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THE BRAID-PERMUTATION GROUP

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We consider the subgroup of the automorphism group of the free group generated by the braid group and the permutation group. This is proved to be the same as the subgroup of automorphisms of permutation-conjugacy type and is represented by generalised braids (braids in which some crossings are allowed to be "welded"). As a consequence of this representation there is a finite presentation which shows the close connection with both the classical braid and permutation groups. The group is isomorphic to the automorphism group of the free quandle and closely related to the automorphism group of the free rack. These automorphism groups are connected with invariants of classical knots and links in the 3-sphere. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

Let F_n denote the free group of rank n with generators $\{x_1, \dots, x_n\}$ and let $\mathrm{Aut}\, F_n$ denote its automorphism group. Let $\sigma_i \in \mathrm{Aut}\, F_n$, $i = 1, 2, \dots, n-1$ be given by

$$\begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j, \quad j \neq i, \ i+1 \end{cases}$$

and let $\tau_i \in \mathrm{Aut}\, F_n$, $i = 1, 2, \dots, n-1$, be given by

$$\begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j, \quad j \neq i, \ i+1. \end{cases}$$

The elements σ_i, $i = 1, 2, \dots, n-1$, generate the *braid subgroup* B_n of $\mathrm{Aut}\, F_n$ which is well known to be isomorphic to the classical braid group of on n strings (for a proof see [1, 1.10] or [2, 7.3]), and the elements τ_i, $i = 1, 2, \dots, n-1$, generate the *permutation subgroup* P_n of $\mathrm{Aut}\, F_n$ which is a copy of the symmetric group S_n of degree n. We shall call the subgroup BP_n of $\mathrm{Aut}\, F_n$ generated by both sets of elements σ_i and τ_i, $i = 1, 2, \dots, n-1$, the *braid-permutation group* and this is the subject of this paper.

Our main result is that this group is isomorphic to a group of generalised braids. These are braids in which some of the crossings are "welded" and the welded and unwelded crossings interact in an intuitively simple way.

As a consequence of this isomorphism, BP_n has the following finite presentation which includes the standard presentations for both B_n and S_n. This result was announced in [3]. *Generators* for BP_n are $\sigma_i, \tau_i, i = 1, \dots, n-1$, as above, and *relations* are:

$$(\textit{Braid group relations}) \quad \begin{cases} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

$$(\textit{Permutation group relations})\quad \begin{cases} \tau_i^2 &= 1 \\ \tau_i\tau_j &= \tau_j\tau_i, \quad |i-j|>1 \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \end{cases}$$

$$(\textit{Mixed relations})\quad \begin{cases} \sigma_i\tau_j &= \tau_j\sigma_i, \quad |i-j|>1 \\ \tau_i\tau_{i+1}\sigma_i &= \sigma_{i+1}\tau_i\tau_{i+1} \\ \sigma_i\sigma_{i+1}\tau_i &= \tau_{i+1}\sigma_i\sigma_{i+1}. \end{cases}$$

The proof of this result occupies Sections 2 and 3; Section 2 contains the definition of welded braids and Section 3 contains the main proof, which is diagramatic.

The group BP_n is interesting for a number of other reasons. Firstly, the elements of BP_n have the following simple algebraic characterisation. Let $\pi \in S_n$ be a permutation and w_i, $i = 1, 2, \ldots, n$ be words in F_n. Then the assignation $x_i \mapsto w_i^{-1}x_{\pi(i)}w_i$ determines a homomorphism of F_n to itself which is in fact injective (see [2, Corollary 8.6]). If it is also surjective (and hence an automorphism) then we call it an automorphism of *permutation-conjugacy type*. The automorphisms of permutation-conjugacy type define a subgroup PC_n of $\operatorname{Aut} F_n$ which is in fact precisely BP_n.

Secondly, BP_n is isomorphic to the automorphism group $\operatorname{Aut} FQ_n$ of the free quandle of rank n, and is closely related to the automorphism group $\operatorname{Aut} FR_n$ of the free rack of rank n (the latter group is the wreath product of BP_n with the integers) and these groups are connected with invariants of classical knots and links in the 3-sphere.

Finally, the interpretation of elements of BP_n as welded braids suggests a natural relationship with the *singular braid group* SB_n, of Birman *et al.*, which has applications to the theory of Vassiliev invariants [4–6]. Initially, this was defined as a monoid. However, it naturally embeds in a group with a common presentation, see [7]. Indeed the groups BP_n and SB_n have a common subgroup (the braid group) and a common quotient. The connection of these groups with the Vassiliev invariants will be explored in a later paper.

The proof of the identity of BP_n and BC_n is in Section 4, where the connection with racks and quandles is also to be found.

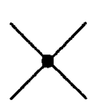
The presentation for BP_n and the proof of this presentation given in this paper are closely related to results for $\operatorname{Aut} FR_n$ given by Krüger [8]. However, his context is rather different from ours and in particular there is no interpretation in terms of welded braids.

2. WELDED BRAIDS

A *welded braid diagram* on n strings is a set of n monotone arcs from n points on a horizontal line at the top of the diagram down to a similar set of n points at the bottom of the diagram. The arcs are allowed to cross each other either in a "crossing" thus:

 or 

or in a "weld" thus:



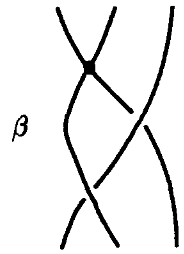


Fig. 1. A welded braid diagram.

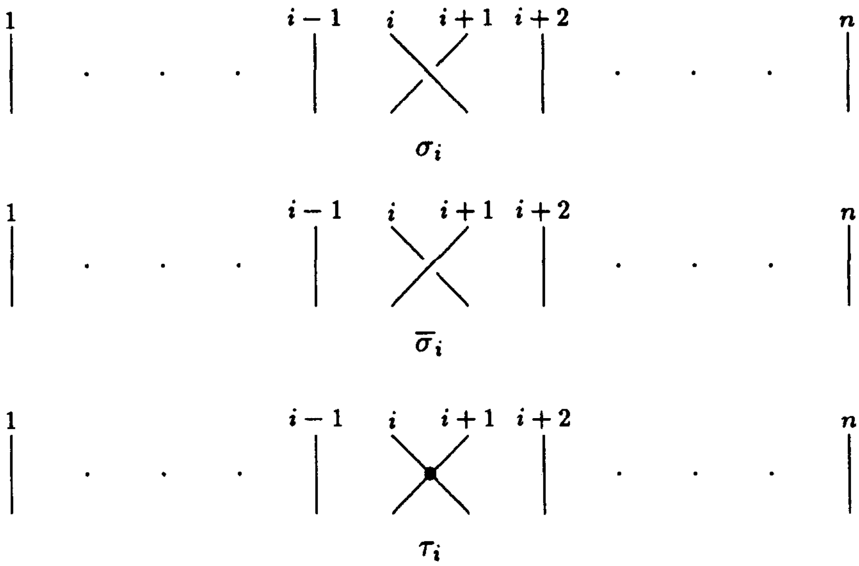


Fig. 2. The three atomic braids.

An example (on three strings) is illustrated in Fig. 1. It is assumed that the crossings and welds all occur on different horizontal levels; thus, a welded braid diagram determines a word in the *atomic* diagram illustrated and labelled in Fig. 2.

2.1. Convention

Braids are read *from top to bottom* and words are read from *left to right*. Thus Fig. 1 determines the word $\tau_1\bar{\sigma}_2\sigma_1$ which is read: first τ_1 then $\bar{\sigma}_2$ then σ_1. We identify braid diagrams which determine the same word in the atomic diagrams (in other words diagrams which differ by a planar isotopy through diagrams).

The set of welded braid diagrams (on n strings) forms a semi-group WD_n with composition given by "stacking": if β_1 and β_2 are diagrams then $\beta_1\beta_2$ is the diagram obtained by placing β_1 above β_2 so that the bottoms of the arcs of β_1 coincide with the tops of the arcs of β_2. There is an identity in WD_n, namely the diagram with no crossings.

The notation for atomic diagrams is intended to be confused with the notation for the generators of BP_n because we now consider the homomorphism $\Phi\colon WD_n \to \operatorname{Aut} F_n$, with image BP_n, defined by mapping σ_i and τ_i to the automorphisms with the same names and $\bar{\sigma}_i$ to σ_i^{-1}.

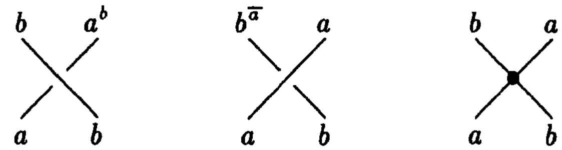


Fig. 3. Rules for labelling subarcs.

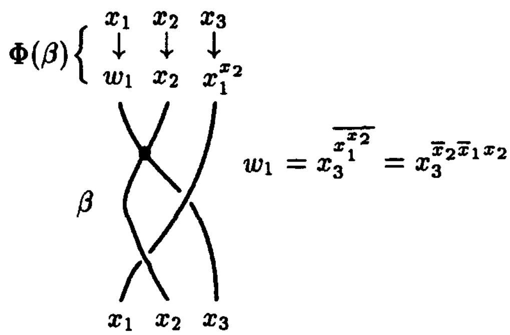


Fig. 4.

Thus, a welded braid diagram β determines an automorphism $\Phi(\beta)$ of F_n (of permutation-conjugacy type). There is a convenient way to read $\Phi(\beta)$ from the diagram β which we now describe.

2.3. Reading the automorphism from the diagram

Label the strings at the bottom of the diagram by the generators $x_1, x_2, \ldots, x_n$ in order and continue to label the subarcs between crossings moving up the diagram and using the rules given in Fig. 3.

2.4. Notation

From now on, until further notice, the notation x^y will mean the product $y^{-1}xy$. In Fig. 3 we have used the convention that $\bar{x}$ means x^{-1}. This is useful to avoid double exponents. The top of the strings are thus labelled by words $w_1, w_2, \ldots, w_n$ and the automorphism determined by the diagram is given by $x_i \mapsto w_i$ for $i = 1, \ldots, n$. An example of this process is given in Fig. 4, where the automorphism corresponding to the diagram in Fig. 1 is calculated.

To see that this process gives the correct result it is merely necessary to observe that it is correct for elementary braids (which follows at once from the rules in Fig. 3) and that the process gives a homomorphism $WD_n \to \operatorname{Aut} F_n$.

To see this consider the effect of stacking the braid β' on top of the braid β. If the ith point at the bottom of β is labelled x_i and at the top is labelled w_i, then the labels at the top of the combined braid are obtained from those for β' by substituting w_i for x_i. But this is precisely how the composition $\Phi(\beta')\Phi(\beta)$ of the two automorphism of F_n is formed.

2.5. Allowable moves on diagrams

We now consider the local changes that can be made to a welded braid diagram β which leave the automorphism $\Phi(\beta)$ unchanged. Four such moves are given in Fig. 5. These are

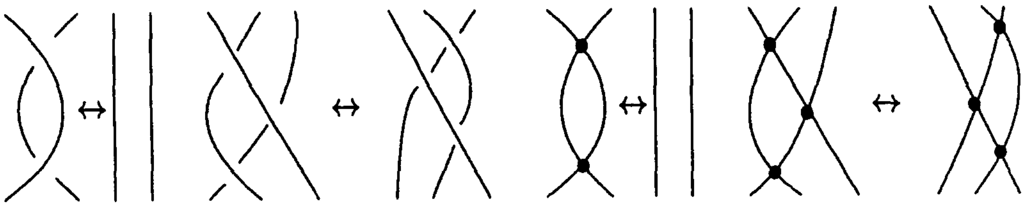


Fig. 5.

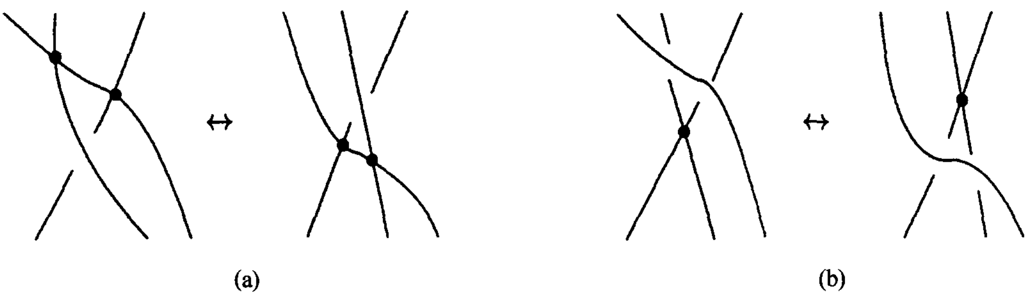


Fig. 6.

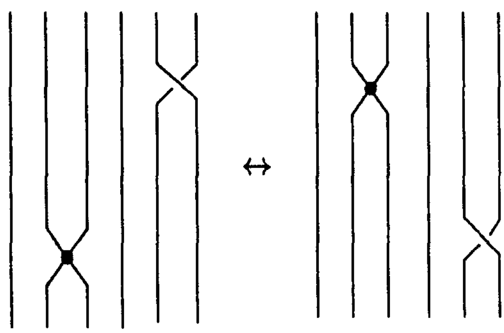


Fig. 7.

moves involving two or three strings which do not mix welded and unwelded crossings. It is easy to check that these leave the induced automorphism unchanged. For the second move in Fig. 5 this follows from the following verified property of conjugation that $a^{bc} = a^{cb^c}$. (Note that the first two moves are Reidemeister moves from knot theory.) If we identify welded braid diagrams that differ by these moves then we can give welded braid diagrams the structure of a group.

However, there are more local changes that do not change the assigned automorphism. In Fig. 6 we present two of them. In addition, there are moves of a general class involving four strings which allow non-interfering crossings to be reordered. One such move is illustrated in Fig. 7.

We call the moves illustrated in Figs 5–7 (and moves similar to Fig. 7) *allowable moves* and we define a *welded braid* to be an equivalence class of welded braid diagrams under allowable moves. Welded braids on n strings form a group which we shall denote WB_n with composition given (as in WD_n) by stacking. The inverse of a welded braid β is obtained by reflecting β in a horizontal line. That this is an inverse follows from the first and third moves in Fig. 5.

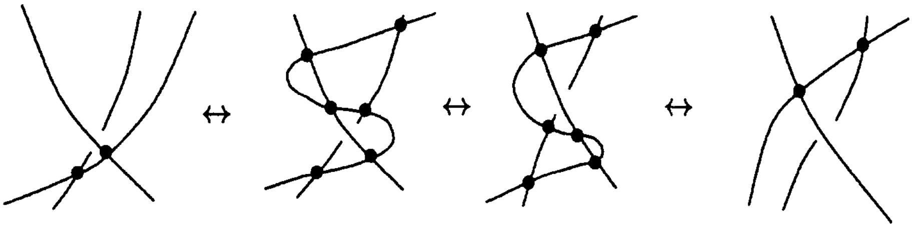


Fig. 8.

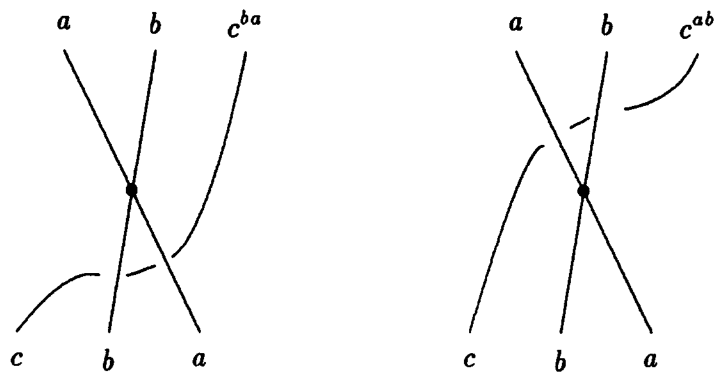


Fig. 9. Two welded braids which do not induce the same element of Aut F_n.

There are other local moves of the same type as those given above which do not also change the induced automorphism. For instance, the moves obtained from the moves in Fig. 6 by reflection in horizontal or vertical lines. However, as the reader can readily check, these other moves can all be obtained as suitable sequences of the allowable moves given in Figs 5 and 6. In Fig. 8, we show as an example how this can be done for a move of the same type as Fig. 6(a). All the allowable moves (and consequent moves such as Fig. 8) are similar to Reidemeister moves, and therefore one could be forgiven for thinking the following. Welded braids can be defined as the two-dimensional projection of three-dimensional welded braids and two welded braid diagrams give the same automorphism of F_n if they are projections of equivalent welded braids — where the equivalence among three-dimensional welded braids is a natural equivalence, for example, isotopy in 3-space. That this is not so can be seen by observing that the local change pictured in Fig. 9 *changes* the assigned automorphism.

There is however a way to regard welded braids as three-dimensional objects if desired. Think of the braids as comprising strings embedded in half of 3-space, namely the half *above* the plane of the diagram, and think of the strings as free to move above the plane except at the welds, which are to be regarded as small spots of glue holding the strings down onto the plane. Thus, strings are not allowed to move behind welds, as happens in Fig. 9.

We can now state the main results of the paper:

MAIN THEOREM 2.1. *Two welded braid diagrams determine the same automorphism of F_n if and only if they can be obtained from each other by a finite sequence of allowable moves (see definition above).*

The proof of the theorem comprises the next section. We finish this section by remarking that the theorem implies the presentation for BP_n given in Section 1. It is not hard to see that the group WB_n of welded braids has this presentation. The elementary braids σ_i and τ_i generate WB_n and the allowable moves each correspond to one of the relations listed in Section 1. For example, the moves in Fig. 6 correspond to the second and third "mixed relations", respectively. Indeed, welded braids can be regarded as a convenient way of illustrating this presentation.

Now we have the homomorphism $\Phi: WD_n \to \operatorname{Aut} F_n$ with image BP_n. Since allowable moves leave the induced automorphism unaltered this factors via a homomorphism $WB_n \to \operatorname{Aut} F_n$ with the same image. The theorem implies that this homomorphism is injective and hence WB_n and BP_n are isomorphic. Moreover, the generators of WB_n and BP_n (with the same names) correspond, hence BP_n has the presentation given.

3. PROOF OF THE MAIN THEOREM

In order to prove the main theorem (Theorem 2.1) it is sufficient to show that any welded braid which induces the identity automorphism of the free group F_n can be reduced to the identity braid by a finite sequence of allowable moves.

Let $C = \{x_i^w \mid w \in F_n\}$ denote the set of conjugates of generators of F_n. If $a = x_i^w$ is in C, let $L(a) = l(w)$ where w is chosen to have minimal length $l(w)$. Note that, for example, $x_i^{x_i w} = x_i^w$. If a, b belong to C then we write $a \prec b$ if $L(b^a) < L(b)$ and we write $\bar{a} \prec b$ if $L(b^{\bar{a}}) < L(b)$. The following technical lemmas will be useful. Let a, b, c be arbitary elements of C.

LEMMA 3.1. *If $a \prec b$ ($\bar{a} \prec b$) and $a = x_i^w$ for some w, then $b = x_j^{u\bar{x}_i w}$ ($b = x_j^{u x_i w}$) for some u where $u\bar{x}_i w$ ($u x_i w$) is reduced.*

Proof. If $b = x_j^v$ then $b^a = x_j^{v\bar{w}x_i w}$. If $l(v\bar{w}x_i w) < l(v)$ it follows that v ends in $\bar{x}_i w$. The case when $\bar{a} \prec b$ is similarly proved. □

LEMMA 3.2. *If $a \prec b$ and $b \prec c$ then $a \prec c$ and $a \prec c^b$.*

Proof. If $a = x_i^w$ then by the above $b = x_j^{u\bar{x}_i w}$ and $c = x_k^{v\bar{x}_j u\bar{x}_i w}$. So $c^a = x_k^{v\bar{x}_j u w}$, $c^{ba} = x_k^{vuw}$ and $c^b = x_k^{vu\bar{x}_i w}$. Using the fact that $l(\bar{x}_i w) > l(w)$ the result follows.

LEMMA 3.3. *If $a \prec b$ and $\bar{b} \prec c$ then $a \prec c$ and $a \prec c^{\bar{b}}$.*

Proof. The proof is similar to the proof of Lemma 3.2. □

LEMMA 3.4. *If $a \prec b$, $c \prec b$ and $a \neq c$ then $L(a) \neq L(c)$. If $L(a) < L(c)$ then $a \prec c$ and $L(b^{ca}) < L(b)$.*

Proof. Let $a = x_i^w$, $c = x_j^u$ in reduced form. From the hypotheses it follows that the exponent of b ends in $\bar{x}_i w$ and also $\bar{x}_j u$. Because $a \neq c$ $L(a) \neq L(c)$. If $L(a) < L(c)$ it follows that $u = s\bar{x}_i w$ and $b = x_k^{v\bar{x}_j s\bar{x}_i w}$, $c = x_j^{s\bar{x}_i w}$. Then $c^a = x_j^{sw}$ so $L(c^a) < L(c)$. We have $b^{ca} = x_k^{vsw}$ which gives the second conclusion. □

LEMMA 3.5. *If $\bar{a} \prec b$, $c \prec b$ and $L(a) < L(c)$, then $\bar{a} \prec c$ and $L(b^{c\bar{a}}) < L(b)$.*

Proof. The proof is similar to the proof of Lemma 3.4. □

LEMMA 3.6. *Each of the following three pairs of conditions is contradictory.*
(1) $a \prec b$ *and* $b \prec a$.
(2) $a \prec b$ *and* $\bar{b} \prec a$.
(3) $a \prec b$ *and* $\bar{a} \prec b$.

Proof. For the pairs (1) and (2), Lemmas 3.2 and 3.3, respectively, imply that $a \prec a$ which is impossible. For (3), Lemma 3.1 implies that $b = x_j^{u_1 \bar{x}_i w} = x_j^{u_2 x_i w}$. A simple argument using the minimal length representation of the exponent shows that $b = x_i^w = a$ which is again a contradiction. $\square$

We can assign a non-negative integer $L(\kappa)$, the *length* to any endomorphism κ of F_n of PC-type by the formula $L(\kappa) = \sum L(\kappa(x_i))$. Clearly, $L(\kappa) = 0$ if and only if κ is a permutation of the generators.

Now let β be a welded braid diagram which represents the identity automorphism. We assume that all (welded and unwelded) crossings have distinct y-coordinates. Let us suppose that the unwelded crossings occur with y-coordinates: $y_1, \ldots, y_{k-1}$. We choose the notation so that

$$0 = y_0 < y_1 < y_2 < \cdots < y_{k-1} < y_k = 1$$

where y_0 is the top level of the braid and y_k is the bottom level of the braid. *Remember the y-coordinate increases as we go down the page.* Any horizontal line whose height is none of these critical values and also does not meet a weld will divide the welded braid β into an upper welded braid β' and a lower welded braid β'' so that $\beta = \beta'\beta''$. If t is the height of the horizontal line define $L(t) = L(\Phi(\beta''))$. Notice that the function $L(t)$ changes only at the critical values. The values of L for $0 < t < y_1$ and for $y_{k-1} < t < 1$ will be 0. If $k > 1$, let us take one of the maximal L-valued intervals, say $[y_s, y_{s+1}]$. The value of s cannot be 0 or $k - 1$. We will show that we can change the welded braid by allowable moves so that the value of L is reduced within this interval and is unaltered elsewhere.

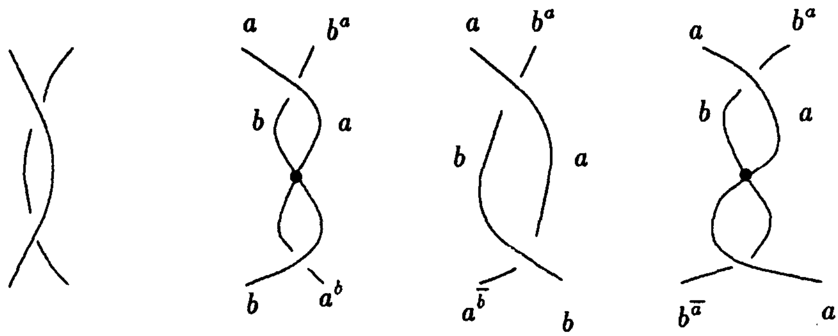
Let ς_i and ς_j be the unwelded crossings at level y_s and y_{s+1}, respectively. Then the braid between levels y_s and y_{s+1} has only welded crossings. In other words, it is a permutation τ say. We will endevour to reduce the number of welds in τ to a minimum.

Let x be the number of strings which are involved with ς_i and ς_j. The integer x can take the values 2, 3 or 4. By allowable moves that do not essentially alter the function L we can assume that the remaining $n - x$ strings do not have any welded crossings in the interval $[y_s, y_{s+1}]$. For example, Figs 6 and 8 show how to move a welded string past an unwelded crossing.

Now we consider the three cases $x = 2, 3, 4$. We will assume that we have used allowable moves to minimize the number of welds in the interval $[y_s, y_{s+1}]$.

Case $x = 2$. The maximum number of welds occurring now is 1. The possible cases up to a simple symmetry are displayed in Fig. 10. The remaining $n - 2$ strings are not shown. The first case can easily be simplified. It is not difficult to show with the help of the above lemmas that the other three cases cannot occur if the interval defines a maximum of the function L. For example, the second picture above gives the conditions $a \prec b$ and $b \prec a$ which contradicts Lemma 4.6(1) (see the notation of the picture).

Case $x = 3$. The number of possibilities to be considered can be reduced up to a simple symmetry to the 9 cases illustrated in Fig. 11. These can be grouped into three sets of 3 by the following description. The first 3 occur where the overcrossing string at the bottom is an overcrossing string at the top. For the next 3 the overcrossing string at the bottom is an


a b^a
b a
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b a
$a^{\bar{b}}$ b
a b^a
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Fig. 10.

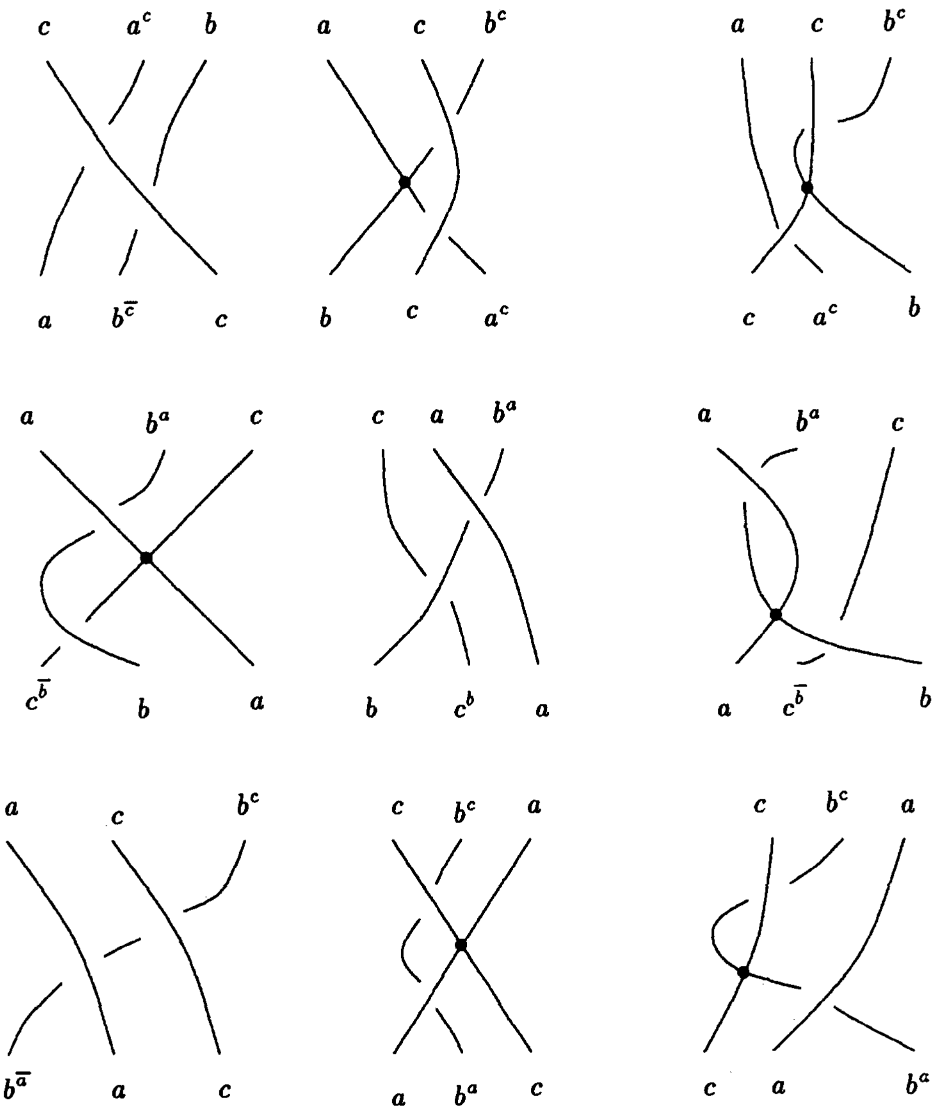
c a^c b
a $b^{\bar{c}}$ c
a c b^c
b c a^c
a c b^c
c a^c b
a b^a c
$c^{\bar{b}}$ b a
c a b^a
b c^b a
a b^a c
a $c^{\bar{b}}$ b
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Fig. 11.

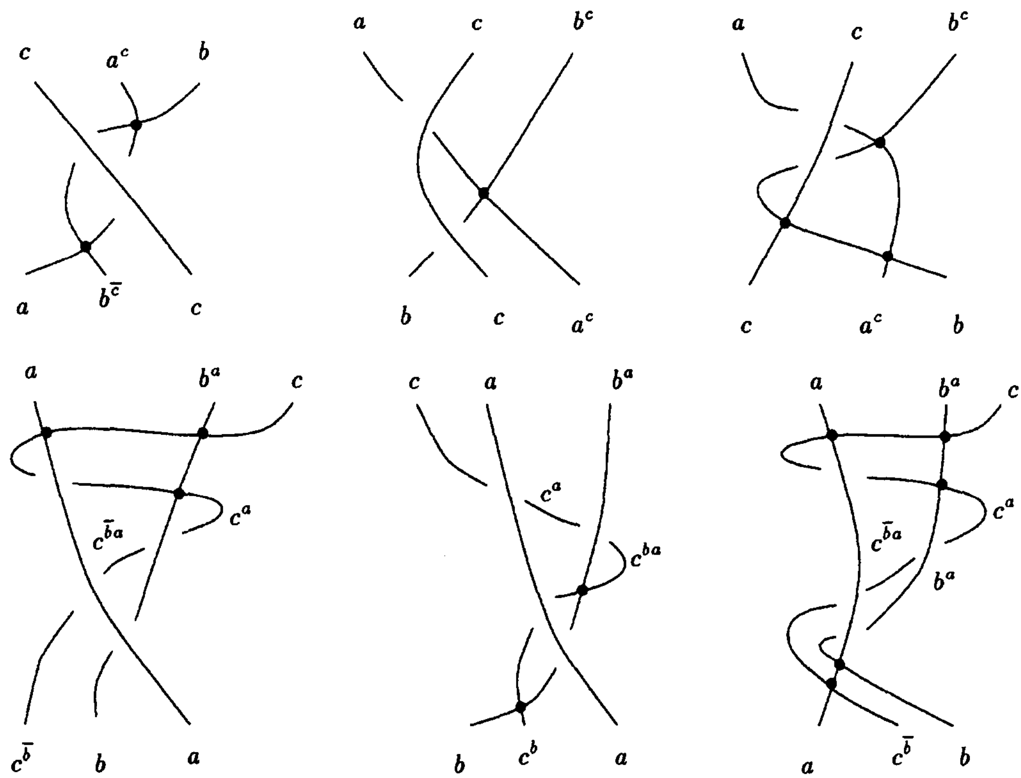

Fig. 12.

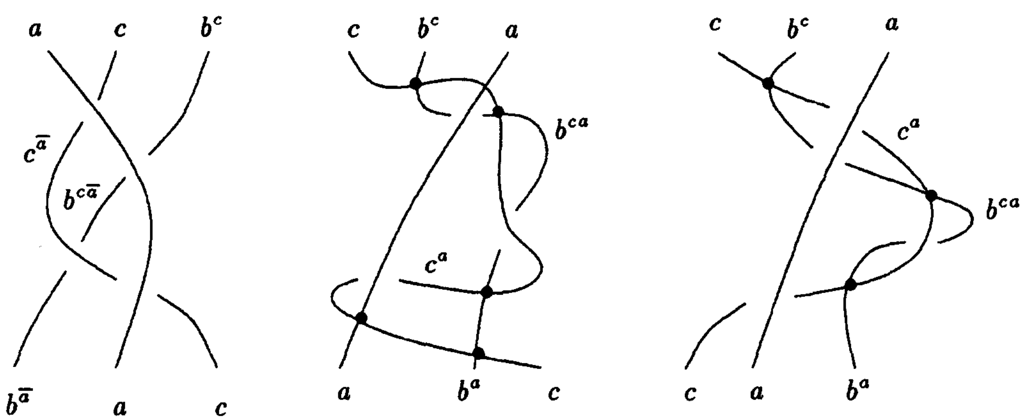

Fig. 13.

undercrossing string at the top and for the last 3 the undercrossing string at the bottom is the undercrossing string at the top.

The fourth possibility is that the undercrossing string at the bottom is an overcrossing string at the top. However, this becomes the middle case by reflection in a horizontal line. We now change the first six cases by allowable moves so that they look like the ones illustrated in Fig. 12. The relevant change to the last three depends on the length of a and c. If the lengths $L(a)$ and $L(c)$ are equal then Lemma 3.4 gives a contradiction. We give the appropriate changes if $L(a) < L(c)$ in Fig. 13. The reader can easily construct the changes in the opposite case. After these changes have been completed it only remains to check that the

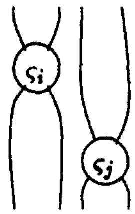

Fig. 14.

value of L has been decreased in the interval. This can be done using the above lemmas and may safely be left to the reader. For convenience, we give the notation in the pictures so that they fit the notations of Lemmas 3.1–3.6.

Case $x = 4$. By minimality there is only one possibility which is illustrated in Fig. 14. We change the situation by interchanging the heights of ς_i and ς_j. This has the curious effect of decreasing the value of L in the interval as the reader may verify.

We now repeat this argument until the value of L is constant (which must be zero). This implies that all crossings are welded and therefore the induced automorphism is a permutation. Since this is the identity permutation we can reduce the welded braid to the identity by a sequence of the two right-hand moves in Fig. 5.

This completes the proof of the main theorem. □

4. FURTHER RESULTS

4.1. Automorphisms of permutation-conjugacy type

The groups BP_n and PC_n were defined in Section 1. We start this section by proving the equality of these two groups. The proof uses standard Nielsen theory techniques and we shall refer to the appendix of [2] for details here; similar results can be found in [9–12].

Since the generators σ_i, τ_i of BP_n are automorphisms of permutation-conjugacy type, BP_n is a subgroup of PC_n; thus, we have to show that any automorphism of permutation-conjugacy type is a product of the "elementary" automorphisms σ_i, τ_i. It is convenient to use rather different elementary automorphisms. We shall call the automorphisms $p_{i,k}, s_\pi \in PC_n$ given as follows, *elementary* automorphisms,

$$p_{i,k}\colon \begin{cases} x_i \mapsto x_i^{x_k} \\ x_j \mapsto x_j, \quad j \neq i \end{cases}$$

$$s_\pi\colon x_i \mapsto x_{\pi(i)}, \quad \pi \in S_n.$$

It is easy to see that $p_{i,k}, s_\pi$ are alternative generators for BP_n indeed:

$$\sigma_i = p_{i+1,i}\tau_i \quad \text{and} \quad \tau_i = s_{t_i}$$

where t_i is the transposition $(i, i+1)$ and

$$p_{i,k} = \begin{cases} \tau_i\tau_{i+1} \dots \tau_{k-1}\sigma_{k-1}\tau_{k-2} \dots \tau_i & \text{if } i < k \\ \tau_{i-1}\tau_{i-2} \dots \tau_{k+1}\sigma_k\tau_k \dots \tau_{i-1} & \text{if } i > k. \end{cases}$$

(Note that products in these formulae are read from left to right.)

THEOREM 4.1. $BP_n = PC_n$.

Proof. We have to prove that if f is an automorphism of F_n of permutation-conjugacy type then f is a product of elementary automorphisms. For the definitions of *sets of PC-type* and *double Nielsen transformations*, see [2, pp. 402, 403].

Let $f(x_i) = u_i$ then $\{u_i\}$ is a set of words of PC-type. Now composition of f with the elementary automorphism $p_{i,k}$ for $i \neq k$ realises a double Nielsen transformation on $\{u_i\}$. (If $i = k$ the automorphism has no effect.) Therefore, by [2, Lemma 8.5] we may assume that $\{u_i\}$ is Nielsen reduced. But by [2, Corollary 8.2] this implies that these words are a permutation of the words $x_i, i = 1, \dots, n$, and therefore f is the elementary automorphism s_π for suitable π. □

4.2. *Racks and quandles*

Finally, we turn to the theory which originally prompted our interest in the subgroups of Aut F_n studied in the previous sections. We have throughout written a^b for the conjugate $b^{-1}ab$ in a group. This leads naturally to the binary operation of a rack which can be thought of as generalised conjugation. A rack or quandle is an algebraic gadget intimately associated with a knot or link. Racks and quandles have been defined by many authors. We refer to [2] for full definitions and examples.

Note that *from now on exponential notation is no longer used as a shorthand for conjugation in groups.*

In [1, pp. 391–393] are defined the automorphism groups of the free rack and quandles (denoted Aut FR_n and Aut FQ_n, respectively) on the basis $\{x_1, \dots, x_n\}$. The elementary properties of these groups and explanation of their connection with knots and links can be found here. In particular, it is proved that Aut FQ_n is isomorphic to a subgroup of Aut F_n [2, p. 393, top], which coincides with BP_n [2, Theorem 8.7].

Let σ_i and τ_i, $i = 1, 2, \dots, n-1$, be the elements of Aut FQ_n and Aut FR_n which are defined in an exactly similar way to their namesakes in Aut F_n. The *elementary automorphisms* $p_{i,k}$ and s_π of FQ_n and FR_n are also defined in exactly the same way as for F_n. By [2, Theorem 8.7] these elementary automorphisms generate the two automorphism groups. Now let ρ_i, $i = 1, 2, \dots, n$, be the elements of Aut FR_n defined by

$$\begin{cases} x_i \mapsto x_i^{x_i} \\ x_j \mapsto x_j, \quad j \neq i. \end{cases}$$

In other words, ρ_i is $p_{i,i}$. It is easily seen that

$$\rho_i\sigma_i = \sigma_i\rho_{i+1}, \qquad \rho_i\tau_i = \tau_i\rho_{i+1} \quad \text{and} \quad \rho_i\rho_j = \rho_j\rho_i.$$

Let η: Aut $FR_n \to$ Aut FQ_n be the natural map. Then the kernel of η is R the subgroup of Aut FR_n generated by the ρ_i. Clearly, R is isomorphic to the lattice group $\mathbb{Z}^n$. Thus, Aut FR_n is a semi-direct product of Aut FQ_n with R; moreover, the action on R permutes the factors. In other words, Aut FR_n is the permutation wreath product of Aut FQ_n with $\mathbb{Z}$.

In terms of presentations we now have the following.

THEOREM 4.2. *The group* Aut FQ_n *has a finite presentation identical to that for* BP_n *given in Section* 1. *The group* Aut FR_n *has a finite presentation with generators* σ_i, τ_i $(i = 1, \dots, n-1)$ *and* ρ_i $(i = 1, \dots, n)$ *as defined above and relations of five types: Braid relations, permutation group relations and mixed relations (identical to those for* BP_n *given in*

Section 1) *and the following two new types of relation*:

$$(\textit{commuting relations}) \qquad \rho_i\rho_j = \rho_j\rho_i$$

$$(\textit{further mixed relations}) \quad \begin{cases} \rho_i\sigma_i = \sigma_i\rho_{i+1} \\ \rho_i\tau_i = \tau_i\rho_{i+1}. \end{cases} \qquad \square$$

4.3. *Final remarks*

The theory of the welded braids developed in this paper has a corresponding theory of welded knots and links. For example, gluing the top of welded braid to the bottom yields a welded link in half 3-space. This can be generalised to yield a theory of welded links in any 3-manifold with boundary. This theory is important because any such link has a fundamental rack: for a glued braid this can be calculated by quotienting the free rack by the automorphism which the welded braid determines. (This automorphism can be computed from the welded braid by the same method as we gave in Section 3 for computing the induced automorphism of the free group.) The theory of racks can then be applied to yield invariants for these generalised links. We shall investigate this further in a subsequent paper.

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REFERENCES

1. J. Birman: Braid links and mapping class groups, *Ann. Math. Stud.* **82**, Princeton (1975).
2. R. Fenn and C. Rourke: Racks and links in codimension two, *J. Knot Theory Ramifications* **1** (1992), 343–406.
3. R. Fenn, R. Rimanyi and C. Rourke: Some remarks on the braid-permutation group, *Proc. the Erzurum Conf.* Turkey, 1992, Kluwer, Dordrecht (1992), pp. 57–68.
4. J. Baez: Link invariants of finite type and permutation theory, *Lett. Maths. Phys.* **2** (1992), 43–51.
5. J. S. Birman: New points of view in knot theory, *Bull. A.M.S.* (1993), 253–287.
6. M. Khovanov: New generalisations of braids and links, preprint.
7. R. Fenn, E. Keyman and C. Rourke: Singular braids embed in a group, preprint.
8. B. Krüger, Automorphe Mengen und die Artinschen Zopfgruppen, *Bonner Math. Schriften* (1990).
9. S. P. Humphries: On weakly distinguished bases and free generating sets of free groups, *Quart. J. Math. Oxford* **920** (1985), 215–219.
10. H. D. Coldewey, E. Vogt and H. Zieschang: Surfaces and planar discontinuous groups, SLNM **835**, Springer, Berlin (1970).
11. D. Collins and H. Zieschang: On the Nielsen method in free products with amalgamated subgroups, *Math. Z.* **197** (1987), 97–118.
12. K. H. Ko: Pseudo-conjugations, *Bull. Korean Math. Soc.* **25** (1988), 247–251.

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