A GENERALIZATION OF BANCHOFF'S TRIPLE POINT THEOREM

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Abstract. Consider an immersion of a surface into S^3. Banchoff's theorem [B] states that the parity of the number of triple points and the parity of the Euler characteristic of the surface coincide. Here we generalize this theorem to codimension 1 immersions of arbitrary even dimensional manifolds in spheres. The proof is an analogue of a proof of Banchoff's theorem circulated in preprint form by R. Fenn and P. Taylor in 1977 [FT].

Let us consider a codimension 1 smooth generic (i.e. self-transverse) immersion f of a closed manifold M^n in the sphere S^{n+1}. Let us recall how a neighborhood of an i-tuple point (in $R^{n+1} \subset S^{n+1}$) looks like in such a self-transverse immersion. Consider the coordinate hyperplanes in R^i and take the direct product of this configuration with R^{n+1-i}. What is obtained is diffeomorphic to the neighborhood of an i-tuple point in the image of f.

For any natural number i, $1 \le i \le n+1$, let us denote by $\tilde{\Delta}_i$ the set of i-tuple points in S^{n+1} i.e.

$$\tilde{\Delta}_i = \{y \in S^{n+1} \mid f^{-1}(y) \text{ consists of } i \text{ different points}\}.$$

As it is well known, *dim* $\tilde{\Delta}_i = n+1-i$, and $\cup_{r=i}^{\infty} \tilde{\Delta}_r$ is an immersed manifold (although it is not in general position i.e. it is the image of a *non*-selftransverse immersion). Let Δ_i be a closed manifold such that $\cup_{r=i}^{\infty} \tilde{\Delta}_r$ is the image of an immersion of Δ_i in S^{n+1}.

Remark. Of course, many different manifolds can be immersed into S^{n+1} so that their images are $\cup_{r=i}^{\infty} \tilde{\Delta}_r$. For example if a possible Δ_i is given, then any of its finite coverings serves as well. We make the choice of Δ_i explicit by assuming that the i-tuple points of f are non-multiple points of the immersion $\Delta_i \looparrowright S^{n+1}$.

We shall call the manifold Δ_i the i-tuple manifold of f. Our theorem claims that for n even the sum of the Euler characteristics of i-tuple manifolds is even. (For $n = 2$ this is exactly Banchoff's theorem.)

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Theorem. *If $n > 0$ is even, then*

$$\sum_{i=1}^{n+1} \chi(\Delta_i) \equiv 0 \ mod\ 2.$$

The following proof is an analogue of the proof in [FT] for Banchoff's triple point theorem.

Proof. Since n is even, we can omit the terms corresponding to even i's, because in those cases the dimension of Δ_i is odd. Now let us triangulate the *image* of f such a way that for any i the set of points of multiplicity i or higher forms a subcomplex of $f(M)$.

Let α_r^i denote the number of i-dimensional simplexes whose interiors lie in $\tilde{\Delta}_r$, and let

$$\beta_r = \alpha_r^0 - \alpha_r^1 + \ldots \pm \alpha_r^{n+1-r}.$$

Observe, that β_r is *not* the Euler characteristic of any complex. However, we have that

$$\chi(\Delta_i) = \sum_{r=i}^{n+1} \binom{r}{i} \beta_r.$$

The coefficient $\binom{r}{i}$ counts the multiplicity of the self-intersection of Δ_i at $\tilde{\Delta}_r$. So

$$\sum_{i=1}^{n+1}{}^{*} \chi(\Delta_i) = \sum_{i=1}^{n+1}{}^{*} \sum_{r=i}^{n+1} \binom{r}{i} \beta_r,$$

where * indicates that the sum is taken only for odd i's. After changing the order of the summations we get:

$$\sum_{r=1}^{n+1} \Big(\sum_{i=1}^{r}{}^{*} \binom{r}{i}\Big) \beta_r = \sum_{r=1}^{n+1} 2^{r-1} \beta_r \equiv \beta_1 \ mod\ 2. \tag{1}$$

Now let us color the complement of $f(M)$ in S^{n+1} in two colors in a chessboard-style, i. e. let any two neighboring domains have different colors (where "neighboring" means that they are separated by a component of $\tilde{\Delta}_1$). This is possible, since $H_n(S^{n+1}; Z_2) = 0$.

Let N be the boundary of an ε-neighborhood of $f(M)$ in the black subset of S^{n+1}. Notice, that from the given triangulation of $f(M)$ we can construct a triangulation of N by pushing the simplexes from $f(M)$ to N in a reasonable way. Simplexes in $\tilde{\Delta}_i$ will have 2^{i-1} counterparts in N (i hyperplanes divide the Euclidean n-space into 2^i parts, half of which are black). Thus:

$$\chi(N) = \sum_{i=1}^{n+1} 2^{i-1} \beta_i \equiv \beta_1 \ mod\ 2.$$

But $\chi(N)$ is even, because N is embedded in codimension 1 (and $n > 0$), so the proof is complete.

Remark 1. As it is clear from the proof, the space S^{n+1} can be replaced by any manifold such that its n^{th} Z_2-homology group is 0.

Remark 2. The above proof does not work for n odd, since the sum $\sum_{i=1}^{r} {}^* \binom{r}{i}$ (where the star this time means summation for even i's) equals to $2^{r-1} - 1$, so the sum in formula (1) gives $\sum_{r=2}^{n+1} \beta_r$ (which is clearly the Euler characteristic of the complex $f(M)$).

The figure 8 immersion of the circle in the plane shows that the statement of the theorem is false for $n = 1$. A theorem of Freedman [F] (and its generalization to unoriented 3-manifolds given in [A]) shows that it is true for $n = 3$. We do not know whether it is true or not for $n > 3$.

Remark 3. If we consider only *oriented* n-manifolds and their codimension 1 immersions in S^{n+1}, and the n^{th} stable homotopy group of spheres has no 2-primary torsion, then the Euler characteristics of the i-tuple manifolds are all even, for any i. (Indeed, for any i $\chi(\Delta_i)$ *mod* 2 defines a homomorphism from the stable homotopy group $\pi_{n+N}(S^N)$, $N >> n$ to Z_2.)

In particular the statement of the theorem is true for $n = 5$ or $n = 13$ for oriented manifolds.

Remark 4. If the dimension $n = 4$, then more is true than it is stated in the theorem, namely all $\chi(\Delta_i)$'s are even, since the stable homotopy group $\pi_5^s(RP^\infty)$ vanishes (see [L]), and this group is isomorphic to the cobordism group of immersions of 4-manifolds into R^5.

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