CONFORMAL BLOCKS AND EQUIVARIANT COHOMOLOGY

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To the memory of Vladimir Arnold

§1. Introduction. On multinomial coefficients

1.1. Let $\bar{V}$ be the m-dimensional vector representation of the complex Lie algebra $\mathfrak{g} = \mathfrak{gl}(m)$; consider its N-fold tensor product $V = \bar{V}^{\otimes N}$. The space V is graded by the set $\mathcal{P}_m(N)$ of m-tuples of natural numbers $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\sum \lambda_i = N$:

$$V = \oplus_{\lambda \in \mathcal{P}_m(N)} V_\lambda \tag{1.1}$$

(for the definition of this gradation see §2 below; the reader may try to figure it out as an exercise). The dimension of V_λ is given by the multinomial coefficient:

$$\dim V_\lambda = C_\lambda := \frac{(\sum \lambda_i)!}{\lambda_1! \ldots \lambda_m!}. \tag{1.2}$$

For example, if $m = 2$, the decomposition (1.1) corresponds the familiar formula $(1+1)^N = \sum_{i=0}^N \binom{N}{i}$.

On the other hand the same numbers appear as the dimensions of certain cohomology. Namely, let X_λ denote the variety of flags of linear subspaces $0 = L_0 \subset L_1 \subset \ldots \subset L_m = \mathbb{C}^N$ where $\dim L_i/L_{i-1} = \lambda_i$; this is a smooth complex projective variety of dimension

$$d_\lambda = \sum_{i<j} \lambda_i \lambda_j.$$

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It has only even complex cohomology; consider the total cohomology space $H^*(X_\lambda)=\oplus_{i=0}^{d_\lambda}H^{2i}(X_\lambda)$ where by definition $H^k(X):=H^k(X,\mathbb{C})$. Then

$$\dim H^*(X_\lambda)=C_\lambda. \tag{1.3}$$

To see (1.3) one can argue as follows, following Weil and Grothendieck. We consider X_λ as the set of $\mathbb{C}$-points of a $\mathbb{Z}$-scheme $\mathfrak{X}_\lambda$. Given a prime power $q=p^k$, the $\mathbb{F}_q$-points of it are by definition flags in $\mathbb{F}_q^N$, so their number is given by the q-multinomial coefficient

$$\#\mathfrak{X}_\lambda(\mathbb{F}_q)=C_\lambda(q):=\frac{[N]_q^!}{\prod_{i=1}^m[\lambda_i]_q^!} \tag{1.4}$$

where

$$[n]_q^!=\prod_{i=1}^n[i]_q,\ \ [n]_q=\frac{q^n-1}{q-1}. \tag{1.5}$$

Now we apply the Lefschetz fixed point formula in ℓ-adic cohomology ($\ell\neq p$) to the $\bar{\mathbb{F}}_q$-variety $X_{\lambda;q}:=\mathfrak{X}_\lambda\otimes_{\mathbb{Z}}\bar{\mathbb{F}}_q$:

$$\#\mathfrak{X}_\lambda(\mathbb{F}_q)=\sum_{i=0}^{d_\lambda}Tr(F_q;H^{2i}(\mathfrak{X}_{\lambda;q},\mathbb{Q}_\ell))=\sum_{i=0}^{d_\lambda}\dim H^{2i}(X_\lambda)q^i \tag{1.6}$$

where F_q is the Frobenius endomorphism; in our case it acts on $H^{2i}(\mathfrak{X}_{\lambda;q},\mathbb{Q}_\ell)$ as the multiplication by q^i; we also use the comparison theorem of complex and ℓ-adic cohomology. This implies that the limit of (1.4) when $q\to 1$ gives the left hand side of (1.3). (In the case of grassmanians ($m=2$) the above argument is contained in André Weil's classical paper [W]; of course the historical logic is opposite...)

1.2. Instead of usual cohomology one can consider the equivariant one, and use another incarnation of the Lefschetz formula — the Atiyah - Bott localization theorem. Namely, the complex torus $T=\mathbb{C}^{*N}$ acts naturally on X_λ with C_λ fixed points. The T-equivariant cohomology $H_T^*(X_\lambda)$ is a commutative $R:=H_T^*(*)$-algebra where the last ring may be identified with the polynomial algebra

$$R=\mathbb{C}[z_1,\ldots,z_N],\ \ z_i=c_1(M_i), \tag{1.7}$$

$M_i=\mathbb{C}$ with T acting through the i-th projection $T\longrightarrow\mathbb{C}^*$. As before, all cohomology is even. One can show $H_T^*(X_\lambda)$ is a free R-module of rank $\dim H^*(X_\lambda)$. The Atiyah - Bott theorem gives a basis of this module after certain localization.

Namely, consider the localized algebra

$$R'=R[D^{-1}],\ \ D=\prod_{i<j}(z_i-z_j). \tag{1.8}$$

Let $i_\lambda:\ X_\lambda^T \hookrightarrow X_\lambda$ be the inclusion of the set of fixed points. The Atiyah - Bott localization theorem [AB] says that the restriction map

$$i_\lambda^*:\ H_T^*(X_\lambda)_{R'} \longrightarrow H_T^*(X_\lambda^T)_{R'} \tag{1.9}$$

is an isomorphism[3]. We have $\# X_\lambda^T = C_\lambda$, whence in particular (1.3). One can say that in the first proof of (1.3) the dimensions have been deformed, whereas in the second proof the vector spaces are deformed.

In fact, we get more. One can define a bijection of X_λ^T with a certain basis in V_λ, so we get an isomorphism of two free R'-modules of rank C_λ

$$\phi_\lambda:\ V_{\lambda;R'} \xrightarrow{\sim} H_T^*(X_\lambda)_{R'} \tag{1.10}$$

by identifying their respective bases. Using these bases one defines a canonical element

$$y_\lambda \in V_\lambda \otimes H_T^*(X_\lambda)_{R'}$$

which we may integrate along X_λ to obtain an element $p_\lambda \in V_\lambda \otimes R'$ which we may interprete as a rational V_λ-valued function in z_i's.

The first observation of the present note (see §2) is that p_λ coincides with the element constructed in [RV] and thus satisfies all the nice properties of the last element. In particular if λ is such that the corresponding "bundle of conformal blocks" is of rank 1, for example $\lambda = (a, a, \ldots, a)$, then p_λ satisfies the Knizhnik - Zamolodchikov differential equations (this is not true for a general λ). It seems also that the element y_λ before the integration has some remarkable properties.

One possible advantage of this construction is that it works for any other cohomology theory satisfying Atiyah - Bott: for example one can replace the usual cohomology by K-theory; in this case one should obtain a "q-difference" version of the picture.

1.3. Secondly, we deal with the situation of rank 1 conformal blocks. In that case we have two natural generating sections of this bundle: the first one coming from the equivariant cohomology and the second one given by a hypergeometric integral from [SV]. These two sections are proportional; the proportionality coefficient ("normalisation constant") is given as usual by a "period": a Selberg type integral, we compute these integrals in §4. These two ways to define conformal blocks are somewhat similar to two ways of defining the Givental hypergeometric functions connected with quantum cohomology of flag spaces: the first one via the integration of a certain canonical element in the cohomology of a quasimaps' space (cf. [G1, Br]), the second — mirror dual — one, via stationary phase integrals, cf. [G2]. This analogy with mirror symmetry was the starting point of our reflections.

[3] for an R-module M, $M_{R'} := M \otimes_R R'$

Finally in the last Section, §5, we define geometrically an action of the Lie algebra of positive currents $\mathfrak{gl}(m)[t]$ on the equivariant cohomology $H^*_T(X_{m,N})$ where $X_{m,N} = \coprod_{\lambda\in\mathcal{P}_m(N)} X_\lambda$ in such a way that the isomorphisms (1.10), summed over all λ, become $\mathfrak{gl}(m)[t]$-equivariant. This action seems to be closely related to the actions studied by Ginzburg, Nakajima and others, cf. 5.4.

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§2. Weight spaces and fixed points

2.1. *The gradation in V.* We identify the fundamental representation $\bar{V}$ of $\mathfrak{g} = \mathfrak{gl}(m)$ with the component of degree 1 in the polynomial algebra $\mathbb{C}[y_1, \ldots, y_m]$ where $\deg y_i = 1$, with the obvious action of $\mathfrak{g}$.

More generally, the $\mathfrak{g}$-module $V = \bar{V}^{\otimes N}$ will be identified with a subspace in the polynomial ring in mN variables y_{ij}, $1 \leq i \leq m,\ 1 \leq j \leq N$ spanned by all monomials

$$y^A = \prod_{i,j} y_{ij}^{a_{ij}}$$

which for each $j = 1, \ldots, N$ contain exactly one character y_{ij}. In other words, the basis $\{y^A\}$ is in one-to-one correspondence with the set $M(m, N)$ of $m \times N$ matrices $A = (a_{ij})$ whose entries are zeros or ones, which contain exactly one 1 in each column.

Given such a matrix A, we can do otherwise, and count the number of 1's in its rows. Set $\lambda(A) = (\lambda_1(A), \ldots, \lambda_m(A))$ where

$$\lambda_i(A) = \#\{j|\ a_{ij} = 1\}.$$

Obviously $\lambda(A) \in \mathcal{P}_m(N)$; we set $M(\lambda) := \{A \in M(m, N)|\ \lambda(A) = \lambda)$. In each set $M(\lambda)$ we define a point $M_\lambda \in M(\lambda)$ to be the matrix with

$$(M_\lambda)_{ij} = 1 \text{ if } \mu_{i-1} \leq j \leq \mu_i$$

where $\mu_i := \sum_{k=1}^{i} \lambda_k$.

For example, for $\lambda = (1, 1, \ldots, 1)$, $M(\lambda) =$ the set of permutation matrices, $M_\lambda =$ the unity matrix.

The symmetric group in N letters S_N acts on $M(\lambda)$ by permutation of columns; this action is transitive and the stabiliser of M_λ is the subgroup
$S_\lambda := S_{\lambda_1} \times \ldots \times S_{\lambda_m}$, which gives rise to a bijection

$$M(\lambda) \cong S(\lambda) := S_N/S_\lambda. \tag{2.1}$$

It follows that $\#M(\lambda) = C_\lambda$.

Another useful set in bijection with $M(\lambda)$ is defined as follows. Denote $[N] = \{1, 2, \ldots, N\}$. Define $\Pi(\lambda)$ as the set of all maps $\pi : [N] \longrightarrow [m]$ such that $\#\pi^{-1}(i) = \lambda_i$ for all i. Given a matrix $M = (m_{ij}) \in M(\lambda)$ let us associate to it a map π as follows: we set $\pi(j) = i$ such that $m_{ij} = 1$; obviously $\pi \in \Pi(\lambda)$ and we've got a bijection

$$M(\lambda) \cong \Pi(\lambda). \tag{2.1a}$$

Given $\lambda \in \mathcal{P}_m(N)$, we define $V_\lambda \subset W$ to be the subspace spanned by the monomials y^A with $A \in M(\lambda)$.

2.2. *Cohomology of flag varieties.* Let $\lambda \in \mathcal{P}_m(N)$. Recall the flag variety X_λ of dimension d_λ. Over it we have the tautological flag of vector bundles

$$0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_{m-1} \subset \mathcal{L}_m = \mathcal{O}_{X_\lambda}^N.$$

Set $\mathcal{M}_i := \mathcal{L}_i/\mathcal{L}_{i-1}$; these are vector bundles of dimension λ_i.

The cohomology ring $H^*(X_\lambda)$ is generated as a graded $\mathbb{C}$-algebra by the Chern classes $c_{ij} := c_j(\mathcal{M}_i) \in H^{2j}(X_\lambda),\ 1 \leq i \leq m,\ 1 \leq j \leq \lambda_i$, the ideal of relations is generated by N relations which follow from the identity

$$\prod_{i=1}^{m} (1 + \sum_{j=1}^{\lambda_j} c_{ij} t^j) = 1 \tag{2.2}$$

(i.e. we equate to 0 all the coefficents of the t-polynomial on left, except the zeroth one).

More generally, the T-equivariant cohomology may be described exactly in the same manner. Recall the coefficient ring $R = H_T^*(pt) = \mathbb{C}[z_1, \ldots, z_N]$. As a graded R-algebra $H_T^*(X_\lambda)$ is generated by the Chern classes $c_{T;ij} := c_j(\mathcal{M}_i) \in H_T^{2j}(X_\lambda),\ 1 \leq i \leq m,\ 1 \leq j \leq \lambda_i$, the ideal of relations is generated by N relations which follow from the identity

$$\prod_{i=1}^{m} (1 + \sum_{j=1}^{\lambda_i} c_{T;ij} t^j) = \prod_{n=1}^{N} (1 + z_n t). \tag{2.3}$$

It follows from this description that $H_T^*(X_\lambda)$ is a free graded R-module of rank $\dim H^*(X_\lambda)$.

2.3. *Fixed points.* The action of T on X_λ has a finite set of fixed points $X_\lambda^T \subset X_\lambda$. To describe them explicitly, let $\{e_1, \ldots, e_N\}$ be the standard basis in $\mathbb{C}^N$. Let $F_e = (0 \subset F_1 \subset \ldots \subset F_{m-1} \subset F_m = \mathbb{C}^N) \in X_\lambda$ be the flag with F_i being the subspace spanned by $e_1, \ldots, e_{\mu_i}$ (recall that $\mu_i = \sum_{j=1}^{i} \lambda_i$). Then F_e is fixed under the action of T.

The symmetric group S_N acts on $\mathbb{C}^N$ by permuting the elements of the standard basis, so it acts on the set of all flags. For all $\sigma \in S_N$ $F_\sigma := \sigma(F_e)$ belongs to X_λ^T and in such a way we get all fixed points. The stabiliser of F_e coincides with S_λ, so the mapping $\sigma \mapsto F_\sigma$ induces a bijection

$$S(\lambda) \cong X_\lambda^T, \tag{2.4}$$

cf. (2.1)

For $w \in S(\lambda)$ we denote by x_w the corresponding fixed point. The tangent space $T_w := T_{X_\lambda, x_w}$ inherits the T-action; we will be interested in its Euler (top Chern) class:

$$e_w := c_{d_\lambda}(T_w) \in H_T^{2d_\lambda}(pt).$$

The explicit formula is as follows. Let $\pi_w \in \Pi(\lambda)$, $\pi_w : [N] \longrightarrow [m]$, be the map corresponding to w (cf. (2.1a)). Then

$$e_w = \prod_{i>j} \prod_{a \in \pi_w^{-1}(i), b \in \pi_w^{-1}(j)} (z_a - z_b). \tag{2.5}$$

Let i_w denote the inclusion $i_w : x_w \hookrightarrow X_\lambda$; it is compatible with the T-action.

Define the elements $y'_w := i_{w*}(1) \in H^{2d_\lambda}(X_\lambda)$. The explicit formula for them is as follows. For each $i \in [m]$ let $\gamma_{ij}, j \in [\lambda_i]$, denote the *Chern roots* of $\mathcal{M}_i$ — the formal symbols such that $c_j(\mathcal{M}_i) = \sigma_j(\gamma_{i1}, \ldots, \gamma_{i\lambda_i})$, the elementary symmetric function. Let π_w be as above. Then

$$y'_w = \prod_{i>j} \prod_{a=1}^{\lambda_i} \prod_{b \in \pi_w^{-1}(j)} (\gamma_{ia} - z_b). \tag{2.6}$$

Here is the main property of these elements which charterizes them:

$$i_w^* y'_{w'} = e_w \delta_{ww'}. \tag{2.7}$$

The restriction map i_w^* acts as follows: if $\pi_w^{-1}(i) = \{k_1, \ldots, k_{\lambda_i}\}$ with $k_1 < \ldots < k_{\lambda_i}$ then

$$i_w^*(\gamma_{ij}) = z_{k_j}.$$

The composition $i_w^* i_{w*}$ equals the multiplication by e_w.

Recall the localized ring R'; all e_w become invertible in R'. The Atiyah - Bott localization theorem says that the restriction map is an isomorphism:

$$i^* : H_T^*(X_\lambda)' := H_T^*(X_\lambda)_{R'} \longrightarrow H_T^*(X_\lambda^T)' = \oplus_{w \in S(\lambda)} R' \cdot 1_w.$$

The elements $y_w := y'_w / e_w$ form a basis of the free R' module $H_T^*(X_\lambda)'$, cf. [AB] (the case $X_{(1,\ldots,1)} = G/B$ is discussed in [S]).

Note that the explicit formula for y_w written using (2.5) and (2.6) is very similar to the master function of a hypergeometric integral connected with a KZ equation.

We shall also need the map $\int_{X_\lambda} = p_* : \ H_T^*(X_\lambda) \longrightarrow H_T^*(pt)$; we have for it $\int_{X_\lambda} y'_w = 1$.

2.3.1. *Example.* For $X_\lambda = \mathbb{P}^{m-1}$ (i.e. $\lambda = (m-1,1)$), $H_T^*(\mathbb{P}^{m-1}) = \mathbb{C}[x, z_1, \ldots, z_m]/(\prod(x-z_i))$ where $x = c_1(\mathcal{O}(1))$. The action of T has m fixed points $x_i,\ i = 1, \ldots, m$; we have $y'_i = \prod_{j\neq i}(x-z_j)$, $e_i = \prod_{j\neq i}(z_j - z_i)$, $y_i = y'_i/e_i$ is nothing else but the i-th Lagrange interpolation polynomial.

2.4. *The canonical element.* Let us identify $M(\lambda)$ with $S(\lambda)$ by means of the bijection defined above. So for each $w \in S(\lambda)$ we will have the corresponding element in the weight subspace $y^w \in V_\lambda$ from 2.1 on the one hand, and the element $y_w \in H_T^*(X_\lambda)'$ on the other hand.

Consider the sum

$$y_\lambda = \sum_{w\in S(\lambda)} y^w \otimes y_w \in V_\lambda \otimes H_T^*(X_\lambda)'.$$

After integration over X_λ we get an element

$$p_\lambda := \int_{X_\lambda} y_\lambda = \sum_{w\in S(\lambda)} \frac{y^w}{e_w} \in V_\lambda \otimes R' = V_\lambda[z_1, \ldots, z_N][D^{-1}].$$

We may consider $p_\lambda = p_\lambda(z)$ as a rational function in variables $z_1, \ldots, z_N$ with poles along the diagonals, taking values in V_λ.

To compare this element with that from [RV], let us recall some notation from there. Let $\mathcal{J}$ denote the set of all decompositions of the set $[N]$ into a disjoint union

$$[N] = \coprod_{j=1}^{m} I_j$$

with $\sharp I_j = \lambda_j$. We set

$$R(z_{I_1}|z_{I_2}|\ldots|z_{I_m}) = \prod_{i<j}\ \prod_{a\in I_i, b\in I_j} (z_a - z_b).$$

Define

$$P_z(\lambda) = \sum_{\mathcal{J}} \frac{\prod_{j=1}^m \prod_{a\in I_j}\ y_{ia}}{R(z_{I_1}|z_{I_2}|\ldots|z_{I_m})},$$

cf. [RV], Defintion 4.1.

2.5. *Theorem. The element $p_\lambda(z)$ coincides with the element $P_z(\lambda)$.*

This is evident after identifying the Euler classes e_w with the elements $R(z_{I_1}|\dots|z_{I_m})$.

Therefore, $p_\lambda(z)$ satisfies all properties proven in *op. cit.* Let us list these properties. Let $\{e_{ij},\ 1\le i,j\le m\}$ be the standard basis of $\mathfrak{g}$. For $x\in\mathfrak{g}$ we shall denote by $x^{(i)}$ the operator on $\bar V^{\otimes N}$ acting as x on the i-th factor and identity on other factors.

We define the *subspace of singular vectors*

$$V^s=\{y\in V|\ e_{ij}y=0,\ 1\le i<j\le m\}.$$

Let us call $\lambda=(\lambda_1,\dots,\lambda_m)\in\mathcal{P}_m(N)$ *a partition* if $\lambda_1\ge\dots\ge\lambda_m$; we denote by $\mathcal{Q}_m(N)\subset\mathcal{P}_m(N)$ the subset of all partitions. Denote $V^s_\lambda=V^s\cap V_\lambda$. We have

$$V^s=\oplus_{\lambda\in\mathcal{Q}_m(N)}V^s_\lambda.$$

We denote $Z=\{(z_1,\dots,z_N)\in\mathbb{C}^N|\ z_i\ne z_j \text{ for } i\ne j\}$. For $z=(z_1,\dots,z_N)\in Z$ we denote by $e(z)$ the operator

$$e(z)=\sum_{i=1}^N z_ie^{(i)}_{1m}$$

acting on V. Given a partition $\lambda=(\lambda_1,\dots,\lambda_m)$ and a natural $\ell\ge\lambda_1-\lambda_m$ (*the level*), one defines *the space of conformal blocks of level* ℓ

$$CB^\ell_\lambda(z)=V^s_\lambda\cap\mathrm{Ker}\ e(z)^{\ell-\lambda_1+\lambda_m+1}.$$

2.6. *Theorem. For all $z\in Z$ and $\lambda\in\mathcal{Q}_m(N)$*

(a) $p_\lambda(z)\in V^s$.

(b) If $\lambda_1>\lambda_m$ then $e(z)p_\lambda(z)=0$; if $\lambda_m=\lambda_1$ then $e(z)^2p_\lambda(z)=0$.

(c) Suppose that $\lambda_1-\lambda_m\le 1$ (in this case $\dim CB^1_\lambda(z)=1$). Then $p_\lambda(z)$ satisfies to the system of Knizhnik - Zamolodchikov differential equations

$$\frac{\partial p_\lambda(z)}{\partial z_i}=\frac{1}{m+1}\sum_{j\ne i}\frac{\pi_{ij}-m\cdot Id}{z_i-z_j}p_\lambda(z)$$

where $\pi\in End(\bar V\otimes\bar V)$, $\pi(x\otimes y)=y\otimes x$ (note that $\pi=\sum_{a<b}\ e_{ab}\otimes e_{ba}$), π_{ij} means the transposition of the i-th and j-th factors.

§3. Hypergeometric solutions

In this section we recall the main construction from [SV].

3.1. *Master function.* Let $\mathfrak{g}$ be a simple complex Lie algebra of rank r. We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, the generators e_i (resp. f_i) of $\mathfrak{n}_+$ (resp. of $\mathfrak{n}_-$), simple roots $\alpha_i \in \mathfrak{h}^*$, $i = 1, \ldots, r$.

Given a nonzero complex number κ, N weights $\Lambda_j \in \mathfrak{h}^*,\ j = 1, \ldots, N$ and a weight

$$\mu = \sum_{j=1}^{N} \Lambda_j - \sum_{i=1}^{r} n_i \alpha_i \tag{3.1}$$

where all $n_i \in \mathbb{N}$, we associate to these data a maultivalued *master function* $\Phi(t, z)$. It depends on two groups of variables: $z = (z_1, \ldots, z_N)$ and $t = (t_{ia},\ 1 \le i \le r,\ 1 \le a \le n_i)$, and is defined by

$$\Phi(t,z) = \prod_{i<j} \prod_{a,b} (t_{ia} - t_{jb})^{(\alpha_i,\alpha_j)/\kappa} \prod_{i} \prod_{a<b} (t_{ia} - t_{ib})^{(\alpha_i,\alpha_i)/\kappa} \cdot$$

$$\cdot \prod_{i,a} \prod_{j} (t_{ia} - z_j)^{-(\alpha_i,\Lambda_j)/\kappa} \cdot \prod_{j<k} (z_j - z_k)^{(\Lambda_j,\Lambda_k)/\kappa}. \tag{3.2}$$

3.2. *Accompanying rational functions.* Let U denote the universal enveloping algebra of the free Lie algebra in generators $\tilde{f}_i,\ 1 \le i \le r$, i.e. the free associative $\mathbb{C}$-algebra in generators $\tilde{f}_i$. Consider its tensor power $U^{\otimes N}$.

This algebra is $\mathbb{N}^r$-graded. Namely, for $\bar{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we denote by $(U^{\otimes N})_{\bar{n}} \subset U^{\otimes N}$ the linear subspace generated by all monomials

$$m = \prod \tilde{f}_j \otimes \ldots \otimes \prod \tilde{f}_k \tag{$*$}$$

which contain n_i times the character $\tilde{f}_i$. We denote $S_{\bar{n}} = \prod_{i=1}^{r}\ S_{n_i}$.

We associate to m a rational function $\psi(m) = \psi(m; t, z)$; here $z = (z_1, \ldots, z_N)$ and $t = (t_{ia})$ is a group of variables as in 3.1. Note that $S_{\bar{n}}$ acts in the evident way on the set $\{t_{ia}\}$; our functions $\psi(m)$ will be symmetric with respect to this action.

First we define their "nonsymmetric" version, rational functions $\tilde{\psi}(m)$.

By definition, $\tilde{\psi}(1 \otimes \ldots \otimes 1) = 1$. We procede the definition by induction on the length of $m := \sum n_i$. We denote by $f_j^{(n)} \in \mathrm{Hom}(U^{\otimes N}, U^{\otimes N})$ the left multiplication by f_j on the n-th factor.

Let $m' = \tilde{f}_j^{(n)} m'$ where $\tilde{\psi}(m')$ is already defined; let $m'_n \in U$ be the n-th factor of m'. We set:

$$\tilde{\psi}(m) = \frac{1}{t_{jn_j} - t} \cdot \tilde{\psi}(m')$$

where $t = z_n$ if $m'_n = 1$, $t = t_{kn_k}$ if $m'_n = \tilde{f}_k y,\ k \neq j$ and $t = t_{j,n_j-1}$ if $m'_n = \tilde{f}_j y$.

Finally we set

$$\psi(m) = \sum_{\sigma \in S_{\bar{n}}} \sigma \tilde{\psi}(m)$$

where the group $S_{\bar{n}}$ acts on functions $\tilde{\psi}(m)$ through variables t_{ia}.

3.3. *Canonical element.* For a fixed $\bar{n} = (n_1, \ldots, n_r)$ let $\{m_\alpha\}_{\alpha \in A}$ be the basis of the space $(U^{\otimes N})_{\bar{n}}$ consisting of monomials of the form (*). To each m_α corresponds a rational function $\psi(m_\alpha) \in \mathbb{C}(t, z)$; denote

$$\omega_\alpha := \psi(m_\alpha) \Phi(t, z) dt_{1n_1} \wedge \ldots \wedge dt_{rn_r}.$$

This is a maultivalued differential form of degree $\ell(n) := \sum_{i=1}^r n_i$ on the complex affine space with corrdinates t_{ia}, z_n with logarithmic singularities along the hyperplanes $t_{ia} = t_{jb},\ t_{ia} = z_n$. Let us denote the space of such forms $\Omega^{\ell(\bar{n})}(t, z)$ and consider an element

$$\tilde{\delta} := \sum_{\alpha \in A} m_\alpha \otimes \omega_\alpha \in (U^{\otimes N})_{\bar{n}} \otimes \Omega^{\ell(\bar{n})}(t, z).$$

Given N weights $\Lambda_1, \ldots, \Lambda_N$ as above, let $L(\Lambda_j)$ denote the irreducible $\mathfrak{g}$-module of highest weight Λ_j, with the vacuum vector 1_j. Let $\pi_j :\ U \longrightarrow L(\Lambda_j)$ be the unique epimorphism such that $\pi_j(1) = 1_j$ and $\pi_j(\tilde{f}_i x) = f_i \pi_j(x)$ for all i and $x \in U$; taking their tensor product we get an epimorphism

$$\pi :\ U^{\otimes N} \longrightarrow L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_N)$$

which maps $(U^{\otimes N})_{\bar{n}}$ onto $(L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_N))_\mu$ where $\mu = \sum_j\ \Lambda_j - \sum_i\ n_i \alpha_i$ as in (3.1).

We set

$$\delta := \pi(\tilde{\delta}) = \sum_{\alpha \in A} \pi(m_\alpha) \otimes \omega_\alpha \in (L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_N))_\mu \otimes \Omega^{\ell(\bar{n})}(t, z).$$

Finally, if $C = \{C(z)\}$ is a family of homology cycles with coefficients in the local system dual to (3.2) which is horizontal with respect to Gauss-Manin connection along z, we can integrate δ along C and get a maultivaluedd function

$$\phi(z) := \int_{C(z)} \delta \in (L(\Lambda_1) \otimes \ldots \otimes L(\Lambda_N))_\mu.$$

The main result of [SV] and [FSV] says that $\phi(z)$ is a section of the subbundle of conformal blocks (in particular, for each $\phi(z)$ is a singular vector for each z), and this section is horizontal with respect to the KZ connection, i.e. it satisfies to the following system of linear differential equations:

$$\kappa\frac{\partial\phi}{\partial z_j}=\sum_{i\neq j}\frac{\Omega_{ij}}{z_j-z_i}\phi(z),\ j=1,\ldots,N \tag{3.3}$$

Here we note that our simple Lie algebra $\mathfrak{g}$ comes equipped with a canonical $\mathfrak{g}$-invariant *Casimir* element $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$ and by definition $\Omega_{ij}\in\mathrm{End}(L(\Lambda_1)\otimes\ldots\otimes L(\Lambda_m))$ denotes an endomorphism acting as Ω on the product of the i-th and the j-th factors, and as the identity on the others.

So we get a family of solutions of KZ equations numbered by the above homology cycles.

3.4. Note that given $\phi(z)$ satisfying (3.3) and any symmetric $m\times m$ complex matrix (c_{ij}), if we put

$$\psi(z)=\prod_{i<j}(z_i-z_j)^{c_{ij}}\phi(z)$$

then this new function satisfies the differential equations

$$\kappa\frac{\partial\psi}{\partial z_j}=\sum_{i\neq j}\frac{\Omega_{ij}+c_{ij}\mathrm{Id}}{z_j-z_i}\psi(z),\ j=1,\ldots,N \tag{3.4}$$

The systems (3.3) and (3.4) are called *gauge equivalent*. We will use this simple remark several times.

3.5. Below we will use the KZ equations for the reductive Lie algebra $\mathfrak{gl}(m)$ which look exactly as in (3.3) but now $\phi(z)\in\otimes_{i=1}^N L_i$ where L_i are some representions or $\mathfrak{gl}(m)$ and Ω_{ij} is defined starting from

$$\Omega=\sum_{a,b=1}^{m}e_{ab}\otimes e_{ba}\in\mathfrak{gl}(m)\otimes\mathfrak{gl}(m)$$

On the tensor square of the vector representation $\bar V\otimes\bar V$ this Ω acts as $\Omega(x\otimes y)=\pi(x\otimes y):=y\otimes x$.

On the other hand, the standard Casimir for the simple Lie algebra $\mathfrak{sl}(m)$ acts on $\bar V\otimes\bar V$ as $\pi-m^{-1}\cdot\mathrm{Id}$.

It follows that if all $L_i=\bar V$ then the KZ equations for $\mathfrak{gl}(m)$ and for $\mathfrak{sl}(m)$ are gauge equivalent.

§4. Selberg integrals

Here we specify the previous construction to our case.

4.1. We have $\mathfrak{g} = \mathfrak{sl}(m),\ r = m-1,\ e_i = e_{i,i+1},\ f_i = e_{i+1,i}$. Let us reconcile our present notation with that from 2.1.

We consider the vector representation $\bar{V} = L(\omega_1)$ where ω_1 is the first fundamental weight. It has a basis $\{y_1, \ldots, y_m\}$ where y_1 is a vacuum vector, i.e. $e_i y_1 = 0$ for all i, and $y_{j+1} = f_j y_j,\ j = 1, \ldots, m-1$. In other words,

$$y_j = f_{j-1} f_{j-2} \ldots f_1 y_1. \tag{4.1}$$

(Alternatively, we can remark that $y_j = e_{j1} y_1$ and

$$e_{j1} = [e_{j,j-1}, [e_{j-1,j-2}, \ldots [e_{32}, e_{21}] \ldots]).$$

(a) *The case G/B.*

4.2. Consider the weight subspace $V_\lambda = (\bar{V}^{\otimes m})_\lambda$ where $\lambda = (1, \ldots, 1) \in \mathcal{P}_m(m)$; its dimension is $m!$ and it admits a basis $\{y_\sigma\},\ \sigma \in S_m$, where $y_\sigma = y_{\sigma(1)} \otimes \ldots \otimes y_{\sigma(m)}$.

The subspace of singular vectors

$$V_\lambda^s = \{x \in V_\lambda |\ e_i x = 0 \text{ for all } i\}$$

is one-dimensional and is spanned by the vector

$$w = \sum_{\sigma \in S_m} (-1)^{|\sigma|} y_\sigma.$$

Let us consider the following V_λ^s-valued Knizhnik-Zamolodchikov equation

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\pi_{ij} - \mathrm{Id}}{z_i - z_j} \Psi,\ 1 \leq i \leq m \tag{4.2}$$

where $\Psi(z) = \psi(z) w \in V_\lambda^s$ and π_{ij} is the permutation of i-th and j-th factor as usual (it is gauge equivalent to the standard V_λ^s-valued KZ equation, cf. 3.5). It has a solution

$$\psi(z_1, \ldots, z_m) = \prod_{1 \leq i < j \leq m} (z_j - z_i)^{-2/\kappa}$$

4.3. We see that after writing the basis vectors in the form (4.1) the weight space V_λ has $m-1$ characters f_1, $m-2$ characters f_2, ..., 1 character f_{m-1}. So in the corresponding hypergeometric integral there is $d := m(m-1)/2$ coordinates t.

Fix on $\mathbb{C}^d$ coordinates

$$t = (t_1^{(1)}, \dots, t_{m-1}^{(1)}, t_1^{(2)}, \dots, t_{m-2}^{(2)}, \dots, t_1^{(m-1)}),$$

the holomorphic volume form

$$\begin{aligned} dt &= dt^{(m-1)} \wedge dt^{(m-2)} \wedge \dots \wedge dt^{(1)} \\ &= dt_1^{(m-1)} \wedge dt_1^{(m-2)} \wedge dt_2^{(m-2)} \wedge \dots \wedge dt_1^{(1)} \wedge \dots \wedge dt_{m-1}^{(1)}, \end{aligned}$$

and the master function

$$\begin{aligned} \Phi(t,z) &= \prod_{i=1}^{m-1} \prod_{j=1}^{m} (t_i^{(1)} - z_j)^{-1} \prod_{1 \le i < j \le m-1} (t_j^{(1)} - t_i^{(1)})^2 \times \\ & \prod_{i=1}^{m-2} \prod_{j=1}^{m-1} (t_i^{(2)} - t_j^{(1)})^{-1} \prod_{1 \le i < j \le m-2} (t_j^{(2)} - t_i^{(2)})^2 \cdots \prod_{i=1}^{1} \prod_{j=1}^{2} (t_i^{(m-1)} - t_j^{(m-2)})^{-1}. \end{aligned}$$

Actions of symmetric groups

Let the group S_m act on functions of t, z by permuting the variables $z_1, \dots, z_m$,

$$(\sigma g)(t, z_1, \dots, z_m) = g(t, z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(m)}).$$

Similarly let the group S_{m-1} act on functions of t, z by permuting the variables $t_1^{(1)}, \dots, t_{m-1}^{(1)}$ and so on. For a function $g(t,z)$ define the symmetrizations

$$\mathrm{Sym}_z g(t,z) = \sum_{\sigma \in S_m} (\sigma h)(t,z), \qquad \mathrm{Sym}_{t^{(1)}} g(t,z) = \sum_{\sigma \in S_{m-1}} (\sigma h)(t,z), \qquad \text{and so on.}$$

Weight functions

Set

$$\begin{aligned} g(t,z) &= \left[\prod_{i=1}^{m-1} (t_i^{(1)} - z_{m-i+1}) \prod_{i=1}^{m-2} (t_i^{(2)} - t_i^{(1)}) \prod_{i=1}^{m-3} (t_i^{(3)} - t_i^{(2)})^{-1} \cdots \prod_{i=1}^{1} (t_i^{(m-1)} - t_i^{(m-2)}) \right]^{-1}, \\ \omega(t,z) &= \mathrm{Sym}_{t^{(1)}} \mathrm{Sym}_{t^{(2)}} \dots \mathrm{Sym}_{t^{(m-2)}} g(t,z). \end{aligned}$$

For any $\sigma \in S_m$, define

$$\omega_\sigma(t,z) = (\sigma \omega)(t,z).$$

Example. For $m = 2$, $\sigma = id \in S_2$ and $\sigma' = (12) \in S_2$ we have

$$\omega_\sigma = (t_1^{(1)} - z_2)^{-1}, \qquad \omega_{\sigma'} = (t_1^{(1)} - z_1)^{-1}.$$

For $m=3$, $\sigma = id \in S_3$ and $\sigma' = (13) \in S_2$ we have

$$\begin{aligned}
\omega_\sigma &= [(t_1^{(1)}-z_3)(t_2^{(1)}-z_2)(t_1^{(2)}-t_1^{(1)})]^{-1} + [(t_2^{(1)}-z_3)(t_1^{(1)}-z_2)(t_1^{(2)}-t_2^{(1)})]^{-1},\\
\omega_{\sigma'} &= [(t_1^{(1)}-z_1)(t_2^{(1)}-z_2)(t_1^{(2)}-t_1^{(1)})]^{-1} + [(t_2^{(1)}-z_1)(t_1^{(1)}-z_2)(t_1^{(2)}-t_2^{(1)})]^{-1}.
\end{aligned}$$

Define

$$\omega(t,z) = \sum_{\sigma \in S_m} \omega_\sigma(t,z) y_\sigma .$$

This is a V_λ-valued function of t, z.

4.4. *Integrals.* Consider the V_λ-valued differential d-form

$$\Phi(t,z)^{1/\kappa}\omega(t,z)dt.$$

Let $\delta(z)$ be a flat section of the homological bundle associated with this differential form, see [SV, V3]. Then by [SV] the V_λ-valued function

$$I(z) = \int_{\delta(z)} \Phi(t,z)^{1/\kappa}\omega(t,z)dt$$

takes values in the space of singular vectors V_λ^s and is a solution of the KZ equations.

4.5. *Gelfand-Zetlin cycle.*

For real $z=(z_1,\dots,z_m)$ with $z_1<z_2<\dots<z_m$ define a d-dimensional cell

$$\gamma_m = \gamma_m(t;z) = \gamma_m(t^{(1)},\dots,t^{(m-1)},z)$$

in $\mathbb{C}^d$ by the conditions

$$\begin{gathered}
t_1^{(m-2)} < t_1^{(m-1)} < t_2^{(m-2)},\\
t_1^{(m-3)} < t_1^{(m-2)} < t_2^{(m-3)} < t_2^{(m-2)} < t_3^{(m-3)},\\
\dots\\
t_1^{(1)} < t_1^{(2)} < t_2^{(1)} < \dots < t_{m-2}^{(1)} < t_{m-2}^{(2)} < t_{m-1}^{(1)},\\
z_1 < t_1^{(1)} < z_2 < \dots < z_{m-1} < t_{m-1}^{(1)} < z_1.
\end{gathered}$$

We denote by $\gamma_m^{m-1}(t^{(m-2)})$ the set of all points $t^{(m-1)}$ satisfying the conditions in the first line of these inequalities. We denote by $\gamma_m^{m-2}(t^{(m-3)})$ the set of all points $t^{(m-2)}$ satisfying the conditions in the second line of these inequalities and so on until we denote by $\gamma_m^1(z)$ the set of all points $t^{(1)}$ satisfying the conditions in the last line of these inequalities.

The Gelfand - Zetlin cell $\gamma_m(t^{(1)},\dots,t^{(m-1)};z)$ has an important *factorization property*:

$\gamma_m(t^{(1)},\ldots,t^{(m-1)};z)$ consists of points $(t^{(1)},\ldots,t^{(m-1)})$ such that $t^{(1)}$ lies in $\gamma^1_m(z)$ and $(t^{(2)},\ldots,t^{(m-1)})$ lies in $\gamma_{m-1}(t^{(2)},\ldots,t^{(m-1)};t^{(1)})$.

4.6. Consider the iterated integral over $\gamma_m(t;z)$,

$$I_\kappa(z)=\int_{\gamma^1_m(z)}dt^{(1)}\int_{\gamma^2_m(t^{(1)})}dt^{(2)}\ldots\int_{\gamma^{m-1}_m(t^{(m-2)})}dt^{(m-1)}\Phi(t,z)^{1/\kappa}\omega(t,z).$$

The function $\Phi^{1/\kappa}$ is multivalued. In order to define the integral (apart from its possible divergence) we need to choose a section over $\gamma_m(t,z)$ of the local system associated with the function $\Phi^{1/\kappa}$. We choose the section

$$\prod_{i=1}^{m-1}\prod_{j=1}^{m}|t^{(1)}_i-z_j|^{-1/\kappa}\prod_{1\le i<j\le m-1}|t^{(1)}_j-t^{(1)}_i|^{2/\kappa}\times$$
$$\prod_{i=1}^{m-2}\prod_{j=1}^{m-1}|t^{(2)}_i-t^{(1)}_j|^{-1/\kappa}\prod_{1\le i<j\le m-2}|t^{(2)}_j-t^{(2)}_i|^{2/\kappa}\cdots\prod_{i=1}^{1}\prod_{j=1}^{2}|t^{(m-1)}_i-t^{(m-2)}_j|^{-1/\kappa}.$$

4.7. *Theorem. Let $z_1<z_2<\cdots<z_m$ and $1/\kappa<0$. Then the integral $I_\kappa(z)$ is convergent and equals $C_m(\kappa)\psi(z)w$, where*

$$C_m(\kappa)=(-1)^{a_m}\frac{\Gamma(1-1/\kappa)^{m(m+1)/2}}{(-1/\kappa)^{m-1}\prod_{i=1}^m\Gamma(1-i/\kappa)},\qquad a_m=m-1+\sum_{j=4}^{m}\binom{m-2}{2}.$$

Moreover, the integral has a well-defined analytic continuation with respect to $1/\kappa$ to the region where the real part of $1/\kappa$ is less than $1/m$.

4.8. *Proof* is by induction on m. For $m=2$ we have

$$\Phi=|t^{(1)}_1-z_1|^{-1}|t^{(1)}_1-z_2|^{-1},$$

$$\begin{aligned}I_\kappa(z_1,z_2)&=\int_{z_1}^{z_2}dt^{(1)}_1\Phi^{1/\kappa}\Big(\frac{v_{id}}{t^{(1)}_1-z_2}+\frac{v_{(21)}}{t^{(1)}_1-z_1}\Big)\\&=(z_1-z_2)^{-2/\kappa}\int_0^1dt\,t^{-1/\kappa}(1-t)^{-1/\kappa}\Big(\frac{v_{id}}{t-1}+\frac{v_{(21)}}{t}\Big)\\&=(z_1-z_2)^{-2/\kappa}\frac{\Gamma(1-1/\kappa)\Gamma(-1/\kappa)}{\Gamma(1-2/\kappa)}(-v_{id}+v_{(21)})\end{aligned}$$

and the first statement of the theorem is proved for $m=2$. To show the required analytic continuation we replace the integral over the interval by the corresponding Pochhammer double loop.

Let $m=3$. The integral is three-dimensional. The first integration over $t^{(2)}_1$ from $t^{(1)}_1$ to $t^{(1)}_2$ is exactly the calculation of the integral for $m=2$ in which $z_1,z_2,t^{(1)}_1$ are replaced with $t^{(1)}_1,t^{(1)}_2,t^{(2)}_1$, respectively. Using the result for $m=2$, we see that after the first integration in the remaining double integral over $t_1^{(1)}$, $t_2^{(1)}$ the factor $(t_2^{(1)} - t_1^{(1)})^{2/\kappa}$ in the master function is canceled with the factor $(t_2^{(1)} - t_1^{(1)})^{2/\kappa}$ obtained after the first integration. Therefore, in the remaining double integral the variables $t_1^{(1)}$, $t_2^{(1)}$ become decoupled. More precisely, we have

$$I_\kappa(z_1, z_2, z_3) = \sum_{\sigma \in S_3} I_{\kappa,\sigma}(z_1, z_2, z_3) y_\sigma$$

where $I_{\kappa,\sigma}$ is the determinant of the 2×2-matrix whose rows are

$$\int_{z_1}^{z_2} \tilde{\Phi}(t_1^{(1)}, z)^{1/\kappa}(t_1^{(1)} - z_a)^{-1} dt_1^{(1)} \qquad \int_{z_1}^{z_2} \tilde{\Phi}(t_1^{(1)}, z)^{1/\kappa}(t_1^{(1)} - z_b)^{-1} dt_1^{(1)} \ ,$$
$$\int_{z_2}^{z_3} \tilde{\Phi}(t_2^{(1)}, z)^{1/\kappa}(t_2^{(1)} - z_a)^{-1} dt_2^{(1)} \qquad \int_{z_2}^{z_3} \tilde{\Phi}(t_2^{(1)}, z)^{1/\kappa}(t_2^{(1)} - z_b)^{-1} dt_2^{(1)}$$

with $a = \sigma^{-1}(3),\ b = \sigma^{-1}(2)$, $\tilde{\Phi}(s, z) = \prod_{i=1}^3 (s - z_i)^{-1}$. By [V1] this determinant equals

$$-(-1)^\sigma \frac{\Gamma(1 - 1/\kappa)\Gamma(1 - 1/\kappa)\Gamma(-1/\kappa)}{\Gamma(1 - 3/\kappa)},$$

see also Section 3.3 in [V2]. Together with the statement for $m = 2$, this formula implies the first statement of the theorem for $m = 3$. To show the required analytic continuation we again replace integrals over the intervals by the corresponding Pochhammer double loops.

For arbitrary m we use the induction hypothesis and reduce the coefficients of the basis vectors to $(m-1) \times (m-1)$-determinants of one-dimensional integrals. Those determinants were calculated in [V1], cf. [V2]. As a result we get the first statement of the theorem for arbitrary m. The second statement is proved by using the Pochhammer double loops. $\square$

(b) *Selberg integrals associated with conformal blocks at level 1*

4.9. *Subbundle of conformal blocks of level 1.* Now for arbitrary N consider the N-th tensor power of $V = \bar{V}^{\otimes N}$ where $\bar{V}$ is the vector representation of $\mathfrak{sl}(m)$ as before. Denote by V_λ its weight subspace of weight

$$\lambda = (\lambda_1, \dots, \lambda_m) = (a+1, \dots, a+1, a \dots, a) = (1, \dots, 1, 0, \dots, 0) + (a, \dots, a)$$

where a is some nonnegative integer, the vector $(1, \dots, 1, 0, \dots, 0)$ has m' ones with $0 \le m' < m$.

For $z = (z_1, \dots, z_N)$ with distinct coordinates, denote by $CB^1(z) \subset V_\lambda$ the corresponding one-dimensional subspace of conformal blocks of level 1; it has rank 1 and admits as a generating section

$$\Psi(z) = p_\lambda(z) = \sum_{w\in S(\lambda)} \frac{y^w}{e_w},$$

cf. 2.4. That section is a solution of the KZ differential equations

$$\frac{\partial\Psi}{\partial z_i} = \frac{1}{m+1}\sum_{j\neq i}\frac{\pi_{ij}-m\cdot\mathrm{Id}}{z_i-z_j}\,\Psi, \qquad i=1,\dots,N.$$

Notice that the coefficient of Id in these equations is different from the coefficient of Id in (4.2).

4.10. *Master function.* Denote $\mu_i = \lambda_{i+1}+\dots+\lambda_m$ for $i=0,\dots,m-1$, and

$$d_N = \lambda_2 + 2\lambda_3 + \dots + (m-1)\lambda_m = \mu_1 + \dots + \mu_{m-1} = a\frac{m(m-1)}{2} + \frac{m'(m'-1)}{2}.$$

Fix on $\mathbb{C}^{d_N}$ coordinates $t = (t^{(1)},\dots,t^{(m-1)})$, where

$$t^{(i)} = (t^{(i)}_1,\dots,t^{(i)}_{\mu_i}), \qquad i=1,\dots,m-1.$$

Fix the holomorphic volume form on $\mathbb{C}^{d_N}$,

$$\begin{aligned} dt &= dt^{(m-1)}\wedge dt^{(m-2)}\wedge\dots\wedge dt^{(1)} \\ &= dt^{(m-1)}_1\wedge\dots\wedge dt^{(m-1)}_{\mu_{m-1}}\wedge\dots\wedge dt^{(1)}_1\wedge\dots\wedge dt^{(1)}_{\mu_1}, \end{aligned}$$

and the master function

$$\begin{aligned} \Phi(t,z) \;=\; & \prod_{1\le i<j\le N}(z_j-z_i)^{1-m}\prod_{i=1}^{\mu_1}\prod_{j=1}^{N}(t^{(1)}_i - z_j)^{-1}\prod_{1\le i<j\le\mu_1}(t^{(1)}_j - t^{(1)}_i)^2\,\times \\ & \prod_{i=1}^{\mu_2}\prod_{j=1}^{\mu_1}(t^{(2)}_i - t^{(1)}_j)^{-1}\prod_{1\le i<j\le\mu_2}(t^{(2)}_j - t^{(2)}_i)^2\cdots\times \\ & \prod_{i=1}^{\mu_{m-1}}\prod_{j=1}^{\mu_{m-2}}(t^{(m-1)}_i - t^{(m-2)}_j)^{-1}\prod_{1\le i<j\le\mu_{m-1}}(t^{(m-1)}_j - t^{(m-1)}_i)^2\ . \end{aligned}$$

4.11. *Actions of symmetric groups.* Let the group S_N act on functions of t, z by permuting the variables $z_1,\dots,z_N$,

$$(\sigma g)(t,z_1,\dots,z_N) = g(t, z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(N)}).$$

Similarly let the group S_{μ_1} act on functions of t, z by permuting the variables $t^{(1)}_1$, …, $t^{(1)}_{\mu_1}$ and so on. For a function $g(t,z)$ define the symmetrizations

$$\mathrm{Sym}_z g(t,z) = \sum_{\sigma\in S_N}(\sigma h)(t,z), \qquad \mathrm{Sym}_{t^{(1)}} g(t,z) = \sum_{\sigma\in S_{\mu_1}}(\sigma h)(t,z), \qquad \text{and so on.}$$

4.12. *Weight functions.* Let v_I be a basis vector of V_λ. We have $I = (I_1, \dots, I_m)$ where $I_1, \dots, I_m$ form a partition of the set $\{1, \dots, N\}$ with $|I_i| = \lambda_i$, $i = 1, \dots, m$.

For every $i = 0, \dots, m-1$, fix a bijection

$$\nu_j : \{1, \dots, \mu_i\} \to I_{i+1} \cup \dots \cup I_m$$

such that the first λ_m elements of $\{1, \dots, \mu_i\}$ are mapped to I_m, the next λ_{m-1} elements of $\{1, \dots, \mu_i\}$ are mapped to I_{m-1} and so on until the last λ_{i+1} elements of $\{1, \dots, \mu_i\}$ are mapped to I_{i+1}.

Denote

$$\begin{aligned}
g_{I,\nu}(t,z) &= \prod_{i=1}^{\mu_1} (t^{(1)}_{\nu_1(i)} - z_{\nu_0(i)})^{-1} \prod_{i=1}^{\mu_2} (t^{(2)}_{\nu_2(i)} - t^{(1)}_{\nu_1(i)})^{-1} \cdots \prod_{i=1}^{\mu_{m-1}} (t^{(m-1)}_{\nu_{m-1}(i)} - t^{(m-2)}_{\nu_{m-2}(i)})^{-1}, \\
\omega_I(t,z) &= \mathrm{Sym}_{t^{(1)}} \mathrm{Sym}_{t^{(2)}} \dots \mathrm{Sym}_{t^{(m-2)}} g_{I,\nu}(t,z), \\
\omega(t,z) &= \sum_I \omega_I(t,z) v_I.
\end{aligned}$$

This is a V_λ-valued function of t, z.

4.13. *Integrals.* Consider the V_λ-valued differential d_N-form

$$\Phi(t,z)^{1/\kappa} \omega(t,z) dt.$$

Let $\delta(z)$ be a flat section of the homological bundle associated with this differential form, see [SV], [V3]. Then by [SV], [FSV] the V_λ-valued function

$$I(z) = \int_{\delta(z)} \Phi(t,z)^{1/\kappa} \omega(t,z) dt$$

is a solution of the KZ equations

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\pi_{ij} - m\mathrm{Id}}{z_i - z_j} I, \qquad i = 1, \dots, N,$$

moreover if $\kappa = m+1$, then $I(z) \in CB^1(z)$.

4.14. *The cycle.* For real $z = (z_1, \dots, z_N)$ with $z_1 < z_2 < \dots < z_N$ we define a d_N-dimensional cell $\gamma = \gamma(t; z) = \gamma(t^{(1)}, \dots, t^{(m-1)}, z)$ in $\mathbb{C}^{d_N}$ as follows.

We split numbers $z_1, \dots, z_N$ into $a+1$ groups

$$\begin{aligned}
z^{(j)} &= (z_{m(j-1)+1}, \dots, z_{m(j-1)+m}), \qquad j = 1, \dots, a, \\
z^{(a+1)} &= (z_{ma+1}, \dots, z_{ma+m'} = z_N).
\end{aligned}$$

We split variables $t^{(i)}$ into $a+1$ groups

$$\begin{aligned} t^{(i,j)} &= (t^{(i)}_{(m-i)(j-1)+1}, \dots, t^{(i)}_{(m-i)(j-1)+(j-1)}), \qquad j = 1, \dots, a, \\ t^{(i,a+1)} &= (t^{(i)}_{(m-i)(j-1)+1}, \dots, t^{(i)}_{\mu_i}). \end{aligned}$$

Note that the last group $t^{(i,a+1)}$ is empty for $i \geq m'$. We define

$$\begin{aligned} \gamma(t;z) &= \gamma_m(t^{(1,1)}, \dots, t^{(m-1,1)}; z^{(1)}) \times \gamma_m(t^{(1,2)}, \dots, t^{(m-1,2)}; z^{(2)}) \times \dots \\ &\quad \dots \times \gamma_m(t^{(1,a)}, \dots, t^{(m-1,a)}; z^{(a)}) \times \gamma_{m'}(t^{(1,a+1)}, \dots, t^{(m'-1,a+1)}; z^{(a+1)}) \end{aligned}$$

where the cells in the right hand side are introduced in Section 4.5.

4.15. Consider the following iterated integral over the cell $\gamma(t;z)$:

$$I_\kappa(z) = \int dt^{(1)} \int dt^{(2)} \dots \int dt^{(m-1)} \Phi(t,z)^{1/\kappa} \omega(t,z),$$

cf. 4.6. The function $\Phi^{1/\kappa}$ is multivalued. In order to define the integral (apart from its possible divergence) we need to choose a section over the cell $\gamma(t;z)$ of the local system associated with the function $\Phi^{1/\kappa}$. We choose the section

$$\begin{aligned} &\prod_{1\leq i<j\leq N} |z_j - z_i|^{(1-m)/\kappa} \prod_{i=1}^{\mu_1} \prod_{j=1}^{N} |t_i^{(1)} - z_j|^{-1/\kappa} \prod_{1\leq i<j\leq \mu_1} |t_j^{(1)} - t_i^{(1)}|^{2/\kappa} \times \\ &\prod_{i=1}^{\mu_2} \prod_{j=1}^{\mu_1} |t_i^{(2)} - t_j^{(1)}|^{-1/\kappa} \prod_{1\leq i<j\leq \mu_2} |t_j^{(2)} - t_i^{(2)}|^{2/\kappa} \dots \times \\ &\prod_{i=1}^{\mu_{m-1}} \prod_{j=1}^{\mu_{m-2}} |t_i^{(m-1)} - t_j^{(m-2)}|^{-1/\kappa} \prod_{1\leq i<j\leq \mu_{m-1}} |t_j^{(m-1)} - t_i^{(m-1)}|^{2/\kappa} . \end{aligned}$$

4.16. *Theorem. For $z_1 < z_2 < \dots < z_N$ and $1/\kappa < 0$ the integral $I_\kappa(z)$ in 4.15 is convergent and is a solution to the KZ equations from 4.9.*

The integral $I_\kappa(z)$ has a well-defined analytic continuation to $\kappa = m+1$. We have

$$I_{m+1}(z) = C p_\lambda(z), \qquad \text{with } C = (C_m(m+1))^a C_{m'}(m+1),$$

where the constants $C_m(m+1)$, $C_{m'}(m+1)$ are defined in Theorem 4.7.

4.17. *Proof.* The first two statements of the theorem follow from §3.

For $\kappa = m+1$, both functions $I_{m+1}(z)$ and $p_\lambda(z)$ are solutions of the KZ equations with values in the one-dimensional bundle of conformal blocks, hence, they are proportional. The coefficient of the proportianality is calculated in the limit $z^{(i)} \to y_i$, $i = 1, \dots, a+1$, where $y_1 < \dots < y_{a+1}$ are some fixed numbers. Comparing the asymptotics of both functions we get the formula for C, cf. Theorem 4.8 in [RV]. □

§5. Action of positive currents

5.1. Let us return to the setup of §2. For each $\lambda \in \mathcal{P}_m(N)$ we have defined an isomorphism of (free) R'-modules

$$H_T^*(X_\lambda)_{R'} \xrightarrow{\sim} (\bar{V}^{\otimes N})_{\lambda;R'}. \tag{5.1}$$

Let us denote by $X_{m,N}$ the variety of all flags

$$0 \subset L_1 \subset \ldots \subset L_m = \mathbb{C}^N \tag{5.2}$$

of length m in $\mathbb{C}^N$, so it is the disjoint union

$$X_{m,N} = \coprod_{\lambda \in \mathcal{P}_m(N)} X_\lambda.$$

Summing up (5.1) over all $\lambda \in \mathcal{P}_m(N)$ we get an isomorphism of R'-modules

$$H_T^*(X_{m,N})_{R'} \xrightarrow{\sim} (\bar{V}^{\otimes N})_{R'} = \bar{V}^{\otimes N} \otimes_{\mathbb{C}} R'. \tag{5.2}$$

The Lie algebra $\mathfrak{gl}(m)$ acts on $(\bar{V}^{\otimes N})_{R'}$ through its action on $\bar{V}^{\otimes N}$. Due to the extension of scalars one can extend this action to an action of the Lie algebra of positive currents $\mathfrak{gl}(m)[t]$. Namely, for $x \in \mathfrak{gl}(m)$ the action of xt^j on $\bar{V}^{\otimes N}$ is defined by the operator

$$xt^j = \sum_{i=1}^{N} x^{(i)} z_i^j$$

(as is usual in Conformal Field Theory, one should imagine the i-th tensor factor of $\bar{V}^{\otimes N}$ as sitting at a point z_i of the Riemann sphere).

In this section we shall define geometrically an action of $\mathfrak{gl}(m)[t]$ on the equivariant cohomology $H_T^*(X_{m,N})$ in such a way that after the extension of scalars to R' the isomorphism (5.1) will be compatible with this action.

5.2. Given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_m(N)$ and $1 \le a < m$, set

$$e_{a,a+1}\lambda = (\lambda_1, \ldots, \lambda_{a-1}, \lambda_a + 1, \lambda_{a+1} - 1, \lambda_{a+2}, \ldots, \lambda_m),$$

this is defined if $\lambda_{a+1} > 0$, and

$$e_{a+1,a}\lambda = (\lambda_1, \ldots, \lambda_{a-1}, \lambda_a - 1, \lambda_{a+1} + 1, \lambda_{a+2}, \ldots, \lambda_m),$$

this is defined if $\lambda_a > 0$. Recall that X_λ parametrizes flags (5.2) with $\mu_i := \dim L_i$ such that $\lambda_i = \mu_i - \mu_{i-1}$.

Define

$$\mu'(\lambda, a) = (\lambda_1, \ldots, \lambda_{a-1}, \lambda_a, 1, \lambda_{a+1} - 1, \lambda_{a+2}, \ldots, \lambda_m) \in \mathcal{P}_{m+1}(N)$$

and

$$\mu''(\lambda, a) = (\lambda_1, \ldots, \lambda_{a-1}, \lambda_a - 1, 1, \lambda_{a+1}, \lambda_{a+2}, \ldots, \lambda_m) \in \mathcal{P}_{m+1}(N)$$

Consider the variety $X'_{\lambda,a} := X_{\mu'(\lambda,a)}$. We have obvious projections

$$X_\lambda \xleftarrow{\pi'_1} X'_{\lambda,a} \xrightarrow{\pi'_2} X_{e_{a,a+1}\lambda}.$$

Let S' (resp. Q') denote the rank 1 (resp. rank $\lambda_{a+1}-1$) vector bundle over $X'_{\lambda,a}$ whose fiber over a flag $L_1 \subset \ldots \subset L_{m+1} = \mathbb{C}^N$ is L_{a+1}/L_a (resp. L_{a+2}/L_{a+1}).

We define the map

$$\rho(e_{a,a+1}t^j):\ H^*_T(X_\lambda) \longrightarrow H^*_T(X_{e_{a,a+1}\lambda})$$

by

$$\rho(e_{a,a+1}t^j)(x) = \pi'_{2*}(\pi'^*_1(x)\cdot e(Hom(S',Q'))\cdot e(S'^{\otimes j})) \tag{5.3$'$}$$

where $e(L)$ denotes the Euler (top Chern) class of a vector bundle L.

Similarly, consider the variety $X''_{\lambda,a} := X_{\mu''(\lambda,a)}$. We have obvious projections

$$X_\lambda \xleftarrow{\pi''_1} X''_{\lambda,a} \xrightarrow{\pi''_2} X_{e_{a+1,a}\lambda}.$$

Let S'' (resp. Q'') denote the rank $\lambda_a - 1$ (resp. rank 1) vector bundle over $X''_{\lambda,a}$ whose fiber over a flag $L_1 \subset \ldots \subset L_{m+1} = \mathbb{C}^N$ is L_a/L_{a-1} (resp. L_{a+1}/L_a).

We define the map

$$\rho(e_{a+1,a}t^j):\ H^*_T(X_\lambda) \longrightarrow H^*_T(X_{e_{a+1,a}\lambda})$$

by

$$\rho(e_{a+1,a}t^j)(x) = \pi''_{2*}(\pi''^*_1(x)\cdot e(Hom(S'',Q''))\cdot e(Q''^{\otimes j})). \tag{5.3$''$}$$

Note that the maps (5.3)$'$ and (5.3)$''$ are $R = H^*_T(pt)$-equivariant, due to the projection formula, so they may be localized to R'.

5.3. *Theorem. The maps* (5.3)$'$ *and* (5.3)$''$ *define an action of the Lie algebra* $\mathfrak{gl}(m)[t]$ *on* $H^*_T(X_{m,N})$ *such that (after extension of scalars to* R'*) the isomorphism (5.1) is* $\mathfrak{gl}(m)[t]$*-equivariant.*

To prove the theorem, one remarks first that we can do the above extension of scalars. After that, one checks that the action of the operators $\rho(e_{a,a+1}t^j)$ and $\rho(e_{a+1,a}t^j)$ transferred to $\bar{V}_{R'}^{\otimes N}$ via the isomorphism (5.1) coincides with the action described in 5.1. The details will appear elsewhere.

5.4. *Remark.* Let us consider the cotangent bundle $T^*X_{m,N}$; it may be realized as the variety of pairs $\{(L_1 \subset \ldots L_m) \in X_{m,N}, A \in \mathrm{End}(\mathbb{C}^N),\ A(L_i) \subset L_{i-1},\ i = 2,\ldots,m\}$. This variety is of course $GL(N)$-equivariantly homotopically equivalent to $X_{m,N}$. However, it admits one more symmetry — an action of $\mathbb{C}^*$ by dilations along the fibers.

The work of Ginzburg, Nakajima, Vasserot, Varagnolo, ... (cf. [CG] [N, §7], [V] and references therein) shows that the equivariant cohomology $H^*_{\mathbb{C}^*\times GL(N)}(T^*X_{m,N})$

admits an action of the Yangian of the loop algebra $Y(\mathfrak{gl}(m)[t,t^{-1}])$. If we forget the action of $\mathbb{C}^*$, we get the cohomology $H^*_{GL(N)}(T^*X_{m,N}) = H^*_{GL(N)}(X_{m,N})$ which is a quotient of $H^*_{\mathbb{C}^*\times GL(N)}(T^*X_{m,N})$ and the above action of the Yangian should factor through a quotient isomorphic to $U(\mathfrak{gl}(m)[t])$ (this was explained to us by Misha Finkelberg).

The spaces $H^*_{GL(N)}(X_{m,N})$ and $H^*_T(X_{m,N})$ are different but closely related. Namely, $T \subset GL(N)$ is a maximal torus, and

$$H^*_T(X_{m,N}) = H^*_{GL(N)}(X_{m,N}) \otimes_{H^*_{GL(N)}(pt)} H^*_T(pt).$$

One should expect that the Ginzburg-Vasserot-Varagnolo action induces the action defined in the previous sections, however we did not verify this.

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