl[\prod_{a=1}^n(1-h^{-1}u/z_a)\big/(1-u/z_a)\bigr]$. Hence, the ideal generated by the elements $b^q_{N,-s}$ for all $s\in\mathbb{Z}_{>0}$ and $\mathbb{C}(h)$ contains all symmetric polynomials in $z_1^{-1},\dots,z_n^{-1}$, in particular, $z_1^{-1}\dots z_n^{-1}$. Therefore, the statement follows from equality (13.13) and Proposition 13.2. □

Lemma 13.9 implies that statement of Corollary 13.8 can serve as a definition of the isomorphism $\mu^q_{\boldsymbol\lambda}$.

Theorem 13.10. *The map*

$$\nu^q_{\boldsymbol\lambda}:\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\,\to\,\tfrac{1}{D}\mathcal{V}_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\,,\qquad f\,\mapsto\sum_{I\in\mathcal{I}_{\boldsymbol\lambda}}\frac{1}{R(\boldsymbol z_I)}\,\mu^q_{\boldsymbol\lambda}(f)\,\xi_I\,, \tag{13.17}$$

cf. (13.11), *is an isomorphism of graded* $\mathbb{C}[\boldsymbol z^{\pm1}]^{S_n}\otimes\mathbb{C}(h)$*-modules.*

Proof. Let $r=\bigl[\prod_{a=1}^N\prod_{i\in I_a}z_i^{n-\lambda^{(a)}}\bigr]\in\mathcal{K}^q_{\boldsymbol\lambda}$. The isomorphism $\widetilde\vartheta^q_{\boldsymbol\lambda}:\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}\otimes\mathbb{C}(h)\to\frac{1}{D}\widetilde{\mathcal{V}}^-_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$, see Theorem 13.5, induces the isomorphism $\tilde\nu^q_{\boldsymbol\lambda}:\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\to\frac{1}{D}\mathcal{V}_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$ such that $\nu^q_{\boldsymbol\lambda}(f)=\tilde\nu^q_{\boldsymbol\lambda}(fr)$. Since the element r is invertible, the statement follows. □

Corollary 13.11. *The* $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$*-module isomorphism* $\nu^q_{\boldsymbol\lambda}:\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\to\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}\otimes\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$ *and the* $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$*-algebra isomorphism* $\mu^q_{\boldsymbol\lambda}:\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\to\phi_{\boldsymbol\lambda}(\mathcal{B}^q)\otimes\mathbb{C}[\boldsymbol z^{\pm1}]$ *identify the* $\phi_{\boldsymbol\lambda}(\mathcal{B}^q)\otimes\mathbb{C}[\boldsymbol z^{\pm1}]$*-module* $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}\otimes\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$ *with the regular representation of* $\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$.

Proof. The statement follows from Theorems 13.7 and 13.10. □

Recall $K_T(\mathcal{X}_n)=\bigoplus_{|\boldsymbol\lambda|=n}K_T(T^*\mathcal{F}_{\boldsymbol\lambda})$, see (2.2). Let

$$\nu_{\boldsymbol\lambda}:K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)\,\to\,\tfrac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}\otimes\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$$

be the graded $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$-module isomorphism obtained by the restriction of the isomorphism (12.1). By Theorems 12.3 and 13.10, the composition $\beta_{\boldsymbol\lambda}=(\nu^q_{\boldsymbol\lambda})^{-1}\nu_{\boldsymbol\lambda}$,

$$\beta_{\boldsymbol\lambda}:K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)\,\to\,\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h) \tag{13.18}$$

is a graded $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$-module isomorphism. Notice $\beta_{\boldsymbol\lambda}(1)=1$ since $\nu^q_{\boldsymbol\lambda}(1)=\nu_{\boldsymbol\lambda}(1)$.

Recall the $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$-action ρ on $K_T(\mathcal{X}_n)\otimes\mathbb{C}(h)$, see formulae (12.3) and Theorem 12.4. Let

$$\rho_{\boldsymbol\lambda}:\mathcal{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak h}\,\to\,\mathrm{End}\bigl(K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\bigr)\otimes\mathbb{C}(h)$$

be the induced $\mathbb{C}(h)$-algebra homomorphism. Recall the generators $b^q_{p,\pm s}$ of the algebra $\mathcal{K}^q_{\boldsymbol\lambda}$, see (13.7), (13.16).

Lemma 13.12. *The assignment* $[b^q_{p,\pm s}]\mapsto\rho_{\boldsymbol\lambda}(B^q_{p,\pm s})$, $p=1,\dots,N,\ s\in\mathbb{Z}_{>0}$, *defines a graded* $\mathbb{C}[\boldsymbol z^{\pm1}]^{S_n}\otimes\mathbb{C}(h)$*-algebra isomorphism* $\alpha_{\boldsymbol\lambda}:\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)\to\rho_{\boldsymbol\lambda}(\mathcal{B}^q)$.

Proof. By the definition of $\rho_{\boldsymbol\lambda}$, we have $\rho_{\boldsymbol\lambda}(X) = (\nu_{\boldsymbol\lambda})^{-1}\rho_{\boldsymbol\lambda}(X)\,\nu_{\boldsymbol\lambda}$ for any $X \in \mathcal{U}(\widetilde{\mathfrak{gl}_N})^{\mathfrak h}$. Thus Corollary 13.8 and Lemma 13.9 imply that

$$\alpha_{\boldsymbol\lambda}(f) \,=\, (\nu_{\boldsymbol\lambda})^{-1}\mu^q_{\boldsymbol\lambda}(f)\,\nu_{\boldsymbol\lambda} \tag{13.19}$$

for any $f \in \mathcal{K}^q_{\boldsymbol\lambda}$, and Lemma 13.12 follows from Theorems 12.3 and 13.7. □

Corollary 13.13. *The* $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$*-module isomorphism* $\beta^{-1}_{\boldsymbol\lambda} : \mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h) \to K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$ *and the* $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$*-algebra isomorphism* $\alpha_{\boldsymbol\lambda} : \mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h) \to \rho_{\boldsymbol\lambda}(\mathcal{B}^q)\otimes\mathbb{C}[\boldsymbol z^{\pm1}]$ *identify the* $\rho_{\boldsymbol\lambda}(\mathcal{B}^q)\otimes\mathbb{C}[\boldsymbol z^{\pm1}]$*-module* $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$ *with the regular representation of* $\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$. □

13.4. **New multiplication on** $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$**.** Define a new commutative associative multiplication $*^q$ on $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$, depending on the parameters $q_1,\dots,q_N$, by the rule:

$$\beta_{\boldsymbol\lambda}(f *^q g) \,=\, \beta_{\boldsymbol\lambda}(f)\,\beta_{\boldsymbol\lambda}(g) \tag{13.20}$$

for any $f,g \in K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$.

Lemma 13.14. *For any* $f \in K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$*, the operator* $f*^q$ *coincides with the operator* $\alpha_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(f)\bigr) \in \rho_{\boldsymbol\lambda}(\mathcal{B}^q)$. *The map*

$$K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h) \to \rho_{\boldsymbol\lambda}(\mathcal{B}^q)\,, \qquad f \mapsto f*^q\,,$$

is an isomorphism of graded $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$*-modules.*

Proof. Using the equality $\nu^q_{\boldsymbol\lambda} = \beta_{\boldsymbol\lambda}\,\nu_{\boldsymbol\lambda}$, formula (13.19), and Corollary 13.11, we have

$$\begin{aligned}\alpha_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(f)\bigr)\,g \,&=\, (\nu_{\boldsymbol\lambda})^{-1}\bigl(\mu^q_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(f)\bigr)\,\nu_{\boldsymbol\lambda}(g)\bigr) \,=\, (\nu_{\boldsymbol\lambda})^{-1}\bigl(\mu^q_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(f)\bigr)\,\nu^q_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(g)\bigr)\bigr)\\ &=\, (\nu_{\boldsymbol\lambda})^{-1}\bigl(\nu^q_{\boldsymbol\lambda}\bigl(\beta_{\boldsymbol\lambda}(f)\,\beta_{\boldsymbol\lambda}(g)\bigr)\bigr) \,=\, (\beta_{\boldsymbol\lambda})^{-1}\bigl(\beta_{\boldsymbol\lambda}(f)\,\beta_{\boldsymbol\lambda}(g)\bigr) \,=\, f *^q g\,.\end{aligned}$$

Since both $\alpha_{\boldsymbol\lambda}$ and $\beta_{\boldsymbol\lambda}$ are graded $\mathbb{C}[\boldsymbol z^{\pm1}]\otimes\mathbb{C}(h)$-module isomorphisms, the statement follows. □

13.5. **Conjecture on the quantum equivariant K-theory.** The quantum deformation of the equivariant K-theory algebra was introduced by Givental and Lee, motivated, in particular, by a study of the relationship between Gromov-Witten theory and integrable systems, see [G, GL]. The quantum multiplication on $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ depends on new parameters $q_2/q_1,\dots,q_N/q_{N-1}$ and tends to the ordinary multiplication on $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})$ as all of these ratios tend to zero.

Conjecture 13.15. *The multiplication* (13.20) *on* $K_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$ *coincides with the quantum multiplication.*

This conjecture is the K-theoretic analog of the main theorem in [MO] that describes the quantum multiplication in the equivariant cohomology of Nakajima varieties. The case of the quantum multiplication in the equivariant cohomologies of the varieties $T^*\mathcal{F}_{\boldsymbol\lambda}$ was considered also in [GRTV, RTV, TV4].

Lemmas 13.12 and 13.14 mean that modulo Conjecture 13.15, the quantum equivariant K-theory algebra $QK_T(T^*\mathcal{F}_{\boldsymbol\lambda})\otimes\mathbb{C}(h)$ is isomorphic to the Bethe algebra $\rho_{\boldsymbol\lambda}(\mathcal{B}^q)$ and is isomorphic to the algebra $\mathcal{K}^q_{\boldsymbol\lambda}\otimes\mathbb{C}(h)$.

13.6. **Limit $h \to 0$.** Let $M^Q(u)$ be the $N \times N$ matrix with entries $M^Q_{i,j} = 0$ for $|i-j| > 1$,

$$M^Q_{i,i}(u) \,=\, \prod_{k=1}^{\lambda_i} (1 - \gamma_{i,k}/u)\,, \qquad i = 1, \dots, N\,,$$

$$M^Q_{i+1,i}(u) \,=\, Q_{i+1}/Q_i\,, \quad M^Q_{i,i+1}(u) \,=\, (-u)^{-\lambda_i}\gamma_{i,1}\dots\gamma_{i,\lambda_i}\,, \qquad i = 1, \dots, N-1\,,$$

depending on parameters $Q_1, \dots, Q_N$. Set $\widetilde{W}^Q(u) = \det M^Q(u)$.

Lemma 13.16. *Let $\lambda_i > 0$ and $q_i = Q_i h^{\lambda_1 + \dots + \lambda_{i-1}}$ for all $i = 1, \dots, N$. Then*

$$W^q(u) \,\to\, \widetilde{W}^Q(u)$$

as $h \to 0$.

Proof. The proof is similar to that of [GRTV, Theorem 7.3]. □

Therefore, in the limit $h \to 0$ such that $q_i h^{-\lambda_1 - \dots - \lambda_{i-1}}$ are fixed for all $i = 1, \dots, N$, the algebra $\mathcal{K}^q_{\boldsymbol{\lambda}}$ turns into the algebra

$$\widetilde{\mathcal{K}}^Q_{\boldsymbol{\lambda}} \,=\, \mathbb{C}[\boldsymbol{\Gamma}^{\pm 1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}^{\pm 1}] \Big/ \Big\langle \widetilde{W}^Q(u) \,=\, \prod_{a=1}^{n} (1 - z_a/u) \Big\rangle\,.$$

For example, if $N = n = 2$ and $\boldsymbol{\lambda} = (1,1)$, then

$$\widetilde{W}^Q(u) \,=\, \det \begin{pmatrix} 1 - \gamma_{1,1}/u & -\gamma_{1,1}/u \\ Q_2/Q_1 & 1 - \gamma_{2,1}/u \end{pmatrix}$$

and the relations in $\widetilde{\mathcal{K}}^Q_{\boldsymbol{\lambda}}$ are

$$\gamma_{1,1}\,(1 - Q_2/Q_1) + \gamma_{2,1} \,=\, z_1 + z_2\,, \qquad \gamma_{1,1}\,\gamma_{2,1} \,=\, z_1 z_2\,,$$

cf. (13.14)

Conjecture 13.17. *The quantum equivariant K-theory algebra $QK_{(\mathbb{C}^\times)^n}(\mathcal{F}_{\boldsymbol{\lambda}})$ is isomorphic to the algebra $\widetilde{\mathcal{K}}^Q_{\boldsymbol{\lambda}}$.*

This conjecture is the K-theoretic version of the observation in [GRTV, Theorem 7.13] on how to obtain the presentation of the quantum equivariant cohomology $QK_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}})$ in [AS, Formula (2.22)] from the presentation of the quantum equivariant cohomology $QK_{GL_n \times \mathbb{C}^\times}(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ in [GRTV, Theorem 7.10].

14. Appendix 1. Weight functions specialize to Grothendieck polynomials

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ be two sequences of variables. Double Grothendieck polynomials $\mathfrak{G}_w(\alpha; \beta)$ were introduced by Lascoux and Schutzenberger in [LS] by the following recursion.

- For the longest permutation $\sigma_0 \in S_n$, define

$$\mathfrak{G}_{\sigma_0} = \prod_{i+j \leqslant n} (1 - \beta_i/\alpha_j)\,. \tag{14.1}$$

- Let s_i be the i-th elementary transposition. If the length of σs_i is larger than the length of σ, then

$$\mathfrak{G}_\sigma = \pi_{\alpha_i,\alpha_{i+1}}(\mathfrak{G}_{\sigma s_i}), \tag{14.2}$$

where $\pi_{\alpha_i,\alpha_{i+1}}$ is the trigonometric difference operator, see (6.22).

Here is the list of double Grothendieck polynomials for $\sigma \in S_3$:

$$\mathfrak{G}_{321} = (1-\beta_1/\alpha_1)\,(1-\beta_2/\alpha_1)\,(1-\beta_1/\alpha_2)\,, \qquad \mathfrak{G}_{231} = (1-\beta_1/\alpha_1)\,(1-\beta_1/\alpha_2)\,,$$
$$\mathfrak{G}_{312} = (1-\beta_1/\alpha_1)\,(1-\beta_2/\alpha_1)\,, \qquad \mathfrak{G}_{213} = 1-\beta_1/\alpha_1,$$
$$\mathfrak{G}_{132} = 1-\beta_1\beta_2/(\alpha_1\alpha_2), \qquad \mathfrak{G}_{123} = 1\,.$$

Note that the usual choice of variables is $x_i = 1-1/\alpha_i$ and $y_i = 1-\beta_i$, see for example [LS]. In those variables the $\mathfrak{G}_w$'s are indeed polynomials.

Consider the substitution $t_a^{(k)} = u_a$, $h = 0$ into the weight function W_I, and denote it by $\overline{W}_I$.

For

$$I = (\{i_1^{(1)} < \ldots < i_{\lambda_1}^{(1)}\}, \{i_1^{(2)} < \ldots < i_{\lambda_2}^{(2)}\}, \ldots, \{i_1^{(N)} < \ldots < i_{\lambda_N}^{(N)}\})$$

in $\mathcal{I}_{\boldsymbol{\lambda}}$, define $\sigma_I \in S_n$ to be the permutation that maps the ordered list $n, n-1, \ldots, 1$ to the ordered list

$$i_{\lambda_1}^{(1)}, i_{\lambda_1-1}^{(1)}, \ldots, i_1^{(1)}, i_{\lambda_2}^{(2)}, i_{\lambda_2-1}^{(2)}, \ldots, i_1^{(2)}, \ldots, i_{\lambda_N}^{(N)}, i_{\lambda_N-1}^{(N)}, \ldots, i_1^{(N)}.$$

Observe that σ_I and σ_J can belong to the same group S_n even if the their $\boldsymbol{\lambda}$'s are different as long as their corresponding n's are the same. For example $\sigma_{(\{1\},\{3\},\{2\})} = \sigma_{(\{1\},\{2,3\})} = 231$.

Theorem 14.1. *We have*

$$\overline{W}_I = \mathfrak{G}_{\sigma_I}(z_n^{-1}, z_{n-1}^{-1}, \ldots; u_1^{-1}, u_2^{-1}, \ldots).$$

Proof. For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$ let

$$\check{I} = (\{i_{\lambda_1}^{(1)}\}, \{i_{\lambda_1-1}^{(1)}\}, \ldots, \{i_1^{(1)}\}, \{i_{\lambda_2}^{(2)}\}, \{i_{\lambda_2-1}^{(2)}\}, \ldots, \{i_1^{(2)}\}, \ldots, \{i_{\lambda_N}^{(N)}\}, \{i_{\lambda_N-1}^{(N)}\}, \ldots, \{i_1^{(N)}\})$$

in $\mathcal{I}_{(1,1,\ldots,1)}$. We have $\sigma_I = \sigma_{\check{I}}$ and $\overline{W}_I = \overline{W}_{\check{I}}$. The first claim is obvious. Although the functions W_I are $W_{\check{I}}$ are rather different (e.g. they depend on different sets of variables) it can be seen from the definition of the weight functions that after the substitutions indicated by $W \mapsto \overline{W}$ they are equal. Hence it is enough to prove the theorem for $\lambda = (1,1,\ldots,1) \in \mathbb{Z}^n_{\geqslant 0}$.

For this special case we will show that initial conditions and recursions for the two sides of the statement agree. Indeed we have

$$\overline{W}_{\{1\},\{2\},\ldots,\{n\}} = \prod_{j=1}^{n-1}\prod_{i=j+1}^{n}(1-z_i/u_j),$$

and $\mathfrak{G}_{n,n-1,\ldots,1}(z_n^{-1}, z_{n-1}^{-1}, \ldots; u_1^{-1}, u_2^{-1}, \ldots)$ is the same expression because of (14.1). Relations (6.20), (6.23) for $\overline{W}$ read

$$\overline{W}_{s_{a,a+1}(I)} = \pi_{z_a,z_{a+1}}\overline{W}_I$$

if $I_a < I_{a+1}$. After the variable change given in the theorem this is equivalent to the recursion (14.2) for Grothendieck polynomials. $\square$

Corollary 14.2. *The* $t_a^{(k)} = u_a$, $h = 0$ *substitution into the formulae* (4.1) *and* (4.4) *give equivariant localization and iterated residue expressions for the double Grothendieck polynomial* $\mathfrak{G}_{\sigma_I}(z_n^{-1}, z_{n-1}^{-1}, \ldots; u_1^{-1}, u_2^{-1}, \ldots)$.

Iterated residue formulae for stable Grothendieck polynomials, as well as their applications to stability and positivity results for quiver polynomials and Thom polynomials are explored in [RSz, Al, AR].

It is well known in Schubert calculus, see for example [LS, Bu], that the class in $K_{(\mathbb{C}^\times)^n}(\mathcal{F}_{\boldsymbol{\lambda}})$ of the structure sheaf of a Schubert variety is represented by a double Grothendieck polynomial. In our Grothendieck polynomial and index conventions the class of the Schubert variety $\overline{\Omega}_{\mathrm{id},I}$ is represented by the polynomial $\mathfrak{G}_{\sigma_I}(z_n^{-1}, z_{n-1}^{-1}, \ldots; \gamma_{1,1}^{-1}, \gamma_{1,2}^{-1}, \ldots) \cdot P_{\mathrm{id},I}^{-1}$. Denote this class by $\mathcal{O}_I$ and hence from Theorem 14.1 we obtain the following.

Corollary 14.3. *We have*

$$\mathcal{O}_I = [\overline{\Omega}_{\mathrm{id},I}] = [\mathfrak{G}_{\sigma_I}(z_n^{-1}, z_{n-1}^{-1}, \ldots; \gamma_{1,1}^{-1}, \gamma_{1,2}^{-1}, \ldots)] \cdot P_{\mathrm{id},I}^{-1} = [W_I(\Gamma, \boldsymbol{z}, 0)] \cdot P_{\mathrm{id},I}^{-1} \in K_{(\mathbb{C}^\times)^n}(\mathcal{F}_{\boldsymbol{\lambda}}).$$

15. Appendix 2. Interpolation definition of K-theory classes of Schubert varieties

In this appendix we give a new axiomatic definition of the classes $\mathcal{O}_I P_{\mathrm{id},I}$, that is, the classes of Grothendieck polynomials. Note that the polarization $P_{\mathrm{id},I}$ is a monomial, an invertible element of $\mathbb{C}[\boldsymbol{z}^{\pm 1}]$.

Let f_1 and f_2 be Laurent polynomials is the $\boldsymbol{z}$-variables, with $\varphi_I(N(f_1)) = [0, m_1]$, $\varphi_I(N(f_2)) = [0, m_2]$. We write $f_1 \prec_I f_2$ if $m_1 < m_2$, cf. the proof of Lemma 6.5. For the purpose of this section define f to be I-small, if

$$f \prec_I \prod_{k<l} \prod_{a \in I_k} \prod_{\substack{b \in I_l \\ b>a}} (1 - z_b/z_a).$$

Theorem 15.1. *The classes* $\mathcal{O}_I P_{\mathrm{id},I}$*, that is, the images of the Grothendieck polynomials in* $K_T(F_{\boldsymbol{\lambda}})$*, are uniquely determined by the properties*

(1) $\mathcal{O}_I|_{x_I} \cdot P_{\mathrm{id},I} = \prod_{k<l} \prod_{a \in I_k} \prod_{\substack{b \in I_l \\ b>a}} (1 - z_b/z_a)$,

(2) $\mathcal{O}_I|_{x_J} \cdot P_{\mathrm{id},I}$ *is* J*-small if* $J \neq I$.

Proof. The uniqueness proof is the obvious modification of the uniqueness proof in Section 3.2. To prove existence we need to show that $\mathcal{O}_I P_{\mathrm{id},I}$ satisfies the two properties. According to Corollary 14.3 we have $\mathcal{O}_I P_{\mathrm{id},I} = [W_I(\Gamma, \boldsymbol{z}, 0)]$. Property (1) follows from either topology (the variety $\Omega_{\mathrm{id},I}$ is smooth at the point x_I) or from the $h = 0$ substitution in Lemma 6.4. We are going to prove property (2). Let $J \neq I$. We proved in Lemma 6.5 that

$$W_I(\boldsymbol{z}_J, \boldsymbol{z}, h) \prec_J W_J(\boldsymbol{z}_J, \boldsymbol{z}, h) = \prod_{k<l} \prod_{a \in I_k} \Big(\prod_{\substack{b \in I_l \\ b>a}} (1 - z_b/z_a) \prod_{\substack{b \in I_l \\ b<a}} (1 - h z_b/z_a) \Big).$$

We also saw in Lemma 6.3 that $W_I(\boldsymbol{z}_J, \boldsymbol{z}, h)$ is divisible by $\prod_{k<l}\prod_{a\in I_k}\prod_{\substack{b\in I_l\\ b<a}}(1-hz_b/z_a)$. Hence for the quotient we have

$$W_I(\boldsymbol{z}_J,\boldsymbol{z},h)/\prod_{k<l}\prod_{a\in I_k}\prod_{\substack{b\in I_l\\ b<a}}(1-hz_b/z_a)\prec_J \prod_{k<l}\prod_{a\in I_k}\prod_{\substack{b\in I_l\\ b>a}}(1-z_b/z_a).$$

Observe that the right hand side does not depend on h, so the same $\prec_J$ inequality also holds after plugging in $h=0$ in the left hand side. The $h=0$ substitution of the left hand side is $[W_I(\Gamma,\boldsymbol{z},0)]|_{x_J}$, hence property (2) is proved. $\square$

16. Appendix 3. Presentations and structure constants of algebras associated with the projective line

In Section 13.5 we described the conjectured presentation of the equivariant quantum K-theory algebra of partial flag manifolds. In this section we describe in detail this algebra for the special case of the projective line $\mathbb{P}^1$. The basic algebra $K(\mathbb{P}^1)$ can be decorated in three independent ways, namely by considering the equivariant version (with $z=(z_1,z_2)$ parameters), the quantum version (with $q=(q_1,q_2)$ parameters), and the cotangent bundle version (with h parameter). We describe all eight possible algebras, shown in the diagram below and obtained by considering or not considering any of the three decorations. In all eight cases we give the presentation, as well as the structure constants in terms of a natural choice of bases. The descriptions of the four algebras on the front face of the cube are known, and the descriptions of the four algebras on the back face are conjectural.

In the diagram, the symbol ♣ means the limit $q_1=Q_1h^{-1}, q_2=Q_2, h\to 0$.

$$\begin{tikzcd}
 & QK_{\mathbb{C}^\times}(T^*\mathbb{P}^1) \arrow[dl, "q_2/q_1\to 0"'] \arrow[dd, "\clubsuit"] & & QK_T(T^*\mathbb{P}^1) \arrow[ll, "z=1"'] \arrow[dl, "q_2/q_1\to 0"'] \arrow[dd, "\clubsuit"] \\
K_{\mathbb{C}^\times}(T^*\mathbb{P}^1) \arrow[dd, "h=0"] & & K_T(T^*\mathbb{P}^1) \arrow[ll, "z=1"'] \arrow[dd, "h=0"] & \\
 & QK(\mathbb{P}^1) \arrow[dl, "Q_2/Q_1\to 0"'] & & QK_{(\mathbb{C}^\times)^2}(\mathbb{P}^1) \arrow[ll, "z=1"'] \arrow[dl, "Q_2/Q_1\to 0"'] \\
K(\mathbb{P}^1) & & K_{(\mathbb{C}^\times)^2}(\mathbb{P}^1) \arrow[ll, "z=1"'] &
\end{tikzcd}$$

- For $K(\mathbb{P}^1)$ we have the presentation: $\mathbb{C}[\gamma^{\pm1},\delta^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in
$$(1-\gamma/u)(1-\delta/u)-(1-1/u)^2.$$
With the choice of basis $\varkappa_0=1$, $\varkappa_1=1-1/\gamma$, the multiplication is
$$\varkappa_0\varkappa_0=\varkappa_0,\qquad \varkappa_0\varkappa_1=\varkappa_1,\qquad \varkappa_1\varkappa_1=0.$$
- For $K_{(\mathbb{C}^\times)^2}(\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1},\delta^{\pm1},z_1^{\pm1},z_2^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in
$$(1-\gamma/u)(1-\delta/u)-(1-z_1/u)(1-z_2/u).$$

With the choice of basis $\varkappa_0 = 1$, $\varkappa_1 = 1 - z_2/\gamma$, the multiplication is

$$\varkappa_0\varkappa_0 = \varkappa_0, \qquad \varkappa_0\varkappa_1 = \varkappa_1, \qquad \varkappa_1\varkappa_1 = \left(1 - \frac{z_2}{z_1}\right)\varkappa_1.$$

- For $QK(\mathbb{P}^1)$ we have the presentation: $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, Q_1^{\pm1}, Q_2^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$\det\begin{pmatrix} 1-\gamma/u & -\gamma/u \\ Q_2/Q_1 & 1-\delta/u \end{pmatrix} - (1-1/u)^2.$$

With the choice of basis $\varkappa_0 = 1$, $\varkappa_1 = 1 - 1/\gamma$ the multiplication is

$$\varkappa_0\varkappa_0 = \varkappa_0, \qquad \varkappa_0\varkappa_1 = \varkappa_1, \qquad \varkappa_1\varkappa_1 = \frac{Q_2}{Q_1}\varkappa_0.$$

- For $QK_{(\mathbb{C}^\times)^2}(\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, z_1^{\pm1}, z_2^{\pm1}, Q_1^{\pm1}, Q_2^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$\det\begin{pmatrix} 1-\gamma/u & -\gamma/u \\ Q_2/Q_1 & 1-\delta/u \end{pmatrix} - (1-z_1/u)(1-z_2/u).$$

With the choice of basis $\varkappa_0 = 1$, $\varkappa_1 = 1 - z_2/\gamma$ the multiplication is

$$\varkappa_0\varkappa_0 = \varkappa_0, \qquad \varkappa_0\varkappa_1 = \varkappa_1, \qquad \varkappa_1\varkappa_1 = \frac{Q_2 z_2}{Q_1 z_1}\varkappa_0 + \left(1 - \frac{z_2}{z_1}\right)\varkappa_1.$$

- For $K_{\mathbb{C}^\times}(T^*\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, h^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$(1-\gamma/u)(1-\delta/u) - (1-1/u)^2.$$

With the choice of basis $\varkappa_0 = 1 - h/\gamma$, $\varkappa_1 = 1 - 1/\gamma$ the multiplication is

$$\varkappa_0\varkappa_0 = (1-h)\varkappa_0 + h(1-h)\varkappa_1, \qquad \varkappa_0\varkappa_1 = (1-h)\varkappa_1, \qquad \varkappa_1\varkappa_1 = 0.$$

- For $K_T(T^*\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, z_1^{\pm1}, z_2^{\pm1}, h^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$(1-\gamma/u)(1-\delta/u) - (1-z_1/u)(1-z_2/u).$$

With the choice of basis $\varkappa_0 = 1 - hz_1/\gamma$, $\varkappa_1 = 1 - z_2/\gamma$ the multiplication is

$$\begin{aligned} \varkappa_0\varkappa_0 =& (1-hz_1/z_2)\varkappa_0 + h(1-h)z_1/z_2\varkappa_1, \\ \varkappa_0\varkappa_1 =& (1-h)\varkappa_1, \\ \varkappa_1\varkappa_1 =& (1-z_2/z_1)\varkappa_1. \end{aligned}$$

- For $QK_{\mathbb{C}^\times}(T^*\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, q_1^{\pm1}, q_2^{\pm1}, h^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$\det\begin{pmatrix} 1-\gamma/u & 1/q_1(1-\gamma/(uh)) \\ q_2(1-h\delta/u) & (1-\delta/u) \end{pmatrix} - (1-q_2/q_1)(1-1/u)^2$$

With the choice of basis $\varkappa_0 = 1 - h/\gamma$, $\varkappa_1 = 1 - 1/\gamma$ the multiplication is (define $q = q_2/q_1$)

$$\begin{aligned}
\varkappa_0\varkappa_0 =& \frac{1-h}{1-qh}\varkappa_0 + \frac{(1-h)h}{1-qh}\varkappa_1\\
\varkappa_0\varkappa_1 =& \frac{q(1-h)}{1-qh}\varkappa_0 + \frac{1-h}{1-qh}\varkappa_1\\
\varkappa_1\varkappa_1 =& \frac{(1-h)q}{h(1-qh)}\varkappa_0 + \frac{q(1-h)}{(1-qh)}\varkappa_1.
\end{aligned}$$

- For $QK_T(T^*\mathbb{P}^1)$ we have the presentation $\mathbb{C}[\gamma^{\pm1}, \delta^{\pm1}, z_1^{\pm1}, z_2^{\pm1}, q_1^{\pm1}, q_2^{\pm1}, h^{\pm1}]/\langle\text{relations}\rangle$, where the ideal of relations is generated by the coefficients of u-powers in

$$\det\begin{pmatrix} 1-\gamma/u & 1/q_1(1-\gamma/(uh)) \\ q_2(1-h\delta/u) & (1-\delta/u)\end{pmatrix} - (1-q_2/q_1)(1-z_1/u)(1-z_2/u)$$

With the choice of basis $\varkappa_0 = 1 - hz_1/\gamma$, $\varkappa_1 = 1 - z_2/\gamma$ the multiplication is (define $q = q_2/q_1$)

$$\begin{aligned}
\varkappa_0\varkappa_0 =& \frac{1-qh(1-z_1/z_2)-z_1h/z_2}{1-qh}\varkappa_0 + \frac{(1-h)hz_1/z_2}{1-qh}\varkappa_1\\
\varkappa_0\varkappa_1 =& \frac{q(1-h)}{1-qh}\varkappa_0 + \frac{1-h}{1-qh}\varkappa_1\\
\varkappa_1\varkappa_1 =& \frac{(1-h)qz_2/z_1}{h(1-qh)}\varkappa_0 + \frac{1-z_2/z_1+q(z_2/z_1-h)}{1-qh}\varkappa_1.
\end{aligned}$$

17. Appendix 4. Equivariant K-theoretic Schubert calculus on the cotangent bundle of partial flag manifolds

In this section we present a result on the structure constants of the algebra $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \otimes \mathbb{C}(\boldsymbol{z}, h)$ with respect to the basis $\{\kappa_I := \kappa_{\mathrm{id},I}\}_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$. In view of Theorem 14.1 of Appendix 1, this result is a natural "h-deformation" of the multiplication rules for double Grothendieck polynomials.

Let $N, n, \mathcal{I}_{\boldsymbol{\lambda}}, \sigma_0$ be as before.

Theorem 17.1. *The following multiplication rules hold in $K_T(T^*\mathcal{F}_{\boldsymbol{\lambda}})\otimes\mathbb{C}(\boldsymbol{z}, h)$. For $A, B \in \mathcal{I}_{\boldsymbol{\lambda}}$, we have $\kappa_A\kappa_B = \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} c^J_{A,B}\kappa_J$, where*

$$c^J_{A,B} = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{\widetilde{W}_A(\boldsymbol{z}_I, \boldsymbol{z}, h)\widetilde{W}_B(\boldsymbol{z}_I, \boldsymbol{z}, h)\widetilde{W}_{\sigma_0,J}(\boldsymbol{z}_I^{-1}, \boldsymbol{z}^{-1}, h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)} h^{p(J)}P(\boldsymbol{z}_I).$$

Proof. Consider the z_I-substitution into $\kappa_A\kappa_B = \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} c^J_{A,B}\kappa_J$. We obtain

$$\widetilde{W}_A(\boldsymbol{z}_I, \boldsymbol{z}, h)\widetilde{W}_B(\boldsymbol{z}_I, \boldsymbol{z}, h) = \sum_J c^J_{A,B}\widetilde{W}_J(\boldsymbol{z}_I, \boldsymbol{z}, h).$$

Choose a $K \in \mathcal{I}_{\boldsymbol{\lambda}}$, and multiply both sides by

$$\frac{\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1}, \boldsymbol{z}^{-1}, h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)} h^{p(K)}P(\boldsymbol{z}_I).$$

We obtain

$$\frac{\widetilde{W}_A(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_B(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1},\boldsymbol{z}^{-1},h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)}h^{p(K)}P(\boldsymbol{z}_I)=$$
$$=\sum_J c_{A,B}^J\frac{\widetilde{W}_J(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1},\boldsymbol{z}^{-1},h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)}h^{p(K)}P(\boldsymbol{z}_I).$$

Add this equation for all $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, then rearrangement gives

$$\sum_I\frac{\widetilde{W}_A(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_B(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1},\boldsymbol{z}^{-1},h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)}h^{p(K)}P(\boldsymbol{z}_I)=$$
$$=\sum_J c_{A,B}^J\sum_I\frac{\widetilde{W}_J(\boldsymbol{z}_I,\boldsymbol{z},h)\widetilde{W}_{\sigma_0,K}(\boldsymbol{z}_I^{-1},\boldsymbol{z}^{-1},h^{-1})}{R(\boldsymbol{z}_I)Q(\boldsymbol{z}_I)}h^{p(K)}P(\boldsymbol{z}_I).$$

According to the orthogonality Theorem 6.6 the last sum $\sum_I(...)$ is $\delta_{J,K}$. Hence the right-hand side is equal to $c_{A,B}^K$. This proves the theorem. $\square$

Example. For $\lambda=(1,1)$ one recovers the multiplication table for $K_T(T^*\mathbb{P}^1)$ in the basis $\varkappa_1=\kappa_{\{1\},\{2\}}$ and $\varkappa_0=\kappa_{\{2\},\{1\}}$ of Section 16. Here are some sample products for $\lambda=(2,2)$:

$$\kappa_{\{1,2\},\{3,4\}}\kappa_{\{1,4\},\{2,3\}} = (1-h)(1-z_3/z_1)(1-z_4/z_1)(1-hz_3/z_2)\kappa_{\{1,2\},\{3,4\}},$$
$$\kappa_{\{1,3\},\{2,4\}}\kappa_{\{1,4\},\{2,3\}} =$$
$$\big((1-h)^2(1-z_4/z_2)(z_2/z_1+h(1-z_2/z_1-z_3/z_1+z_2/z_3-z_2^2/(z_1z_3)))\big)\kappa_{\{1,2\},\{3,4\}}+$$
$$(1-h)(1-z_2/z_1)(1-z_4/z_1)(1-hz_2/z_3)\kappa_{\{1,3\},\{2,4\}}.$$

The $h=0$ substitution in these formulae gives the multiplication of the equivariant K-theory classes of structure sheaves of Schubert varieties (up to the polarization) in the Grassmannian of 2-planes in $\mathbb{C}^4$.

18. Appendix 5. Bethe algebra of the XXZ model

Let $\mathcal{U}'(\widetilde{\mathfrak{gl}_N})$ be the subalgebra of $\mathcal{U}(\widetilde{\mathfrak{gl}_N})$ generated over $\mathbb{C}[h^{\pm1}]$ by elements (10.5), (10.6). Given a complex number $c\neq 0,1$, set $\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})=\mathcal{U}'(\widetilde{\mathfrak{gl}_N})/\langle h=c\rangle$. Let $\mathcal{B}_c^q$ be the subalgebra of $\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})$ generated by the images of elements (10.10) and the elements $B_{p,\pm s}^q$, see (10.17), under the canonical projection $\mathcal{U}'(\widetilde{\mathfrak{gl}_N})\to\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})$.

By Proposition 11.1, there is a $\mathbb{C}[h^{\pm1}]$-algebra homomorphism

$$\phi:\mathcal{U}'(\widetilde{\mathfrak{gl}_N})\to\operatorname{End}\big((\mathbb{C}^N)^{\otimes n}\big)\otimes\mathbb{C}[\boldsymbol{z}^{\pm1},h^{\pm1}].$$

Given $\boldsymbol{b}\in(\mathbb{C}^\times)^n$ and $c\neq 0,1$, the evaluation at $\boldsymbol{z}=\boldsymbol{b}$, $h=c$ induces an algebra homomorphism

$$\phi_{\boldsymbol{b},c}:\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})\to\operatorname{End}\big((\mathbb{C}^N)^{\otimes n}\big)$$

The homomorphism $\phi_{\boldsymbol{b},c}$ makes $(\mathbb{C}^N)^{\otimes n}$ into a $\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})$-module denoted $(\mathbb{C}^N)^{\otimes n}(\boldsymbol{b},c)$ and called the tensor product of vector representations with evaluation parameters $\boldsymbol{b}$. The algebra $\mathcal{B}_{\boldsymbol{b},c}^q=\phi_{\boldsymbol{b},c}(\mathcal{B}_c^q)$ is called the Bethe algebra of the associated XXZ model on $(\mathbb{C}^N)^{\otimes n}$.

Remark. Usually the *XXZ* model on $(\mathbb{C}^N)^{\otimes n}$ is defined by considering $(\mathbb{C}^N)^{\otimes n}$ as a module over the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_N})$. Under this definition, the Bethe subalgebra of the *XXZ* model, coincides with $\mathcal{B}^q_{\boldsymbol{b},q^{-2}}$.

Recall the space $\frac{1}{D}\mathcal{V}^-$, see Section 9.2. Let $\boldsymbol{b}=(b_1,\dots,b_n)$ and c be such that $b_i\neq cb_j$ for all $i,j=1,\dots,n$. The evaluation at $\boldsymbol{z}=\boldsymbol{b}$, $h=c$ defines a homomorphism of $\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})$-modules

$$\iota_{\boldsymbol{b},c}:\tfrac{1}{D}\mathcal{V}^-/\langle h=c\rangle\,\to\,(\mathbb{C}^N)^{\otimes n}(\boldsymbol{b},c)\,.$$

Proposition 18.1. *Let $b_i\neq cb_j$ for all $i,j=1,\dots,n$. Then $\iota_{\boldsymbol{b},c}$ is an epimorphism.*

Proof. The proof is similar to the proof of [MTV3, Proposition 10.4]. The image of $\iota_{\boldsymbol{b},c}$ is not zero, while by [AK], the $\mathcal{U}_c(\widetilde{\mathfrak{gl}_N})$-module $(\mathbb{C}^N)^{\otimes n}(\boldsymbol{b},c)$ is irreducible if $b_i\neq cb_j$ for all $i,j=1,\dots,n$, see also [NT2, Theorem 3.4]. □

The elements of $\mathcal{B}^q_{\boldsymbol{b},c}$ preserve each of the subspaces $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. Let $\mathcal{B}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}\subset\operatorname{End}\bigl((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\bigr)$ be the subalgebra induced by the action of $\mathcal{B}^q_{\boldsymbol{b},c}$ on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$.

Given c and $\boldsymbol{\lambda}$, let $\mathcal{K}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}=\mathcal{K}^q_{\boldsymbol{\lambda}}/\langle\boldsymbol{z}=\boldsymbol{b},\ h=c\rangle$. The algebra $\mathcal{K}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$ is the algebra of functions on the fiber of the Wronski map.

Corollary 18.2. *Let $\boldsymbol{b}=(b_1,\dots,b_n)$ and c be such that $b_i\neq cb_j$ for all $i,j=1,\dots,n$. Then isomorphisms* (13.15), (13.17), *and the evaluation at $\boldsymbol{z}=\boldsymbol{b}$, $h=c$ induce an algebra isomorphism $\mu^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}:\mathcal{K}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}\to\mathcal{B}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$ and an isomorphism of vector spaces $\nu^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}:\mathcal{K}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}\to(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. The isomorphisms $\mu^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$ and $\nu^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$ identify the regular representation of $\mathcal{K}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$ and the $\mathcal{B}^q_{\boldsymbol{b},c,\boldsymbol{\lambda}}$-module $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$.* □

Corollary 18.3. *The algebra $\mathcal{B}^q_{\boldsymbol{b},c}$ is a maximal commutative subalgebra of* $\operatorname{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$. □

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