PARTITION IDENTITIES AND QUIVER REPRESENTATIONS

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ABSTRACT. We present a particular connection between classical partition combinatorics and the theory of quiver representations. Specifically, we give a bijective proof of an analogue of A. L. Cauchy's Durfee square identity to multipartitions. We then use this result to give a new proof of M. Reineke's identity in the case of quivers $\mathcal{Q}$ of Dynkin type A. Our identity is stated in terms of the lacing diagrams of S. Abeasis–A. Del Fra, which parameterize orbits of the representation space of $\mathcal{Q}$ for a fixed dimension vector.

CONTENTS

1. INTRODUCTION

The main goal of this paper is to establish a specific connection between classical partition combinatorics and the theory of quiver representations. Our motivation is to give an elementary proof for a family of identities introduced by M. Reineke [Rei10]. The identities are closely related to cluster algebras (see e.g., work of V. V. Fock–A. B. Goncharov [FG09] and references therein), wall crossing phenomena (see e.g., the paper [DM16] of B. Davison–S. Meinhardt as well as the references therein), and Donaldson-Thomas invariants and Cohomological Hall Algebras (see, e.g., the work of M. Kontsevich–Y. Soibelman [KS11]). This paper is intended to be an initial step towards understanding the rich combinatorics encoded by advanced dilogarithm identities, such as B. Keller's identities [Kel11]. We give a new explanation for M. Reineke's identities in type A via generating series arguments.

Following the conventions of [Rim13], we define the **quantum dilogarithm series**

(1)
$$\mathbb{E}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k q^{k^2/2}}{(1-q)(1-q^2)\dots(1-q^k)}.$$

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In each term of (1), the denominator may be written more compactly using the q-shifted factorial,

$$(q)_k = (1-q)(1-q^2)\cdots(1-q^k).$$

This has an interpretation in terms of partitions; the reciprocal of $(q)_k$ is the generating series for partitions with at most k parts [And84, Theorem 1.1].

There are many interesting identities among quantum dilogarithms. We highlight the following, which specializes to the *pentagon identity* of Rogers' dilogarithm.

Theorem 1.1 ([Sch53] [FV93], [FK94])**.** *Suppose x and y are formal variables so that $yx = qxy$. Then*

$$\mathbb{E}(x)\mathbb{E}(y) = \mathbb{E}(y)\mathbb{E}(-q^{1/2}xy)\mathbb{E}(x). \tag{2}$$

M. Reineke extended (2) to give a family of identities, one for each *Dynkin quiver* ([Rei10], [Kel11]). The quantum pentagon identity corresponds to the quiver which has two vertices connected by a single edge. In the present work, we show that Theorem 1.1 can actually be proven using the combinatorial tool of *Durfee rectangles*. In fact, we give a proof of M. Reineke's identity in type A by proving related identities using iterated Durfee rectangles on *multipartitions*. To state these identities, we first give some necessary background on quivers.

A **quiver**, $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a directed graph with vertex set $\mathcal{Q}_0$ and arrows $\mathcal{Q}_1$. Throughout, we will assume $\mathcal{Q}$ has finitely many vertices and identify Q_0 with $[n] = \{1, 2, \ldots, n\}$. For $a \in \mathcal{Q}_1$, let $h(a)$ be the head of the arrow and $t(a)$ its tail. The **Euler form**

$$\chi_{\mathcal{Q}} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

is defined by

$$\chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \sum_{i \in Q_0} \mathbf{d}_1(i)\mathbf{d}_2(i) - \sum_{a \in Q_1} \mathbf{d}_1(t(a))\mathbf{d}_2(h(a)). \tag{3}$$

Define

$$\lambda_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \chi_{\mathcal{Q}}(\mathbf{d}_2, \mathbf{d}_1) - \chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2).$$

Write $\mathbb{N}$ for the set of nonnegative integers. Following [Rim13], the **quantum algebra** $\mathbb{A}_{\mathcal{Q}}$ is generated over $\mathbb{Q}(q^{1/2})$ by

$$\{z_{\mathbf{d}} : \mathbf{d} \in \mathbb{N}^n\}$$

with multiplication given by

$$z_{\mathbf{d}_1} z_{\mathbf{d}_2} = -q^{1/2\lambda_{\mathbf{Q}}(\mathbf{d}_1, \mathbf{d}_2)} z_{\mathbf{d}_1 + \mathbf{d}_2}.$$

Reineke's identities are among quantum dilogarithms evaluated on elements of $\mathbb{A}_{\mathcal{Q}}$. To state them, we require some background regarding quiver representations. We briefly recall the relevant facts here. For a self contained introduction to quiver representations, see [Bri08]. Throughout, we take $\mathbb{C}$ to be our ground field. A representation V of $\mathcal{Q}$ is an assignment of a vector space V_i to each $i \in \mathcal{Q}_0$ and a linear transformation

$$\mathsf{V}_a : \mathsf{V}_{t(a)} \to \mathsf{V}_{\mathsf{h(a)}}$$

for each arrow $a \in \mathcal{Q}_1$. Each representation V of $\mathcal{Q}$ has an associated **dimension vector**

$$\mathbf{d}_{\mathsf{V}} = (\mathbf{d}_{\mathsf{V}}(1), \ldots, \mathbf{d}_{\mathsf{V}}(n)) \in \mathbb{N}^n, \text{ where } \mathbf{d}_{\mathsf{V}}(i) = \dim \mathsf{V}_i.$$

A **morphism** $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ is a collection of linear transformations $(\mathsf{T}_i : \mathsf{V}_i \to \mathsf{W}_i)_{i \in \mathcal{Q}_0}$ such that

$$\mathsf{T}_{h(a)}\mathsf{V}_a = \mathsf{W}_a\mathsf{T}_{t(a)} \text{ for every arrow } a \in \mathcal{Q}_1.$$

If each of the T_i's are isomorphisms, then V and W are **isomorphic** representations.

A representation is **simple** if it has no proper sub-representation. A representation is **indecomposable** if it does not admit a nontrivial decomposition as a direct sum of two representations. A quiver is **Dynkin** if its underlying undirected graph is a Dynkin diagram of type ADE. The representation theory of Dynkin quivers is particularly well behaved; if $\mathcal{Q}$ is Dynkin, it has finitely many isomorphism classes of simple and indecomposable representations. Furthermore, these classes are uniquely determined by their dimension vectors.

Theorem 1.2 ([Rei10]). *If $\mathcal{Q}$ is Dynkin, there exists an ordering on the dimension vectors for the simple representations $\alpha_1, \dots, \alpha_n$ and the indecomposable representations $\beta_1, \dots, \beta_N$ so that*

$$\mathbb{E}(z_{\alpha_1}) \cdots \mathbb{E}(z_{\alpha_n}) = \mathbb{E}(z_{\beta_1}) \cdots \mathbb{E}(z_{\beta_N}). \tag{4}$$

Proving Theorem 1.2 is equivalent to showing that for every $\mathbf{d} \in \mathbb{N}^n$ the coefficient of $z_{\mathbf{d}}$ is equal on both sides of the expression (4). This calculation of these coefficients is carried out in [Rim13]. Here, the identity is restated in terms of the geometry of quiver representations.

Let $\mathsf{Mat}(m, n)$ be the space of $m \times n$ matrices. The **representation space** is

$$\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d}) := \bigoplus_{a \in \mathcal{Q}_1} \mathsf{Mat}(\mathbf{d}(h(a)), \mathbf{d}(t(a))).$$

A matrix in $\mathsf{Mat}(m, n)$ determines a map from $\mathbb{C}^n$ to $\mathbb{C}^m$. As such, points of $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ determine $\mathbf{d}$ dimensional representations of $\mathcal{Q}$. Conversely, any $\mathbf{d}$ dimensional representation is isomorphic to some $\mathsf{V} \in \mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$. Let

$$\mathsf{GL}_{\mathcal{Q}}(\mathbf{d}) := \prod_{x \in \mathcal{Q}_0} \mathsf{GL}(\mathbf{d}(x)).$$

$\mathsf{GL}_{\mathcal{Q}}(\mathbf{d})$ acts on $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ by base change. Write $\mathcal{O}_{\mathcal{Q}}(\mathbf{d})$ for the set of orbits in $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$. Given $\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$, let $\operatorname{codim}_{\mathbb{C}}(\gamma)$ denote the complex codimension of γ in $\mathsf{Rep}_{Q}(\mathbf{d})$. Pick any representation $\mathsf{V} \in \gamma$. Then by complete reducibility,

$$\mathsf{V} \cong \bigoplus_{i=1}^{N} \mathsf{V}_{\beta_i}^{\oplus m_{\beta_i}},$$

where V_{β_i} is an indecomposable representation so that $\dim(\mathsf{V}_{\beta_i}) = \beta_i$. In fact, any $\mathsf{V}' \in \gamma$ has this same irreducible decomposition; the m_{β_i}'s are constant on orbits. So we define $m_{\beta_i}(\gamma)$ to be the multiplicity of V_{β_i} in the irreducible decomposition of any $\mathsf{V} \in \gamma$.

Theorem 1.3 ([Rim13]). *For each dimension vector $\mathbf{d} = (\mathbf{d}(1), \mathbf{d}(2), \dots, \mathbf{d}(n))$,*

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma)}} \tag{5}$$

We now restrict our focus to a special case. Assume $\mathcal{Q}$ is a type A quiver, i.e. its underlying graph is just a path on n vertices. We label the vertices from left to right with the set $\{1, 2, \dots, n\}$. A **lacing diagram** [ADF80] $\mathcal{L}$ is a graph so that:

(1) the vertices are arranged in n columns labeled $1, 2, \ldots, n$ (left to right) and
(2) the edges between adjacent columns form a partial matching.

A **strand** is a connected component of $\mathcal{L}$. A strand is of **type** $[i, j]$ if it starts in column i and ends in column j. Write

$$m_{[i,j]}(\mathcal{L}) = \#\{\text{strands of type } [i, j] \text{ in } \mathcal{L}\}. \tag{6}$$

There is an explicit dictionary between representations of $\mathcal{Q}$ and lacing diagrams. Each lacing diagram may be interpreted as a sequence of **partial permutation matrices**. This sequence defines a representation $\mathsf{V}_{\mathcal{L}} \in \mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$. We do not give the details here, as we are not concerned with the representations themselves, but merely their dimension vectors. See [KMS06] for the equioriented case and [BR07] for quivers of arbitrary orientation.

Let $\mathcal{L}_{[i,j]}$ be the lacing diagram that has a single strand of type $[i, j]$. Define $\mathsf{V}_{[i,j]} := \mathsf{V}_{\mathcal{L}_{[i,j]}}$. Each $\mathsf{V}_{[i,j]}$ is indecomposable. In fact, up to isomorphism, these are the only indecomposable representations of $\mathcal{Q}$. This is a consequence of Gabriel's theorem. The simple representations are the special case $\mathsf{V}_{[i,i]}$.

The strands in $\mathcal{L}$ immediately reveal the irreducible decomposition of $\mathsf{V}_{\mathcal{L}}$:

$$\mathsf{V}_{\mathcal{L}} \cong \bigoplus_{1 \le i \le j \le n} \mathsf{V}_{[i,j]}^{\oplus m_{[i,j]}(\mathcal{L})}. \tag{7}$$

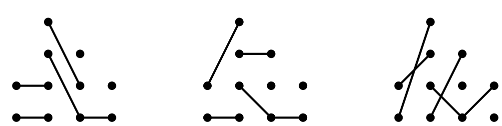
We associate a dimension vector to $\mathcal{L}$. Write

$$\mathbf{dim}(\mathcal{L}) = (\mathbf{d}_{\mathcal{L}}(1), \ldots, \mathbf{d}_{\mathcal{L}}(n))$$

where $\mathbf{d}_{\mathcal{L}}(k)$ is the number of vertices in column k of $\mathcal{L}$. Equivalently, by counting the number of strands which use a vertex of column k, we have

$$\mathbf{d}_{\mathcal{L}}(k) = \sum_{1 \le i \le k \le j \le n} m_{[i,j]}(\mathcal{L}). \tag{8}$$

Translating from lacing diagrams to representations, we have $\mathbf{dim}(\mathcal{L}) = \mathbf{dim}(V_{\mathcal{L}})$.



Two lacing diagrams are **equivalent** if they only differ by reordering of vertices within columns. For example, the lacing diagrams pictured above are all equivalent. Alternatively, we may say

$$[\mathcal{L}] = [\mathcal{L}'] \text{ if and only if } m_{[i,j]}(\mathcal{L}) = m_{[i,j]}(\mathcal{L}') \text{ for all } 1 \le i \le j \le n.$$

Therefore, we will write $m_{[i,j]}([\mathcal{L}]) := m_{[i,j]}(\mathcal{L})$. Using (7), it follows that isomorphism classes of representations are in bijection with equivalence classes of lacing diagrams:

$$V_{\mathcal{L}} \cong V_{\mathcal{L}'} \text{ if and only if } [\mathcal{L}] = [\mathcal{L}'].$$

Let

$$\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) = \{[\mathcal{L}] : \mathbf{dim}(\mathcal{L}) = \mathbf{d}\}$$

denote the set of equivalence classes of $\mathbf{d}$ dimensional lacing diagrams. Given $\eta = [\mathcal{L}] \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$, write $\gamma_\eta \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$ for the orbit which contains $\mathsf{V}_{\mathcal{L}}$. The map $\eta \mapsto \gamma_\eta$ defines a bijection from $\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) \to \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$.

We now associate certain statistics to η. Set parameters

$$s_i^k(\eta) = m_{[i,k-1]}(\eta), \text{ and} \tag{9}$$

$$t_j^k(\eta) = m_{[j,k]}(\eta) + m_{[j,k+1]}(\eta) + \ldots + m_{[j,n]}(\eta). \tag{10}$$

Let $\mathfrak{S}_i$ denote the ith symmetric group. Fix a sequence of permutations

$$\mathbf{w} = (w^{(1)}, \ldots, w^{(n)}), \text{ where } w^{(i)} \in \mathfrak{S}_i \text{ and } w^{(i)}(i) = i. \tag{11}$$

The partition combinatorics behind Theorem 1.4 below suggests the **Durfee statistic**:

$$r_{\mathbf{w}}(\eta) = \sum_{1 \le i < j \le k \le n} s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta). \tag{12}$$

With these definitions, we now state our main theorem.

Theorem 1.4 (Quiver Durfee Identity). *For $\mathbf{d} = (\mathbf{d}(1), \ldots, \mathbf{d}(n))$ and $\mathbf{w}$ as in (11),*

$$\prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} = \sum_{\eta \in \mathcal{C}_Q(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta) + s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q . \tag{13}$$

Here

$$\begin{bmatrix} i+j \\ j \end{bmatrix}_q = \frac{(q)_{i+j}}{(q)_i (q)_j}$$

is the q-**binomial coefficient**, the generating series for partitions with at most i rows and j columns [And84, Theorem 3.1]. Indeed, we will show in Lemma 3.1 that each side of (13) has an interpretation as the generating series of a set of multipartitions. By doing some algebraic cancellations, Theorem 1.4 implies the following:

Corollary 1.5.

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\eta \in \mathcal{C}_Q(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{1 \le i \le j \le n} \frac{1}{(q)_{m_{[i,j]}(\eta)}} \tag{14}$$

This is our link to Reineke's identity. In Definition 4.1, we assign each type A quiver a sequence of permutations $\mathbf{w}_Q$. We then show this choice satisfies

Theorem 1.6.

$$r_{\mathbf{w}_Q}(\eta) = \operatorname{codim}_{\mathbb{C}}(\gamma_\eta).$$

For type A, Theorem 1.3 follows as a consequence of Corollary 1.5 and Theorem 1.6.

The paper is organized as follows. In Section 2, we recall some background on generating series. In Section 3, we define sets S and T so that the left hand side of (13) is a generating series for S and the right hand side is a generating series for T. We give an explicit bijection between S and T, thus proving Theorem 1.4. By simple algebraic cancellations, we prove Corollary 1.5. Finally, in Section 4, we prove Theorem 1.6, thus completing our proof.

2. GENERATING SERIES FOR PARTITIONS

First we recall some background on generating series. Let A be a set equipped with a weight function

$$\mathtt{wt}_A : A \to \mathbb{N}.$$

Suppose

$$a_n := |a \in A : \mathtt{wt}(a) = n| < \infty$$

for each n. Then the **generating series** for A is

$$G(A, q) := \sum_{a \in A} q^{\mathtt{wt}_A(a)}. \tag{15}$$

Equivalently, by collecting like terms,

$$G(A, q) = \sum_{i=0}^{\infty} a_i q^i. \tag{16}$$

Generating series are well behaved under taking products and disjoint unions of sets. Define

$$\mathtt{wt}_{A\times B}(a, b) = \mathtt{wt}_A(a) + \mathtt{wt}_B(b).$$

Then

$$G(A \times B, q) = G(A, q)G(B, q). \tag{17}$$

For disjoint unions, the generating series is additive:

$$G(A \sqcup B, q) = G(A, q) + G(B, q). \tag{18}$$

Here, we focus on generating series for *multipartitions*. A **partition** is a finite sequence of weakly decreasing, nonnegative integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0).$$

The λ_i are the **parts** of λ. Define the **length** of λ to be $\ell(\lambda) = \ell$, the number of positive parts of λ. We represent λ visually by its **Young diagram**, a collection of boxes arranged in rows so that the number of boxes in row i equals λ_i. Each partition has an associated weight

$$\mathtt{wt}(\lambda) = |\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i. \tag{19}$$

Equivalently, $\mathtt{wt}(\lambda)$ is the total number of boxes in the Young diagram of λ. A **multipartition** is simply a tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(i)})_{i\in I}$. We weight $\boldsymbol{\lambda}$ by defining

$$\mathtt{wt}(\boldsymbol{\lambda}) = \sum_{i\in I} \mathtt{wt}(\lambda^{(i)}).$$

Let $p_k = \{\lambda : \mathtt{wt}(\lambda) = k\}$. Famously due to L. Euler, the generating series for the set of all partitions is

$$\sum_{k=0}^{\infty} p_k q^k = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}. \tag{20}$$

Throughout, we will be interested in subsets of partitions which have constraints placed on the total number of rows or columns in their Young diagram. Let

$$\mathcal{P}(i,j) = \{\lambda : \ell(\lambda) \leq i \text{ and } \lambda_1 \leq j\}.$$

Here we allow for i or j to be infinite. When i and j are finite,

$$G(\mathcal{P}(i,j),q) = \begin{bmatrix} i+j \\ i \end{bmatrix}_q. \tag{21}$$

The generating series for $\mathcal{P}(\infty,k)$, as well as $\mathcal{P}(k,\infty)$, is obtained by truncating the product in (20):

$$\frac{1}{(q)_k} = \prod_{i=1}^{k} \frac{1}{1-q^i}. \tag{22}$$

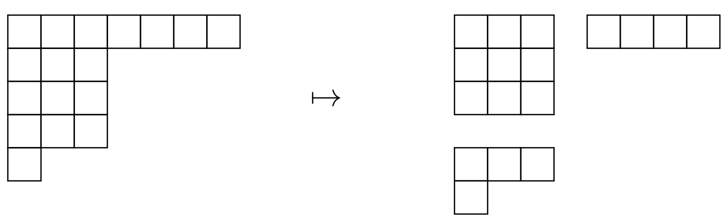
Write $i \times j$ for the rectangular partition with i parts of size j and let $\mathcal{R}(i,j) = \{i \times j\}$. Immediately from (15),

$$G(\mathcal{R}(i,j),q) = q^{ij}. \tag{23}$$

The following identity is due to Euler:

$$\frac{1}{(q)_\infty} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{((q)_j)^2}. \tag{24}$$

We sketch a textbook bijective proof. The **Durfee square** $D(\lambda)$ is the largest $j \times j$ square partition that fits inside λ. Draw $D(\lambda)$ inside of λ so that it is justified against the top left corner. By cutting λ along the boundary of $D(\lambda)$, we may divide λ into three smaller partitions, as pictured below.



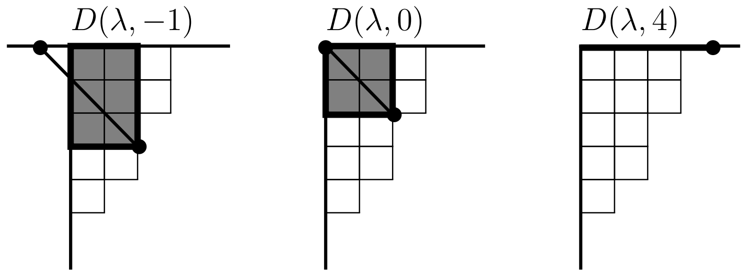
This decomposition defines a bijection:

$$\mathcal{P}(\infty,\infty) \xrightarrow{\sim} \bigcup_{j=0}^{\infty} \mathcal{R}(j,j) \times \mathcal{P}(j,\infty) \times \mathcal{P}(\infty,j).$$

See [And84, pp 27-28] for details and related identities.

The present work uses a generalization of the Durfee square. Fix $r \in \mathbb{Z}$. The **Durfee rectangle** $D(\lambda,r)$ is the largest $i \times (i+r)$ rectangular partition contained in λ. By convention, we say *any* 0-width or 0-height rectangle is contained in λ. Equivalently, $D(\lambda,r)$ is the rectangle with top left corner positioned at $(0,0)$ and bottom right corner where the line $x+y=r$ intersects the (infinite) boundary line of the partition.

Example 2.1. Let $\lambda = (3,3,2,2,1)$. Pictured below are the Durfee rectangles $D(\lambda,r)$ for $r = -1,0,4$.



Notice that $D(\lambda, 4) = 0 \times 4$ rectangle since the line $x + y = 4$ intersects the boundary of λ at the point $(4, 0)$. $\square$

Decomposing λ using $D(\lambda, r)$ gives a proof of the following identity of B. Gordon and L. Houten [GH68, pp. 91-92]:

$$\frac{1}{(q)_\infty} = \sum_{i=\max\{0,-r\}}^{\infty} \frac{q^{i(r+i)}}{(q)_i (q)_{r+i}}. \tag{25}$$

The A_2 case of Theorem 1.4 can be proved using a truncated version of (25). We sketch the explicit connection here. Fix $r \leq k$. We can split $\lambda \in \mathcal{P}(\infty, k)$ into three partitions using $D(\lambda, r)$. This defines a bijection

$$\mathcal{P}(\infty, k) \xrightarrow{\sim} \bigcup_{i=\max\{0,-r\}}^{k-r} \mathcal{R}(i, r+i) \times \mathcal{P}(\infty, r+i) \times \mathcal{P}(i, k-(r+i))$$

which corresponds to the following identity of generating series:

$$\frac{1}{(q)_k} = \sum_{i=\max\{0,-r\}}^{k-r} \frac{q^{i(r+i)}}{(q)_{r+i}} \begin{bmatrix} k-(r+i)+i \\ i \end{bmatrix}_q. \tag{26}$$

We may rephrase (26) in the language of lacing diagrams. Set $n = 2$ and fix a dimension vector $\mathbf{d} = (k - r, k)$. Choose a $\mathbf{d}$-dimensional lacing diagram $\mathcal{L}$ such that $m_{[1,1]}(\mathcal{L}) = i$. Since $m_{[1,1]}(\mathcal{L}) + m_{[1,2]}(\mathcal{L}) = k - r$, necessarily $m_{[1,2]}(\mathcal{L}) = k - r - i$. Similarly, $m_{[2,2]}(\mathcal{L}) = r + i$.

We reindex the sum in (26) and obtain

$$\frac{1}{(q)_{\mathbf{d}(2)}} = \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{m_{[1,1]}(\eta) m_{[2,2]}(\eta)}}{(q)_{m_{[2,2]}(\eta)}} \begin{bmatrix} m_{[1,1]}(\eta) + m_{[2,2]}(\eta) \\ m_{[1,1]}(\eta) \end{bmatrix}_q. \tag{27}$$

For any η, we have $t_1^1(\eta) = \mathbf{d}(1)$. Dividing both sides of (27) by $(q)_{\mathbf{d}(1)}$ and using the equations (9) and (10) gives

$$\frac{1}{(q)_{\mathbf{d}(1)} (q)_{\mathbf{d}(2)}} = \frac{1}{(q)_{\mathbf{d}(1)}} \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{s_1^2(\eta) t_2^2(\eta)}}{(q)_{t_2^2(\eta)}} \begin{bmatrix} s_1^2(\eta) + t_2^2(\eta) \\ s_1^2(\eta) \end{bmatrix}_q. \tag{28}$$

We have $\mathbf{d}(1) = t_1^1(\eta)$ for any $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. So we obtain

$$\frac{1}{(q)_{\mathbf{d}(1)} (q)_{\mathbf{d}(2)}} = \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{s_1^2(\eta) t_2^2(\eta)}}{(q)_{t_1^1(\eta)} (q)_{t_2^2(\eta)}} \begin{bmatrix} s_1^2(\eta) + t_2^2(\eta) \\ s_1^2(\eta) \end{bmatrix}_q. \tag{29}$$

This is the $n = 2$ case of Theorem 1.4.

For $n > 2$, the proof of Theorem 1.4 uses multiple Durfee rectangles. This technique is similar to the *Durfee dissections* of A. Schilling [SW98]. See also the work of C. Boulet on successive Durfee rectangles [Bou10]. We also note the resemblance to the *Durfee systems* of P. Bouwknegt [Bou02]. Also see the references to *loc. cit.* for other work on generalized Durfee square identities. Our main point of difference is that these identities do not directly concern lacing diagrams.

3. Proof of Theorem 1.4

Throughout this section, fix a dimension vector $\mathbf{d} = (\mathbf{d}(1), \dots \mathbf{d}(n))$ and a sequence permutations as in (11):

$$\mathbf{w} = (w^{(1)}, \dots, w^{(n)}) \text{ with } w^{(i)} \in \mathfrak{S}_i \text{ and } w^i(i) = i.$$

Define

$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \ldots \times \mathcal{P}(\infty, \mathbf{d}(n)). \tag{30}$$

Let

$$R(\eta) = \{\boldsymbol{\mu} = (\mu^k_{i,j}) : \mu^k_{i,j} \in \mathcal{R}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(j)}(\eta)), 1 \le i < j \le k \le n\} \tag{31}$$

which consists of a single element, a tuple of rectangles. For ease of notation, we write $s^k_k(\eta) = \infty$ for each k. Let

$$P(\eta) = \{\boldsymbol{\nu} = (\nu^k_i) : \nu^k_i \in \mathcal{P}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(i)}(\eta)), 1 \le i \le k \le n\}. \tag{32}$$

Define

$$T(\eta) = R(\eta) \times P(\eta). \tag{33}$$

Finally, we let

$$T = \bigcup_{\eta \in \mathcal{C}_Q(\mathbf{d})} T(\eta). \tag{34}$$

Weight $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}) \in S$ by defining

$$\mathtt{wt}_S(\boldsymbol{\lambda}) = \sum_{k=1}^{n} |\lambda^{(k)}|.$$

Assign $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$ the weight

$$\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{1 \le i < j < k \le n} |\mu^k_{i,j}| + \sum_{1 \le i \le k \le n} |\nu^k_i|.$$

Lemma 3.1. (1) *The generating series for S is*

$$G(S, q) = \prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}}.$$

(2) *The generating series for T is*

$$G(T, q) = \sum_{\eta \in \mathcal{C}_Q(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta) + s^k_i(\eta) \\ s^k_i(\eta) \end{bmatrix}_q.$$

Proof. (1) By (30),

$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \ldots \times \mathcal{P}(\infty, \mathbf{d}(n)).$$

Then,

$$\begin{aligned} G(S,q) &= \prod_{k=1}^{n} G(\mathcal{P}(\infty, \mathbf{d}(k)), q) && \text{(by (17))} \\ &= \prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} && \text{(by (22))} \end{aligned}$$

(2) First, observe that

$$\begin{aligned} G(R(\eta),q) &= \prod_{1\le i<j\le k\le n} G(\mathcal{R}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(j)}(\eta)), q) && \text{(by (17) and (31))} \\ &= \prod_{1\le i<j\le k\le n} q^{s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta)} && \text{(by (23))} \\ &= q^{r_{\mathbf{w}}(\eta)} && \text{(by (12))} \end{aligned}$$

Now,

$$\begin{aligned} G(P(\eta),q) &= \prod_{1\le i\le k\le n} G(\mathcal{P}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(i)}(\eta)), q) && \text{(by (32))} \\ &= \prod_{1\le i\le k\le n} G(\mathcal{P}(s^k_i(\eta), t^k_i(\eta)), q) && \text{(by permuting indices)} \\ &= \prod_{k=1}^{n} G(\mathcal{P}(s^k_k(\eta), t^k_k(\eta)), q) \prod_{i=1}^{k-1} G(\mathcal{P}(s^k_i(\eta), t^k_i(\eta)), q) \\ &= \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta) + s^k_i(\eta) \\ s^k_i(\eta) \end{bmatrix}_q && \text{(by (22) and (21))} \end{aligned}$$

Therefore,

$$\begin{aligned} G(T,q) &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(T(\eta), q) && \text{(by (18) and (34))} \\ &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(R(\eta)\times P(\eta), q) && \text{(by (33))} \\ &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(R(\eta), q) G(P(\eta), q) && \text{(by (17))} \\ &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta) + s^k_i(\eta) \\ s^k_i(\eta) \end{bmatrix}_q && \square \end{aligned}$$

We now define the general "cutting" operation we use to map from S to T. Fix two weakly increasing sequences of nonnegative integers

$$\mathbf{m} = (m_0 \le m_1 \le \ldots \le m_{k_m}) \text{ and } \mathbf{n} = (n_0 \le n_1 \le \ldots \le n_{k_n}).$$

Given a partition λ, let $\lambda^{(i,j)}(\mathbf{m},\mathbf{n})$ be the partition formed by restricting the Young diagram of λ to rows $[m_{i-1}+1, m_i]$ and columns $[n_{j-1}+1, n_j]$. Here, we allow for infinite m_{k_m} and n_{k_n}. Immediately from the definition,

$$\lambda^{(i,j)}(\mathbf{m},\mathbf{n}) \in \mathcal{P}(m_i - m_{i-1}, n_j - n_{j-1}). \tag{35}$$

Furthermore,

$$\lambda^{(i,j)}(\mathbf{m},\mathbf{n}) \in \mathcal{R}(m_i - m_{i-1}, n_j - n_{j-1}) \tag{36}$$

if and only if the Young diagram of λ has a box in position (m_i, n_j).

The following lemma describes how the size of $D(\lambda, r)$ varies as r changes.

Lemma 3.2. *Fix λ and suppose $r' \leq r$. If $D(\lambda, r) = s \times (s+r)$ and $D(\lambda, r') = s' \times (s'+r')$ then*

(1) *$s \leq s'$ and*
(2) *$s' + r' \leq s + r$.*

Proof. (1) Suppose $s+r' < 0$. We have $0 \leq s'+r'$, since it is the width of $D(\lambda, r')$. Therefore, $s \leq s'$. Otherwise, if $s + r' \geq 0$, the rectangle

$$s \times (s + r') \subseteq s \times (s + r) \subseteq \lambda.$$

Since $D(\lambda, r') = s' \times (s' + r')$, we have $s \leq s'$.
(2) If $s = s'$ then

$$s' + r' \leq s' + r = s + r.$$

Then suppose $s < s'$. Since $s + 1 \leq s'$ and

$$D(\lambda, r') = s' \times (r' + s') \subseteq \lambda,$$

we have

$$(s + 1) \times (r' + s') \subseteq \lambda.$$

Since $D(\lambda, r) = s \times (s + r)$, by definition, $(s + 1) \times (s + 1 + r) \not\subseteq \lambda$. And so

$$s' + r' \leq \lambda_{s+1} < s + 1 + r,$$

i.e. $s' + r' \leq s + r$. □

Define a map $\Psi_k : T \to \mathcal{P}(\infty, \mathbf{d}(k))$ by "gluing" the partitions of T with superscript k as indicated in Figure 1. Then let $\Psi = \Psi_1 \times \ldots \times \Psi_n$.

The proposed inverse $\Phi : S \to T$ is defined as follows. We will recursively define parameters

$$t_j^k(\boldsymbol{\lambda}) \text{ for } 1 \leq j \leq k \leq n$$

by induction on k. Our initial condition is that $t_1^1(\boldsymbol{\lambda}) = \mathbf{d}(1)$. Assume the sequence

$$t_1^{k-1}(\boldsymbol{\lambda}), \ldots, t_{k-1}^{k-1}(\boldsymbol{\lambda})$$

has been previously determined and that

$$t_j^{k-1}(\boldsymbol{\lambda}) \geq 0 \text{ for all } 1 \leq j \leq k-1.$$

Let

$$\delta_i^k(\boldsymbol{\lambda}) = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{\ell=1}^{i} t_{w^{(k)}(\ell)}^{k-1}(\boldsymbol{\lambda})) \text{ for } i = 0, \ldots, k-1. \tag{37}$$

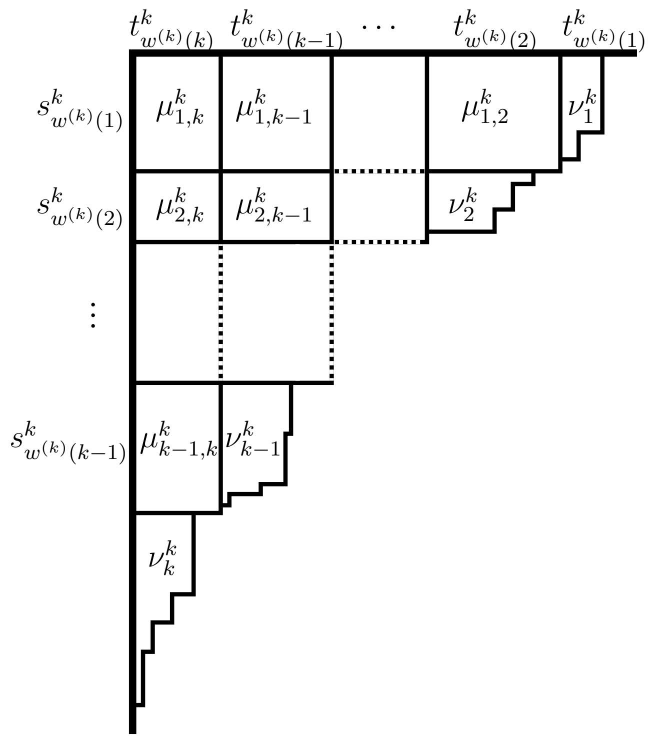


FIGURE 1. Description of the map $\Psi_k : T \to S$.

Note in particular that $\delta_0^k(\boldsymbol{\lambda}) = 0 \times \mathbf{d}(k)$ for all $1 \le k \le n$. Suppose

(38) $$\delta_i^k(\boldsymbol{\lambda}) = a_i^k(\boldsymbol{\lambda}) \times b_i^k(\boldsymbol{\lambda}) \text{ rectangle.}$$

For ease of indexing, write $b_k^k(\boldsymbol{\lambda}) = 0$. Let

(39) $$t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = b^k_{i-1}(\boldsymbol{\lambda}) - b^k_i(\boldsymbol{\lambda}) \text{ for } i = 1, \dots, k$$

We also define

(40) $$s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = a^k_i(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda}) \text{ for } i = 1, \dots, k-1$$

By the hypothesis, $t_j^{k-1}(\boldsymbol{\lambda}) \ge 0$ for all $1 \le j \le k-1$. Therefore,

$$\mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})) \le \mathbf{d}(k) - \sum_{\ell=1}^{i-1} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})) \text{ for all } i\text{'s} .$$

Then we may apply Lemma 3.2 to the δ_i^k's, to obtain sequences

(41) $$\mathbf{a}^k(\boldsymbol{\lambda}) = (a_0^k(\boldsymbol{\lambda}) \le a_1^k(\boldsymbol{\lambda}) \le \cdots \le a_{k-1}^k(\boldsymbol{\lambda}) \le a_k^k(\boldsymbol{\lambda}))$$

with $a_k^k(\boldsymbol{\lambda}) = \infty$ and

(42) $$\mathbf{b}^k(\boldsymbol{\lambda}) = (b_k^k(\boldsymbol{\lambda}) \le b_{k-1}^k(\boldsymbol{\lambda}) \le \cdots \le b_1^k(\boldsymbol{\lambda}) \le b_0^k(\boldsymbol{\lambda})).$$

By (41) and (42), the $s_i^k(\boldsymbol{\lambda})$'s and $t_j^k(\boldsymbol{\lambda})$'s are all nonnegative. Continue until $k = n$.

We then map $\boldsymbol{\lambda} \mapsto (\boldsymbol{\mu}, \boldsymbol{\nu})$ where

$$\mu^k_{i,j} = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1}$$

and

$$\nu_i^k = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-i+1}.$$

In the proof, we will justify this map is well defined, i.e. $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$. This involves finding a class $\eta(\boldsymbol{\lambda}) \in \mathcal{C}_Q(\mathbf{d})$ so that $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta(\boldsymbol{\lambda}))$. We define our candidate now.

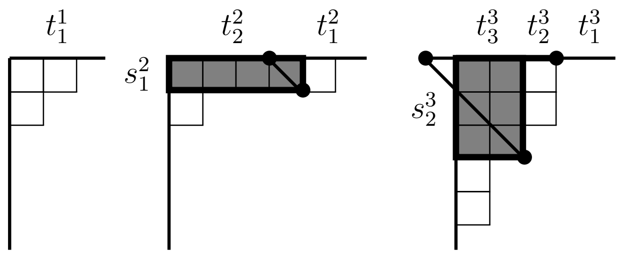
Definition 3.3. Let $\eta(\boldsymbol{\lambda})$ be the equivalence class of a lacing diagram uniquely defined by:

- $m_{[i,j]}(\eta(\boldsymbol{\lambda})) = s_i^{j+1}(\boldsymbol{\lambda})$ for $1 \le i \le j \le n-1$;
- $m_{[i,n]}(\eta(\boldsymbol{\lambda})) = t_i^n(\boldsymbol{\lambda})$ for $i = 1 \ldots n$.

Since each $m_{[i,j]}(\eta(\boldsymbol{\lambda})) \ge 0$, we have that $\eta(\boldsymbol{\lambda})$ is well defined.

Example 3.4. Assume $\mathbf{w} = (1, 12, 123)$. Fix a dimension vector $\mathbf{d} = (3, 6, 5)$ and partitions

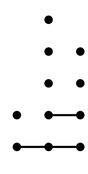
$$\lambda^{(1)} = (2,1), \lambda^{(2)} = (5,1), \text{ and } \lambda^{(3)} = (3,3,2,1,1).$$



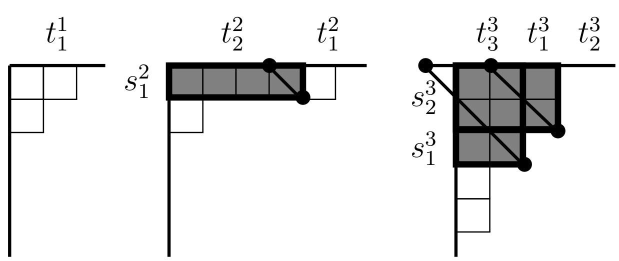
Then $\delta_1^2(\boldsymbol{\lambda}) = D(\lambda^{(2)}, 6-3) = 1 \times 4$ and so $t_1^2(\boldsymbol{\lambda}) = 2$, and $t_2^2(\boldsymbol{\lambda}) = 4$. From this, we have

$$\delta_1^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-2) = 0 \times 3 \text{ and } \delta_2^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-2-4) = 3 \times 2.$$

So $t_1^3(\boldsymbol{\lambda}) = 2$, $t_2^3(\boldsymbol{\lambda}) = 1$, and $t_3^3(\boldsymbol{\lambda}) = 2$. This corresponds to $\eta(\boldsymbol{\lambda}) = [\mathcal{L}]$ where

$\mathcal{L} =$ 

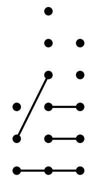
Alternatively, suppose $\mathbf{w} = (1, 12, 213)$. Keeping the same $\mathbf{d}$ and $\lambda^{(i)}$'s gives



As before, $\delta_1^2(\boldsymbol{\lambda}) = D(\lambda^{(2)}, 6-3) = 1 \times 4$. Consequently,

$$\delta_1^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-4) = 2 \times 3 \text{ and } \delta_2^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-4-2) = 3 \times 2.$$

This yields $\eta(\boldsymbol{\lambda}) = [\mathcal{L}']$, where

$\mathcal{L}' =$ 

$\square$

Immediately from the definitions (9) and (10), we have

$$t_i^k(\eta) + s_i^k(\eta) = t_i^{k-1}(\eta). \tag{43}$$

for any $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. We show the parameters defined in (39) and (40) satisfy the same recursion.

Lemma 3.5. *$t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) + s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = t^{k-1}_{w^{(k)}(i)}(\boldsymbol{\lambda})$ for $1 \leq i < k \leq n$.*

Proof. By (38) and the definition of a Durfee rectangle,

$$b_i^k(\boldsymbol{\lambda}) - a_i^k(\boldsymbol{\lambda}) = \mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda}). \tag{44}$$

Applying (39) and (40),

$$\begin{aligned}
t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) + s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) &= b^k_{i-1}(\boldsymbol{\lambda}) - b^k_i(\boldsymbol{\lambda}) + a^k_i(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda}) \\
&= (b^k_{i-1}(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda})) - (b^k_i(\boldsymbol{\lambda}) - a^k_i(\boldsymbol{\lambda})) \\
&= \left(\mathbf{d}(k) - \sum_{\ell=1}^{i-1} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})\right) - \left(\mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})\right) \\
&= t^{k-1}_{w^{(k)}(i)}(\boldsymbol{\lambda}).
\end{aligned}$$

$\square$

The next lemma collects various properties $\eta(\boldsymbol{\lambda})$. In particular, it justifies our choice in notation for $s_i^k(\boldsymbol{\lambda})$ and $t_j^k(\boldsymbol{\lambda})$.

Lemma 3.6. (1) $s_i^k(\eta(\boldsymbol{\lambda})) = s_i^k(\boldsymbol{\lambda})$
(2) $t_j^k(\eta(\boldsymbol{\lambda})) = t_j^k(\boldsymbol{\lambda})$
(3) $\eta(\boldsymbol{\lambda})) \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$.

Proof. (1) This is immediate from Definition 3.3.
(2) By Lemma 3.5,

$$t_i^k(\boldsymbol{\lambda}) = t_i^{k+1}(\boldsymbol{\lambda}) + s_i^{k+1}(\boldsymbol{\lambda}).$$

Iterating, we obtain

$$\begin{aligned}
t_i^k(\boldsymbol{\lambda}) &= t_i^{k+2}(\boldsymbol{\lambda}) + s_i^{k+2}(\boldsymbol{\lambda}) + s_i^{k+1}(\boldsymbol{\lambda}) \\
&= \dots \\
&= t_i^n(\boldsymbol{\lambda}) + \sum_{\ell=k+1}^{n} s_i^{\ell}(\boldsymbol{\lambda}) \\
&= t_i^n(\eta(\boldsymbol{\lambda})) + \sum_{\ell=k+1}^{n} s_i^{\ell}(\eta(\boldsymbol{\lambda})) && \text{(by Definition 3.3)}
\end{aligned}$$

$$= m_{[i,n]}(\eta(\boldsymbol{\lambda})) + \sum_{\ell=k+1}^{n} m_{[i,\ell-1]}(\eta(\boldsymbol{\lambda})) \qquad \text{(by (10) and (9))}$$
$$= t_i^k(\eta(\boldsymbol{\lambda})) \qquad \text{(by (10))}.$$

(3) For each k,

$$\begin{aligned}
\mathbf{d}(k) &= b_0^k(\boldsymbol{\lambda}) - b_k^k(\boldsymbol{\lambda}) \\
&= \sum_{i=1}^{k} b_{i-1}^k(\boldsymbol{\lambda}) - b_i^k(\boldsymbol{\lambda}) && \text{(by (39))} \\
&= \sum_{i=1}^{k} t_{w^{(k)}(i)}^k(\boldsymbol{\lambda}) \\
&= \sum_{i=1}^{k} t_i^k(\boldsymbol{\lambda}) && \text{(permute the terms of the sum)} \\
&= \sum_{i=1}^{k} t_i^k(\eta(\boldsymbol{\lambda})) && \text{(by part (2))} \\
&= \sum_{1\le i\le k\le j\le n} m_{[i,j]}(\eta(\boldsymbol{\lambda})) && \text{(by (10)}
\end{aligned}$$

By (8), we have $\eta(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. ☐

Theorem 3.7. *$\Psi : T \to S$ is a weight-preserving bijection, i.e.,* $\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathtt{wt}_S(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}))$.

Proof. Ψ is weight-preserving: That $\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu})) = \mathtt{wt}_S(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}))$ is clear since Ψ preserves the total number of boxes.

Ψ is well-defined: If $\mathbf{dim}(\eta) = (\mathbf{d}(1), \dots, \mathbf{d}(n))$ then

$$\mathbf{d}(k) = \sum_{i=1}^{k} \sum_{j=k}^{n} m_{[i,j]}(\eta) = \sum_{i=1}^{k} t_i^k(\eta) \text{ for } k = 1, \dots, n.$$

Therefore, $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu})$ has parts of size at most $\mathbf{d}(k)$ for each k, i.e. $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{P}(\infty, \mathbf{d}(k))$ for each k. Therefore, $\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}) \in S$.

Φ is well-defined:

By (35),

$$\lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1} \in \mathcal{P}(a_i^k(\boldsymbol{\lambda}) - a_{i-1}^k(\boldsymbol{\lambda}), b_{j-1}^k(\boldsymbol{\lambda}) - b_j^k(\boldsymbol{\lambda})).$$

By (40) and (39),

$$s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})) = a_i^k(\boldsymbol{\lambda}) - a_{i-1}^k(\boldsymbol{\lambda}) \text{ and } t_{w^{(k)}(j)}^k(\eta(\boldsymbol{\lambda})) = b_{j-1}^k(\boldsymbol{\lambda}) - b_j^k(\boldsymbol{\lambda}).$$

Therefore,

$$\lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1} \in \mathcal{P}(s_{w^{(k)}(i)}^k(\eta(\boldsymbol{\lambda})), t_{w^{(k)}(j)}^k(\eta(\boldsymbol{\lambda}))).$$

By definition,

$$\nu_i^k = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-i+1}$$

and so

$$\nu_i^k \in P(s^k_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})))$$

as desired.

Similarly, by (35),

$$\mu^k_{i,j} = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1}$$

and so

$$\mu^k_{i,j} \in \mathcal{P}(s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(j)}(\eta(\boldsymbol{\lambda}))).$$

Since $\delta^k_i(\boldsymbol{\lambda}) \subset \lambda^{(k)}$, the box $(a^k_i(\boldsymbol{\lambda}), b^k_i(\boldsymbol{\lambda})) \in \lambda^{(k)}$. Likewise, since $\delta^k_{k-j+1}(\boldsymbol{\lambda}) \subset \lambda^{(k)}$, we have

$$(a^k_{k-j+1}(\boldsymbol{\lambda}), b^k_{k-j+1}(\boldsymbol{\lambda})) \in \lambda^{(k)}.$$

Therefore, $(a^k_i(\boldsymbol{\lambda}), b^k_{k-j+1}(\boldsymbol{\lambda})) \in \lambda^{(k)}$ So, in fact, by (36),

$$\mu^k_{i,j} \in \mathcal{R}(s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(j)}(\eta(\boldsymbol{\lambda}))).$$

Therefore, $\Phi(\boldsymbol{\lambda}) \in T(\eta(\boldsymbol{\lambda})) \subseteq T$.

$\Psi \circ \Phi = \mathrm{Id}$:

Φ acts by cutting the $\lambda^{(k)}$'s into various pieces and Ψ glues these shapes together into their original configurations. So for every $\boldsymbol{\lambda} \in S$, we have $\Psi(\Phi(\boldsymbol{\lambda})) = \boldsymbol{\lambda}$.

$\Phi \circ \Psi = \mathrm{Id}$:

Fix $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$. Then in particular, $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta)$ for some $\eta \in \mathcal{C}_Q(\mathbf{d})$. Let $\boldsymbol{\lambda} := \Psi(\boldsymbol{\mu}, \boldsymbol{\nu})$. We must argue $\eta = \eta(\boldsymbol{\lambda})$. If so, $\Phi(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu})) = (\boldsymbol{\mu}, \boldsymbol{\nu})$.

Since $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta)$, each $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu})$ contains a rectangle

$$\epsilon^k_j = \left(\sum_{i=1}^{j} s^k_{w^{(k)}(i)}(\eta)\right) \times \left(\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta)\right) \tag{45}$$

for all $1 \leq j < k$ as in Figure 1.

By definition, $\mathbf{dim}(\eta) = \mathbf{d}$. Then it follows

$$\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta) = \mathbf{d}(k) - \left(\sum_{i=1}^{j} t^k_{w^{(k)}(i)}(\eta)\right).$$

As in (43), $t^k_i(\eta) + s^k_i(\eta) = t^{k-1}_i(\eta)$. So substituting we have

$$\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta) = \mathbf{d}(k) - \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta) + \sum_{i=1}^{j} s^k_{w^{(k)}(i)}(\eta). \tag{46}$$

Substitution of (46) into (45) yields

$$\epsilon^k_j = s \times (s + \mathbf{d}(k) - \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta))$$

contained in $\lambda^{(k)}$. Here, $s = \sum_{i=1}^{j} s^k_i(\eta)$. In particular, by construction, the bottom right corner of ϵ^k_j intersects the boundary of $\lambda^{(k)}$ (see Figure 1), i.e. s is the maximum value for which $\epsilon_j^k \subseteq \lambda^{(k)}$. So by the definition of a Durfee rectangle,

$$\epsilon_j^k = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta)).$$

By (37) and Claim 3.6 part (2),

$$\delta_j^k(\boldsymbol{\lambda}) = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta(\boldsymbol{\lambda})).$$

We seek to show $\delta_j^k(\boldsymbol{\lambda}) = \epsilon_j^k$ for all $1 \leq j < k \leq n$. Our argument is by induction on k. By definition, $t_1^1(\eta) = \mathbf{d}(1) = t_1^1(\eta(\boldsymbol{\lambda}))$. Then

$$\begin{aligned}\delta_1^2(\boldsymbol{\lambda}) &= D(\lambda^{(2)}, \mathbf{d}(2) - t_1^1(\eta)) \\ &= D(\lambda^{(2)}, \mathbf{d}(2) - t_1^1(\eta(\boldsymbol{\lambda})) \\ &= \epsilon_1^2,\end{aligned}$$

so the Durfee rectangles agree. Assume $\delta_j^{k-1}(\boldsymbol{\lambda}) = \epsilon_j^{k-1}$ for all $1 \leq j < k-1$. Then in particular, $t_i^{k-1}(\eta) = t_i^{k-1}(\eta(\boldsymbol{\lambda}))$ for all $1 \leq i \leq k-1$.

$$\sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta) = \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta(\boldsymbol{\lambda})), \tag{47}$$

it follows that $\delta_j^k = \epsilon_j^k$ since both are Durfee rectangles defined by the *same* parameter. Hence, $\delta_j^k = \epsilon_j^k$. Therefore,

$$s_i^k(\eta) = s_i^k(\eta(\boldsymbol{\lambda})) \text{ for all } 1 \leq i < k \leq n$$

and

$$t_i^k(\eta) = t_i^k(\eta(\boldsymbol{\lambda})) \text{ for all } 1 \leq i \leq k \leq n.$$

Hence $\eta = \eta(\boldsymbol{\lambda})$. □

We now conclude the proof of Theorem 1.4.

Proof. By Theorem 3.7, S and T are in weight preserving bijection. Therefore,

$$G(S, q) = G(T, q).$$

Applying Lemma 3.1 gives the result. □

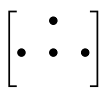
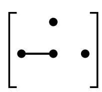
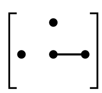
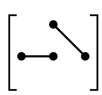
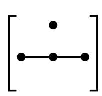
Example 3.8. Let $n = 3$ and $\mathbf{d} = (1, 2, 1)$ and $\mathbf{w} = (1, 12, 123)$. Then

$$r_{\mathbf{w}}(\eta) = (s_1^2(\eta)t_2^2(\eta)) + (s_1^3(\eta)t_2^3(\eta) + s_1^3(\eta)t_3^3(\eta) + s_2^3(\eta)t_3^3(\eta))$$

and

$$G(P(\eta), q) = \frac{1}{(q)_{t_1^1(\eta)}} \frac{1}{(q)_{t_2^2(\eta)}} \begin{bmatrix} t_1^2(\eta) + s_1^2(\eta) \\ s_1(\eta)^2 \end{bmatrix}_q \frac{1}{(q)_{t_3^3(\eta)}} \begin{bmatrix} t_1^3(\eta) + s_1^3(\eta) \\ s_1^3(\eta) \end{bmatrix}_q \begin{bmatrix} t_2^3(\eta) + s_2^3(\eta) \\ s_2^3(\eta) \end{bmatrix}_q.$$

The table below gives the equivalence classes for $\mathbf{d} = (1, 2, 1)$ and their corresponding terms on the right hand side of (13).

$\eta = [\mathcal{L}]$	$(s_j^k(\eta))$	$(t_j^k(\eta))$	$G(T(\eta), q)$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 1 & 2 \\ 2 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 2 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{array}$	$q^4 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_2}\right)\left(\frac{1}{(q)_1}\right) = \frac{q^4}{(1-q)^3(1-q^2)}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 1 & 1 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \end{array}$	$q^2 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{q^2}{(1-q)^3}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 1 & 2 \\ 1 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{array}$	$q^2 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_2}\right)\left(\begin{bmatrix}2\\1\end{bmatrix}_q\right) = \frac{q^2}{(1-q)^3}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 0 & 1 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}$	$q \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{q}{(1-q)^2}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 1 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}$	$\left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{1}{(1-q)^2}$

We then verify,

$$\begin{aligned} G(T,q) &= \frac{q^4}{(1-q)^3(1-q^2)} + \frac{q^2}{(1-q)^3} + \frac{q^2}{(1-q)^3} + \frac{q}{(1-q)^2} + \frac{1}{(1-q)^2} \\ &= \frac{1}{(1-q)^3(1-q^2)}(q^4 + 2q^2(1-q^2) + q(1-q)(1-q^2) + (1-q)(1-q^2)) \\ &= \frac{1}{(q)_1(q)_2(q)_1} \\ &= G(S,q). \end{aligned}$$

∎

We now give the proof of Corollary 1.5.

Proof. By (43),

$$t_i^k(\eta) + s_i^k(\eta) = t_i^{k-1}(\eta). \tag{48}$$

Furthermore by (9) and (10),

$$s_i^k(\eta) = m_{[i,k-1]}(\eta) \text{ and } t_i^n(\eta) = m_{[i,n]}(\eta).$$

Thus,

$$\begin{aligned} \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta) + s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q &= \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^k(\eta)+s_i^k(\eta)}}{(q)_{t_i^k(\eta)}(q)_{s_i^k(\eta)}} \\ &= \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^{k-1}(\eta)}}{(q)_{t_i^k(\eta)}(q)_{s_i^k(\eta)}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^{k-1}(\eta)}}{(q)_{t_i^k}(\eta)}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}}\right) \\
&= \left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{(q)_{t_i^k(\eta)}}\right) \left(\prod_{k=2}^{n}\prod_{i=1}^{k-1} (q)_{t_i^{k-1}(\eta)}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}}\right) \\
&= \left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{(q)_{t_i^k(\eta)}}\right) \left(\prod_{k=1}^{n-1}\prod_{i=1}^{k} (q)_{t_i^{k}(\eta)}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}}\right) \\
&= \left(\prod_{i=1}^{n} \frac{1}{(q)_{t_i^n(\eta)}}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}}\right) \\
&= \left(\prod_{i=1}^{n} \frac{1}{(q)_{m_{[i,n]}(\eta)}}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k-1} \frac{1}{(q)_{m_{[i,k-1]}(\eta)}}\right) \\
&= \prod_{1\le i\le j\le n} \frac{1}{(q)_{m_{[i,j]}(\eta)}}. \qquad \square
\end{aligned}
$$

The proof of Theorem 3.7 implies an enriched form of Theorem 1.4. Let

$$(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}).$$

For $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$, let $u_j(\eta)$ be the number of strands that terminate at column j in some (equivalently any) lace diagram $\mathcal{L} \in \eta$. That is,

$$u_j(\eta) = \sum_{i=1}^{j} s_i^{j+1}(\eta). \tag{49}$$

Corollary 3.9 (of Theorem 3.7)**.**

$$\prod_{k=1}^{n} \frac{1}{(qz;q)_{\mathbf{d}(k)}} = \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} z^{u_{k-1}(\eta)} \frac{1}{(qz;q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta)+s_i^k(\eta) \\ s_i^k(\eta)\end{bmatrix}_q. \tag{50}$$

Proof. The lefthand side of (50) is the generating series for S with respect to the weight that uses q to mark the number of boxes and z to mark length of the partitions involved. Now, suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in S$. Under the indicated decomposition in Figure 1,

$$\ell(\lambda^{(k)}) = \ell(\nu_k^k) + \sum_{i=1}^{k-1} s_{w^{(k)}(i)}^k(\eta(\boldsymbol{\lambda})) = \ell(\nu_k^k) + u_{k-1}(\eta(\boldsymbol{\lambda})),$$

where the second equality holds by (49) and reordering terms. The corollary follows immediately from this and Theorem 3.7 combined. $\square$

Theorem 1.4 is the $z=1$ case of Corollary 3.9. By analysis as in Section 2, we obtain as a special case this Durfee rectangle identity:

$$\frac{1}{(qz;q)_k} = \sum_{i=\max\{0,-r\}}^{k-r} \frac{z^i q^{i(r+i)}}{(qz;q)_{r+i}} \begin{bmatrix} k-(r+i)+i \\ i \end{bmatrix}_q.$$

From Corollary 3.9 one can deduce an enriched form of Theorem 1.3.

4. Proof of Theorem 1.6

Assume $\mathcal{Q}$ is a type A quiver. Label its vertices from left to right with the numbers $1, 2, \ldots, n$. Write a_i for the arrow whose left endpoint is vertex i. Let $\mathcal{I}$ be the set of intervals in $\mathcal{Q}$, i.e.

$$\mathcal{I} = \{[i, j] : i \leq j \text{ and } i, j \in [n]\}.$$

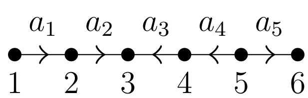
We associate a sequence of permutations to $\mathcal{Q}$ as follows:

Definition 4.1. Let $w_{\mathcal{Q}}^{(1)} = 1$ and $w_{\mathcal{Q}}^{(2)} = 12$. For $i \geq 3$ let ι be the natural inclusion from $\mathfrak{S}_{i-1}$ to $\mathfrak{S}_i$ and let $w_0^{(i-1)}$ denote the longest permutation in $\mathfrak{S}_{i-1}$. Set

$$w_{\mathcal{Q}}^{(i)} = \begin{cases} \iota(w_{\mathcal{Q}}^{(i-1)}) & \text{if } a_{i-2} \text{ and } a_{i-1} \text{ point in the same direction} \\ \iota(w_{\mathcal{Q}}^{(i-1)} w_0^{(i-1)}) & \text{if } a_{i-2} \text{ and } a_{i-1} \text{ point in opposite directions.} \end{cases}$$

Write $\mathbf{w}_{\mathcal{Q}} := (w_{\mathcal{Q}}^{(1)}, \ldots, w_{\mathcal{Q}}^{(n)})$. By construction, $\mathbf{w}_{\mathcal{Q}}$ is of the form (11).

Example 4.2. Let $\mathcal{Q}$ be the quiver pictured below.



Then $\mathbf{w}_{\mathcal{Q}} = (1, 12, 123, 3214, 32145, 541236)$. $\square$

Definition 4.1 is our link between $\operatorname{codim}_{\mathbb{C}}(\gamma_\eta)$ and the Durfee statistic. The outline of the proof of Theorem 1.6 is as follows. We start by defining two subsets of $\mathcal{I} \times \mathcal{I}$, $\texttt{BoxStrands}(\mathbf{w})$ and $\texttt{ConditionStrands}(\mathcal{Q})$. In Proposition 4.3, we show that

$$r_{\mathbf{w}}(\eta) = \sum_{(I,J) \in \texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta).$$

Proposition 4.10 states

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J) \in \texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta) m_J(\eta).$$

In Proposition 4.13, we show

$$\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q}).$$

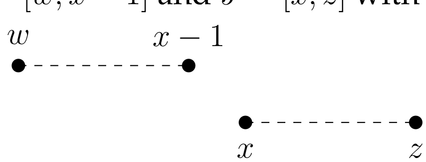
Combining these propositions completes the proof.

Given a sequence $\mathbf{w} = (w^{(1)}, \ldots, w^{(n)})$ which satisfies (11), define

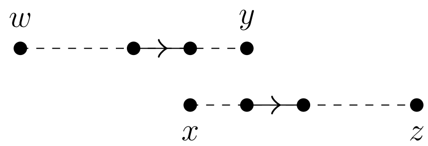
$$\texttt{BoxStrands}(\mathbf{w}) = \{([w^{(k)}(i), k-1], [w^{(k)}(j), \ell]) : 1 \leq i < j \leq k \leq \ell \leq n)\} \subseteq \mathcal{I} \times \mathcal{I}. \tag{51}$$

To define $\texttt{ConditionStrands}(\mathcal{Q})$, we consider pairs of intervals $(I, J) \in \mathcal{I} \times \mathcal{I}$ of the following three types:

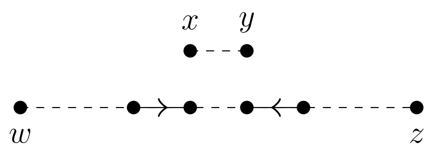
(I) $I = [w, x-1]$ and $J = [x, z]$ with $w < x \leq z$



(II) $I = [w, y]$ and $J = [x, z]$ with $w < x \leq y < z$ and the arrows a_{x-1} and a_y point in the same direction, e.g.,



(III) $I = [x, y]$ and $J = [w, z]$ with $w < x \leq y < z$ and the arrows a_{x-1} and a_y point in different directions, e.g.,



With this, we let

$$\texttt{ConditionStrands}(\mathcal{Q}) = \{(I, J) : \ (I, J) \text{ satisfies (I), (II), or (III)}\}. \tag{52}$$

The set $\texttt{BoxStrands}(\mathbf{w})$ has an immediate connection to the Durfee statistic $r_{\mathbf{w}}(\eta)$.

Proposition 4.3.

$$r_{\mathbf{w}}(\eta) = \sum_{(I,J)\in\texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta).$$

Proof. By (12),

$$r_{\mathbf{w}}(\eta) = \sum_{1\leq i<j\leq k\leq n} s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta).$$

Using (9) and (10), we have:

$$\begin{aligned} r_{\mathbf{w}}(\eta) &= \sum_{1\leq i<j\leq k\leq n} m_{[w^{(k)}(i),k-1]}(\eta) \left(\sum_{\ell=k}^{n} m_{[w^{(k)}(j),\ell]}(\eta) \right) \\ &= \sum_{1\leq i<j\leq k\leq \ell\leq n} m_{[w^{(k)}(i),k-1]}(\eta) m_{[w^{(k)}(j),\ell]}(\eta) \\ &= \sum_{(I,J)\in\texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta). \end{aligned}$$

□

We now recall some more facts from the representation theory of quivers. Write $\mathrm{Hom}(\mathsf{V}, \mathsf{W})$ for the space of morphisms from V to W. Given V and W an **extension** of V by W is a short exact sequence of morphisms

$$0 \to \mathsf{W} \to \mathsf{E} \to \mathsf{V} \to 0.$$

Two extensions are **equivalent** if the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{W} \arrow[r] \arrow[d] & \mathsf{E} \arrow[r] \arrow[d, "\wr"] & \mathsf{V} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathsf{W} \arrow[r] & \mathsf{E}' \arrow[r] & \mathsf{V} \arrow[r] & 0 \end{tikzcd}$$

Write $\mathrm{Ext}^1(\mathsf{V},\mathsf{W})$ for the space of extensions of V by W up to equivalence. $\mathrm{Hom}(\mathsf{V},\mathsf{W})$ and $\mathrm{Ext}^1(\mathsf{V},\mathsf{W})$ are finite dimensional vector spaces. Recall the **Euler form**, defined by

$$\chi_{\mathcal{Q}}(\mathbf{d}_1,\mathbf{d}_2) = \sum_{i\in Q_0} \mathbf{d}_1(i)\mathbf{d}_2(i) - \sum_{a\in Q_1} \mathbf{d}_1(t(a))\mathbf{d}_2(h(a)).$$

We often use the abbreviation

$$\chi_{\mathcal{Q}}(\mathsf{V},\mathsf{W}) := \chi_{\mathcal{Q}}(\mathbf{dim}\mathsf{V},\mathbf{dim}\mathsf{W}).$$

The Euler form satisfies the following:

$$\chi_{\mathcal{Q}}(\mathsf{V},\mathsf{W}) = \mathrm{dimHom}(\mathsf{V},\mathsf{W}) - \mathrm{dimExt}^1(\mathsf{V},\mathsf{W}), \tag{53}$$

(see [Bri08, Corollary 1.4.3]).

Given $I \in \mathcal{I}$, let V_I be an irreducible representation indexed by I. Let

$$\mathsf{V}_\eta = \bigoplus_{I\in\mathcal{I}} \mathsf{V}_I^{\oplus m_I(\eta)}. \tag{54}$$

Each point in the orbit $\gamma_\eta \subseteq \mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ is isomorphic to V_η. The codimension of γ_η may be expressed in terms of extensions of V_η. This is Voigt's lemma (see [Rin80, Lemma 2.3]):

Lemma 4.4 (Voigt)**.**

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \mathrm{dimExt}^1(\mathsf{V}_\eta,\mathsf{V}_\eta).$$

Here, we give an alternate expression for $\mathrm{codim}_{\mathbb{C}}(\gamma_\eta)$ in terms of the Euler form. Define

$$U = \{(I,J) : \chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J) < 0\}.$$

Lemma 4.5.

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in U} m_I(\eta)m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J)).$$

Proof. By [Rei01], Section 2, there exists a total order on $\mathcal{I}$ so that

$$\mathrm{Hom}(\mathsf{V}_I,\mathsf{V}_J) \text{ and } \mathrm{Ext}^1(\mathsf{V}_J,\mathsf{V}_I) = 0 \text{ whenever } I < J \text{ and } I \neq J. \tag{55}$$

Indecomposables for Dynkin quivers have no nontrivial self extensions, that is,

$$\mathrm{Ext}^1(\mathsf{V}_I,\mathsf{V}_I) = 0 \text{ for all } I \in \mathcal{I},$$

[Bri08, Theorem 2.4.3]. So $\mathrm{dimExt}^1(\mathsf{V}_I,\mathsf{V}_J) = 0$ whenever $I \geq J$.

Writing

$$\mathsf{V}_\eta = \bigoplus_{I\in\mathcal{I}} \mathsf{V}_I^{\oplus m_I(\eta)}$$

as a direct sum of indecomposables, we have

$$\mathrm{Ext}^1(\mathsf{V}_\eta,\mathsf{V}_\eta) \cong \bigoplus_{(I,J)\in\mathcal{I}\times\mathcal{I}} \mathrm{Ext}^1(\mathsf{V}_I,\mathsf{V}_J)^{\oplus m_I(\eta)m_J(\eta)}.$$

Then

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \mathrm{dimExt}^1(\mathsf{V}_\eta,\mathsf{V}_\eta) = \sum_{(I,J)\in\mathcal{I}\times\mathcal{I}} m_I(\eta)m_J(\eta)\mathrm{dimExt}^1(\mathsf{V}_I,\mathsf{V}_J).$$

Since $\mathrm{Ext}^1(\mathsf{V}_\eta,\mathsf{V}_\eta)$ vanishes when $I \geq J$,

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J):I<J} m_I(\eta)m_J(\eta)\mathrm{dimExt}^1(\mathsf{V}_I,\mathsf{V}_J),$$

(see [Rim13]). Combining (53) and (55) gives

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J):I<J} m_I(\eta)m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)). \tag{56}$$

Using the ordering on $\mathcal{I}$ and (53), it follows that

$$\text{if } I < J, \text{ then } \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) \leq 0 \text{ and } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) \geq 0. \tag{57}$$

Since $\mathcal{Q}$ is a Dynkin quiver, if $I = J$, then $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) > 0$ [Bri08]. Thus we may reindex the sum, taking only those (I, J) for which $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$. Therefore,

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in U} m_I(\eta)m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)).$$

□

Lemma 4.6. *Fix intervals I and J. If $[x, y] \subseteq I, J$ then*

$$\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) = 1 \tag{58}$$

Proof. Since $[x, y] \subseteq I, J$, $\mathbf{d}_I(i) = \mathbf{d}_J(i) = 1$ for all $i \in [x, y]$. Therefore,

$$\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) = y - x + 1. \tag{59}$$

Regardless of the orientation of a_i, if $i \in [x, y-1]$ then $t(a_i), h(a_i) \in [x, y]$. Because $[x, y] \subseteq I, J$, we have $\mathbf{d}_I(t(a_i)) = \mathbf{d}_J(h(a_i)) = 1$. So

$$\sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) = (y-1) - (x+1). \tag{60}$$

Subtracting (60) from (59) gives (58). □

Let

$$\texttt{StrandPairs} = \{(I, J) = ([x_1, x_2], [y_1, y_2]) \in \mathcal{I} \times \mathcal{I} : x_2 \leq y_2\}.$$

From (51) and the definitions (I)-(III), it follows that

$$\texttt{ConditionStrands}(\mathcal{Q}) \subset \texttt{StrandPairs}.$$

Lemma 4.7. *Let $(I, J) \in \texttt{StrandPairs}$. Then*

$$(I, J) \in \texttt{ConditionStrands}(\mathcal{Q}) \iff \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) < 0.$$

Moreover, if $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) < 0$, then $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1$ and likewise $\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) < 0$ implies $\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$.

Proof. Since we have assumed $\mathcal{Q}$ is a type A quiver, we have

$$\chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \sum_{i=1}^{n} \mathbf{d}_1(i)\mathbf{d}_2(i) - \sum_{i=1}^{n-1} \mathbf{d}_1(t(a_i))\mathbf{d}_2(h(a_i)). \tag{61}$$

Given an interval I, write $\mathbf{d}_I$ for the dimension vector of V_I. By (61), we have

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = \chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) = \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)).$$

We analyze this expression repeatedly throughout our argument.

($\Rightarrow$) By direct computation, we will show if $(I,J) \in \texttt{ConditionStrands}(\mathcal{Q})$ then

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1,$$

which is the last assertion of the claim.

Case 1: $(I,J) = ([w, x-1], [x, z])$ is of type (I).

Subcase i: a_{x-1} points to the right.

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) &= \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= -\sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \quad \text{(since } I \cap J = \emptyset\text{)} \\
&= -\mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) \\
&= -\mathbf{d}_I(x-1)\mathbf{d}_J(x) \\
&= -1
\end{aligned}$$

Subcase ii: a_{x-1} points to the left.

Let $\mathcal{Q}^{\mathrm{op}}$ be the quiver obtained by reversing the direction of all arrows in $\mathcal{Q}$. Then $\chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) = \chi_{\mathcal{Q}^{\mathrm{op}}}(\mathbf{d}_I, \mathbf{d}_J)$. Therefore,

$$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = \chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) = \chi_{Q}^{\mathrm{op}}(\mathbf{d}_I, \mathbf{d}_J) = -1$$

by Subcase 1.i.

Case 2: $(I,J) = ([w, y], [x, z])$ is of type (II).

Subcase i: a_{x-1} and a_y point to the right.

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) &= \sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x-1}^{y} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= \left(\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \right) - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) \\
&\quad - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \\
&= 1 - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \quad \text{(Lemma 4.6)} \\
&= 1 - \mathbf{d}_I(x-1)\mathbf{d}_J(x) - \mathbf{d}_I(y)\mathbf{d}_J(y+1) \\
&= -1
\end{aligned}$$

Subcase ii: a_{x-1} and a_y point to the left.

$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$ by the $\mathcal{Q}^{\mathrm{op}}$ argument, as in Subcase 1.i.

Case 3: $(I,J) = ([x, y], [y, z])$ is of type (III).

Subcase i: a_{x-1} points right and a_y points left.

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = \sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x-1}^{y} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i))$$

$$= \left(\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i))\right) - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1}))$$
$$- \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y))$$
$$= 1 - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \quad \text{(Lemma 4.6)}$$
$$= 1 - \mathbf{d}_I(x-1)\mathbf{d}_J(x) - \mathbf{d}_I(y-1)\mathbf{d}_J(y)$$
$$= -1$$

Subcase ii: a_{x-1} points left and a_y points right.

$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$ by the $\mathcal{Q}^{\mathrm{op}}$ argument, as in Subcase 1.i.

Thus we have shown whenever $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$,

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1.$$

($\Leftarrow$) Let $(I, J) = ([x_1, x_2], [y_1, y_2]) \in \texttt{StrandPairs}$ and first assume $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$.

Case 1: $I \cap J = \emptyset$. Then $\mathbf{d}_I(i) = 0$ or $\mathbf{d}_J(i) = 0$ for all $i \in [1, n]$ and so

$$\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) = -\sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)).$$

Since $\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) < 0$ there must exist an arrow a_i with $t(a_i) \in [x_1, x_2]$ and $h(a_i) \in [y_1, y_2]$. Then $i = x_2$, a_i points to the right, and $y_1 = x_2 + 1$. This implies (I, J) is of type (I).

Case 2: Assume $I \cap J \neq \emptyset$. Since we assume $x_2 \leq y_2$

$$I \cap J = [x_1, x_2] \cap [y_1, y_2] = [z, x_2]$$

where $z \in \{x_1, y_1\}$. Then

$$\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) = \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i))$$
$$= \sum_{i=z}^{x_2} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=z-1}^{x_2} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \quad \text{(Lemma 4.6)}$$
$$= 1 - \mathbf{d}_I(t(a_{z-1}))\mathbf{d}_J(h(a_{z-1})) - \mathbf{d}_I(t(a_{x_2}))\mathbf{d}_J(h(a_{x_2})).$$

Since $\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) < 0$, we must have

$$\mathbf{d}_I(t(a_{z-1})) = \mathbf{d}_J(h(a_{z-1})) = \mathbf{d}_I(t(a_{x_2})) = \mathbf{d}_J(h(a_{x_2})) = 1.$$

Therefore,

$$t(a_{z-1}), t(a_{x_2}) \in I = [x_1, x_2] \tag{62}$$

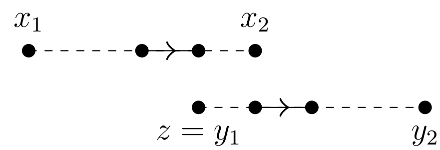
and

$$h(a_{z-1}), h(a_{x_2}) \in J = [y_1, y_2]. \tag{63}$$

If an arrow a_i points to the right, then $h(a_i) = i + 1$ and $t(a_i) = i$. If a_i points left, $h(a_i) = i$ and $t(a_i) = i + 1$. We proceed by analyzing the direction of a_{x_2} and a_{z-1} . First consider a_{x_2}. If a_{x_2} points left, then $t(a_{x_2}) = x_2 + 1$ and so $x_2 + 1 \in [x_1, x_2]$, which is a contradiction. Therefore, we may assume a_{x_2} points right.

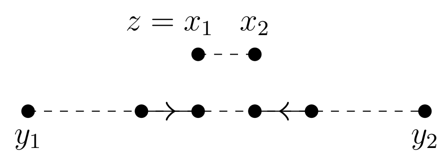
Now consider the direction of a_{z-1}.

If a_{z-1} points to the right, then $t(a_{z-1}) = z - 1 \in [x_1, x_2]$ by (62) and so $z > x_1$. Since $z \in \{x_1, y_1\}$, we must have $z = y_1$.



Therefore (I, J) is of type (II).

If a_{z-1} points left, now we have by (63) $h(a_{z-1}) = z - 1 \in [y_1, y_2]$. Therefore $z - 1 > y_1$ and so $z \neq y_1$ which implies $z = x_1$. Hence we have:



So (I, J) is of type (III).

By near identical arguments, $\chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) < 0$ when

(1) a_{z-1} and a_{x_2} both point left, $z = y_1$, and $x_2 < y_2$; i.e., (I, J) is of type (II)
(2) a_{z-1} points right, a_{x_2} points left, $z = x_1$ and $x_2 < y_2$ so (I, J) is of type (III).

□

In particular, we have the following corollary.

Corollary 4.8. *If* $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) < 0$ *then* $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) = -1$

Proof. If $(I, J) \in \texttt{StrandPairs}$, this is immediate by Lemma 4.7. Otherwise, $(J, I) \in \texttt{StrandParis}$. Then by Lemma 4.7 $(J, I) \in \texttt{ConditionStrands}(\mathcal{Q})$. As such,

$$\chi_Q(\mathsf{V}_I, \mathsf{V}_J) = -1.$$

□

Recall $U = \{(I, J) : \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0\}$. We let

$$U_1 = \{(I, J) = ([x_1, x_2], [y_1, y_2]) : (I, J) \in U \text{ and } x_2 \leq y_2\}, \text{ and}$$

$$U_2 = \{(I, J) = ([x_1, x_2], [y_1, y_2]) : (I, J) \in U \text{ and } x_2 > y_2\}.$$

Trivially,

$$U = U_1 \sqcup U_2 \tag{64}$$

Let

$$\widetilde{U_2} = \{(J, I) : (I, J) \in U_2\}.$$

Lemma 4.9. $\texttt{ConditionStrands}(\mathcal{Q}) = U_1 \sqcup \widetilde{U_2}$.

Proof. If $(I,J) \in \widetilde{U_2}$, then $(J,I) \in U_2 \subset U$ and so $\chi_Q(\mathsf{V}_J,\mathsf{V}_I) < 0$. Therefore $\chi_Q(\mathsf{V}_I,\mathsf{V}_J) \geq 0$ and hence $(I,J) \notin U$. So $(I,J) \notin U_1$. Therefore, $U_1 \cap \widetilde{U_2} = \emptyset$.

($\subseteq$) If $(I,J) \in \texttt{ConditionStrands}(\mathcal{Q})$, by Lemma 4.7, $\chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J) < 0$ or $\chi_{\mathcal{Q}}(\mathsf{V}_J,\mathsf{V}_I) < 0$. In the first case, from the definition, $(I,J) \in U_1$. In the second case, again by definition, $(J,I) \in U_2$, which implies $(I,J) \in \widetilde{U}_2$.

($\supseteq$) We have $U_1, \widetilde{U_2} \subseteq \texttt{StrandPairs}$. Thus by Lemma 4.7, $U_1, \widetilde{U_2} \subseteq \texttt{ConditionStrands}(\mathcal{Q})$. ☐

Proposition 4.10.

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in\texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta)m_J(\eta).$$

Proof. If $(I,J) \in U$, then $\chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J) < 0$. Applying Corollary 4.8, $\chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J) = -1$. Then by Lemma 4.5,

$$\begin{aligned}\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) &= \sum_{(I,J)\in U} m_I(\eta)m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I,\mathsf{V}_J)) \\ &= \sum_{(I,J)\in U} m_I(\eta)m_J(\eta)\end{aligned}$$

Therefore, applying (64)

$$\begin{aligned}\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) &= \sum_{(I,J)\in U_1} m_I(\eta)m_J(\eta) + \sum_{(I,J)\in U_2} m_I(\eta)m_J(\eta) \\ &= \sum_{(I,J)\in U_1} m_I(\eta)m_J(\eta) + \sum_{(I,J)\in \widetilde{U_2}} m_I(\eta)m_J(\eta) \\ &= \sum_{(I,J)\in\texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta)m_J(\eta) \qquad \text{(Lemma 4.9)},\end{aligned}$$

as claimed. ☐

Our final goal is to show $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$. We start with a lemma.

Lemma 4.11. *All elements of* $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$ *and* $\texttt{ConditionStrands}(\mathcal{Q})$ *may be written in the form:*

$$(I,J) = ([x,k-1],[y,\ell]), \text{ with } x \neq y,\ k \leq \ell. \tag{65}$$

Proof. If

$$([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(j), \ell]) \in \texttt{Boxstrands}(\mathbf{w}_{\mathcal{Q}}),$$

then

$$w_{\mathcal{Q}}^{(k)}(i) \neq w_{\mathcal{Q}}^{(k)}(j) \text{ and } k \leq \ell.$$

Hence, by setting $x = w_{\mathcal{Q}}^{(k)}(i)$ and $y = w_{\mathcal{Q}}^{(k)}(j)$, we are done.

Now suppose

$$([x_1,x_2],[y_1,y_2]) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

By definition (I)-(III), $x_1 \neq y_1$ and $x_2 < y_2$. So set $x = x_1$, $y = y_1$, $k = x_2 + 1$ and $\ell = y_2$. ☐

With Lemma 4.11 in mind, to prove $\texttt{Boxstrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$, it is enough to show given (I, J) of the form in (65),

$$(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff (I,J) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

We first handle the special case when I and J are disjoint.

Lemma 4.12. *Let (I, J) be as in (65) and suppose $I \cap J = \emptyset$. Then $(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$ if and only if $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.*

Proof. If $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$, then by the disjointness hypothesis it must be of type (I), i.e.

$$(I,J) = ([x, k-1], [k, \ell]).$$

Now, since $x \leq k-1$ and $w_{\mathcal{Q}}^{(k)} \in \mathfrak{S}_k$ with $w_{\mathcal{Q}}^{(k)}(k) = k$, there exists $i < k$ such that $w_{\mathcal{Q}}^{(k)}(i) = x$. So

$$([x, k-1], [k, \ell]) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(k), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}).$$

Conversely, assume

$$(I,J) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(j), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$$

and $I \cap J = \emptyset$. Then $w_{\mathcal{Q}}^{(k)}(j) > k-1$ which means $w_{\mathcal{Q}}^{(k)}(j) = k$ and $j = k$ by the definition of $w_{\mathcal{Q}}^{(k)}$. Furthermore, $w_{\mathcal{Q}}^{(k)}(i) \leq k-1$ since $i < j = k$. So

$$(I,J) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [k, \ell])$$

is of type (I), and hence in $\texttt{ConditionStrands}(\mathcal{Q})$. □

We now show:

Proposition 4.13. $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$.

Proof. Let (I, J) be as in (65). We seek to show

$$(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff (I,J) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

We will proceed by induction on k. In the base case $k = 2$, we must have $x = 1$ and so $y \geq 2$. As such, $I \cap J = \emptyset$ and so we are done by Lemma 4.12. Fix $k > 2$ and assume the claim holds for $k-1$. That is, given a pair of intervals $([x', k-2], [y', \ell'])$ so that x', y' and ℓ' satisfy $x' \neq y'$ and $k-1 \leq \ell'$ we have

(66)
$$([x', k-2], [y', \ell']) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff ([x', k-2], [y', \ell']) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

Now let (I, J) be as in (65), i.e.,

$$(I,J) = ([x, k-1], [y, \ell]), \text{ with } x \neq y, k \leq \ell.$$

Again, by Lemma 4.12, if $I \cap J = \emptyset$ we are done, so assume $I \cap J \neq \emptyset$. Then $y < k$.

Now, since $1 \leq x, y \leq k$, there exist i and j such that

$$1 \leq i, j \leq k \text{ with } x = w^{(k)}(i) \text{ and } y = w^{(k)}(j).$$

So from (51)

$$(I,J) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(j), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff i < j. \tag{67}$$

Throughout, when $x \leq k-2$ we write $I' := [x, k-2]$. We will break the argument into two main cases.

Case 1: a_{k-2} and a_{k-1} point in the same direction.

By definition, $w_{\mathcal{Q}}^{(k)} = \iota(w_{\mathcal{Q}}^{(k-1)})$. Then if $x \leq k-2$, it follows that

$$\begin{aligned}(I', J) &= ([x, k-2], [y, \ell]) \\ &= ([w_{\mathcal{Q}}^{k-1}(i), k-2], [w_{\mathcal{Q}}^{k-1}(j), \ell])\end{aligned}$$

and so

$$(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ if and only if } i < j. \tag{68}$$

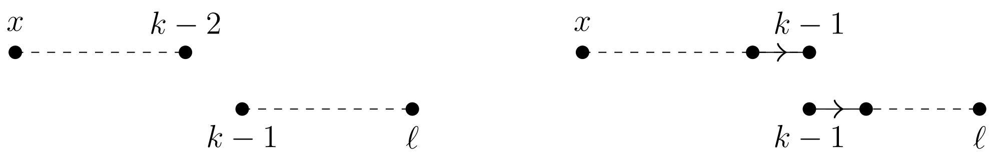
We have four possible subcases, based on the relative values of x and y.

Subcase i: $x < y = k-1$.

(I, J) is of type (II), and hence $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$. Furthermore, note that

$$(I', J) = ([x, k-2], [k-1, \ell])$$

is of type (I), and so in $\texttt{ConditionStrands}(\mathcal{Q})$. The intervals for (I', J) and (I, J) look like this:



By the inductive hypothesis (66), $(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. By (68), $i < j$. Therefore, by (67), $(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

Therefore, (I, J) is in both $\texttt{ConditionStrands}(\mathcal{Q})$ and $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

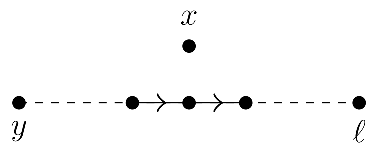
Subcase ii: $x < y < k-1$.

$$\begin{aligned}(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \quad \text{by (67)} \\ &\iff (I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (68)} \\ &\iff (I', J) \in \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (66)} \\ &\iff a_{x-1} \text{ points in the same direction as } a_{k-2} \\ &\iff a_{x-1} \text{ points in the same direction as } a_{k-1} \\ &\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).\end{aligned}$$

The following picture depicts (I', J) and (I, J) respectively when (I', J) and (I, J) are in $\texttt{ConditionStrands}(\mathcal{Q})$.

Subcase iii: $y < x = k-1$.

Pictured below are the intervals I and J.



Since $y < x$ and this case assumes a_{k-2} and a_{k-1} point in the same direction, (I, J) cannot be of type (III) and is not in $\texttt{ConditionStrands}(\mathcal{Q})$. Since

$$w_{\mathcal{Q}}^{(k)} = \iota w_{\mathcal{Q}}^{(k-1)} \text{ and } w_{\mathcal{Q}}^{(k-1)}(k-1) = k-1,$$

it follows that $i = k - 1$. Since

$$y = w_{\mathcal{Q}}^{(k)}(j) = w_{\mathcal{Q}}^{(k-1)}(j) < k - 1,$$

it follows that $i > j$, and so by (67)

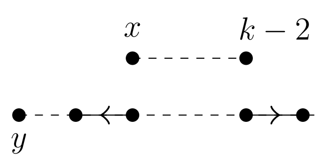
$$(I, J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}).$$

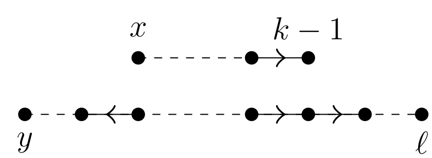
Therefore, (I, J) is in neither $\texttt{ConditionStrands}(\mathcal{Q})$ nor $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

Subcase iv: $y < x < k - 1$.

$$\begin{aligned}
(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \quad \text{by (67)} \\
&\iff (I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (68)} \\
&\iff (I', J) \in \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (66)} \\
&\iff a_{x-1} \text{ points in the opposite direction as } a_{k-2} \\
&\iff a_{x-1} \text{ points in the opposite direction as } a_{k-1} \\
&\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).
\end{aligned}$$

Below are $(I'J)$ and (I, J) respectively, in the case $(I', J), (I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.





Case 2: a_{k-2} and a_{k-1} point in opposite directions.

By definition,

$$w_{\mathcal{Q}}^{(k)} = \iota(w_{\mathcal{Q}}^{(k-1)} w_0^{(k-1)}).$$

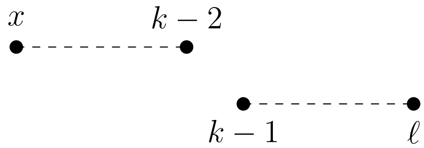
If $x \leq k - 2$, and $y \leq k - 1$ it follows that

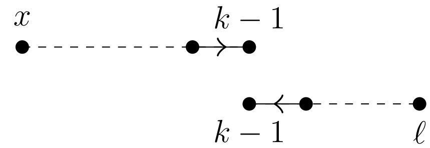
$$\begin{aligned}
(I', J) &= ([x, k-2], [y, \ell]) \\
&= ([w_{\mathcal{Q}}^{(k-1)}(k-i), k-2], [w_{\mathcal{Q}}^{(k-1)}(k-j), \ell])
\end{aligned}$$

and so

(69) $\qquad (I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$ if and only if $k - i < k - j$ if and only if $i > j$.

Subcase i: $x < y = k - 1$.





Since a_{k-2} and a_{k-1} point in opposite directions, $(I,J) \notin \texttt{ConditionStrands}(\mathcal{Q})$. The assumption $y = k-1$ implies $(I',J) \in \texttt{ConditionStrands}(\mathcal{Q})$. By (66) $(I',J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. Since $x, y < k$, we have

$$x = w_{\mathcal{Q}}^{(k)}(i) = w_{\mathcal{Q}}^{(k-1)}(k-i) \text{ and } y = w_{\mathcal{Q}}^{(k)}(j) = w_{\mathcal{Q}}^{(k-1)}(k-j).$$

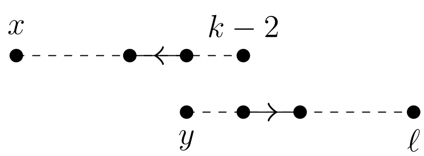
Then $k-i < k-j$, so $i > j$ and $(I,J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$, by (67).

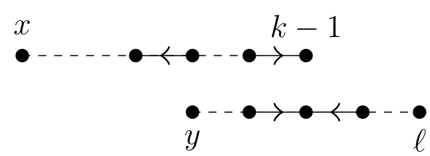
Hence (I,J) is neither in $\texttt{ConditionStrands}(\mathcal{Q})$ nor $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

Subcase ii: $x < y < k-1$.

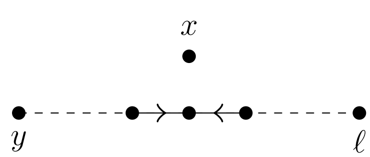
$$\begin{aligned}
(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \text{ by (67)} \\
&\iff (I',J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (69)} \\
&\iff (I',J) \notin \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (66)} \\
&\iff a_{y-1} \text{ points in the opposite direction as } a_{k-2} \\
&\iff a_{y-1} \text{ points in the same direction as } a_{k-1} \\
&\iff (I,J) \in \texttt{ConditionStrands}(\mathcal{Q}).
\end{aligned}$$

Below, we have $(I',J) \notin \texttt{ConditionStrands}(\mathcal{Q})$ and $(I,J) \in \texttt{ConditionStrands}(\mathcal{Q})$.





Subcase iii: $y < x = k-1$. Here (I,J) looks like:



Since Case 2 assumes a_{k-2} and a_{k-1} point in opposite directions, (I,J) is type (II) and so in $\texttt{ConditionStrands}(\mathcal{Q})$. Now,

$$k-1 = x = w_{\mathcal{Q}}^{(k)}(i) = w_{\mathcal{Q}}^{(k-1)}(k-i)$$

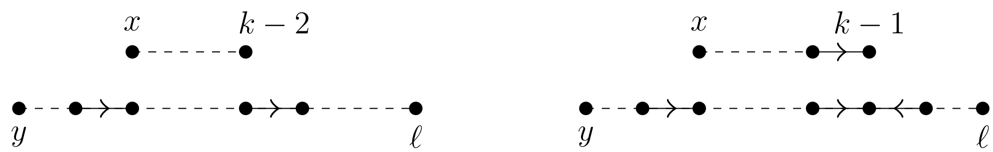
which implies $i = 1$. Then $j > i$, so $(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. So (I,J) is both in $\texttt{ConditionStrands}(\mathcal{Q})$ and $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

Subcase iv: $y < x < k-1$.

$$\begin{aligned}
(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \text{ by (67)} \\
&\iff (I',J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (69)} \\
&\iff (I',J) \notin \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (66)}
\end{aligned}$$

$$\iff a_{x-1} \text{ points in the same direction as } a_{k-2}$$
$$\iff a_{x-1} \text{ points the opposite direction as } a_{k-1}$$
$$\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

Pictured below are (I', J) and (I, J), in the case that $(I', J) \notin \texttt{ConditionStrands}(\mathcal{Q})$ and $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.



Thus, we have $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$. $\square$

.

Proof of Theorem 1.6.

$$\begin{aligned}
r_{\mathbf{w}_{\mathcal{Q}}}(\eta) &= \sum_{(I,J)\in\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})} m_I(\eta) m_J(\eta) && \text{(by Proposition 4.3)} \\
&= \sum_{(I,J)\in\texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta) m_J(\eta) && \text{(by by Proposition 4.13)} \\
&= \operatorname{codim}_{\mathbb{C}}(\gamma_\eta) && \text{(by Proposition 4.10)}
\end{aligned}$$

$\square$

We conclude by giving our proof of Reineke's identity for type A quivers.

Proof of Theorem 1.3 (in type A). The map $\eta \mapsto \gamma_\eta$ defines a bijection from $\mathcal{C}_{\mathcal{Q}}(\mathbf{d})$ to $\mathcal{O}_{\mathcal{Q}}(\mathbf{d})$. Since $\mathcal{Q}$ is type A, there is a bijection between $\{\beta_1, \dots, \beta_N\}$ and $\mathcal{I}$. Furthermore, whenever $I \mapsto \beta_i$, we have $m_I(\eta) = m_{\beta_i}(\gamma_\eta)$. Then starting from Corollary 1.5, we have

$$\begin{aligned}
\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}_{\mathcal{Q}}}(\eta)} \prod_{1\le i\le j\le n} \frac{1}{(q)_{m_{[i,j]}(\eta)}} \\
&= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma_\eta)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma_\eta)}} \\
&= \sum_{\gamma\in\mathcal{O}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma)}}
\end{aligned}$$

$\square$

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