

COHOMOLOGY CLASSES OF CONORMAL BUNDLES OF SCHUBERT VARIETIES AND YANGIAN WEIGHT FUNCTIONS

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ABSTRACT. We consider the conormal bundle of a Schubert variety S_I in the cotangent bundle $T^*\text{Gr}$ of the Grassmannian Gr of k -planes in \mathbb{C}^n . This conormal bundle has a fundamental class κ_I in the equivariant cohomology $H_{\mathbb{T}}^*(T^*\text{Gr})$. Here $\mathbb{T} = (\mathbb{C}^*)^n \times \mathbb{C}^*$. The torus $(\mathbb{C}^*)^n$ acts on $T^*\text{Gr}$ in the standard way and the last factor \mathbb{C}^* acts by multiplication on fibers of the bundle. We express this fundamental class as a sum Y_I of the Yangian $Y(\mathfrak{gl}_2)$ weight functions $(W_J)_J$. We describe a relation of Y_I with the double Schur polynomial $[S_I]$.

A modified version of the κ_I classes, named κ'_I , satisfy an orthogonality relation with respect to an inner product induced by integration on the non-compact manifold $T^*\text{Gr}$. This orthogonality is analogous to the well known orthogonality satisfied by the classes of Schubert varieties with respect to integration on Gr .

The classes $(\kappa'_I)_I$ form a basis in the suitably localized equivariant cohomology $H_{\mathbb{T}}^*(T^*\text{Gr})$. This basis depends on the choice of the coordinate flag in \mathbb{C}^n . We show that the bases corresponding to different coordinate flags are related by the Yangian R-matrix.

1. INTRODUCTION

The equivariant cohomology of the cotangent bundle of a Grassmannian has hidden Yangian symmetries, see for example [Vas1, Vas2, GRTV]. In this paper we give another example of that symmetry.

We study the conormal bundle of a Schubert variety S_I in the cotangent bundle $T^*\text{Gr}$ of the Grassmannian Gr of k -planes in \mathbb{C}^n . We consider a resolution \tilde{S}_I of S_I , which lies in a flag variety Fl . The resolution map is the restriction of the natural forgetful map $\pi : \text{Fl} \rightarrow \text{Gr}$. The conormal bundle $C\tilde{S}_I \subset T^*\text{Fl}$ of \tilde{S}_I has an equivariant fundamental cohomology class $[C\tilde{S}_I]$ in the torus equivariant cohomology ring $H_{\mathbb{T}}^*(T^*\text{Fl}) = H_{\mathbb{T}}^*(\text{Fl})$. Here $\mathbb{T} = (\mathbb{C}^*)^n \times \mathbb{C}^*$. The torus $(\mathbb{C}^*)^n$ acts on $T^*\text{Fl}$ in the standard way and the last factor \mathbb{C}^* acts by multiplication on fibers of the bundle. Our main object of study is

$$\kappa_I = \pi_*([C\tilde{S}_I]) \in H_{\mathbb{T}}^*(\text{Gr}) = H_{\mathbb{T}}^*(T^*\text{Gr}),$$

which we call the *equivariant fundamental cohomology class* of the cotangent bundle of S_I .

In Theorem 4.4 we express $e_h \cdot \kappa_I$ as a sum Y_I of the Yangian $Y(\mathfrak{gl}_2)$ weight functions $(W_J)_J$. Here e_h is an explicit cohomology class independent of I , and e_h is not a zero-divisor.

In Section 5 we consider the fundamental class $[S_I]$ of the Schubert variety in the equivariant cohomology $H_{(\mathbb{C}^*)^n}^*(\text{Gr})$ and obtain $[S_I]$ as a suitable leading coefficient of the class κ_I . The classes $[S_I]$ are key objects in Schubert calculus. They are represented by the double Schur polynomials

(in the associated Chern roots). Proposition 5.4 and Corollary 5.5 say that the double Schur polynomials $[S_I]$ are leading coefficients of Yangian weight functions W_I , and of their sums Y_I .

In Section 7 we define a modified version κ'_I of κ_I . The transition matrix from $(\kappa_I)_I$ to $(\kappa'_I)_I$ is an upper triangular matrix with integer coefficients and ones in the diagonal. For every I the difference $\kappa'_I - \kappa_I$ is supported on $p^{-1}(S_I - S_I^o)$, where S_I^o is the Schubert cell, and p is the projection of the bundle $p : T^*\text{Gr} \rightarrow \text{Gr}$. We show that the κ'_I classes satisfy an orthogonality relation with respect to an inner product induced by integration on the non-compact manifold $T^*\text{Gr}$. This orthogonality is analogous to the well known orthogonality satisfied by the classes of Schubert varieties with respect to integration on Gr .

The classes $(\kappa'_I)_I$ form a basis in the suitably localized equivariant cohomology $H_{\mathbb{T}}^*(T^*\text{Gr})$. This basis depends on the choice of the coordinate flag in \mathbb{C}^n . In Section 8 we show that the bases corresponding to different coordinate flags are related by the Yangian R-matrix.

The weight functions were used in [TV2] to describe q -hypergeometric solutions of the qKZ equations associated with the Yangian $Y(\mathfrak{gl}_2)$. The q -hypergeometric solutions are of the form

$$(1) \quad I_\gamma(z_1, \dots, z_n) = \sum_J \left(\int_\gamma \Phi(t_1, \dots, t_k, z_1, \dots, z_n) W_J(t_1, \dots, t_k, z_1, \dots, z_n) dt_1 \dots dt_k \right) v_J$$

where (v_J) is a basis of a vector space, γ is an integration cycle parametrizing solutions, $\Phi(t_1, \dots, t_k, z_1, \dots, z_n)$ is a (master) function independent of the index J , the functions $W_J(t_1, \dots, t_k, z_1, \dots, z_n)$ are the weight functions. To express the fundamental class κ_I we replace in the weight functions the integration variables t_1, \dots, t_k with the Chern roots of the canonical bundle over Gr .

As explained in [TV2], the q -hypergeometric solutions (1) identify the qKZ equations with a suitable discrete Gauss-Manin connection. In [TV2], the weight functions were identified with the cohomology classes of a discretization of a suitable de Rham cohomology group. Our formula for the cohomology class of the conormal bundle of a Schubert variety in terms of weight functions indicates a connection between that discrete de Rham cohomology group of [TV2] and the equivariant cohomology of the Grassmannian.

This paper is a part of our project of identification of the equivariant cohomology of partial flag varieties with Bethe algebras of quantum integrable systems, see [V, RV, RSV, RTVZ, GRTV]. On this subject see also for example [BMO, NS].

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2. PRELIMINARIES

2.1. Blocks. The positive integers $k \leq n$ will be fixed throughout the paper.

In Sections 3 and 4 we will parameterize various objects in representation theory and geometry by k element subsets I of $\{1, \dots, n\}$. We will use some notation on the ‘blocks’ in I , as follows.

Write I as a disjoint union

$$\begin{aligned} I &= I_1 \cup I_2 \cup \dots \cup I_l \\ &= \{i_1, \dots, i_{v(1)}\} \cup \{i_{v(1)+1}, \dots, i_{v(2)}\} \cup \dots \cup \{i_{v(l-1)+1}, \dots, i_{v(l)}\} \end{aligned}$$

of maximal intervals of consecutive numbers, that is, we assume

- $i_1 < i_2 < \dots < i_{v(l)}$ ($v(l) = k$),
- $i_{v(c-1)+1}, \dots, i_{v(c)}$ is an interval of consecutive integers for $c = 1, \dots, l$ (we put $v(0) = 0$),
- $i_{v(c)} + 1 < i_{v(c)+1}$ for all $c = 1, \dots, l - 1$.

The subsets I_1, \dots, I_l will be called the blocks of I . The lengths of the blocks will be denoted by $m_c = |I_c| = v(c) - v(c - 1)$, $c = 1, \dots, l$. Let $\mathbf{m} = (m_1, \dots, m_l)$. For $a \in \{1, \dots, k\}$ let $i_{\hat{a}}$ be the largest element of the block containing i_a . That is, $\hat{a} = v(c)$ for some $c \in \{1, \dots, l\}$. We set $I! = m_1! \dots m_l!$.

$$\text{Let } \ell(I) = \sum_{a=1}^k (i_a - a).$$

2.2. Symmetrizer operations. For a function f of the variables t_1, \dots, t_k we set

$$\text{Sym}_{S_k} f = \sum_{\sigma \in S_k} f(t_{\sigma(1)}, \dots, t_{\sigma(k)}).$$

If a vector $\mathbf{m} = (m_1, \dots, m_l) \in \{0, 1, 2, \dots\}^l$ is given with $v(c) = \sum_{d=1}^c m_d$, $v(l) = k$, then let $S_{\mathbf{m}}$ be the subgroup of S_k permuting the groups of variables

$$(2) \quad \{t_1, \dots, t_{v(1)}\}, \quad \{t_{v(1)+1}, \dots, t_{v(2)}\}, \quad \dots \quad \{t_{v(l-1)+1}, \dots, t_{v(l)}\}$$

independently. We set

$$\text{Sym}_{S_{\mathbf{m}}} f = \sum_{\sigma \in S_{\mathbf{m}}} f(t_{\sigma(1)}, \dots, t_{\sigma(k)}).$$

If the function $f(t_1, \dots, t_k)$ is symmetric in the groups of variables (2), then we define

$$\text{Sym}_{S_k/S_{\mathbf{m}}} f = \frac{1}{\prod_{c=1}^l m_c!} \text{Sym}_{S_k} f.$$

$$\text{Clearly } \text{Sym}_{S_k} f = \text{Sym}_{S_k/S_{\mathbf{m}}} (\text{Sym}_{S_{\mathbf{m}}} f).$$

3. REPRESENTATION THEORY: THE WEIGHT FUNCTIONS

In this section let $z_1, \dots, z_n, h, t_1, \dots, t_k$ be variables.

3.1. Weight functions. Let $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, $i_1 < \dots < i_k$. Following [TV2] we call the function

$$(3) \quad W_I(t_1, \dots, t_k) = h^k \text{Sym}_{S_k} \left(\prod_{a=1}^k \left(\prod_{u=1}^{i_a-1} (t_a - z_u + h) \prod_{u=i_a+1}^n (t_a - z_u) \prod_{b=a+1}^k \frac{t_a - t_b + h}{t_a - t_b} \right) \right)$$

a weight function. For example

- for $k = 1$, $1 \leq i \leq n$ we have $W_{\{i\}} = h \prod_{u=1}^{i-1} (t_1 - z_u + h) \prod_{u=i+1}^n (t_1 - z_u)$

- for $n = 4$ we have

$$W_{\{1,2\}} = h^2(t_1 - z_3)(t_1 - z_4)(t_2 - z_3)(t_2 - z_4) ((h + t_1 + t_2)(h - z_1 - z_2) + 2(t_1 t_2 + z_1 z_2)).$$

For $I = \{i_1 < \dots < i_k\}$ and $J = \{j_1 < \dots < j_k\}$ we define $I \geq J$ if $i_a \geq j_a$ for all a . The following interpolation property of weight functions follows directly from the definition.

Lemma 3.1. *We have $W_J|_{t_a=z_{j_a}} \neq 0$. If $I \not\geq J$ then $W_I|_{t_a=z_{j_a}} = 0$.* \square

Let $\mathcal{R} = \mathbb{C}[z_1, \dots, z_n, h]((z_i - z_j + h)^{-1})_{i,j=1,\dots,n}$. Note that $i = j$ is allowed, hence $z_1 - z_1 + h = h$ is invertible in \mathcal{R} . Consider the \mathcal{R} -submodule $M_{k,n}$ of $\mathcal{R}[t_1, \dots, t_n]$, spanned by the weight functions $W_I(t_1, \dots, t_n)$ for all $I \subset \{1, \dots, n\}$, $|I| = k$.

Lemma 3.2. *The module $M_{k,n}$ is a free module with basis W_I .*

Proof. In an \mathcal{R} -linear relation among the W_I -functions there is a term W_J , such that for all other terms W_I in the relation we have $I \not\geq J$. Then the $t_a = z_{j_a}$ substitution cancels all the terms but W_J , see Lemma 3.1. \square

3.2. R-matrix. For $1 \leq a < k$, $I \subset \{1, \dots, n\}$, $|I| = k$ let $I^{(a,a+1)}$ be the k -element subset of $\{1, \dots, n\}$ in which the roles of a and $a+1$ are replaced. That is

- $a \in I$ if and only if $a+1 \in I^{(a,a+1)}$, and $a+1 \in I$ if and only if $a \in I^{(a,a+1)}$,
- for $b \notin \{a, a+1\}$ we have $b \in I^{(a,a+1)}$ if and only if $b \in I$.

Proposition 3.3. *For $1 \leq a < k$ we have*

$$(4) \quad \begin{aligned} \frac{h}{z_{a+1} - z_a + h} W_I + \frac{z_{a+1} - z_a}{z_{a+1} - z_a + h} W_{I^{(a,a+1)}} &= W_I|_{z_a \leftrightarrow z_{a+1}} \\ \frac{z_{a+1} - z_a}{z_{a+1} - z_a + h} W_I + \frac{h}{z_{a+1} - z_a + h} W_{I^{(a,a+1)}} &= W_{I^{(a,a+1)}}|_{z_a \leftrightarrow z_{a+1}} \end{aligned}$$

Proof. If $I^{(a,a+1)} = I$ then the statement reduces to the fact that in this case W_I is symmetric in z_a and z_{a+1} . Otherwise the statement of the proposition follows from the special case of $k = 1$, $n = 2$, $a = 1$ by simple manipulations of the formulas. This special case, namely,

$$(5) \quad \left(\begin{array}{cc} h & z_2 - z_1 \\ \frac{z_2 - z_1 + h}{z_2 - z_1} & \frac{z_2 - z_1 + h}{h} \end{array} \right) \begin{pmatrix} h(t_1 - z_2) \\ h(t_1 - z_1 + h) \end{pmatrix} = \begin{pmatrix} h(t_1 - z_1) \\ h(t_1 - z_2 + h) \end{pmatrix},$$

follows by direct calculation. \square

Consider the Lie algebra \mathfrak{gl}_2 with its standard generators $e_{i,j}$ for $i, j \in \{1, 2\}$. We denote the basis of its vector representation \mathbb{C}^2 by v_+, v_- . We have $e_{2,1}v_+ = v_-$, $e_{2,1}v_- = 0$.

For a subset $I \subset \{1, \dots, n\}$ define $v_I = v_{i_1} \otimes \dots \otimes v_{i_n}$ where $i_u = -$ for $u \in I$ and $i_u = +$ for $u \notin I$. The collection of vectors v_I for all subsets I of $\{1, \dots, n\}$ form a basis of the vector space $(\mathbb{C}^2)^{\otimes n}$. Hence we have the isomorphism of free \mathcal{R} -modules

$$\bigoplus_{k=0}^n M_{k,n} \cong (\mathbb{C}^2)^{\otimes n} \otimes \mathcal{R} \\ W_I \leftrightarrow v_I.$$

Proposition 3.3 means that under this identification the following two natural operations on the two sides are identified.

- Substituting $z_a \leftrightarrow z_{a+1}$ on the left-hand side;
- Acting by the ‘R-matrix’

$$\frac{h \cdot \text{Id} + (z_{a+1} - z_a) \cdot P^{(a,a+1)}}{z_{a+1} - z_a + h}$$

on the right-hand side, where $P^{(a,a+1)}$ is the transposition of the a th and $a + 1$ st factors of $(\mathbb{C}^2)^{\otimes n}$.

3.3. Y -functions. The main object of this section is a certain linear combination of the weight functions. Consider again the \mathfrak{gl}_2 representation $(\mathbb{C}^2)^{\otimes n}$. Let $e_{2,1}^{(j)}$ denote the action of $e_{2,1} \in \mathfrak{gl}_2$ on the j 'th factor of $(\mathbb{C}^2)^{\otimes n}$. Define

$$\Sigma_j = e_{2,1}^{(1)} + \dots + e_{2,1}^{(j)}.$$

For an $I \subset \{1, \dots, n\}$ recall the notation on the block-structure of I from Section 2.1 and set

$$\Sigma_I = \left(\frac{1}{m_1!} \Sigma_{v(1)} \right) \left(\frac{1}{m_2!} \Sigma_{v(2)} \right) \dots \left(\frac{1}{m_l!} \Sigma_{v(l)} \right) \in \text{End}((\mathbb{C}^2)^{\otimes n}).$$

Definition 3.4. For $I \subset \{1, \dots, n\}$, let $\Sigma_I(v_\emptyset) = \sum_J c_J v_J$ be the result of application of the operator Σ_I to the vector v_\emptyset . Define

$$Y_I = \sum_J c_J W_J.$$

Example 3.5. For $k = 1$, $i \leq n$ we have $Y_{\{i\}} = \sum_{j=1}^i W_{\{j\}}$.

For $k = 2$, $n = 4$ we have

$$\begin{aligned} \Sigma_{\{2,4\}}(v_+ \otimes v_+ \otimes v_+ \otimes v_+) &= \left((e_{2,1}^{(1)} + e_{2,1}^{(2)})(e_{2,1}^{(1)} + e_{2,1}^{(2)} + e_{2,1}^{(3)} + e_{2,1}^{(4)}) \right) (v_+ \otimes v_+ \otimes v_+ \otimes v_+) \\ &= 2v_{\{1,2\}} + v_{\{1,3\}} + v_{\{1,4\}} + v_{\{2,3\}} + v_{\{2,4\}}, \end{aligned}$$

hence

- $Y_{\{2,4\}} = 2W_{\{1,2\}} + W_{\{1,3\}} + W_{\{1,4\}} + W_{\{2,3\}} + W_{\{2,4\}}$.

The other Y -functions for $k = 2$, $n = 4$ are

- $Y_{\{1,2\}} = W_{\{1,2\}}$
- $Y_{\{1,3\}} = W_{\{1,2\}} + W_{\{1,3\}}$
- $Y_{\{1,4\}} = W_{\{1,2\}} + W_{\{1,3\}} + W_{\{1,4\}}$
- $Y_{\{2,3\}} = W_{\{1,2\}} + W_{\{1,3\}} + W_{\{2,3\}}$
- $Y_{\{3,4\}} = W_{\{1,2\}} + W_{\{1,3\}} + W_{\{1,4\}} + W_{\{2,3\}} + W_{\{2,4\}} + W_{\{3,4\}}$.

3.4. Formulas for Y -functions.

Lemma 3.6. *Using the notation of Section 2.1 we have*

$$(6) \quad Y_I = \frac{1}{I!} \text{Sym}_{S_k} \left[\prod_{a=1}^k \left(\prod_{u=1}^{i_{\hat{a}}} (t_a - z_u + h) \prod_{b=1}^{a-1} \frac{t_a - t_b - h}{t_a - t_b} - \prod_{u=1}^{i_{\hat{a}}} (t_a - z_u) \prod_{b=1}^{a-1} \frac{t_a - t_b + h}{t_a - t_b} \right) \prod_{u=i_{\hat{a}}+1}^n (t_a - z_u) \right].$$

In addition, let

$$(7) \quad N_I = \left(\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (t_a - t_b - h) \right) \cdot \prod_{a=1}^k \left(\prod_{u \leq i_{\hat{a}}} (t_a - z_u + h) \prod_{u > i_{\hat{a}}} (t_a - z_u) \right).$$

Then

$$(8) \quad Y_I = \text{Sym}_{S_k/S_m} \frac{N_I(t_1, \dots, t_k) + M_I(t_1, \dots, t_k)}{\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (t_a - t_b)},$$

where M is a polynomial symmetric in the groups of variables (2) and $M(z_{\sigma(1)}, \dots, z_{\sigma(k)}) = 0$ for any permutation $\sigma \in S_k$.

Proof. Formula (6) is a corollary of Lemma 2.21 in [TV2].

Now consider the function F in []-brackets in (6). Our goal is to write its S_m symmetrization in the form

$$\text{Sym}_{S_m} F = I! \cdot \frac{N_I + M_I}{\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (t_a - t_b)}$$

with N_I and M_I having the required properties.

The function F is a product of k factors Q_1, \dots, Q_k , with each factor being a difference of two rational functions. For example the first such factor is

$$Q_1 = \prod_{u=1}^{i_{\hat{1}}} (t_1 - z_u + h) \prod_{u=i_{\hat{1}}+1}^n (t_1 - z_u) - \prod_{u=1}^n (t_1 - z_u),$$

and the last such factor is

$$Q_k = \prod_{u=1}^{i_{\hat{k}}} (t_k - z_u + h) \prod_{u=i_{\hat{k}}+1}^n (t_k - z_u) \prod_{b=1}^{k-1} \frac{t_k - t_b - h}{t_k - t_b} - \prod_{u=1}^n (t_k - z_u) \prod_{b=1}^{k-1} \frac{t_k - t_b + h}{t_k - t_b}.$$

When this product of k factors is distributed, we have 2^k terms. Each of these terms are of the form

$$\frac{p(t_1, \dots, t_k)}{\prod_{1 \leq b \leq a \leq k} (t_a - t_b)},$$

where p is a polynomial. Hence, the S_m symmetrization of this term is of the form

$$\frac{q(t_1, \dots, t_k)}{\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (t_a - t_b)}$$

for an appropriate polynomial q .

We first claim that $q(z_{\sigma(1)}, \dots, z_{\sigma(k)}) = 0$ for any permutation $\sigma \in S_k$, unless the term we are considering comes from the choice of choosing the *first term from each* Q_1, Q_2, \dots, Q_k . Indeed, the second term of Q_a is divisible by $\prod_{u=1}^n (t_a - z_u)$. Hence p is divisible by this. Therefore q can be written as a sum of terms, each having a factor $\prod_{u=1}^n (t_b - z_u)$ (for different b 's). This proves our first claim. The sum of the q polynomials corresponding to these $2^k - 1$ choices will be $I! \cdot M_I$.

Now consider the product of the first terms of Q_1, \dots, Q_k . It is

$$\prod_{a=1}^k \left(\prod_{u=1}^{i_a} (t_a - z_u + h) \prod_{u=i_a+1}^n (t_a - z_u) \right) \prod_{1 \leq b \leq a \leq k} \frac{t_a - t_b - h}{t_a - t_b}.$$

Its S_m symmetrization is equal

$$\prod_{a=1}^k \left(\prod_{u=1}^{i_a} (t_a - z_u + h) \prod_{u=i_a+1}^n (t_a - z_u) \right) \text{Sym}_{S_m} \prod_{1 \leq b \leq a \leq k} \frac{t_a - t_b - h}{t_a - t_b}.$$

Observe that

$$(9) \quad \text{Sym}_{S_m} \prod_{1 \leq b \leq a \leq k} \frac{t_a - t_b - h}{t_a - t_b} = \prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} \frac{t_a - t_b - h}{t_a - t_b} \cdot \text{Sym}_{S_m} \prod_{\substack{a>b \\ i_a=i_b}} \frac{t_a - t_b - h}{t_a - t_b}.$$

Applying the simple identity

$$\text{Sym}_{S_r} \prod_{1 \leq b \leq a \leq r} \frac{t_a - t_b - h}{t_a - t_b} = r!$$

for $r = m_1, m_2, \dots$, we find that (9) equals

$$\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} \frac{t_a - t_b - h}{t_a - t_b} \cdot I!.$$

Hence the q polynomial corresponding to the choice of the first factors from Q_1, \dots, Q_k is $I! \cdot N_I$. This proves the second part of the Lemma. \square

4. GEOMETRY: CONORMAL BUNDLES OF SCHUBERT VARIETIES

4.1. The Schubert variety and its resolution. Let $\epsilon_1, \dots, \epsilon_n$ be the standard basis of \mathbb{C}^n . Consider the Grassmannian $\text{Gr} = \text{Gr}_k \mathbb{C}^n$ of k dimensional subspaces of \mathbb{C}^n , and the standard flag

$$(10) \quad \mathbb{C}^1 \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^n,$$

where $\mathbb{C}^a = \text{span}(\epsilon_1, \dots, \epsilon_a)$. For $I = \{i_1 < \dots < i_k\} \subset \{1, \dots, n\}$ we define the Schubert variety

$$S_I = \{W^k \subset \mathbb{C}^n : \dim(W^k \cap \mathbb{C}^{i_a}) \geq a \text{ for } a = 1, \dots, k\} \subset \text{Gr}.$$

The dimension of S_I is $\ell(I) = \sum_{a=1}^k (i_a - a)$. Recall the notations of the block-structure of I from Section 2.1. In particular, there are indexes $v(1), \dots, v(l)$ determined by I . The definition of S_I can be rephrased (see e.g. [M, Sect. 3.2]) as

$$S_I = \{W^k \subset \mathbb{C}^n : \dim(W^k \cap \mathbb{C}^{i_{v(c)}}) \geq v(c) \text{ for } c = 1, \dots, l\} \subset \text{Gr}.$$

Consider the partial flag variety $\text{Fl} = \text{Fl}_{v(1), v(2), \dots, v(l)} \mathbb{C}^n$ of nested subspaces

$$L_1 \subset L_2 \subset \dots \subset L_l$$

of \mathbb{C}^n , where $\dim L_c = v(c)$. Note that $v(1), \dots, v(l)$ depend on I , hence Fl depends on I . The natural forgetful map $\text{Fl} \rightarrow \text{Gr}$, $(L_1 \subset \dots \subset L_l) \mapsto L_l$ will be denoted by π . The Schubert variety

$$(11) \quad \tilde{S}_I = \{(L_1 \subset \dots \subset L_l) : L_c \subset \mathbb{C}^{i_{v(c)}} \text{ for } c = 1, \dots, l\} \subset \text{Fl}$$

is a resolution of S through the map π [M, Section 3.4].

Since \tilde{S}_I is a smooth subvariety of Fl , we can consider its conormal bundle

$$(12) \quad C\tilde{S}_I = \{\alpha \in T_p^* \text{Fl} : p \in \tilde{S}_I, \alpha(T_p \tilde{S}_I) = 0\} \subset T^* \text{Fl}.$$

It is a rank $\dim \text{Fl} - \dim \tilde{S}_I$ subbundle of the cotangent bundle of Fl restricted to \tilde{S}_I , hence $\dim C\tilde{S}_I = \dim \text{Fl}$. It is well known that $C\tilde{S}_I$ is a Lagrangian subvariety of $T^* \text{Fl}$ with its usual symplectic form.

4.2. Torus equivariant cohomology. Consider the torus $\mathbb{T} = (\mathbb{C}^*)^n \times \mathbb{C}^*$. Let L_i be the tautological line bundle over the i 'th component of the classifying space $B\mathbb{T} = (\mathbb{P}^\infty)^n \times \mathbb{P}^\infty$. We define $z_i = c_1(L_i)$ for $i = 1, \dots, n$, and $h = c_1(L_{n+1})$. We have $H_{\mathbb{T}}^*(\text{one-point space}) = H^*(B\mathbb{T}) = \mathbb{C}[z_1, \dots, z_n, h]$. The \mathbb{T} equivariant cohomology ring of any \mathbb{T} space is a module over this polynomial ring.

Consider the action of \mathbb{T} on \mathbb{C}^n given by $(\alpha_1, \dots, \alpha_n) \cdot (v_1, \dots, v_n) = (\alpha_1 v_1, \dots, \alpha_n v_n)$. This action induces an action of \mathbb{T} on Gr and Fl . We will be concerned with the action of \mathbb{T} on the cotangent bundles $T^* \text{Gr}$ and $T^* \text{Fl}$ defined as follows: the action of the subgroup $(\mathbb{C}^*)^n$ is induced from the action of \mathbb{T} on Gr and Fl , while the extra \mathbb{C}^* factor acts by multiplication in the fiber direction. We have the diagram of maps in \mathbb{T} -equivariant cohomology

$$\begin{array}{ccc} H_{\mathbb{T}}^*(\text{Fl}) & \cong & H_{\mathbb{T}}^*(T^* \text{Fl}) \\ \pi_* \downarrow & & \\ H_{\mathbb{T}}^*(\text{Gr}) & \cong & H_{\mathbb{T}}^*(T^* \text{Gr}) \end{array}$$

where π_* is the push-forward map (also known as Gysin map) in cohomology. The isomorphisms $H_{\mathbb{T}}^*(X) \cong H_{\mathbb{T}}^*(T^* X)$ are induced by the equivariant homotopy equivalences $T^* X \rightarrow X$. In notation we will not distinguish between $H_{\mathbb{T}}^*(X)$ and $H_{\mathbb{T}}^*(T^* X)$, in particular π_* will denote the map $H_{\mathbb{T}}^*(T^* \text{Fl}) \rightarrow H_{\mathbb{T}}^*(T^* \text{Gr})$ as well.

There are natural bundles over Gr , whose fibers over the point $W \in \text{Gr}$ are $W, \mathbb{C}^n/W$. Let the Chern roots of these bundles be denoted by

$$(13) \quad \underbrace{t_1, \dots, t_k}_k, \quad \underbrace{\tilde{t}_1, \dots, \tilde{t}_{n-k}}_{n-k}.$$

Let the group S_k act by permuting the t_a 's, and S_{n-k} by permuting the \tilde{t}_b 's. We have

$$H_{\mathbb{T}}^*(\text{Gr}) = \mathbb{C}[t_1, \dots, t_k, \tilde{t}_1, \dots, \tilde{t}_{n-k}, z_1, \dots, z_n, h]^{S_k \times S_{n-k}} / I_{\text{Gr}},$$

where the ideal I_{Gr} is generated by the coefficients of the following polynomial in ξ :

$$\prod_{a=1}^k (1 + t_a \xi) \prod_{b=1}^{n-k} (1 + \tilde{t}_b \xi) - \prod_{u=1}^n (1 + z_u \xi).$$

There are natural bundles over Fl , whose fibers over the point $(L_1, L_2, \dots, L_l) \in \text{Fl}$ are $L_1, L_2/L_1, L_3/L_2, \dots, L_l/L_{l-1}, \mathbb{C}^n/L_l$. Let the Chern roots of these bundles be denoted by

$$(14) \quad \underbrace{t_1, \dots, t_{v(1)}}_{m_1}, \quad \underbrace{t_{v(1)+1}, \dots, t_{v(2)}}_{m_2}, \quad \dots \quad \underbrace{t_{v(l-1)+1}, \dots, t_{v(l)}}_{m_l}, \quad \underbrace{\tilde{t}_1, \dots, \tilde{t}_{n-k}}_{n-k}.$$

Recall that S_m permutes the variables t_a as in Section 2.2, S_{n-k} permutes the variables \tilde{t}_b . We have

$$(15) \quad H_{\mathbb{T}}^*(\text{Fl}) = \mathbb{C}[t_1, \dots, t_k, \tilde{t}_1, \dots, \tilde{t}_{n-k}, z_1, \dots, z_n, h]^{S_m \times S_{n-k}} / I_{\text{Fl}},$$

where the ideal I_{Fl} is generated by the coefficients of the following polynomial in ξ :

$$\prod_{a=1}^k (1 + t_a \xi) \prod_{b=1}^{n-k} (1 + \tilde{t}_b \xi) - \prod_{u=1}^n (1 + z_u \xi).$$

4.3. Equivariant localization. We recall some facts from the theory of equivariant localization, specified for Fl and Gr . Our reference is [AB], a more recent account is e.g. [CG, Ch. 5]. Let \mathcal{F}_{Fl} be the set of fixed points of the \mathbb{T} action on Fl , and let \mathcal{F}_{Gr} be the set of fixed points of the \mathbb{T} action on Gr . The sets $\mathcal{F}_{\text{Fl}}, \mathcal{F}_{\text{Gr}}$ are finite. Consider the restriction maps

$$(16) \quad H_{\mathbb{T}}^*(\text{Fl}) \rightarrow H_{\mathbb{T}}^*(\mathcal{F}_{\text{Fl}}) = \bigoplus_{f \in \mathcal{F}_{\text{Fl}}} H_{\mathbb{T}}^*(f), \quad H_{\mathbb{T}}^*(\text{Gr}) \rightarrow H_{\mathbb{T}}^*(\mathcal{F}_{\text{Gr}}) = \bigoplus_{f \in \mathcal{F}_{\text{Gr}}} H_{\mathbb{T}}^*(f).$$

A key fact of the theory of equivariant localization is that these restriction maps are injective. The explicit form of these restriction maps is as follows.

- A fixed point $f \in \mathcal{F}_{\text{Gr}}$ is a coordinate k -plane in \mathbb{C}^n , hence it corresponds to a subset $I \subset \{1, \dots, n\}$, $|I| = k$. The restriction map $H_{\mathbb{T}}^*(\text{Gr}) \rightarrow H_{\mathbb{T}}^*(f) = \mathbb{C}[z]$ to the fixed point corresponding to I is

$$[p(t, \tilde{t}, z, h)] \mapsto p(z_I, z_{\bar{I}}, z, h),$$

where $t = (t_1, \dots, t_k)$, $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{n-k})$, $z = (z_1, \dots, z_n)$; and z_I stands for the list of z_u 's with $u \in I$; and $z_{\bar{I}}$ stands for the list of z_u 's with $u \notin I$.

- Recall that Fl depends on I , and the block structure of I determines numbers l, m_c as in Section 2.1. A fixed point $f \in \text{Fl}$ corresponds to a decomposition $\{1, \dots, n\} = K_1 \cup \dots \cup K_l \cup K_{l+1}$ into disjoint subsets K_1, \dots, K_l, K_{l+1} with $|K_c| = m_c$ for $c = 1, \dots, l$, and $|K_{l+1}| = n - k$. The restriction map $H_{\mathbb{T}}^*(\text{Fl}) \rightarrow H_{\mathbb{T}}^*(f) = \mathbb{C}[z]$ to the fixed point corresponding to this decomposition is

$$[p(t, \tilde{t}, z, h)] \mapsto p(z_{K_1}, \dots, z_{K_l}, z_{K_{l+1}}, z, h).$$

We will also need a formula for the push-forward map π_* .

Proposition 4.1. *Let $[p] \in H_{\mathbb{T}}^*(\text{Fl})$, where $p \in \mathbb{C}[t_1, \dots, t_k, \tilde{t}_1, \dots, \tilde{t}_{n-k}, z_1, \dots, z_n]^{S_m \times S_{n-k}}$ (see (15)). Then*

$$(17) \quad \pi_*([p]) = \left[\text{Sym}_{S_k/S_m} \frac{p}{\prod_{c>d} \prod_{t_a \in I_c} \prod_{t_b \in I_d} (t_a - t_b)} \right].$$

The expression on the right-hand side is formally a sum of fractions. However, in the sum the denominators cancel: the sum is a polynomial.

Proof. We sketch a proof of this formula well known in localization theory. Let f_I be the fixed point on Gr corresponding to the subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. We have

$$(18) \quad \pi_*([p])|_{f_I} = (\pi|_{\pi^{-1}(f_I)})_*([p]_{\pi^{-1}(f_I)}) = (\pi|_{\pi^{-1}(f_I)})_* \left(\sum_{f \in \pi^{-1}(f_I) \cap \mathcal{F}_{\text{Fl}}} \frac{j_{f*}([p]_f)}{e(T_f \pi^{-1}(f))} \right).$$

Here j_f is the inclusion $\{f\} \subset \pi^{-1}(f_I)$. In the last equality we used the formula [AB, p.9.] which describes how one can recover an equivariant cohomology class from its restrictions to the fixed points. Using the fact that $(\pi|_{\pi^{-1}(f_I)} \circ j_f)_* : \mathbb{Z}[z, h] \rightarrow \mathbb{Z}[z, h]$ is the identity map, we obtain that (18) is equal to

$$\left[\text{Sym}_{S_k/S_m} \frac{p}{\prod_{c>d} \prod_{t_a \in I_c} \prod_{t_b \in I_d} (t_a - t_b)} \right] \Big|_{t_a = z_{i_a}, \tilde{t} = z_{\bar{I}}}.$$

The restriction of the right-hand-side of (17) to f_I is clearly the same formula. This proves the proposition. \square

4.4. Equivariant fundamental class of the conormal bundle on Fl. The variety $C\tilde{S}_I$ is invariant under the \mathbb{T} action on $T^*\text{Fl}$. Hence it has an equivariant fundamental cohomology class, an element $[C\tilde{S}_I] \in H_{\mathbb{T}}^{2 \dim \text{Fl}}(T^*\text{Fl}) = H_{\mathbb{T}}^{2 \dim \text{Fl}}(\text{Fl})$. Our goal in this section is to express the geometrically defined class $[C\tilde{S}_I]$ in terms of the Chern roots z_u, h , and t_a .

Let $\tilde{\gamma}$ be the bundle over Fl whose fiber over (L_1, \dots, L_l) is L_l . Consider the \mathbb{T} action on the bundle $\tilde{\gamma}^* \otimes \tilde{\gamma}$ where the $(\mathbb{C}^*)^n$ action is induced by the action on Fl, and the extra \mathbb{C}^* acts by multiplication in the fiber direction. The Euler class of this bundle will be denoted by $e_h(\tilde{\gamma}^* \otimes \tilde{\gamma})$. We have

$$e_h(\tilde{\gamma}^* \otimes \tilde{\gamma}) = \prod_{a=1}^k \prod_{b=1}^k (t_a - t_b + h).$$

Theorem 4.2. For $I \subset \{1, \dots, n\}$, $|I| = k$ we use the notations of Section 2.1. Let

$$\operatorname{sgn}(I) = (-1)^{\operatorname{codim} S_I} = (-1)^{k(n-k)-\ell(I)} = (-1)^{\sum_{a=1}^k (n-i_a) - \sum_{1 \leq c < d \leq l} m_c m_d}.$$

Consider the cohomology class

$$(19) \quad N_I = \left(\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (t_a - t_b - h) \right) \cdot \prod_{a=1}^k \left(\prod_{u=1}^{i_a} (t_a - z_u + h) \prod_{u=i_a+1}^n (t_a - z_u) \right) \in H_{\mathbb{T}}^*(\mathbb{F}l).$$

Then

$$(20) \quad \operatorname{sgn}(I) \cdot e_h(\tilde{\gamma}^* \otimes \tilde{\gamma}) \cdot [C\tilde{S}_I] = N_I \in H_{\mathbb{T}}^*(\mathbb{F}l).$$

Note that expressions (7) and (19) coincide. However, in Lemma 3.6, N_I is a polynomial, while in Theorem 4.2, N_I is a cohomology class (that is, an element of the quotient ring (15)).

Note also, that the element $e_h(\tilde{\gamma}^* \otimes \tilde{\gamma})$ is not a zero-divisor in the ring $H_{\mathbb{T}}^*(\mathbb{F}l)$ (because none of its fixed point restrictions vanish), hence equation (20) uniquely determines $[C\tilde{S}_I]$.

Proof. We will show that the restrictions of the two sides of (20) to the torus fixed points on $\mathbb{F}l$ agree.

Let us pick a torus fixed point $f \in \mathbb{F}l$. It corresponds to a decomposition

$$\{1, \dots, n\} = K_1 \cup \dots \cup K_l \cup K_{l+1}$$

into disjoint subsets with $|K_c| = m_c$ for $c = 1, \dots, l$ and $|K_{l+1}| = n - k$. First listing the elements of K_1 (in any order), then the elements of K_2 (in any order), etc, lastly the elements of K_l (in any order), we obtain a list of numbers j_1, \dots, j_k . Restricting $[p(t_1, \dots, t_k, z, h)] \in H_{\mathbb{T}}^*(\operatorname{Gr})$ to the fixed point f amounts to substituting $t_a = z_{j_a}$ into p .

Observe that $N_I|_{t_a=z_{j_a}}$ is 0 unless $j_a \leq i_a$ for all a . If $j_a \leq i_a$ for all a then

$$\begin{aligned} N_I|_{t_a=z_{j_a}} &= \left(\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (z_{j_a} - z_{j_b} - h) \right) \cdot \prod_{a=1}^k \left(\prod_{u=1}^{i_a} (z_{j_a} - z_u + h) \prod_{u=i_a+1}^n (z_{j_a} - z_u) \right) \\ &= (-1)^{\sum_{c<d} m_c m_d} \left(\prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (z_{j_b} - z_{j_a} + h) \right) \cdot \prod_{a=1}^k \left(\prod_{u=1}^{i_a} (z_{j_a} - z_u + h) \prod_{u=i_a+1}^n (z_{j_a} - z_u) \right) \\ &= (-1)^{\sum_{c<d} m_c m_d} \left(\prod_{a=1}^k \prod_{b=1}^k (z_{j_a} - z_{j_b} + h) \right) \cdot \prod_{a=1}^k \left(\prod_{\substack{u \in \{1, \dots, i_a\} \\ -\{j_1, \dots, j_a\}}} (z_{j_a} - z_u + h) \prod_{u=i_a+1}^n (z_{j_a} - z_u) \right). \end{aligned}$$

Now we turn to the study of $[C\tilde{S}_I]|_f$.

The point f is not contained in \tilde{S}_I unless $j_a \leq i_a$ for all $a = 1, \dots, k$, hence in this case $[C\tilde{S}_I]|_f = 0$. Let us now assume $j_a \leq i_a$ for all $a = 1, \dots, k$. The weights of the action of \mathbb{T} on the tangent plane $T_f \mathbb{F}l$ are

$$z_u - z_{j_a} \quad \text{for } a = 1, \dots, k, \quad u \in \{1, \dots, n\} - \{j_1, \dots, j_a\}.$$

Among these weights the ones that correspond to eigenvectors in the tangent space to \tilde{S}_I are

$$z_u - z_{j_a} \quad \text{for } a = 1, \dots, k, \quad u \in \{1, \dots, i_{\hat{a}}\} - \{j_1, \dots, j_{\hat{a}}\}.$$

Hence the weights of the normal space to \tilde{S}_I at f are $z_u - z_{j_a}$ for $u > i_{\hat{a}}$. The weights of the normal space to $C\tilde{S}_I|_f \subset T^*\text{Fl}|_f$ are hence $z_{j_a} - z_u + h$ for $u \in \{1, \dots, i_{\hat{a}}\} - \{j_1, \dots, j_{\hat{a}}\}$.

Therefore we have

$$[C\tilde{S}_I]|_f = \prod_{a=1}^k \left(\prod_{\substack{u \in \{1, \dots, i_{\hat{a}}\} \\ - \{j_1, \dots, j_{\hat{a}}\}}} (z_{j_a} - z_u + h) \prod_{u=i_{\hat{a}}+1}^n (z_u - z_{j_a}) \right).$$

Since the restriction of $e_h(\tilde{\gamma}^* \otimes \tilde{\gamma})$ to the fixed point f is $\prod_{a=1}^k \prod_{b=1}^k (z_{j_a} - z_{j_b} + h)$, we proved that the two sides of (20) are equal restricted to the fixed point f . \square

4.5. Equivariant fundamental class of the conormal bundle on Gr.

Definition 4.3. *We define the equivariant fundamental cohomology class of the conormal bundle of $S_I \subset \text{Gr}$ to be $\pi_*([C\tilde{S}_I])$, and we will denote this class by κ_I .*

Let γ be the bundle over Gr whose fiber over $W \subset \mathbb{C}^n$ is W . Consider the \mathbb{T} action on the bundle $\gamma^* \otimes \gamma$ where the $(\mathbb{C}^*)^n$ action is induced by the action on Gr, and the extra \mathbb{C}^* acts by multiplication in the fiber direction. The Euler class of this bundle will be denoted by $e_h(\gamma^* \otimes \gamma)$. We have

$$e_h(\gamma^* \otimes \gamma) = \prod_{a=1}^k \prod_{b=1}^k (t_a - t_b + h).$$

Recall the definition of the Y_I -function from Section 3.3. Interpret the variables t , z , and h of this function according to the definitions of the present section, that is, as equivariant cohomology classes in Gr. Then $Y_I \in H_{\mathbb{T}}^*(\text{Gr})$. Recall that $\text{sgn}(I) = (-1)^{\text{codim } S_I}$.

Theorem 4.4. *We have*

$$\text{sgn}(I) \cdot e_h(\gamma^* \otimes \gamma) \cdot \kappa_I = Y_I \quad \in H_{\mathbb{T}}^*(\text{Gr}).$$

Note that $e_h(\gamma^* \otimes \gamma)$ is not a zero-divisor (because none of its fixed point restrictions vanish), hence the equation in Theorem 4.4 uniquely determines κ_I .

Proof. Let us apply π_* to (20). The left hand side will map to $\text{sgn}(I)e_h(\gamma^* \otimes \gamma)\kappa_I$, because the bundle $\tilde{\gamma}$ over Fl is the pullback of the bundle γ over Gr.

Recall the definition of M_I from (8). As a cohomology class in $H_T^*(\text{Fl})$, M_I is zero, because it restricts to 0 at every \mathbb{T} fixed points of Fl. Hence, π_* applied to the right hand side of (20) is

$$\pi_*(N_I) = \pi_*(N_I + M_I) = Y_I.$$

The last equality holds because of the comparison of (8) and (17). \square

Let $p : T^*\text{Gr} \rightarrow \text{Gr}$ be the projection of the bundle. Let $\text{Sing } S_I \subset S_I$ be the subvariety of singular points of S_I and $S_I^{\text{sm}} = S_I - \text{Sing } S_I$ the smooth part. Define the conormal bundle of the smooth part to be

$$(21) \quad CS_I^{\text{sm}} = \{\alpha \in T_x^* \text{Gr} : x \in S_I^{\text{sm}}, \alpha(T_x S_I) = 0\} \subset p^{-1}(\text{Gr} - \text{Sing } S_I).$$

Denote $[CS_I^{\text{sm}}] \in H_{\mathbb{T}}^*(p^{-1}(\text{Gr} - \text{Sing } S_I))$ the equivariant fundamental cohomology class of CS_I^{sm} . Consider the embedding $j : p^{-1}(\text{Gr} - \text{Sing } S_I) \hookrightarrow T^*\text{Gr}$.

Theorem 4.5. *We have $j^*(\kappa_I) = [CS_I^{\text{sm}}] \in H_{\mathbb{T}}^*(p^{-1}(\text{Gr} - \text{Sing } S_I))$.*

Proof. To prove Theorem 4.5 first we recall two lemmas from Schubert calculus that are probably known to the specialists, but we sketch their proofs because we did not find exact references.

Lemma 4.6. *Let f_J be a smooth \mathbb{T} fixed point on S_I . Then there is exactly one \mathbb{T} fixed point in $\tilde{S}_I \cap \pi^{-1}(f_J)$.*

Proof. Recall the notation of the blocks in I , and assume $J = \{j_1 < \dots < j_k\}$. Assume that $f_J \in S_I$, hence $j_a \leq i_a$ for all $a = 1, \dots, k$. The description of the components of the singular locus of S_I in [M, Thm. 3.4.4] can be rephrased to our language as follows: The \mathbb{T} fixed point f_J is a singular point on S_I if and only if $j_{v(c)+1} \leq i_{v(c)}$ for some c . If this does not happen for any c then

$$|\{a : j_a \leq v(c)\}| = v(c)$$

for all $c = 1, \dots, l$. In this case there is only one choice for a fixed point in $\tilde{S}_I \cap \pi^{-1}(f_J)$, namely

$$\text{span}\{\epsilon_{j_1}, \dots, \epsilon_{j_{v(1)}}\} \subset \text{span}\{\epsilon_{j_1}, \dots, \epsilon_{j_{v(2)}}\} \subset \dots \subset \text{span}\{\epsilon_{j_1}, \dots, \epsilon_{j_{v(l)}}\}.$$

□

Lemma 4.7. *The localization map*

$$H_{\mathbb{T}}^*(\text{Gr} - \text{Sing } S_I) \rightarrow \bigoplus_f H_{\mathbb{T}}^*(f), \quad \alpha \mapsto (\alpha|_f),$$

where \bigoplus runs for the \mathbb{T} fixed points f in $\text{Gr} - \text{Sing } S_I$, is injective.

Proof. The injectivity of the localization map is usually phrased for compact manifolds (see for example the original [AB]). However, here we sketch an argument for $\text{Gr} - \text{Sing } S_I$. Starting with the open Schubert cell, we add the cells of $\text{Gr} - \text{Sing } S_I$ one by one, in order of codimension. At each step we can use a Mayer-Vietoris argument to show that if the localization map was injective before adding the cell, then it is injective after adding the cell. A strictly analogous argument is shown in the proof of [FR1, Lemma 5.3]. The only condition for such a step-by-step Mayer-Vietoris argument to work is that the equivariant Euler-class of the normal bundle of each cell is not a zero-divisor. This calculation is done for Grassmannians, for example, in [FR2, Sect. 5]. □

Now we can prove Theorem 4.5. Let f_J be a smooth \mathbb{T} fixed point of S_I , and $\tilde{f}_J \in \text{Fl}$ the unique \mathbb{T} fixed point in $\tilde{S}_I \cap \pi^{-1}(f_J)$. An analysis of the \mathbb{T} representations on $T_{f_J} \text{Gr}$ and $T_{\tilde{f}_J} \text{Fl}$, similar to the one in the proof of Theorem 4.2, gives

$$(22) \quad [C\tilde{S}_I]_{\tilde{f}_J} = [CS_I^{\text{sm}}]_{f_J} \cdot \prod_{c>d} \prod_{i_a \in I_c} \prod_{i_b \in I_d} (z_{i_a} - z_{i_b}).$$

Therefore we have

$$\kappa_I|_{f_J} = \pi_* \left([C\tilde{S}_I]_{\tilde{f}_J} \right) = [CS_I^{\text{sm}}]_{f_J}.$$

If f is a \mathbb{T} fixed point in $\text{Gr} - S_I$ then obviously both $\kappa_I|_f$ and $[CS_I^{\text{sm}}]_f$ are 0. Thus we found that the localization map of Lemma 4.7 maps $j^*(\kappa_I) - [CS_I^{\text{sm}}]$ to 0. This proves the theorem. \square

5. SCHUR POLYNOMIALS

Definition 5.1. For $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, \dots, n\}$ we define the double Schur polynomial by

$$\Delta_I(t_1, \dots, t_k, z_1, \dots, z_n) = (-1)^{\text{codim } S_I} \det \left(\prod_{u=i_a+1}^n (t_\beta - z_u) \right)_{\alpha, \beta=1, \dots, k} \frac{1}{\prod_{1 \leq a < b \leq k} (t_a - t_b)}.$$

Remark 5.2. This definition of double Schur polynomials is the so-called *bialternant* definition. Other definitions include a (generalized) Jacobi-Trudi determinant, an interpolation definition, and the fact that double Schur polynomials are special cases of double Schubert polynomials that are described recursively using divided difference operators. For references see [M1], [M2, Ex.20, Section I.3, p.54], [F, Lecture 8], or [FR2].

In Schubert calculus it is well known that equivariant classes of Schubert varieties in Grassmannians are represented by double Schur polynomials.

Theorem 5.3. (See [F, Lecture 8] and references therein.) The $(\mathbb{C}^*)^n$ -equivariant fundamental class $[S_I]$ of S_I in $H_{(\mathbb{C}^*)^n}^*(\text{Gr})$ is represented by the double Schur polynomial $\Delta_I(t, z)$.

What connects this fact with the objects of the present paper is the following observation.

Proposition 5.4. Recall that $\ell(I) = \sum_{a=1}^k (i_a - a)$ is the dimension of S_I . Consider the weight function W_I as a polynomial in h . We have

$$W_I = \text{sgn}(I) \Delta_I(t, z) \cdot h^{k^2 + \ell(I)} + \text{lower degree terms.}$$

Proof. The statement follows from the explicit formulae for W_I and Δ_I . \square

Corollary 5.5. We have

$$\begin{aligned} Y_I &= \text{sgn}(I) \Delta_I(t, z) h^{k^2 + \ell(I)} + \text{lower degree terms.} \\ \kappa_I &= [S_I] h^{\ell(I)} + \text{lower degree terms.} \end{aligned}$$

Proof. The first statement follows from the fact $Y_I = W_I +$ sums of W_J -functions with $J < I$. The h -degrees of these W_J functions are strictly less than the h -degree of W_I . The second statement is the consequence of Theorems 4.4 and 5.3. Note that the second statement could also be proved geometrically. \square

6. MODIFIED EQUIVARIANT CLASS OF THE CONORMAL BUNDLE

Recall that

$$Y_I = W_I + \sum_{J < I} c_{IJ} W_J,$$

where c_{IJ} are some positive integer coefficients. In other words, the transition matrix from the W_I functions to the Y_I functions is triangular (with 1's in the diagonal). Hence the the same is true for its inverse:

$$W_I = Y_I + \sum_{J < I} c'_{IJ} Y_J,$$

where c'_{IJ} are some integer coefficients. Comparing this expression with Theorem 4.4 we obtain that in $H_{\mathbb{T}}^*(\text{Gr})$ we have

$$\begin{aligned} W_I &= \text{sgn}(I) e_h(\gamma^* \otimes \gamma) \kappa_I + \sum_{J < I} c'_{IJ} \text{sgn}(J) e_h(\gamma^* \otimes \gamma) \kappa_J \\ &= \text{sgn}(I) e_h(\gamma^* \otimes \gamma) \left(\kappa_I + \sum_{J < I} c'_{IJ} \frac{\text{sgn}(J)}{\text{sgn}(I)} \kappa_J \right). \end{aligned}$$

Definition 6.1. We define the modified equivariant fundamental class of the conormal bundle of the Schubert variety S_I to be

$$\kappa'_I = \kappa_I + \sum_{J < I} c'_{IJ} \frac{\text{sgn}(J)}{\text{sgn}(I)} \kappa_J.$$

Notice that $J < I$ if and only if S_J is a proper subvariety of S_I (hence, in particular, $\dim S_J < \dim S_I$). Therefore the difference of κ'_I and κ_I is a linear combination of fundamental classes of conormal bundles of proper subvarieties of S_I .

Example 6.2. For $k = 1$ we have $Y_{\{i\}} = \sum_{j \leq i} W_{\{j\}}$. Hence for $i > 1$

$$\begin{aligned} W_{\{i\}} &= Y_{\{i\}} - Y_{\{i-1\}} \\ &= (-1)^{n-i} e_h(\gamma^* \otimes \gamma) \kappa_{\{i\}} - (-1)^{n-(i-1)} e_h(\gamma^* \otimes \gamma) \kappa_{\{i-1\}} \\ &= (-1)^{n-i} e_h(\gamma^* \otimes \gamma) (\kappa_{\{i\}} + \kappa_{\{i-1\}}). \end{aligned}$$

Therefore $\kappa'_{\{i\}} = \kappa_{\{i\}} + \kappa_{\{i-1\}} = [CS_{\{i\}}] + [CS_{\{i-1\}}]$.

For example, for $k = 1$, $n = 2$, let $p : T^*\text{Gr} \rightarrow \text{Gr}$ be the projection of the bundle as before, and $f_{\{1\}}$ the \mathbb{T} fixed point with homogeneous coordinate $(1 : 0)$ on $\text{Gr} = \text{Gr}_1 \mathbb{C}^2 = \mathbb{P}^1$. Then

$$\begin{aligned} \kappa'_{\{1\}} &= \kappa_{\{1\}} = [p^{-1}(f_{\{1\}})] = z_2 - t_1, \\ \kappa'_{\{2\}} &= \kappa_{\{2\}} + \kappa_{\{1\}} = [\text{Gr}] + [p^{-1}(f_{\{1\}})] = (2t_1 - z_1 - z_2 + h) + (z_2 - t_1) = t_1 - z_1 + h. \end{aligned}$$

For $k = 2, n = 4$ calculation shows

$$\begin{aligned} \kappa'_{\{1,2\}} &= \kappa_{\{1,2\}}, & \kappa'_{\{1,3\}} &= \kappa_{\{1,3\}} + \kappa_{\{1,2\}}, & \kappa'_{\{1,4\}} &= \kappa_{\{1,4\}} + \kappa_{\{1,3\}}, & \kappa'_{\{2,3\}} &= \kappa_{\{2,3\}} + \kappa_{\{1,3\}}, \\ \kappa'_{\{2,4\}} &= \kappa_{\{2,4\}} + \kappa_{\{2,3\}} + \kappa_{\{1,4\}} + \kappa_{\{1,3\}} + \kappa_{\{1,2\}}, & \kappa'_{\{3,4\}} &= \kappa_{\{3,4\}} + \kappa_{\{2,4\}} + \kappa_{\{1,2\}}. \end{aligned}$$

Let us compare the modified fundamental class with the earlier, non-modified κ_I classes. We have

$$(23) \quad \begin{aligned} \operatorname{sgn}(I) e_h(\gamma^* \otimes \gamma) \kappa_I &= Y_I, \\ \operatorname{sgn}(I) e_h(\gamma^* \otimes \gamma) \kappa'_I &= W_I. \end{aligned}$$

Recall from Theorem 4.5 that $j^*(\kappa_I) = [CS_I^{\text{sm}}] \in H_{\mathbb{T}}^*(p^{-1}(\operatorname{Gr} - \operatorname{Sing} S_I))$. The class κ'_I does not satisfy this property over the whole $\operatorname{Gr} - \operatorname{Sing} S_I$, but only over the Schubert cell S_I^o which is a dense open subset of S_I .

Theorem 6.3. *Let $p : T^* \operatorname{Gr} \rightarrow \operatorname{Gr}$ be the projection of the bundle. Consider the Schubert cell*

$$S_I^o = \{W^k \subset \mathbb{C}^n : \dim(W^k \cap \mathbb{C}^{i_a}) = a \text{ for } a = 1, \dots, k\} \subset \operatorname{Gr},$$

and its conormal bundle $CS_I^o \subset p^{-1}(\operatorname{Gr} - \cup_{J < I} S_J)$. Let i denote the embedding

$$i : p^{-1}(\operatorname{Gr} - \cup_{J < I} S_J) \hookrightarrow T^* \operatorname{Gr}.$$

Then

$$i^*(\kappa'_I) = [CS_I^o] \in H^*(p^{-1}(\operatorname{Gr} - \cup_{J < I} S_J)).$$

Proof. The difference $\kappa'_I - \kappa_I$ is supported on $p^{-1}(\cup_{J < I} S_J)$. Therefore $i^*(\kappa'_I) = i^*(\kappa_I)$, so it is enough to show that $i^*(\kappa_I) = [CS_I^o]$. This follows from the diagram

$$\begin{array}{ccccc} & & i & & \\ & & \curvearrowright & & \\ p^{-1}(\operatorname{Gr} - \cup_{J < I} S_J) & \hookrightarrow & p^{-1}(\operatorname{Gr} - \operatorname{Sing} S_I) & \xrightarrow{j} & T^* \operatorname{Gr} \\ & & & & \\ i^*(\kappa_I) & \longleftarrow & j^*(\kappa_I) & \longleftarrow & \kappa_I \\ & & \parallel & & \\ [CS_I^o] & \longleftarrow & [CS_I^{\text{sm}}] & & \end{array}$$

□

Some advantages of κ'_I over κ_I are discussed in Sections 7 and 8.

7. ORTHOGONALITY

7.1. Orthogonality on Gr. Recall that $\epsilon_1, \dots, \epsilon_n$ is the standard basis of \mathbb{C}^n , and we used the standard flag to define the Schubert variety

$$S_I = \{W^k \subset \mathbb{C}^n : \dim(W^k \cap \text{span}(\epsilon_1, \dots, \epsilon_{i_a})) \geq a \text{ for } a = 1, \dots, k\} \subset \text{Gr}.$$

Considering the opposite flag, we may define the opposite Schubert variety

$$\check{S}_I = \{W^k \subset \mathbb{C}^n : \dim(W^k \cap \text{span}(\epsilon_{n+1-i_a}, \dots, \epsilon_n)) \geq a \text{ for } a = 1, \dots, k\} \subset \text{Gr}.$$

Consider the bilinear form on $H_{\mathbb{T}}^*(\text{Gr})$ defined by

$$(f, g) \mapsto \langle f, g \rangle = \int_{\text{Gr}} fg,$$

where the equivariant integral on Gr can be expressed via localization by

$$\int_{\text{Gr}} \alpha(t_1, \dots, t_k) = \sum_I \frac{\alpha(z_I)}{\prod_{u \in I, v \notin I} (z_v - z_u)}.$$

Here the sum runs for k -element subsets I of $\{1, \dots, n\}$. Note that the denominator is the equivariant Euler class of the tangent space to Gr at the fixed point corresponding to I .

Definition 7.1. For $I = \{i_1 < \dots < i_k\}$ define $\check{I} = \{(n+1) - i_k, (n+1) - i_{k-1}, \dots, (n+1) - i_1\}$.

It is well known in Schubert calculus that

$$\langle [S_I], [\check{S}_J] \rangle = \delta_{I, J}.$$

7.2. Orthogonality on $T^*\text{Gr}$. Our goal is to describe similar orthogonality relations involving equivariant classes of conormal bundles.

Consider the bilinear form on $H_{\mathbb{T}}^*(T^*\text{Gr}) = H_{\mathbb{T}}^*(\text{Gr})$ defined by

$$(f, g) \mapsto \langle\langle f, g \rangle\rangle = \int_{T^*\text{Gr}} fg,$$

where the equivariant integral on $T^*\text{Gr}$ is defined via localization by

$$\int_{T^*\text{Gr}} \alpha(t_1, \dots, t_k) = \sum_I \frac{\alpha(z_I)}{\prod_{u \in I, v \notin I} (z_v - z_u)(z_u - z_v + h)}.$$

Here again, the sum runs for k -element subsets I of $\{1, \dots, n\}$. Note that the denominator is the equivariant Euler class of the tangent space to $T^*\text{Gr}$ at the fixed point corresponding to I . This bilinear form takes values in the ring of rational functions in z_1, \dots, z_n and h . For more details, see [GRTV, Sect. 5.2].

In Section 4, starting with the Schubert variety S_I we defined the equivariant fundamental classes κ_I and κ'_I . Similarly, starting with the opposite Schubert variety \check{S}_I , we may define the opposite equivariant fundamental classes $\check{\kappa}_I$ and $\check{\kappa}'_I$. Arguments analogous to the ones in the sections above prove the following theorem.

Theorem 7.2. *We have*

$$(24) \quad \text{sgn}(\check{I})e_h(\gamma^* \otimes \gamma)\check{\kappa}'_I = \check{W}_I,$$

where

$$(25) \quad \check{W}_I(t_1, \dots, t_k) = h^k \text{Sym}_{S_k} \left(\prod_{a=1}^k \left(\prod_{u=1}^{i_a-1} (t_a - z_u) \prod_{u=i_a+1}^n (t_a - z_u + h) \prod_{b=a+1}^k \frac{t_a - t_b - h}{t_a - t_b} \right) \right)$$

is the “dual” weight function.

The following lemma is a special case of Theorem C.9 of [TV1].

Lemma 7.3. *For $A = \{a_1, \dots, a_k\} \subset \{1, \dots, n\}$ let $p(z_A)$ denote the substitution of z_{a_1}, \dots, z_{a_k} into the variables t_1, \dots, t_k of the polynomial p . For k -element subsets I, J of $\{1, \dots, n\}$ we have*

$$\sum_A \frac{W_I(z_A)\check{W}_J(z_A)}{\prod_{u \in A, v \in \bar{A}} (z_u - z_v)(z_u - z_v + h) \cdot \prod_{u, v \in A} (z_u - z_v + h)^2} = \delta_{I, J},$$

where the summation runs for all k -element subsets A of $\{1, \dots, n\}$.

Now we are ready to prove the orthogonality relations for fundamental classes of conormal bundles.

Theorem 7.4. *We have*

$$\ll \kappa'_I, \check{\kappa}'_J \gg = \delta_{I, J}.$$

Proof. Tracing back the definitions, as well as formulas (23) and (24), give

$$\ll \kappa'_I, \check{\kappa}'_J \gg = \text{sgn}(I) \text{sgn}(\check{J}) \sum_A \frac{W_I(z_A)\check{W}_J(z_A)}{\prod_{u \in A, v \notin A} (z_v - z_u)(z_u - z_v + h) \cdot (e_h(\gamma^* \otimes \gamma)(z_A))^2}.$$

According to Lemma 7.3 this further equals

$$\text{sgn}(I) \text{sgn}(\check{J}) (-1)^{k(n-k)} \delta_{I, J} = \delta_{I, J}.$$

□

8. R-MATRIX IN COHOMOLOGY

Recall that $\mathcal{R} = \mathbb{C}[z_1, \dots, z_n, h]((z_i - z_j + h)^{-1})_{i, j}$.

Proposition 8.1. *The following are bases of the free module $H_{\mathbb{T}}^*(\text{Gr}) \otimes \mathcal{R}$:*

- (1) the classes $[S_I]$ for $I \subset \{1, \dots, n\}$, $|I| = k$;
- (2) the classes W_I for $I \subset \{1, \dots, n\}$, $|I| = k$;
- (3) the classes Y_I for $I \subset \{1, \dots, n\}$, $|I| = k$;
- (4) the classes κ_I for $I \subset \{1, \dots, n\}$, $|I| = k$;
- (5) the classes κ'_I for $I \subset \{1, \dots, n\}$, $|I| = k$.

Proof. The Leray-Hirsch Theorem (e.g. [BT, Thm. 5.11]) implies that the $\mathbb{C}[z_1, \dots, z_n, h]$ -module $H_{\mathbb{T}}^*(\text{Gr})$ is free, with basis $[S_I]$. This implies (1). Statement (5) can be proved from Theorem 7.4 and some extra analysis of the denominators, or from the fact that the transition matrix from $[S_I]$ to κ'_I is upper triangular with diagonal entries

$$\prod_{a \in I, b \notin I, b < a} (z_a - z_b + h).$$

Details will be given elsewhere. The equivalence of (5) and (2) follows from equivariant localization and the fact $\pm e_h(\gamma^* \otimes \gamma)\kappa'_I = W_I$. The transition matrix between (2) and (3), as well as the transition matrix between (4) and (5) are upper triangular with 1's in the diagonal. \square

Recall that S_I was defined using the standard flag

$$\text{span}(\epsilon_1) \subset \text{span}(\epsilon_1, \epsilon_2) \subset \dots \subset \text{span}(\epsilon_1, \dots, \epsilon_n).$$

Let $\sigma \in S_n$ be a permutation. We may, redefine our geometric objects $S_I, \tilde{S}_I, [C\tilde{S}_I], \kappa_I, \kappa'_I$ using the complete flag

$$\text{span}(\epsilon_{\sigma(1)}) \subset \text{span}(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}) \subset \dots \subset \text{span}(\epsilon_{\sigma(1)}, \dots, \epsilon_{\sigma(n)}).$$

We call the resulting objects $S_I^\sigma, \tilde{S}_I^\sigma, [C\tilde{S}_I^\sigma], \kappa_I^\sigma, \kappa'_I{}^\sigma$.

Proposition 8.2. *We have*

$$\begin{aligned} \kappa_I^\sigma &= \kappa_I|_{z_u \mapsto z_{\sigma(u)}}, \\ \kappa'_I{}^\sigma &= \kappa'_I|_{z_u \mapsto z_{\sigma(u)}}. \end{aligned}$$

Proof. The statement follows from equivariant localization (see Section 4.3). \square

Comparing this result with Section 3.2 we obtain the following remarkable fact. Let σ be an elementary transposition in S_n . Then the geometrically defined action $\text{sgn}(I)\kappa'_I \mapsto \text{sgn}(I)\kappa'_I{}^\sigma$ on $H_{\mathbb{T}}^*(\text{Gr}) \otimes \mathcal{R}$ can be expressed by an R-matrix.

Example 8.3. Let $n = 2, k = 1$, and let σ be the transposition in S_2 . Then we have

$$\begin{pmatrix} h & z_2 - z_1 \\ \frac{z_2 - z_1 + h}{z_2 - z_1} & \frac{z_2 - z_1 + h}{h} \end{pmatrix} \cdot \begin{pmatrix} -\kappa'_{\{1\}} \\ \kappa'_{\{2\}} \end{pmatrix} = \begin{pmatrix} -\kappa'_{\{1\}}{}^\sigma \\ \kappa'_{\{2\}}{}^\sigma \end{pmatrix},$$

which can be directly verified by substituting

$$\begin{aligned} \text{sgn}(\{1\}) &= -1, & \kappa'_{\{1\}} &= \kappa_{\{1\}} = z_2 - t_1, \\ \text{sgn}(\{2\}) &= +1, & \kappa'_{\{2\}} &= \kappa_{\{2\}} + \kappa_{\{1\}} = (2t_1 - z_1 - z_2 + h) + (z_2 - t_1) = t_1 - z_1 + h. \end{aligned}$$

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