COHOMOLOGY OF A FLAG VARIETY AS A BETHE ALGEBRA

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To the memory of V.I. Arnold

Abstract. We interpret the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda},\mathbb{C})$ of a partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing subspaces $0=F_0\subset F_1\subset\dots\subset F_N=\mathbb{C}^n$, $\dim F_i/F_{i-1}=\lambda_i$, as the Bethe algebra $\mathcal{B}^\infty(\mathcal{V}^\pm_{\boldsymbol\lambda})$ of the $\mathfrak{gl}_N$-weight subspace $\mathcal{V}^\pm_{\boldsymbol\lambda}$ of a $\mathfrak{gl}_N[t]$-module $\mathcal{V}^\pm$.

1. Introduction

A Bethe algebra of a quantum integrable model is a commutative algebra of linear operators (Hamiltonians) acting on the space of states of the model. An interesting problem is to describe the Bethe algebra as the algebra of functions on a suitable scheme. Such a description can be considered as an instance of the geometric Langlands correspondence, see [MTV2], [MTV3]. The $\mathfrak{gl}_N$ Gaudin model is an example of a quantum integrable model [G1], [G2]. The Bethe algebra $\mathcal{B}^K$ of the $\mathfrak{gl}_N$ Gaudin model is a commutative subalgebra of the current algebra $U(\mathfrak{gl}_N[t])$. The algebra $\mathcal{B}^K$ depends on the parameters $K=(K_1,\dots,K_N)\in\mathbb{C}^N$. Having a $\mathfrak{gl}_N[t]$-module M, one obtains the commutative subalgebra $\mathcal{B}^K(M)\subset\operatorname{End}(M)$ as the image of $\mathcal{B}^K$. The geometric interpretation of the algebra $\mathcal{B}^K(M)$ as the algebra of functions on a scheme leads to interesting objects. For example, the Bethe algebra $\mathcal{B}^{K=0}((\otimes_{s=1}^n L_{\boldsymbol\Lambda_s}(z_s))^{sing}_{\boldsymbol\lambda})$ of the subspace of singular vectors of the $\mathfrak{gl}_N$-weight $\boldsymbol\lambda$ of the tensor product of finite dimensional evaluation modules $\otimes_{s=1}^n L_{\boldsymbol\Lambda_s}(z_s)$ is interpreted as the space of functions on the intersection of suitable Schubert cycles in a Grassmannian variety, see [MTV2]. This interpretation gives a relation between representation theory and Schubert calculus useful in both directions.

One of the most interesting $\mathfrak{gl}_N[t]$-modules is the vector space $\mathcal{V}=V^{\otimes n}\otimes\mathbb{C}[z_1,\dots,z_n]$ of $V^{\otimes n}$-valued polynomials in $z_1,\dots,z_n$, where $V=\mathbb{C}^N$ is the standard vector representation

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of $\mathfrak{gl}_N$. The Lie algebra $\mathfrak{gl}_N[t]$ naturally acts on $\mathcal{V}$ as well as the symmetric group S_n, which permutes the factors of $V^{\otimes n}$ and variables $z_1, \dots, z_n$ simultaneously. We denote by $\mathcal{V}^+$ and $\mathcal{V}^-$ the S_n-invariant and antiinvariant subspaces of $\mathcal{V}$, respectively. The actions of $\mathfrak{gl}_N[t]$ and S_n on $\mathcal{V}$ commute, so $\mathcal{V}^+$ and $\mathcal{V}^-$ are $\mathfrak{gl}_N[t]$-submodules of $\mathcal{V}$. The Bethe algebra $\mathcal{B}^K$ preserves the $\mathfrak{gl}_N$-weight decompositions $\mathcal{V}^+ = \oplus_{\boldsymbol{\lambda}} \mathcal{V}^+_{\boldsymbol{\lambda}}$ and $\mathcal{V}^- = \oplus_{\boldsymbol{\lambda}} \mathcal{V}^-_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$. The Bethe algebra $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$ was described in [MTV3] as the algebra of functions on a suitable space of quasiexponentials $\{e^{K_i u}(u^{\lambda_i} + \Sigma_{i1} u^{\lambda_i - 1} + \dots + \Sigma_{i\lambda_i}),\ i = 1, \dots, N\}$. In this paper we give a similar description for $\mathcal{B}^K(\mathcal{V}^-_{\boldsymbol{\lambda}})$ and study the limit of the algebras $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\mathcal{B}^K(\mathcal{V}^-_{\boldsymbol{\lambda}})$ as all coordinates of the vector K tend to infinity so that $K_i/K_{i+1} \to \infty$ for all i. We show that in this limit both Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol{\lambda}})$ can be identified with the algebra of the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C})$ of the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n,$$

$\dim F_i/F_{i-1} = \lambda_i$. This identification was motivated for us by the considerations in [RV], [RSV] where the equivariant cohomology of the partial flag varieties were used to construct certain conformal blocks in $V^{\otimes n}$.

Our identification of the Bethe algebra with the algebra of multiplication operators of the equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C})$ can be considered as a degeneration of the recent description in [O] of the equivariant quantum cohomology of the partial flag varieties as the Bethe algebra of a suitable Yangian model associated with $V^{\otimes n}$, cf. [BMO].

In Section 2 we introduce the Bethe algebra. Section 3 contains the main results — Theorems 3.3, 3.4. Theorems 3.3 identifies the algebra of equivariant cohomology $H^*_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C})$ and the Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol{\lambda}})$. Theorem 3.4 says that the Shapovalov pairing of $\mathcal{V}^+_{\boldsymbol{\lambda}}$ and $\mathcal{V}^-_{\boldsymbol{\lambda}}$ is nondegenerate. In Section 4 we show that the isomorphisms of Theorem 3.3 are limiting cases of a geometric Langlands correspondence. In Section 5 we explain how the Bethe algebras $\mathcal{B}^\infty(\mathcal{V}^+_{\boldsymbol{\lambda}})$, $\mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol{\lambda}})$ are related to the quantum equivariant cohomology $QH_{GL_n \times \mathbb{C}^*}(T^*\mathcal{F}_{\boldsymbol{\lambda}})$ of the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of the flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$. Appendix contains the topological description of $\mathfrak{gl}_N[t]$-actions on $\oplus_{\boldsymbol{\lambda}} H^*_{GL_n}(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C})$.

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2. Representations of current algebra $\mathfrak{gl}_N[t]$

2.1. **Lie algebra $\mathfrak{gl}_N$.** Let e_{ij}, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js} e_{ik} - \delta_{ik} e_{sj}$. We denote by $\mathfrak{h} \subset \mathfrak{gl}_N$ the subalgebra generated by e_{ii}, $i = 1, \dots, N$. For a Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

A vector v of a $\mathfrak{gl}_N$-module M has weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{ii} v = \lambda_i v$ for $i = 1, \dots, N$. We denote by $M_{\boldsymbol{\lambda}} \subset M$ the weight subspace of weight $\boldsymbol{\lambda}$.

Let $V = \mathbb{C}^N$ be the standard vector representation of $\mathfrak{gl}_N$ with basis $v_1, \dots, v_N$ such that $e_{ij} v_k = \delta_{jk} v_i$ for all i, j, k. A tensor power $V^{\otimes n}$ of the vector representation has a basis given by the vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$, where $i_j \in \{1, \dots, N\}$.

Every sequence $(i_1, \dots, i_n)$ defines a decomposition $I = (I_1, \dots, I_N)$ of $\{1, \dots, n\}$ into disjoint subsets $I_1, \dots, I_N$: $I_j = \{k \mid i_k = j\}$. We denote the basis vector $v_{i_1} \otimes \dots \otimes v_{i_n}$ by v_I.

Let

$$V^{\otimes n} = \bigoplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},\,|\boldsymbol{\lambda}|=n} (V^{\otimes n})_{\boldsymbol{\lambda}}$$

be the weight decomposition. Denote $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all indices I with $|I_j| = \lambda_j,\ \ j = 1,\dots N$. The vectors $\{v_I, I \in \mathcal{I}_{\boldsymbol{\lambda}}\}$, form a basis of $(V^{\otimes n})_{\boldsymbol{\lambda}}$. The dimension of $(V^{\otimes n})_{\boldsymbol{\lambda}}$ equals the multinomial coefficient $d_{\boldsymbol{\lambda}} := \frac{n!}{\lambda_1!\dots\lambda_N!}$.

Let $\mathcal{S}$ be the bilinear form on $V^{\otimes n}$ such that the basis $\{v_I\}$ is orthonormal. We call $\mathcal{S}$ the Shapovalov form.

2.2. **Current algebra $\mathfrak{gl}_N[t]$.** Let $\mathfrak{gl}_N[t] = \mathfrak{gl}_N \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_N$-valued polynomials with pointwise commutator. We identify $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$. Hence, any $\mathfrak{gl}_N[t]$-module has the canonical structure of a $\mathfrak{gl}_N$-module.

The Lie algebra $\mathfrak{gl}_N[t]$ has a basis $e_{ij}\otimes t^r$, $i,j = 1,\dots,N$, $r \in \mathbb{Z}_{\geqslant 0}$, such that

$$[e_{ij}\otimes t^r, e_{sk}\otimes t^p] \,=\, \delta_{js}e_{ik}\otimes t^{r+p} - \delta_{ik}e_{sj}\otimes t^{r+p}\,.$$

It is convenient to collect elements of $\mathfrak{gl}_N[t]$ in generating series of a variable u. For $g \in \mathfrak{gl}_N$, set $g(u) = \sum_{s=0}^{\infty}(g\otimes t^s)u^{-s-1}$.

The subalgebra $\mathfrak{z}_N[t] \subset \mathfrak{gl}_N[t]$ with basis $\sum_{i=1}^N e_{ii}\otimes t^r$, $r \in \mathbb{Z}_{\geqslant 0}$, is central.

2.3. **The $\mathfrak{gl}_N[t]$-modules $\mathcal{V}^{\pm}$.** Let S_n be the permutation group on n elements. For an S_n-module M we denote by M^+ (resp. M^-) the subspace of S_n-invariants (resp. antiinvariants).

The group S_n acts on $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1,\dots,z_n]$ by permuting the variables. Denote by $\sigma_s(\boldsymbol{z})$, $s = 1,\dots,n$, the sth elementary symmetric polynomial in $z_1,\dots,z_n$.

Let $\mathcal{V}$ be the vector space of polynomials in variables $\boldsymbol{z}$ with coefficients in $V^{\otimes n}$:

$$\mathcal{V} \,=\, V^{\otimes n}\otimes_{\mathbb{C}}\mathbb{C}[\boldsymbol{z}]\,.$$

The symmetric group S_n acts on $\mathcal{V}$ by permuting the factors of $V^{\otimes n}$ and the variables $\boldsymbol{z}$ simultaneously,

$$\sigma\big(v_1\otimes\dots\otimes v_n\otimes p(z_1,\dots,z_n)\big) \,=\, v_{(\sigma^{-1})_1}\otimes\dots\otimes v_{(\sigma^{-1})_n}\otimes p(z_{\sigma_1},\dots,z_{\sigma_n})\,,\quad \sigma\in S_n\,.$$

We are interested in the subspaces $\mathcal{V}^+,\mathcal{V}^-\subset\mathcal{V}$ of S_n-invariants and antiinvariants.

The space $\mathcal{V}$ is a $\mathfrak{gl}_N[t]$-module,

$$g\otimes t^r\,\big(v_1\otimes\dots\otimes v_n\otimes p(\boldsymbol{z})\big) \,=\, \sum_{s=1}^n v_1\otimes\dots\otimes gv_s\otimes\dots\otimes v_n\otimes z_s^r p(\boldsymbol{z})\,.$$

The image of the subalgebra $U(\mathfrak{z}_N[t])\subset U(\mathfrak{gl}_N[t])$ in $\mathrm{End}(\mathcal{V})$ is the algebra of operators of multiplication by elements of $\mathbb{C}[\boldsymbol{z}]^+$. The $\mathfrak{gl}_N[t]$-action on $\mathcal{V}$ commutes with the S_n-action. Hence, $\mathcal{V}^+$ and $\mathcal{V}^-$ are $\mathfrak{gl}_N[t]$-submodules of $\mathcal{V}$. The subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$ are free $\mathbb{C}[\boldsymbol{z}]^+$-modules of rank N^n.

Consider the $\mathfrak{gl}_N$-weight decompositions

$$\mathcal{V}^+ = \oplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},|\boldsymbol{\lambda}|=n}\mathcal{V}^+_{\boldsymbol{\lambda}},\qquad \mathcal{V}^- = \oplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},|\boldsymbol{\lambda}|=n}\mathcal{V}^-_{\boldsymbol{\lambda}}\,.$$

For any $\boldsymbol{\lambda}$, the subspaces $\mathcal{V}^+_{\boldsymbol{\lambda}}$, and $\mathcal{V}^-_{\boldsymbol{\lambda}}$ are free $\mathbb{C}[\boldsymbol{z}]^+$-modules of rank $d_{\boldsymbol{\lambda}}$.

Denote by $\frac{1}{D}\mathcal{V}^-$ the vector space of all $V^{\otimes n}$-valued rational functions of the form $\frac{1}{D}x$, $x \in \mathcal{V}^-$, $D = \prod_{1\leqslant i<j\leqslant n}(z_j - z_i)$. The Shapovalov form induces a $\mathbb{C}[\boldsymbol{z}]^+$-bilinear map

$$\mathcal{S}_{+-} : \mathcal{V}^+ \otimes \frac{1}{D}\mathcal{V}^- \to \mathbb{C}[\boldsymbol{z}]^+.$$

The $\mathfrak{gl}_N[t]$-module structures on $\mathcal{V}^+$ and $\frac{1}{D}\mathcal{V}^-$ are contravariantly related through the Shapovalov form,

$$\mathcal{S}_{+-}\big((e_{ij}\otimes t^r)x, \frac{1}{D}\,y\big) \,=\, \mathcal{S}_{+-}\big(x, (e_{ji}\otimes t^r)\frac{1}{D}\,y\big) \qquad \text{for all } \ i, j, x, y.$$

2.4. **Bethe algebra.** Given an $N\times N$ matrix A with possibly noncommuting entries a_{ij}, we define its row determinant to be

$$\operatorname{rdet} A \,=\, \sum_{\sigma\in S_N} (-1)^\sigma\, a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{N\sigma(N)}\,.$$

Let $K = (K_1, \dots, K_N)$ be a sequence of distinct complex numbers. Let ∂ be the operator of differentiation in a variable u. Define the universal differential operator $\mathcal{D}^K$ by

$$\mathcal{D}^K \,=\, \operatorname{rdet}\begin{pmatrix} \partial - K_1 - e_{11}(u) & -\,e_{21}(u) & \dots & -\,e_{N1}(u)\\ -\,e_{12}(u) & \partial - K_2 - e_{22}(u) & \dots & -\,e_{N2}(u)\\ \dots & \dots & \dots & \dots\\ -\,e_{1N}(u) & -\,e_{2N}(u) & \dots & \partial - K_N - e_{NN}(u)\end{pmatrix}.$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $U(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^K \,=\, \partial^N + \sum_{i=1}^N B^K_i(u)\,\partial^{N-i}, \qquad B^K_i(u) \,=\, \sum_{j=0}^\infty B^K_{ij}\,u^{-j}$$

and $B^K_{ij} \in U(\mathfrak{gl}_N[t])$ for $i = 1, \dots, N,\ j \geqslant 0$.

Denote by $\mathcal{B}^K$ the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by B^K_{ij}, with $i = 1, \dots, N$, $j \geqslant 0$. The subalgebra $\mathcal{B}^K$ is called the Bethe algebra with parameters K.

Theorem 2.1 ([T], [CT], [MTV1])**.** *The algebra $\mathcal{B}^K$ is commutative. The algebra $\mathcal{B}^K$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$. If $K = 0$, then the algebra $\mathcal{B}^{K=0}$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$.* □

Each element B^K_{ij} is a polynomial in $K_1, \dots, K_N$. We define $\mathcal{B}^\infty$ to be the unital subalgebra of $U(\mathfrak{gl}_N[t])$ generated by the leading terms of the elements B^K_{ij}, $i = 1, \dots, N,\ j \geqslant 0$, as K tends to infinity so that $K_i/K_{i+1} \to \infty$ for all i.

Lemma 2.2. *The algebra $\mathcal{B}^\infty$ is the unital subalgebra generated by the elements $e_{ii}\otimes t^j$ with $i = 1, \dots, N,\ j \geqslant 0$.*

Proof. We have $B^K_{i0} \,=\, (-1)^i\, K_1\dots K_i\,\big(1 + o(1)\big)\,,$ and

$$B^K_{ij} \,=\, (-1)^i\, K_1\dots K_{i-1}\,\Big(\sum_{m=i}^N e_{mm}\otimes t^{j-1} + o(1)\Big)$$

for $j > 0$, where $o(1)$ stands for the terms vanishing as K tends to infinity. □

Remark. There are $N!$ asymptotic zones labeled by elements of S_N in which K may tend to infinity. For $\sigma \in S_N$ we may assume that all coordinates of K tend to infinity and $K_{\sigma_i}/K_{\sigma_{i+1}} \to \infty$ for all i. It is easy to see that the limiting Bethe algebra $\mathcal{B}^\infty$ does not depend on σ.

The algebra $\mathcal{B}^\infty$ is commutative and contains $U(\mathfrak{z}_N[t])$. The algebra $\mathcal{B}^\infty$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_N[t])$.

As a subalgebra of $U(\mathfrak{gl}_N[t])$, the Bethe algebra $\mathcal{B}^K$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}^K$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces $M_{\boldsymbol\lambda}$. If $K = 0$, then $\mathcal{B}^{K=0}$ preserves the singular weight subspaces $M^{sing}_{\boldsymbol\lambda}$. We will study the action of $\mathcal{B}^\infty$ on the weight subspaces $\mathcal{V}^+_{\boldsymbol\lambda}$, $\mathcal{V}^-_{\boldsymbol\lambda}$.

Lemma 2.3. *The element $\sum_{i=1}^N e_{ii} \otimes t^r \in U(\mathfrak{z}_N[t])$ acts on $\mathcal{V}$ as the operator of multiplication by $\sum_{s=1}^n z_s^r$.*

If $L \subset M$ is a $\mathcal{B}^K$-invariant subspace, then the image of $\mathcal{B}^K$ in $\mathrm{End}(L)$ will be called the Bethe algebra of H and denoted by $\mathcal{B}^K(L)$.

3. Equivariant cohomology of partial flag varieties

3.1. **Partial flag varieties.** For $\boldsymbol\lambda \in \mathbb{Z}^N_{\geqslant 0}, |\boldsymbol\lambda| = n$, consider the partial flag variety $\mathcal{F}_{\boldsymbol\lambda}$ parametrizing subspaces

$$0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i,\ i = 1, \dots, N$.

Let $T^n \subset GL_n$ be the torus of diagonal matrices. The groups $T^n \subset GL_n(\mathbb{C})$ act on $\mathbb{C}^n$ and hence on $\mathcal{F}_{\boldsymbol\lambda}$. The fixed points $\mathcal{F}^{T^n}_{\boldsymbol\lambda}$ of the torus action are the coordinate flags $F_I = (F_0 \subset \cdots \subset F_N)$, $I = (I_1, \dots, I_N) \in \mathcal{I}_{\boldsymbol\lambda}$, where F_i is the span of the basis vectors $v_j \in \mathbb{C}^n$ with $j \in I_1 \cup \cdots \cup I_i$. The fixed points are in a one-to-one correspondence with the set $\mathcal{I}_{\boldsymbol\lambda}$ and hence with the basis in $V_{\boldsymbol\lambda}$.

We consider the $GL_n(\mathbb{C})$-equivariant cohomology

$$H_{\boldsymbol\lambda} = H^*_{GL_n}(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C}).$$

Denote by $\Gamma_i = \{\gamma_{i1}, \dots, \gamma_{i\lambda_i}\}$ the set of the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol\lambda}$ with fiber F_i/F_{i-1}. Denote by $\boldsymbol z = \{z_1, \dots, z_n\}$ the Chern roots corresponding to the factors of the torus T^n. Then

$$H_{\boldsymbol\lambda} = \mathbb{C}[\boldsymbol z; \Gamma_1; \dots; \Gamma_N]^{S_N \times S_{\lambda_1} \times \cdots \times S_{\lambda_N}} \Big/ \Big\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (1 + u\gamma_{ij}) = \prod_{i=1}^n (1 + uz_i) \Big\rangle . \tag{3.1}$$

The cohomology $H_{\boldsymbol\lambda}$ is a module over $H^*_{GL_n}(pt, \mathbb{C}) = \mathbb{C}[\boldsymbol z]^+$.

Let $J_H \subset H_{\boldsymbol\lambda}$ be the ideal generated by the polynomials $\sigma_i(\boldsymbol z),\ i = 1, \dots, n$. Then $H_{\boldsymbol\lambda}/J_H = H^*(\mathcal{F}_{\boldsymbol\lambda}, \mathbb{C})$.

3.2. **Integration over $\mathcal{F}_{\boldsymbol{\lambda}}$.** We will need the integration map $\int : H_{\boldsymbol{\lambda}} \to H^*_{GL_n}(pt, \mathbb{C})$. The following formula (3.2) gives the integration map in terms of the fixed point set $\mathcal{F}_{\boldsymbol{\lambda}}^{T^n}$.

For a subset $A \subset \{1, \dots, N\}$ denote $\boldsymbol{z}_A = \{z_a,\, a \in A\}$. For $I = (I_1, \dots, I_N) \in \mathcal{I}_{\boldsymbol{\lambda}}$ denote

$$R(\boldsymbol{z}_{I_1}|\boldsymbol{z}_{I_2}|\dots|\boldsymbol{z}_{I_m}) = \prod_{i<j} \prod_{a\in I_i,\, b\in I_j} (z_b - z_a).$$

The Atiyah-Bott equivariant localization theorem [AB] says that for any $[h(\boldsymbol{z}, \Gamma_1, \dots, \Gamma_N)] \in H_{\boldsymbol{\lambda}}$,

$$\int [h] = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{h(\boldsymbol{z}, \boldsymbol{z}_{I_1}, \dots, \boldsymbol{z}_{I_N})}{R(\boldsymbol{z}_{I_1}|\boldsymbol{z}_{I_2}|\dots|\boldsymbol{z}_{I_N})}\,. \tag{3.2}$$

Clearly, the right hand side in (3.2) lies in $\mathbb{C}[\boldsymbol{z}]^+$. The integration map induces the pairing

$$(\,,\,) : H_{\boldsymbol{\lambda}} \otimes H_{\boldsymbol{\lambda}} \to \mathbb{C}[\boldsymbol{z}]^+, \qquad [h]\otimes[g] \mapsto \int [hg].$$

After factorization by the ideal J_H we obtain the nondegenerate Poincare pairing

$$(\,,\,) : H^*(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C}) \otimes H^*(\mathcal{F}_{\boldsymbol{\lambda}}, \mathbb{C}) \to \mathbb{C}.$$

3.3. **$H_{\boldsymbol{\lambda}}$ and $\mathcal{V}^{\pm}$.**

Lemma 3.1. *The maps*

$$i^+_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \mathcal{V}^+_{\boldsymbol{\lambda}}, \qquad [h(\boldsymbol{z}, \Gamma_1, \dots, \Gamma_N)] \mapsto \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes h(\boldsymbol{z}, \boldsymbol{z}_{I_1}, \dots, \boldsymbol{z}_{I_N}),$$

$$i^-_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \to \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}, \qquad [h(\boldsymbol{z}, \Gamma_1, \dots, \Gamma_N)] \mapsto \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes \frac{h(\boldsymbol{z}, \boldsymbol{z}_{I_1}, \dots, \boldsymbol{z}_{I_N})}{R(\boldsymbol{z}_{I_1}|\boldsymbol{z}_{I_2}|\dots|\boldsymbol{z}_{I_N})}$$

are well-defined isomorphisms of $\mathbb{C}[\boldsymbol{z}]^+$-modules. □

Proof. If h belongs to the ideal of relations in (3.1) then $h(z, z_{I_1}, ..., z_{I_N}) = 0$ for any I, because the $\Gamma_i = z_{I_i}$ substitution makes the generators of the ideal identities. This proves well-definedness.

Consider the $\mathbb{C}[\boldsymbol{z}]^+$-module $\mathbb{C}[\boldsymbol{z}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$ of polynomials symmetric in the first λ_1 variables, the next λ_2 variables, etc. In Schubert calculus it is known that this module is free of rank $d_{\boldsymbol{\lambda}}$, and that it is isomorphic to $H_{\boldsymbol{\lambda}}$ under the correspondence

$$p \in \mathbb{C}[\boldsymbol{z}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}} \longleftrightarrow \big[p(\Gamma_1, \dots, \Gamma_N)\big] \in H_{\boldsymbol{\lambda}}. \tag{3.3}$$

An element $\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes p_I(\boldsymbol{z})$ of $\mathcal{V}_{\boldsymbol{\lambda}}$ belongs to $\mathcal{V}^+_{\boldsymbol{\lambda}}$, if and only if $p_I(\boldsymbol{z}) = p(z_{I_1}, \dots, z_{I_N})$ for a polynomial $p \in \mathbb{C}[\boldsymbol{z}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$. This shows that $\mathcal{V}^+_{\boldsymbol{\lambda}}$ is isomorphic to $\mathbb{C}[\boldsymbol{z}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}$, and that $i^+_{\boldsymbol{\lambda}}$ is the composition of this isomorphism with (3.3).

A similar argument shows that $i^-_{\boldsymbol{\lambda}}$ is also an isomorphism. □

Corollary 3.2. *The Shapovalov form and the Poincare pairing are related by the formula*

$$\mathcal{S}_{+-}(i_+[h], i_-[g]) = \int [h][g]\,.$$

□

Let A be a commutative algebra. The algebra A considered as an A-module is called the regular representation of A. Here is our main result.

Theorem 3.3.

(i) *The maps* $\xi^{\pm}_{\boldsymbol{\lambda}} : e_{ii}\otimes t^r|_{\mathcal{V}^{\pm}_{\boldsymbol{\lambda}}} \mapsto \sum_{j=1}^{\lambda_i}\gamma^r_{ij}$ *define isomorphisms of the algebras* $\mathcal{B}^\infty(\mathcal{V}^{\pm}_{\boldsymbol{\lambda}})$ *and* $H_{\boldsymbol{\lambda}}$.

(ii) *The maps* $\xi^{+}_{\boldsymbol{\lambda}}$, $i^{+}_{\boldsymbol{\lambda}}$ *identify the* $\mathcal{B}^\infty(\mathcal{V}^{+}_{\boldsymbol{\lambda}})$*-module* $\mathcal{V}^{+}_{\boldsymbol{\lambda}}$ *with the regular representation of* $H_{\boldsymbol{\lambda}}$.

(iii) *The maps* $\xi^{-}_{\boldsymbol{\lambda}}$, $i^{-}_{\boldsymbol{\lambda}}$ *identify the* $\mathcal{B}^\infty(\mathcal{V}^{-}_{\boldsymbol{\lambda}})$*-module* $\mathcal{V}^{-}_{\boldsymbol{\lambda}}$ *with the regular representation of* $H_{\boldsymbol{\lambda}}$.

The theorem follows from Lemmas 2.2, 2.3 and 3.1.

3.4. **Cohomology as $\mathfrak{gl}_N[t]$-modules.** Let J be the ideal of $\mathbb{C}[\boldsymbol{z}]^+$ generated by the elementary symmetric functions $\sigma_i(\boldsymbol{z})$, $i=1,\dots,n$. Define $J^+ = J\mathcal{V}^+$ and $J^- = \frac{1}{D}J\mathcal{V}^-$. Clearly, J^+ is a $\mathfrak{gl}_N[t]$-submodule of $\mathcal{V}^+$ and J^- is a $\mathfrak{gl}_N[t]$-submodule of $\frac{1}{D}\mathcal{V}^-$. The $\mathfrak{gl}_N[t]$-module $\mathcal{V}^+/J^+$ is graded and has dimension N^n over $\mathbb{C}$, see [MTV2]. Similarly, $\frac{1}{D}\mathcal{V}^-/J^-$ is a graded $\mathfrak{gl}_N[t]$-module of the same dimension.

Theorem 3.4. *The Shapovalov form establishes a nondegenerate pairing*

$$\mathcal{S}_{+-}\ :\ \mathcal{V}^+/J^+\otimes\frac{1}{D}\mathcal{V}^-/J^-\ \to\ \mathbb{C}.$$

The theorem follows from Lemmas 3.1, 3.2 and the nondegeneracy of the Poincare pairing.

Corollary 3.5. *The* $\mathfrak{gl}_N[t]$*-modules* $\mathcal{V}^+/J^+$ *and* $\frac{1}{D}\mathcal{V}^-/J^-$ *are contravariantly related through the Shapovalov form,* $\mathcal{S}_{+-}\big((e_{ij}\otimes t^r)x,\frac{1}{D}y\big)=\mathcal{S}_{+-}\big(x,(e_{ji}\otimes t^r)\frac{1}{D}y\big)$ *for all* i,j,x,y.

Let W_n be the $\mathfrak{gl}_N[t]$-module generated by a vector w_n with the defining relations:

$$\begin{aligned} e_{ii}(u)w_n &= \delta_{1i}\,\frac{n}{u}\,w_n\,, && i=1,\dots,N\,,\\ e_{ij}(u)w_n &= 0\,, && 1\leqslant i<j\leqslant N\,,\\ (e_{ji}\otimes 1)^{n\delta_{1i}+1}w_n &= 0\,, && 1\leqslant i<j\leqslant N\,. \end{aligned}$$

As an $\mathfrak{sl}_N[t]$-module, the module W_n is isomorphic to the Weyl module from [CL], [CP], corresponding to the weight $n\omega_1$, where ω_1 is the first fundamental weight of $\mathfrak{sl}_N$.

In [MTV2] an isomorphism of $\mathcal{V}^+/J^+$ and the Weyl module W_n is constructed.

Corollary 3.6. *The Shapovalov form* $\mathcal{S}_{+-}$ *establishes an isomorphism of* $\frac{1}{D}\mathcal{V}^-/J^-$ *and the contravariantly dual of the Weyl module* W_n.

Here is an application of this fact. For $\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}|=n$, $\lambda_1\geqslant\dots\geqslant\lambda_N$, denote

$$\Big(\frac{1}{D}\mathcal{V}^-/J^-\Big)^{sing}_{\boldsymbol{\lambda}}\ =\ \{v\in\frac{1}{D}\mathcal{V}^-/J^-\ |\ e_{ij}v=0 \text{ for } i<j,\ e_{ii}v=\lambda_i v\ \text{ for }\ i=1,\dots,N\}\,.$$

This is a graded space. Denote by $\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big)_k$ the subspace of all elements of $\boldsymbol{z}$-degree k. Define the graded character by the formula

$$\operatorname{ch}\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big) \;=\; \sum_k q^k \dim((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}})_k .$$

Corollary 3.7. *We have*

$$\operatorname{ch}\big((\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}\big) \;=\; \frac{(q)_n \prod_{1\leqslant i<j\leqslant N}(1-q^{\lambda_i-\lambda_j+j-i})}{\prod_{i=1}^N (q)_{\lambda_i+N-i}}\; q^{-\sum_{i=1}^N (i-1)\lambda_i}, \tag{3.4}$$

where $(q)_a = \prod_{j=1}^a (1-q^j)$.

The corollary follows from Lemma 2.2 in [MTV2] and Corollary 3.6.

The isomorphisms

$$i^+ = \bigoplus_{\boldsymbol{\lambda}} i^+_{\boldsymbol{\lambda}} \,:\, \bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}} \,\to\, \mathcal{V}^+, \qquad i^- = \bigoplus_{\boldsymbol{\lambda}} i^-_{\boldsymbol{\lambda}} \,:\, \bigoplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}} \,\to\, \frac{1}{D}\mathcal{V}^- \tag{3.5}$$

induce two graded $\mathfrak{gl}_N[t]$-module structures on $\oplus_{\boldsymbol{\lambda}} H_{\boldsymbol{\lambda}}$ denoted by ρ^+ and ρ^-, respectively. These module structures descend to two graded $\mathfrak{gl}_N[t]$-module structures on the cohomology with constant coefficients

$$H(\mathbb{C}) \,:=\, \bigoplus_{\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0},\,|\boldsymbol{\lambda}|=n} H^*(\mathcal{F}_{\boldsymbol{\lambda}},\mathbb{C})\,,$$

denoted by the same letters ρ^+ and ρ^-.

Corollary 3.8. *The $\mathfrak{gl}_N[t]$-module $H(\mathbb{C})$ with the ρ^+-structure is isomorphic to the Weyl module W_n. The $\mathfrak{gl}_N[t]$-module $H(\mathbb{C})$ with the ρ^--structure is isomorphic to the contravariant dual of the Weyl module W_n.*

The $\rho^{\pm}$ structures can be defined topologically, see [RSV] and Appendix. The ρ^--structure appears to be more preferable. It was used in [RV], [RSV] to construct conformal blocks in the tensor power $V^{\otimes n}$.

4. Isomorphisms $i^{\pm}_{\boldsymbol{\lambda}}$ as a geometric Langlands correspondence

4.1. **The $\mathcal{V}^+_{\boldsymbol{\lambda}}$ case.** The following geometric description of the $\mathcal{B}^K$-action on $\mathcal{V}^+_{\boldsymbol{\lambda}}$ was given in [MTV3] as an example of the geometric Langlands correspondence.

Let $K=(K_1,\dots,K_N)$ be a sequence of distinct complex numbers. Let $\boldsymbol{\lambda}\in\mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}|=n$. Introduce the polynomial algebras

$$\mathbb{C}[\boldsymbol{\Sigma}] \,:=\, \mathbb{C}[\Sigma_{ij},\, i=1,\dots,N,\, j=1,\dots,\lambda_i]\,, \qquad \mathbb{C}[\boldsymbol{\sigma}] \,:=\, \mathbb{C}[\sigma_1,\dots,\sigma_n]\,.$$

Define

$$\Sigma_i(u) \,=\, e^{K_i u}\,(u^{\lambda_i} + \Sigma_{i1}u^{\lambda_i-1} + \dots + \Sigma_{i\lambda_i})\,, \qquad i=1,\dots,N\,.$$

For arbitrary functions $g_1(u),\dots,g_N(u)$, introduce the Wronskian determinant by the formula

$$\mathrm{Wr}(g_1(u),\dots,g_N(u)) \,=\, \det\begin{pmatrix} g_1(u) & g_1'(u) & \dots & g_1^{(N-1)}(u)\\ g_2(u) & g_2'(u) & \dots & g_2^{(N-1)}(u)\\ \dots & \dots & \dots & \dots\\ g_N(u) & g_N'(u) & \dots & g_N^{(N-1)}(u)\end{pmatrix}.$$

We have

$$\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u)) \,=\, e^{\sum_{i=1}^N K_i u}\prod_{1\leqslant i<j\leqslant N}(K_j-K_i)\cdot\Big(u^n+\sum_{s=1}^n(-1)^s A_s^K(\boldsymbol{\Sigma})\,u^{n-s}\Big)\,,$$

where $A_1^K(\boldsymbol{\Sigma}),\dots,A_n^K(\boldsymbol{\Sigma})\in\mathbb{C}[\boldsymbol{\Sigma}]$. Define an algebra homomorphism

$$\mathcal{W}^K:\,\mathbb{C}[\boldsymbol{\sigma}]\to\mathbb{C}[\boldsymbol{\Sigma}],\qquad \sigma_s\mapsto A_s^K(\boldsymbol{\Sigma})\,.$$

The homomorphism defines a $\mathbb{C}[\boldsymbol{\sigma}]$-module structure on $\mathbb{C}[\boldsymbol{\Sigma}]$.

Define a differential operator $\mathcal{D}_{\boldsymbol{\Sigma}}^K$ by

$$\mathcal{D}_{\boldsymbol{\Sigma}}^K=\frac{1}{\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u))}\ \mathrm{rdet}\begin{pmatrix}\Sigma_1(u) & \Sigma_1'(u) & \dots & \Sigma_1^{(N)}(u)\\ \Sigma_2(u) & \Sigma_2'(u) & \dots & \Sigma_2^{(N)}(u)\\ \dots & \dots & \dots & \dots\\ 1 & \partial & \dots & \partial^N\end{pmatrix}.$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $\mathbb{C}[\boldsymbol{\Sigma}]$,

$$\mathcal{D}_{\boldsymbol{\Sigma}}^K=\partial^N+\sum_{i=1}^N F_i^K(u)\,\partial^{N-i},\qquad F_i^K(u)\,=\,\sum_{j=0}^\infty F_{ij}^K\,u^{-j}\,, \tag{4.1}$$

and $F_{ij}^K\in\mathbb{C}[\boldsymbol{\Sigma}],\ i=1,\dots,N,\ j\geqslant 0$.

Theorem 4.1 ([MTV3])**.** *The map*

$$\tau_{\boldsymbol{\lambda}}^{K+}\ :\ B_{ij}^K|_{\mathcal{V}_{\boldsymbol{\lambda}}^+}\ \mapsto\ F_{ij}^K$$

defines an isomorphism of the Bethe algebra $\mathcal{B}^K(\mathcal{V}_{\boldsymbol{\lambda}}^+)$ *and the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]$*. The isomorphism* $\tau_{\boldsymbol{\lambda}}^{K+}$ *becomes an isomorphism of the* $U(\mathfrak{z}_N[t])|_{\mathcal{V}_{\boldsymbol{\lambda}}^+}$*-module* $\mathcal{B}^K(\mathcal{V}_{\boldsymbol{\lambda}}^+)$ *and the* $\mathbb{C}[\boldsymbol{\sigma}]$*-module* $\mathbb{C}[\boldsymbol{\Sigma}]$ *if we identify the algebras* $U(\mathfrak{z}_N[t])|_{\mathcal{V}_{\boldsymbol{\lambda}}^+}$ *and* $\mathbb{C}[\boldsymbol{\sigma}]$ *by the map* $.\sigma_s[\boldsymbol{z}]\mapsto\sigma_s,\ s=1,\dots,n$*.*

Denote

$$v^+\,=\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}v_I\in\mathcal{V}_{\boldsymbol{\lambda}}^+.$$

Theorem 4.2 ([MTV3])**.** *The map*

$$\mu_{\boldsymbol{\lambda}}^{K+}\ :\ B_{ij}^K v^+\ \mapsto\ F_{ij}^K\,,$$

defines a linear isomorphism $\mathcal{V}_{\boldsymbol{\lambda}}^+\to\mathbb{C}[\boldsymbol{\Sigma}]$*. The maps* $\tau_{\boldsymbol{\lambda}}^{K+},\mu_{\boldsymbol{\lambda}}^{K+}$ *give an isomorphism of the* $\mathcal{B}^K(\mathcal{V}_{\boldsymbol{\lambda}}^+)$*-module* $\mathcal{V}_{\boldsymbol{\lambda}}^+$ *and the regular representation of the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]$*.*

4.2. **The limit of $\tau_{\boldsymbol\lambda}^{K+}$ and $\mu_{\boldsymbol\lambda}^{K+}$ as $K\to\infty$.** Let all the coordinates of the vector K tend to infinity so that $K_i/K_{i+1}\to\infty$ for $i=1,\dots,N-1$. Then the homomorphism $\mathcal{W}^K$ has a limit $\mathcal{W}^\infty$. Namely, define $A_s^\infty(\boldsymbol\Sigma)$ by the formula

$$\prod_{i=1}^N (u^{\lambda_i}+\Sigma_{i1}u^{\lambda_i-1}+\dots+\Sigma_{i\lambda_i}) \,=\, u^n+\sum_{s=1}^n (-1)^s A_s^\infty(\boldsymbol\Sigma)\,u^{n-s}\,.$$

Then

$$\mathcal{W}^\infty:\,\mathbb{C}[\boldsymbol\sigma]\,\to\,\mathbb{C}[\boldsymbol\Sigma],\qquad \sigma_i\,\mapsto\, A_s^\infty(\boldsymbol\Sigma)\,. \tag{4.2}$$

Define algebra isomorphisms

$$\mathbb{C}[\boldsymbol\sigma]\,\to\,\mathbb{C}[\boldsymbol z]^+,\qquad \eta:\mathbb{C}[\boldsymbol\Sigma]\,\to\,H_{\boldsymbol\lambda}\,, \tag{4.3}$$

by the agreement that the first isomorphism sends σ_s to $\sigma_s(\boldsymbol z)$ for all s, and the second one sends $(-1)^s\Sigma_{is}$ to the sth elementary symmetric function of $\gamma_{i1},\dots,\gamma_{i\lambda_i}$ for all i,s.

Lemma 4.3. *The isomorphisms* (4.3) *identify the $\mathbb{C}[\boldsymbol\sigma]$-module $\mathbb{C}[\boldsymbol\Sigma]$ defined by formula* (4.2) *and the $\mathbb{C}[\boldsymbol z]^+$-module $H_{\boldsymbol\lambda}$.*

Let $p_i(u)\,=\,u^{\lambda_i}+\Sigma_{i1}u^{\lambda_i-1}+\dots+\Sigma_{i\lambda_i}$ for all $i=1,\dots N$. Notice that

$$\eta\bigl(p_i(u)\bigr)\,=\,\prod_{j=1}^{\lambda_i}(u-\gamma_{ij})\,,\qquad \eta\Bigl(\frac{p_i'(u)}{p_i(u)}\Bigr)\,=\,\sum_{r=0}^\infty\sum_{j=1}^{\lambda_i}\gamma_{ij}^r\,u^{-r-1}\,. \tag{4.4}$$

Lemma 4.4. *We have* $F_{i0}^K\,=\,(-1)^i K_1\dots K_i\,\bigl(1+o(1)\bigr)$, *and*

$$\sum_{j=1}^\infty F_{ij}^K u^{-j}\,=\,(-1)^i K_1\dots K_{i-1}\Bigl(\sum_{m=i}^N\frac{p_m'(u)}{p_m(u)}+o(1)\Bigr)\,,$$

where $o(1)$ stands for the terms vanishing as K tends to infinity.

Proof. Let $y_i(u)\,=\,\mathrm{Wr}\bigl(\Sigma_i(u),\dots\Sigma_N(u)\bigr)$, $i\,=\,1,\dots N$. Then the operator $\mathcal{D}_{\boldsymbol\Sigma}^K$ can be factorized:

$$\mathcal{D}_{\boldsymbol\Sigma}^K\,=\,\Bigl(\partial-\frac{y_1'(u)}{y_1(u)}+\frac{y_2'(u)}{y_2(u)}\Bigr)\,\dots\,\Bigl(\partial-\frac{y_{N-1}'(u)}{y_{N-1}(u)}+\frac{y_N'(u)}{y_N(u)}\Bigr)\Bigl(\partial-\frac{y_N'(u)}{y_N(u)}\Bigr)\,. \tag{4.5}$$

Since $y_i(u)\,=\,(-1)^{(N-i)(N-i-1)/2}K_i^{N-i}\dots K_{N-1}\bigl(p_i(u)\dots p_N(u)\,+\,o(1)\bigr)\,e^{\sum_{m=i}^N K_m u}$ as K tends to infinity, the claim follows from formulae (4.1) and (4.5). □

Theorem 4.5.

(i) *The map $\eta\circ\tau_{\boldsymbol\lambda}^{K+}:\mathcal{B}^K(\mathcal{V}_{\boldsymbol\lambda}^+)\to H_{\boldsymbol\lambda}$ tends to the isomorphism $\xi_{\boldsymbol\lambda}^+:\mathcal{B}^\infty(\mathcal{V}_{\boldsymbol\lambda}^+)\to H_{\boldsymbol\lambda}$, see Theorem 3.3, as K tends to infinity.*

(ii) *The map $\eta\circ\mu_{\boldsymbol\lambda}^{K+}:\mathcal{V}_{\boldsymbol\lambda}^+\,\to\,H_{\boldsymbol\lambda}$ tends to the isomorphism $(i_{\boldsymbol\lambda}^+)^{-1}:\mathcal{V}_{\boldsymbol\lambda}^+\,\to\,H_{\boldsymbol\lambda}$, see Lemma 3.1, as K tends to infinity.*

Proof. The statement follows from the definitions of the maps, Lemma 4.4, formulae (4.4), and the proof of Lemma 2.2. □

4.3. **The $\mathcal{V}^-_{\boldsymbol{\lambda}}$ case.** Theorem 3.4 allows us to establish a geometric description of the $\mathcal{B}^K$-action on $\frac{1}{D}\mathcal{V}^-$ which is analogous to the description of the $\mathcal{B}^K$-action on $\mathcal{V}^+$.

Theorem 4.6. *The map*

$$\tau^{K-}_{\boldsymbol{\lambda}} \; : \; B^K_{ij}|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}} \;\mapsto\; F^K_{ij}$$

defines an isomorphism of the Bethe algebra $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ and the algebra $\mathbb{C}[\boldsymbol{\Sigma}]$. The isomorphism $\tau^{K-}_{\boldsymbol{\lambda}}$ becomes an isomorphism of the $U(\mathfrak{z}_N[t])|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}}$-module $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$ and the $\mathbb{C}[\boldsymbol{\sigma}]$-module $\mathbb{C}[\boldsymbol{\Sigma}]$ if we identify the algebras $U(\mathfrak{z}_N[t])|_{\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}}$ and $\mathbb{C}[\boldsymbol{\sigma}]$ by the map $\sigma_s[\boldsymbol{z}] \mapsto \sigma_s$, $s = 1, \dots, n$.

Denote

$$v^- = \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes \frac{1}{R(\boldsymbol{z}_{I_1}|\boldsymbol{z}_{I_2}|\dots|\boldsymbol{z}_{I_N})} \in \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}\,.$$

Theorem 4.7. *The map*

$$\mu^{K-}_{\boldsymbol{\lambda}} \; : \; B^K_{ij} v^- \;\mapsto\; F^K_{ij}\,,$$

defines a linear isomorphism $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} \to \mathbb{C}[\boldsymbol{\Sigma}]$. The maps $\tau^{K-}_{\boldsymbol{\lambda}}, \mu^{K-}_{\boldsymbol{\lambda}}$ give an isomorphism of the $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}}$ and the regular representation of the algebra $\mathbb{C}[\boldsymbol{\Sigma}]$.

The proofs of Theorems 4.6 and 4.7 are basically word by word the same as the proofs of Theorems 4.1 and 4.2 in [MTV3].

It is interesting to note that the element v^- becomes a conformal block under certain conditions and satisfies a *KZ* equation with respect to $\boldsymbol{z}$, see [V], [RV], [RSV].

4.4. **The limit of $\tau^{K-}_{\boldsymbol{\lambda}}$ and $\mu^{K-}_{\boldsymbol{\lambda}}$ as $K \to \infty$.** Let all the coordinates of the vector K tend to infinity so that $K_i/K_{i+1} \to \infty$ for $i = 1, \dots, N-1$.

Theorem 4.8.

(i) *The map $\eta \circ \tau^{K-}_{\boldsymbol{\lambda}} : \mathcal{B}^K(\mathcal{V}^-_{\boldsymbol{\lambda}}) \to H_{\boldsymbol{\lambda}}$ tends to the isomorphism $\xi^-_{\boldsymbol{\lambda}} : \mathcal{B}^\infty(\mathcal{V}^-_{\boldsymbol{\lambda}}) \to H_{\boldsymbol{\lambda}}$, see Theorem 3.3, as K tends to infinity.*

(ii) *The map $\eta \circ \mu^{K-}_{\boldsymbol{\lambda}} : \mathcal{V}^-_{\boldsymbol{\lambda}} \to H_{\boldsymbol{\lambda}}$ tends to the isomorphism $(i^-_{\boldsymbol{\lambda}})^{-1} : \mathcal{V}^-_{\boldsymbol{\lambda}} \to H_{\boldsymbol{\lambda}}$, see Lemma 3.1, as K tends to infinity.*

The proof is similar to the proof of Theorem 4.5.

4.5. **The $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$ case.** Formula (3.4) for the graded character of $(\frac{1}{D}\mathcal{V}^-/J^-)^{sing}_{\boldsymbol{\lambda}}$ is the analog of the formula for the graded character of $(\mathcal{V}^+/J^+)^{sing}_{\boldsymbol{\lambda}}$ in [MTV2]. The latter formula was used in [MTV2] to obtain a geometric description of the $\mathcal{B}^{K=0}$-action on $(\mathcal{V}^+)^{sing}_{\boldsymbol{\lambda}}$. Using formula (3.4) we can obtain a similar geometric description of the $\mathcal{B}^{K=0}$-action on $(\frac{1}{D}\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$.

Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $\lambda_1 \geqslant \dots \geqslant \lambda_N, |\boldsymbol{\lambda}| = n$. Introduce $P = \{d_1, \dots, d_N\}$, $d_i = \lambda_i + N - i, i = 1, \dots, N$. Let

$$\Sigma_i(u) = u^{d_i} + \sum_{j=1,\ d_i - j\notin P} \Sigma_{ij} u^{d_i - j}.$$

Consider the polynomial algebras

$$\mathbb{C}[\boldsymbol{\Sigma}] := \mathbb{C}[\Sigma_{ij}, i = 1, \dots, N, j \in \{1, \dots, d_i\}, d_i - j \notin P], \qquad \mathbb{C}[\boldsymbol{\sigma}] := \mathbb{C}[\sigma_1, \dots, \sigma_n].$$

We have

$$\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u)) \,=\, \prod_{1\leqslant i<j\leqslant N}(d_j-d_i)\cdot\Bigl(u^n+\sum_{s=1}^{n}(-1)^s A_s(\boldsymbol{\Sigma})\,u^{n-s}\Bigr)\,,$$

where $A_1(\boldsymbol{\Sigma}),\dots,A_n(\boldsymbol{\Sigma})\in\mathbb{C}[\boldsymbol{\Sigma}]$. Define an algebra homomorphism

$$\mathcal{W}\,:\,\mathbb{C}[\boldsymbol{\sigma}]\,\to\,\mathbb{C}[\boldsymbol{\Sigma}]\,,\qquad \sigma_s\,\mapsto\,A_s(\boldsymbol{\Sigma})\,.$$

The homomorphism defines a $\mathbb{C}[\boldsymbol{\sigma}]$-module structure on $\mathbb{C}[\boldsymbol{\Sigma}]$. Define a differential operator $\mathcal{D}_{\boldsymbol{\Sigma}}$ by

$$\mathcal{D}_{\boldsymbol{\Sigma}}=\frac{1}{\mathrm{Wr}(\Sigma_1(u),\dots,\Sigma_N(u))}\,\mathrm{rdet}\begin{pmatrix}\Sigma_1(u) & \Sigma_1'(u) & \dots & \Sigma_1^{(N)}(u)\\ \Sigma_2(u) & \Sigma_2'(u) & \dots & \Sigma_2^{(N)}(u)\\ \dots & \dots & \dots & \dots\\ 1 & \partial & \dots & \partial^N\end{pmatrix}.$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $\mathbb{C}[\boldsymbol{\Sigma}]$,

$$\mathcal{D}_{\boldsymbol{\Sigma}}=\,\partial^N+\sum_{i=1}^{N}F_i(u)\,\partial^{N-i}\,,\qquad F_i(u)\,=\,\sum_{j=i}^{\infty}F_{ij}\,u^{-j}\,,$$

and $F_{ij}\in\mathbb{C}[\boldsymbol{\Sigma}],\ i=1,\dots,N,\ j\geqslant i.$

Theorem 4.9. *The map*

$$\tau^-_{\boldsymbol{\lambda}}\,:\,B^{K=0}_{ij}|_{(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}\,\mapsto\,F_{ij}$$

defines an isomorphism of the Bethe algebra $\mathcal{B}^{K=0}\bigl((\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\bigr)$ *and the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]$. *The isomorphism* $\tau^-_{\boldsymbol{\lambda}}$ *becomes an isomorphism of the* $U(\mathfrak{z}_N[t])|_{(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}$*-module* $\mathcal{B}^{K=0}\bigl((\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\bigr)$ *and the* $\mathbb{C}[\boldsymbol{\sigma}]$*-module* $\mathbb{C}[\boldsymbol{\Sigma}]$ *if we identify the algebras* $U(\mathfrak{z}_N[t])|_{(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}}$ *and* $\mathbb{C}[\boldsymbol{\sigma}]$ *by the map* $\sigma_s[\boldsymbol{z}]\mapsto\sigma_s,\ s=1,\dots,n.$

Fix a vector $v^-\in(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$ of degree $\sum_{i=1}^N(1-i)\lambda_i$. By formula (3.4) such a vector is unique up to proportionality.

Theorem 4.10. *The map*

$$\mu^-_{\boldsymbol{\lambda}}\,:\,B^{K=0}_{ij}v^-\,\mapsto\,F_{ij},$$

defines a linear isomorphism $(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\to\mathbb{C}[\boldsymbol{\Sigma}]$. *The maps* $\tau^-_{\boldsymbol{\lambda}},\mu^-_{\boldsymbol{\lambda}}$ *give an isomorphism of the* $\mathcal{B}^{K=0}\bigl((\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}\bigr)$*-module* $(\frac1D\mathcal{V}^-)^{sing}_{\boldsymbol{\lambda}}$ *and the regular representation of the algebra* $\mathbb{C}[\boldsymbol{\Sigma}]$.

The proofs of Theorems 4.9 and 4.10 are word by word the same as the proofs of Theorems 5.3 and 5.6 in [MTV2].

5. Relations with quantum cohomology

In lectures [O] Okounkov, in particular, considers the equivariant quantum cohomology $QH_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol\lambda})$ of the cotangent bundle $T^*F_{\boldsymbol\lambda}$ of a flag variety $F_{\boldsymbol\lambda}$. More precisely, he considers the standard equivariant cohomology $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol\lambda})$ as a module over the algebra of quantum multiplication and described this module as the Yangian Bethe algebra of the XXX model associated with $V^{\otimes n}$.

The algebra $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol\lambda})$ has $n{+}1$ equivariant parameters $z_1,\dots,z_n,u$. The parameters $z_1,\dots,z_n$ correspond to the GL_n-action on $T^*F_{\boldsymbol\lambda}$ and u corresponds of the $\mathbb{C}^*$-action on $T^*F_{\boldsymbol\lambda}$ stretching the cotangent vectors. The operators of quantum multiplication depend on additional parameters $q_1,\dots,q_N$ corresponding to quantum deformation.

It is well-known how the Yangian Bethe algebra degenerates into the Gaudin Bethe algebra, see for example [T], [MTV1]. This degeneration construction gives us the following fact. Introduce new parameters $K_1,\dots,K_N$ by the formula $q_i = 1+K_iu, i=1,\dots,N$, and consider the limit of the algebra of quantum multiplication on $H^*_{GL_n\times\mathbb{C}^*}(T^*F_{\boldsymbol\lambda})$ as $u\to 0$. Then this limit is isomorphic to the $\mathcal{B}^K(\mathcal{V}^+_{\boldsymbol\lambda})$-module $\mathcal{V}^+_{\boldsymbol\lambda}$. This limit is also isomorphic to the $\mathcal{B}^K(\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda})$-module $\frac{1}{D}\mathcal{V}^-_{\boldsymbol\lambda}$.

Appendix: Topological description of the $\mathfrak{gl}_N[t]$-module structure on the cohomology of flag manifolds

Given $\boldsymbol\lambda\in\mathbb{Z}^N_{\geqslant 0}$ define

$$\begin{aligned} e_{a,a+1}\boldsymbol\lambda &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a+1,\lambda_{a+1}-1,\lambda_{a+2},\dots,\lambda_N)\,,\\ e_{a+1,a}\boldsymbol\lambda &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a-1,\lambda_{a+1}+1,\lambda_{a+2},\dots,\lambda_N)\,,\\ \boldsymbol\lambda' &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a,1,\lambda_{a+1}-1,\lambda_{a+2},\dots,\lambda_N)\,,\\ \boldsymbol\lambda'' &= (\lambda_1,\dots,\lambda_{a-1},\lambda_a-1,1,\lambda_{a+1},\lambda_{a+2},\dots,\lambda_N)\,. \end{aligned}$$

Let A' (resp. B',C') be the rank λ_a (resp. rank 1, $\lambda_{a+1}-1$) bundle over $\mathcal{F}_{\boldsymbol\lambda'}$ whose fiber over the flag $L_1\subset\ldots\subset L_{N+1}$ is L_a/L_{a-1} (resp. L_{a+1}/L_a, L_{a+2}/L_{a+1}). Let A'' (resp. B'',C'') be the rank λ_a-1 (resp. rank 1, λ_{a+1}) bundle over $\mathcal{F}_{\boldsymbol\lambda''}$ whose fiber over the flag $L_1\subset\ldots\subset L_{N+1}$ is L_a/L_{a-1} (resp. L_{a+1}/L_a, L_{a+2}/L_{a+1}).

Consider the obvious projections

$$\mathcal{F}_{\boldsymbol\lambda}\xleftarrow{\pi_1'}\mathcal{F}_{\boldsymbol\lambda'}\xrightarrow{\pi_2'}\mathcal{F}_{e_{a,a+1}\boldsymbol\lambda}\qquad\text{and}\qquad \mathcal{F}_{\boldsymbol\lambda}\xleftarrow{\pi_1''}\mathcal{F}_{\boldsymbol\lambda''}\xrightarrow{\pi_2''}\mathcal{F}_{e_{a+1,a}\boldsymbol\lambda}\,.$$

For an equivariant map f (eg. $f=\pi_1'$ or π_1'') the induced pull-back map on equivariant cohomology will be denoted by f^*. For an equivariant fibration f (eg. $f=\pi_2'$ or π_2'') its Gysin map (a.k.a. *push-forward* map, or *integration along the fibers* map) will be denoted by f_*. The equivariant Euler class of a vector bundle X will be denoted by $e(X)$. The following theorem was announced in [RSV].

Theorem A.1.

(i) *The map* $\rho^-(e_{a,a+1}\otimes t^j): H_{\boldsymbol\lambda}\to H_{e_{a,a+1}\boldsymbol\lambda}$

$$x\;\mapsto\;\pi'_{2*}\Big(\pi_1'^{*}(x)\cdot e\big(\mathrm{Hom}(B',C')\big)\cdot e(B')^j\Big)$$

makes the diagram

$$\begin{array}{ccc} H_{\boldsymbol{\lambda}} & \xrightarrow{\rho^-(e_{a,a+1}\otimes t^j)} & H_{e_{a,a+1}\boldsymbol{\lambda}} \\ \big\downarrow i^- & & \big\downarrow i^- \\ \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} & \xrightarrow{e_{a,a+1}\otimes t^j} & \frac{1}{D}\mathcal{V}^-_{e_{a,a+1}\boldsymbol{\lambda}} \end{array}$$

commutative.

(ii) *The map* $\rho^-(e_{a+1,a}\otimes t^j) : H_{\boldsymbol{\lambda}} \to H_{e_{a+1,a}\boldsymbol{\lambda}}$

$$x \mapsto \pi''_{2*}\Big(\pi_1''^{*}(x)\cdot e\big(\mathrm{Hom}(A'',B'')\big)\cdot e(B'')^j\Big)$$

makes the diagram

$$\begin{array}{ccc} H_{\boldsymbol{\lambda}} & \xrightarrow{\rho^-(e_{a+1,a}\otimes t^j)} & H_{e_{a+1,a}\boldsymbol{\lambda}} \\ \big\downarrow i^- & & \big\downarrow i^- \\ \frac{1}{D}\mathcal{V}^-_{\boldsymbol{\lambda}} & \xrightarrow{e_{a+1,a}\otimes t^j} & \frac{1}{D}\mathcal{V}^-_{e_{a+1,a}\boldsymbol{\lambda}} \end{array}$$

commutative.

Proof. We will prove part (i), the proof of part (ii) is similar. Let K be the index in $\mathcal{I}_{e_{a,a+1}\boldsymbol{\lambda}}$ with $K_1 = \{1,\dots,(e_{a,a+1}\boldsymbol{\lambda})_1\}$, $K_2 = \{(e_{a,a+1}\boldsymbol{\lambda})_1+1,\dots,(e_{a,a+1}\boldsymbol{\lambda})_1+(e_{a,a+1}\boldsymbol{\lambda})_2\}$, etc.

Consider $x = [h(\boldsymbol{z},\Gamma_1,\dots,\Gamma_N)] \in H_{\boldsymbol{\lambda}}$. Its i^--image is

$$\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} v_I \otimes \frac{h(\boldsymbol{z},\boldsymbol{z}_{I_1},\dots,\boldsymbol{z}_{I_N})}{R(\boldsymbol{z}_{I_1}|\dots|\boldsymbol{z}_{I_N})}\,.$$

The coefficient of v_K of the $e_{a,a+1}\otimes t^j$-image of this is

$$\text{(A.1)}\qquad \sum_{i\in K_a} \frac{h(\boldsymbol{z},\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_{a-1}},\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\boldsymbol{z}_{K_{a+2}},\dots,\boldsymbol{z}_{K_N})\,z_i^j}{R(\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_{a-1}},\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\boldsymbol{z}_{K_{a+2}},\dots,\boldsymbol{z}_{K_N})} =$$

$$= \frac{1}{R(\boldsymbol{z}_{K_1}|\dots|\boldsymbol{z}_{K_N})}\sum_{i\in K_a}\frac{h(\boldsymbol{z},\boldsymbol{z}_{K_1},\dots,\boldsymbol{z}_{K_a-i},\boldsymbol{z}_{K_{a+1}\cup i},\dots,\boldsymbol{z}_{K_N})\,z_i^j\,R(z_i|\boldsymbol{z}_{K_{a+1}})}{R(\boldsymbol{z}_{K_a-i},z_i)}\,.$$

On the other hand, the $\rho^-(e_{a,a+1}\otimes t^j)$-image of x (using a version of the Atiyah-Bott localization formula for π'_{2*}) is

$$\sum_{\delta\in\Delta_a}\frac{h(\boldsymbol{z},\Delta_1,\dots,\Delta_{a-1},\Delta_a-\delta,\delta,\Delta_{a+1},\dots,\Delta_N)R(\delta|\Delta_{a+1})\delta^j}{R(\Delta_a-\delta|\delta)},$$

where we denoted the Chern roots of the natural bundles over $\mathcal{F}_{e_{a,a+1}\boldsymbol{\lambda}}$ by $\Delta_1,\dots,\Delta_N$. The coefficient of v_K of its i^--image is (A.1). Thus the theorem is proved. □

The topological interpretation of generators of the ρ^+-representation is similar, its proof is left to the reader.

Theorem A.2.

(i) *For the map* $\rho^+(e_{a,a+1}\otimes t^j) : H_{\boldsymbol\lambda} \to H_{e_{a,a+1}\boldsymbol\lambda}$

$$x \;\mapsto\; \pi'_{2*}\Big(\pi_1'^{*}(x)\cdot e\big(\mathrm{Hom}(A',B')\big)\cdot e(B')^j\Big)$$

we have $i^+\circ\rho^+(e_{a,a+1}\otimes t^j) \,=\, (e_{a,a+1}\otimes t^j)\circ i^+$.

(ii) *For the map* $\rho^+(e_{a+1,a}\otimes t^j) : H_{\boldsymbol\lambda} \to H_{e_{a+1,a}\boldsymbol\lambda}$

$$x \mapsto \pi''_{2*}\left(\pi_1''^{*}(x)\cdot e(\mathrm{Hom}(B'',C''))\cdot e(B'')^j\right)$$

we have $i^+\circ\rho^+(e_{a+1,a}\otimes t^j) \,=\, (e_{a+1,a}\otimes t^j)\circ i^+$.

The $\mathfrak{gl}_N[t]$-module structures $\rho^\pm$ on $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ descend to $\mathfrak{gl}_N[t]$-module structures on $H(\mathbb{C})$, also denoted by $\rho^\pm$ in Section 3.4. The topological interpretation of the actions of $e_{a,a+1}\otimes t^j$ and $e_{a+1,a}\otimes t^j$ for these representations is the same as that for $\bigoplus_{\boldsymbol\lambda} H_{\boldsymbol\lambda}$ given in Theorems A.1 and A.2.

References

[AB] M. F. Atiyah, R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28

[BMO] A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Preprint (2010), 1–35, `arXiv:1001.0056`

[CL] V. Chari, S. Loktev, *Weyl, Fusion and Demazure modules for the current algebra of* $\mathfrak{sl}_{r+1}$, Adv. Math. 207 (2006), no. 2, 928–960

[CP] V. Chari, A. Pressley *Weyl Modules for Classical and Quantum Affine algebras*, Represent. Theory 5 (2001), 191–223 (electronic)

[CT] A. Chervov, D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, Preprint (2006), 1–54; `hep-th/0604128`

[G1] M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, J. Physique **37** (1976), no. 10, 1089–1098

[G2] M. Gaudin, *La fonction d'onde de Bethe*, Collection du Commissariat à l'Énergie Atomique: Série Scientifique, Masson, Paris, 1983

[MTV1] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe Eigenvectors of Higher Transfer Matrices*, J. Stat. Mech. (2006), no. 8, P08002, 1–44

[MTV2] E. Mukhin, V. Tarasov, A. Varchenko, *Schubert calculus and representations of general linear group*, Preprint (2007), 1–32; `arXiv:0711.4079`

[MTV3] E. Mukhin, V. Tarasov, A. Varchenko, *Spaces of quasi-exponentials and representations of* $\mathfrak{gl}_N$, Preprint (2008), 1–29, `arXiv:0801.3120`

[O] A. Okounkov, *Quantum Groups and Quantum Cohomology*, Lectures at the 15th Midrasha Mathematicae on: "Derived categories of algebro-geometric origin and integrable systems", December 19–24, 2010, Jerusalem

[RV] R. Rimányi, A. Varchenko, *Conformal blocks in the tensor product of vector representations and localization formulas*, Preprint (2009), 1–21, `arXiv:0911.3253`

[RSV] R. Rimányi, V. Schechtman, A. Varchenko, *Conformal blocks and equivariant cohomology*, Preprint (2010), 1–23, arXiv:1007.3155

[T] D. Talalaev, *Quantization of the Gaudin System*, Preprint (2004), 1–19; hep-th/0404153

[V] A. Varchenko, *A Selberg integral type formula for an* $\mathfrak{sl}_2$ *one-dimensional space of conformal blocks*, Mosc. Math. J. **10** (2010), no. 2, 469–475, 480