A THEOREM ON REMOVING Σ^r SINGULARITIES

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Abstract. As an application of the generalized Pontrjagin-Thom construction ([2]) and a theorem of Golubjatnikov [1] here we prove a result on removing Σ^r singularities in a certain cobordism class of smooth mappings of positive codimension.

The integer $k > 0$ will be fixed throughout the paper. Let $\eta : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{n+k}, 0)$ be a smooth map germ. By a suspension of η we mean a germ $\Sigma\eta : (\mathbb{R}^{n+v}, 0) \longrightarrow (\mathbb{R}^{(n+k)+v}, 0)$ defined by $(x, u) \mapsto (\eta(x), u)$ — otherwise we will use the standard notions and notations of singularity theory, see e.g. [3]. Now consider stable smooth maps between smooth manifolds of codimension k. For such a map $f : N^m \longrightarrow P^{m+k}$ we define the submanifolds

$$\eta(f) = \{\ y \in P \mid \quad f^{-1}(y) \text{ has only one element and the germ of } f \text{ at } f^{-1}(y) \text{ is } \mathcal{A}\text{-equivalent to a suspension of } \eta\ \},$$

$$\Sigma^r(f) = \{\ x \in N \mid \text{the germ of } f \text{ at } x \text{ is of Thom-Boardman type } \Sigma^r.\ \}.$$

Let $\eta_r : (\mathbb{R}^{r^2+rk}, 0) \longrightarrow (\mathbb{R}^{r^2+rk+k}, 0)$ denote the miniversal unfolding of the germ $\zeta_r : (\mathbb{R}^r, 0) \longrightarrow (\mathbb{R}^{r+k}, 0)$ defined by

$$(x_1, \dots, x_r) \mapsto (x_1^2, \dots, x_r^2, x_1x_2, x_1x_3, \dots, x_{r-1}x_r, 0, \dots, 0),$$

where there are $t := k - \binom{r}{2}$ 0's at the end. The Thom-Boardman type of η_r is $\Sigma^{r,0}$, and, in fact, this is the "simplest" among the germs of codimension k and of Thom-Boardman type Σ^r. That is, if $f : N^m \longrightarrow P^{m+k}$ is a map, then the closure of the submanifold $f^{-1}(\eta_r(f)) \subset N$ contains $\Sigma^r(f)$.

Definition 1. *If $g : N^m \longrightarrow P^{m+k} \times \mathbb{R}^r$ is an immersion then the composition $f = pr_P \circ g : N^m \longrightarrow P \times \mathbb{R}^r \longrightarrow P$ will be called a prim-Σ^r map. If, in addition, f does not have other Σ^r singularities than η_r, then we will call it a prim-η_r map. If f does not have Σ^r points at all, then we call it prim-$\emptyset$.*

The word *prim* stands for *pro*jected *im*mersion, and Σ^r and η_r refers to types of the most difficult singularities such a map may have. Now let us fix $\mathbb{R}^{m+k}$ and consider prim-η_r (prim-Σ^r, prim-$\emptyset$) maps of m-manifolds into it. We call two such map $f_1 : N_1^m \longrightarrow \mathbb{R}^{m+k}$ and $f_2 : N_2^m \longrightarrow \mathbb{R}^{m+k}$ cobordant if there is an abstract manifold W^{m+1} with boundary $N_1 \cup N_2$ and a prim-η_r (prim-Σ^r, prim-$\emptyset$) map

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$F : W \longrightarrow \mathbb{R}^{m+k} \times [0,1]$ whose restriction to N_i is f_i $(i = 1, 2)$. Cobordism is clearly an equivalence relation and the set of its classes will be denoted by

$$Cob_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r) \qquad \Big(Cob_m(\mathbb{R}^{m+k}, \text{prim-}\Sigma^r), \qquad Cob_m(\mathbb{R}^{m+k}, \text{prim-}\emptyset)\Big).$$

All three sets form a group under the operation of "remote disjoint union".

The following theorem states that the obvious necessary condition for the realization of a class in $Cob_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r)$ with a map with no η_r points at all (so with no Σ^r points at all) is also sufficient — in a certain dimension range.

THEOREM 2. *Let r be even, $m < (r+1)k + (r^2 - 1)$, and let $f : N^m \longrightarrow \mathbb{R}^{m+k}$ be a stable prim-η_r map. Then the following conditions are equivalent:*

(1) there is a prim-η_r map $g : M^n \longrightarrow \mathbb{R}^{n+k}$ cobordant to f with $\eta_r(g) = \emptyset$;

(2) the abstract manifold $\eta_r(f)$ is nullcobordant.

PROOF. The implication (1)⇒(2) is clear. Indeed, if the cobordism between f and g is given by $F : W \longrightarrow \mathbb{R}^{(n+k)+r}$ then the manifold $\eta_r(F)$ is a cobordism between $\eta_r(f)$ and the emptyset.

For the converse implication we need some notions and results from [2].

Consider the set of stable map germs $(\mathbb{R}^*, \text{finite set}) \longrightarrow (\mathbb{R}^{*+k}, 0)$. The set of equivalence classes of this set under the equivalence relation generated by $\mathcal{A}$-equivalence and suspension is called T. There is an obvious hierarchy on T, whose top element is the class of (k-codimensional) embeddings, and right under this is the class of immersions with a double point, etc. Let τ be an ascending subset of T. A map $f : N^m \longrightarrow P^{m+k}$ is called a τ-map if for all $y \in f(N)$ the germ of f at $f^{-1}(y)$ is from τ. For more details and examples see [2]. If for two τ-maps $f_i : N_i \longrightarrow P^{m+k}$ $(i = 0, 1)$ an abstract cobordism W is given between N_0 and N_1, as well as a τ-map $F : W \longrightarrow P^{m+k} \times [0,1]$ with $F|_{N_i} = f_i \times \{i\}$, then we call f_0 and f_1 cobordant. The set of cobordism classes is denoted by: $Cob_m(P^{m+k}; \tau)$.

DEFINITION 3. *The space X is called a classifying space for τ-maps, if for any closed manifold P^{m+k} there is a bijection between*

$$Cob_m(P^{m+k}, \tau) \qquad \text{and} \qquad [P, X] = \text{homotopy classes of maps } P \longrightarrow X.$$

Now let $\tau = \tau' \cup [\eta]$, where τ and τ' are ascending subsets of T. Suppose also that η is the "simplest" in its equivalence class, that is suppose that $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is not the suspension of any other germ — germs having this property are called isolated. Let G be the maximal compact subgroup of the symmetry group

$$Aut_{\mathcal{A}}\eta = \{\ (\varphi, \phi) \in Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^{n+k}, 0) \mid \phi \circ \eta \circ \varphi^{-1} = \eta\ \},$$

with the representations λ_1 and λ_2 on $\mathbb{R}^n$ and $\mathbb{R}^{n+k}$ (both of which can be supposed to be linear). The vector bundle associated to the universal principal G-bundle using the representation λ_i will be denoted by $E\lambda_i \longrightarrow BG$ and its disc bundle by $D\lambda_i$. The following theorem is proved in [2].

THEOREM 4. *If $X\tau'$ is a classifying space for τ'-maps then the space $X\tau = X\tau' \cup_\rho D\lambda_2$ is a classifying space for τ-maps (for some $\rho : \partial D\lambda_2 \longrightarrow X\tau'$).* □

Since the one-point-space is a classifying space for $\emptyset$-maps, and any other τ can be build up from $\emptyset$ by consecutively adding new $[\eta]$'s, the above theorem can be considered as a construction of a classifying space for τ-maps for all τ. Well, almost... In fact, to carry out this procedure we need some knowledge of the group $Aut_{\mathcal{A}}\eta$, its maximal compact subgroup and its representations λ_1, λ_2. This problem is also essentially solved in [2], we will come back to these results in the concrete examples where we need them.

Now turn back to prim-η_r maps to $\mathbb{R}^{m+k}$. Their cobordism group (defined above) is not $Cob_m(\mathbb{R}^{m+k}, \tau)$ for any τ, but one can evidently extend the notion of classifying space for prim-η_r (as well as prim-Σ^r and prim-$\emptyset$) maps. And the theorem quoted above remains true with a minor modification. For this we need some notation. The germ $\eta_r : \mathbb{R}^{r^2+kr} \longrightarrow \mathbb{R}^{r^2+kr+k}$ is of Thom-Boardman type Σ^r, so the kernel K of its differential is r-dimensional. Now let $\bar{G}$ be the subgroup of $G = MC\ Aut_{\mathcal{A}}\eta_r$ whose induced action on K is trivial. The restriction of λ_i to $\bar{G}$ will be called $\bar{\lambda}_i$, and $E\bar{\lambda}_i \longrightarrow B\bar{G}$ ($D\bar{\lambda}_i \longrightarrow B\bar{G}$) denotes the vector bundle (disc bundle) associated to the universal principal $\bar{G}$-bundle using the representation $\bar{\lambda}_i$. The following theorem is analoguos to Theorem 4. We will not give a proof for it, since it goes the same way.

Theorem 5. *Let X' be a classifying space for prim-$\emptyset$ maps. Then $X = X' \cup_\rho D\bar{\lambda}_2$ is a classifying space for prim-η_r maps (for some $\rho : \partial D\bar{\lambda}_2 \longrightarrow X'$).* $\square$

Now let us consider a portion of the homotopy exact sequence of the pair (X, X'):

$$\begin{array}{ccccc} Cob_m(\mathbb{R}^{m+k}, \text{prim-}\emptyset) & & Cob_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r) & & \\ || & & || & & \\ \pi_{m+k}(X') & \xrightarrow{i_*} & \pi_{m+k}(X) & \xrightarrow{\delta} & \pi_{m+k}(X, X'). \end{array}$$

The statement (2)$\Rightarrow$(1) in this terms is the following: if $[f] \in Cob_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r)$ is such that $\eta_r(f)$ is null-cobordant, then $[f]$ is in the image of i_* — or, what is the same, $\delta([f]) = 0$. Now let us study the group $\pi_{m+k}(X, X')$. First observe that

a) X' is $(k-1)$-connected, since $\pi_i(X') = Cob_{i-k}(S^i, \text{prim-}\emptyset) = 0$ if $i - k < 0$; and

b) the pair (X, X') is $r^2 + rk + k - 1$-connected, since $H^i(X, X') = H^i(T\bar{\lambda}_2) = 0$ for $i = 1, \dots, r^2 + rk + k - 1$, because $rk\ \bar{\lambda}_2 = r^2 + rk + k$.

Now, due to the *homotopy excision theorem* and our dimension restrictions we have the natural isomorphism $\pi_{m+k}(X, X') = \pi_{m+k}(X/X')$ which latter is $\pi_{m+k}(T\bar{\lambda}_2)$. Now we have to analyse the group $\bar{G}$ and the representation $\bar{\lambda}_2$.

Lemma 6.

$$\bar{G} = O(t) \qquad \bar{\lambda}_2 = r \cdot \rho_t \oplus (r^2 - r + r\binom{r+1}{2}) \cdot 1,$$

where $t = k - \binom{r}{2}$, ρ_t is the standard t-dimensional representation of $O(t)$, and 1 is the trivial 1-dimensional representation.

Proof of Lemma 6. First we recall from [2] some results about the maximal compact automorphism group of η_r:

$$MC\ Aut_{\mathcal{A}}\eta_r = MC\ Aut_{\mathcal{K}}\zeta_r \leq Aut\ Q_{\zeta_r} \times O(k-d) =$$

$= MC\ Aut\ \mathbb{R}[[x_1,\dots,x_r]]/(x_1^2,\dots,x_r^2,x_1x_2,\dots,x_{r-1}x_r)\times O(k-t) = O(r)\times O(t),$

where Q_{ζ_r} is the local algebra of ζ_r and d is its defect. In fact $O(r)\times O(t)$ acts as an $\mathcal{A}$-equivalence group (and therefore as an $\mathcal{K}$-equivalence group) of ζ_r, so there there is equation instead of $\leq$ in the formula. So $G = O(r)\times O(t)$.

To determine $\bar{G}\leq G$ and the representation $\bar{\lambda}_2$ we recall some more notions and results from [2] and [4]. Since the germ η_r is a miniversal unfolding of ζ_r with $d\zeta_r(0) = 0$, therefore η_r is $\mathcal{A}$-equivalent to

$$\mathbb{R}^r\times V \longrightarrow \mathbb{R}^{r+k}\times V$$

$$(x,\phi)\mapsto (x+\phi(x),\phi),$$

where V is a complement of the subspace $t\zeta_r(\theta_r)+\zeta_r^*(\mathfrak{m}(r+k))\theta_{\zeta_r}$ in the vector space θ_{ζ_r}. Since G actually acts as an $\mathcal{A}$ automorphism, so it has representations α and β on $\mathbb{R}^r$ and $\mathbb{R}^{r+k}$ respectively. The group G also acts on θ_{ζ_r} by $(\alpha,\beta)\cdot\phi = \beta\circ\phi\circ\alpha^{-1}$ — leaving $t\zeta_r(\theta_a)+\zeta_r^*(\mathfrak{m}(a+k))\theta_{\zeta_r}$ invariant. If V is chosen to be G-invariant (G compact, so it is possible) then G also acts on V. Let this action be γ. A theorem in [2] proves that the maximal compact subgroup of $Aut_{\mathcal{A}}\eta_r$ is G with the representations $\lambda_1 := \alpha\oplus\gamma$, $\lambda_2 := \beta\oplus\gamma$ on the source ($\mathbb{R}^r\times V$) and target ($\mathbb{R}^{r+k}\times V$) spaces, respectively.

Now observe that α is $\rho_r\circ pr_{O(r)}$, where ρ_r is the standard r-dimensional representation of $O(r)$. Observe also that the kernel of $d\eta_r$ is (the tangent space to) $\mathbb{R}^r$, so the subgroup $\bar{G}\leq G$ must be $O(t)$. If we choose V to be spanned by

$$(x_1,\dots,x_r)\mapsto \quad (0,\dots,0,x_i,0,\dots,0) \qquad i=1,\dots,r \text{ the coordinate is } j=1,\dots,r,\ i\neq j$$

$$(x_1,\dots,x_r)\mapsto \quad (0,\dots,0,x_i,0,\dots,0) \qquad i=1,\dots,r \text{ the coordinate is } j=r+1,\dots,r+k,$$

then V will be $O(r)\times O(t)$-invariant, and using the definition of α, β, γ above we can compute

$$\alpha|_{\bar{G}} = r\cdot 1 \qquad \beta|_{\bar{G}} = \rho_t\oplus(\binom{r}{2}+r)\cdot 1 \qquad \gamma|\bar{G} = r\cdot\rho_t\oplus(r(r-1)+r\binom{r}{2})\cdot 1,$$

which proves the lemma. □

Now, according to the original Thom-construction, the group $\pi_{m+k}(T\bar{\lambda}_2)$ is isomorphic to the cobordism group of embeddings of closed $m-r^2-rk$-manifolds into $\mathbb{R}^{m+k}$ with a fixed splitting of the normal bundle to the direct sum of $r+1$ isomorphic t-dimensional bundles and a trivial bundle. If the dimension $n-r^2-rk$ of the embedded manifold is smaller than t (which holds in the dimension range of the Thoerem) then these bundles are already stable normal bundles, so the group $\pi_{m+k}(T\bar{\lambda}_2)$ is isomorphic to the group $\Omega^{(r+1)\gamma}_{m-r^2-rk}$ defined by Golubjatnikov[1]

[1]the notation $\mathfrak{N}^{(r+1)\gamma}_{m-r^2-rk}$ would be perhap better

in [1]. Golubjatnikov also proves that in case $r+1$ is odd then the forgetful map $\Omega^{(r+1)\gamma}_{m-r^2-rk} \longrightarrow \mathfrak{N}_{m-r^2-rk}$ to the abstract cobordism group is an isomorphism.

Putting all these together we see that $\pi_{m+k}(X, X') \cong \mathfrak{N}_{m-r^2-rk}$, and it is easy to see that the image of $\delta([f])$ in $\mathfrak{N}_{m-r^2-rk}$ is the abstract cobordism class of $\eta_r(f)$. Since $\eta_r(f)$ is null-cobordant, we have proved the theorem. $\square$

Remark. In fact, the dimension restriction $m < (r+1)k + (r^2-1)$ in the theorem implies that a stable map $N^m \longrightarrow \mathbb{R}^{m+k}$ does not have any other singularities of type Σ^r than η_r. Therefore the condition "$f : N^m \longrightarrow \mathbb{R}^{m+k}$ is a stable prim-η_r map" can be weakened as "$f : N^m \longrightarrow \mathbb{R}^{m+k}$ is a stable prim-Σ^r map"; so Theorem 2 can actually be considered as a theorem on removing Σ^r singularities.

References

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