ELLiptic classes of Schubert cells via Bott-Samelson resolution

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Abstract. We study the equivariant elliptic characteristic classes of Schubert varieties of the generalized full flag variety $G/B$. For this first we need to twist the notion of elliptic characteristic class of Borisov-Libgober by a line bundle, and thus allow the elliptic classes to depend on extra variables. Using the Bott-Samelson resolution of Schubert varieties we prove a BGG-type recursion for the elliptic classes, and study the Hecke algebra of our elliptic BGG operators. For $G = \text{GL}_n(\mathbb{C})$ we find representatives of the elliptic classes of Schubert varieties in natural presentations of the K theory ring of $G/B$, and identify them with the Tarasov-Varchenko weight function. As a byproduct we find another recursion, different from the known R-matrix recursion for the fixed point restrictions of weight functions. On the other hand the R-matrix recursion generalizes for arbitrary reductive group $G$.

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1. Introduction

The study of Schubert varieties and their singularities is a field where topology, algebraic geometry, and representation theory meet. An effective strategy to study Schubert varieties is assigning characteristic classes to them. However, the most natural characteristic class, the fundamental class, does not exist beyond K theory [BE90]—that is, e.g. the elliptic fundamental class depends on choices (e.g. choice of resolution, or choice of basis in a Hecke algebra) [LZ15]. Important deformations of the fundamental class appeared recently in the center stage in cohomology and K theory, under the names of Chern-Schwartz-MacPherson class and Motivic Chern class, partially due to their relation to Okounkov’s stable envelope classes. In this paper we study the elliptic analogue of the CSM and MC classes, the elliptic characteristic class of Schubert varieties, which unlike the fundamental class, does not depend on choices.

Borisov and Libgober have defined elliptic classes $E_{\ell\ell}$ for so called Kawamata log-terminal pairs. Initially we would like to compute elliptic classes of the pairs $(X_\omega, \partial X_\omega)$ where $X_\omega$ is a Schubert variety in the generalized full flag variety $G/B$. To make sense of such object, it turns out that the original notion of elliptic class needs to be twisted by a line bundle $L_\lambda$ for $\lambda \in t^*$. We find that the thus defined twisted elliptic class of a Schubert variety depends meromorphically on $\lambda$. Hence we treat the elliptic class

$$E(X_\omega, \lambda) := E_{\ell\ell}(X_\omega, \partial X_\omega - L_\lambda)$$
as a function on $\lambda$ (and we omit $\lambda$ in the notation). The classes $E(X_\omega)$ also depend on an additional parameter $h$ and on a modular parameter $q = e^{2\pi i \tau}$.

Right from the beginning we have the freedom of choosing the generalized cohomology theory in which our elliptic classes live. The original Borisov-Libgober classes were defined in rational cohomology (extended by additional parameters). Our classes $E(X_\omega)$ live in equivariant K theory. When considered in elliptic cohomology, treated as a generalized cohomology theory, rationally isomorphic to K theory $K(G/B)$ with the complex orientation defined by the theta function, the elliptic classes $\tilde{E}(X_\omega)$ differs by a correcting factor:

$$E(X_\omega) = \frac{e^K(TG/B)}{e^E(TG/B)} \tilde{E}(X_\omega),$$

where $e^E$ and $e^K$ are Euler classes in elliptic cohomology and in K theory. This is a classical approach to elliptic cohomology and passage from elliptic cohomology to K theory is a Riemann-Roch type transformation.

Our first result is a recursive formula for the elliptic class in equivariant elliptic cohomology of $G/B$. Let $\alpha \in t^*$ be a simple root, then

**Formula 1.1.**

$$\delta \left( \mathcal{L}_\alpha, h^{\alpha^\vee} \right) \cdot s_\alpha^h \tilde{E}(X_\omega) - \delta \left( \mathcal{L}_\alpha, h \right) \cdot s_\alpha^\gamma s_\alpha^\mu \tilde{E}(X_\omega) = \begin{cases} \tilde{E}(X_{\omega s_\alpha}) & \text{if } \ell(\omega s_\alpha) = \ell(\omega) + 1, \\ \delta(h^{\alpha^\vee}, h) \delta(h^{-\alpha^\vee}, h) \cdot \tilde{E}(X_\omega) & \text{if } \ell(\omega s_\alpha) = \ell(\omega) - 1. \end{cases}$$

Here
- $\alpha^\vee$ is the dual root, the expression $h^{\alpha^\vee}$ is a function on $t^*$,
- $\mathcal{L}_\alpha = G \times_B \mathbb{C}_{-\alpha}$ is the line bundle associated to the root $\alpha$,
- $s_\alpha \in W$ is a simple reflection,
- $s_\alpha^h$ is the action of $s_\alpha$ on the arguments of $E(X_\omega) : t^* \to K_T(G/B)(h)$,
- $s_\alpha^\gamma$ is the action of $s_\alpha$ on the $K_T(G/B)$ induced by the right action of the Weyl group; in localization formulas it coincides with the permutation of the fixed points,
- $\delta(x, y) = \frac{\varphi(1)\varphi(xy)}{\varphi(x)\varphi(y)}$ a certain function defied by the multiplicative version of the Jacobi theta function.

The proof relies on the study of the Bott-Samelson inductive resolution of the Schubert variety.

The recursive operator above can be studied in the framework of Hecke algebras, in particular we present a version of a Hecke algebra acting on the elliptic classes. Here we take the opportunity to explore the various degenerations of elliptic classes and the corresponding Hecke operations, such as Chern-Schwartz-MacPherson classes, motivic Chern classes, and cohomological and K theoretic fundamental classes.

Certain classes in elliptic cohomology theory are special, those that have a fixed transformation properties with respect to modular group. These transformation properties are determined by a
quadratic form. If a class is special then it defines a section of a line bundle over a spectrum of elliptic cohomology. We show a formula describing the transformation properties of $\tilde{E}(X_\omega)$.

We prove that, in addition to the Bott-Samelson recursion above, elliptic classes of Schubert varieties satisfy another “dual” recursion in equivariant elliptic cohomology of $G/B$:

**Formula 1.2.**

$$\delta \left( e^{-\alpha}, h^{\omega-1}\alpha^\vee \right) \cdot \tilde{E}(X_\omega) - \delta \left( e^{\alpha}, h \right) \cdot s_\alpha^* \tilde{E}(X_\omega) =$$

$$= \begin{cases} 
\tilde{E}(X_{s_\alpha\omega}) & \text{if } \ell(s_\alpha\omega) = \ell(\omega) + 1, \\
\delta(h^{\omega-1}\alpha^\vee, h)\delta(h^{\omega-1}\alpha^\vee, h) \cdot \tilde{E}(X_{s_\alpha\omega}) & \text{if } \ell(s_\alpha\omega) = \ell(\omega) - 1,
\end{cases}$$

where $s_\alpha^*$ denotes the action of the simple reflection $s_\alpha$ on the equivariant parameters.

Remarkably, if $G = GL_n$ then this recursion is equivalent to a three term identity in [RTV17] (and references therein), where this three term identity is interpreted as the R-matrix identity for an elliptic quantum group. Hence we will call this second dual recursion the R-matrix recursion. In [RTV17] certain special functions called weight functions are defined that satisfy the R-matrix recursion. These weight functions are the elliptic analogues of cohomological and K theoretic weight functions whose origin goes back to [TV97]. The three versions of weight functions play an important role in representation theory, in the theory of hypergeometric solutions of various KZ differential equations, and also turn up in Okounkov’s theory of stable envelopes.

The fact that elliptic classes of Schubert varieties satisfy the R-matrix recursion allows us to prove

**Theorem 1.3.** *The elliptic classes $\tilde{E}(X_\omega)$ for $G = GL_n$ are represented by weight functions $\tilde{w}_\omega$, that is,*

$$\tilde{E}(X_\omega) = \eta(\tilde{w}_\omega),$$

*where $\eta : K_{T \times T}(\text{End}(\mathbb{C}^n)) \to K_{T \times T}(GL_n) \simeq K_T(GL_n/B)$ is the canonical surjection.*

In Schubert calculus one often studies formulas that express a characteristic class of a Schubert variety in terms of some basic classes. The umbrella term for such formulas is Giambelli formulas. In this language the theorem above can be viewed as the Giambelli formula in equivariant elliptic cohomology of the cotangent bundle of $G/B$.

Finally, for an arbitrary linear reductive group $G$ both recursions can be uniformly written in terms of rescaled elliptic classes as follows. Let

$$\mathcal{E}(X_\omega) = \prod_{\alpha \in \Sigma_- \cap \omega^{-1}\Sigma_+} \delta(h^{-\alpha^\vee}, h)^{-1} \cdot E(X_\omega),$$

where $\Sigma_{\pm}$ is the set of positive/negative roots. The Formulas 1.1 and 1.2 translate to the following compact forms:

**Theorem 1.4** (Bott-Samelson recursion). *Let $\alpha$ be a simple root. Then *

$$\mathcal{E}(X_{\omega s_\alpha}) = \frac{\delta(\mathcal{L}_\alpha, h^{\alpha^\vee})}{\delta(h^{-\alpha^\vee}, h)} \cdot s_\alpha^* \mathcal{E}(X_\omega) - \frac{\delta(\mathcal{L}_\alpha, h)}{\delta(h^{-\alpha^\vee}, h)} \cdot s_\alpha^* s_\alpha^* \mathcal{E}(X_\omega).$$
If $G = \text{GL}_n$, then in the language of weight functions
\[ \mathcal{L}_{\alpha_k} = \frac{\gamma_{k+1}}{\gamma_k}, \quad h^{\alpha_k} = \frac{\mu_{k+1}}{\mu_k}. \]

**Theorem 1.5** (R-matrix recursion). Let $\alpha$ be a simple root. Then
\[ E_s(X_{s_{\alpha}\omega}) = \frac{\delta(e^{-\alpha}, h^{\omega^{-1}\alpha^\vee})}{\delta(h^{\omega^{-1}\alpha^\vee}, h)} \cdot E(X_\omega) + \frac{\delta(e^\alpha, h)}{\delta(h^{\omega^{-1}\alpha^\vee}, h)} \cdot s_{\alpha}^* E(X_\omega), \]
\[ e^\alpha \in K_T(pt) \simeq R(T) \text{ is the character corresponding to the root } \alpha. \]
If $G = \text{GL}_n$, then $e^{\alpha_k} = \frac{z_k}{z_{k+1}}$.

An interesting by-product of the fact that $\tilde{E}(X_\omega)$ classes satisfy two seemingly unrelated recursions is the fact that weight functions, besides satisfying the known R-matrix recursions, also satisfy a so far unknown recursion coming from the Bott-Samelson induction. This will be presented in Section 11, and will be interpreted as the R-matrix relation for the 3d mirror dual variety in a followup paper.

Throughout the paper we heavily use equivariant localization, that is, we work with torus fixed point restrictions of the elliptic classes scaled by an Euler class:
\[ E_\sigma(X_\omega) = \frac{E(X_\omega)|_\sigma}{e(T_\sigma(G/B))}, \]
where $\sigma$ is a torus fixed point. Thus our proofs are achieved by calculations of restricted classes that live in the equivariant K theory of fixed points. We should emphasize that many of our formulas encode deep identities among theta functions (occasionally we will remark on Fay’s three term identity, and a four term identity). For us, these identities will come for free, we will not need to prove them—this is the power of Borisov-Libgober’s theory, that their classes are well defined.

**Conventions.** We work with varieties over $\mathbb{C}$; for example by $\text{GL}_n$ we mean $\text{GL}_n(\mathbb{C})$. By K-theory we mean algebraic K theory of coherent sheaves, which in our case is isomorphic to the topological equivariant K theory, see the Appendix of [FRW18]. The Weyl group of $\text{GL}_n$ will be identified with the group of permutations. A permutation $\omega : \{1, \ldots, n\} \to \{1, \ldots, n\}$ will be denoted by a list $\omega = \omega(1) \ldots \omega(n)$. The permutation switching $k$ and $k + 1$ will be denoted by $s_k$. Permutations will be multiplied by the convention that $\omega \sigma$ is the permutation obtained by first applying $\sigma$ then $\omega$; for example $231 \cdot 213 = 321$.

For a general reductive group $G$ we fix a maximal torus $T$ and a Borel group $B$ containing $T$. The set of roots of $B$, according to our convention, are positive. For an integral weight $\lambda \in \mathfrak{t}^*$ the line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ is such that the torus $T$ acts via the character $-\lambda$ on the fiber at $eB \in G/B$. With this convention $\mathcal{L}_\lambda$ is ample for $\lambda$ belonging to the interior of the dominant Weyl chamber. The half sum of positive roots is denoted by $\rho$. The canonical divisor $K_{G/B}$ is isomorphic to $\mathcal{L}_\rho^{-2}$.

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2. THE EQUIVARIANT ELLIPTIC CHARACTERISTIC CLASS TWISTED BY A LINE BUNDLE

First we recall the notion of the elliptic characteristic class of a singular variety, defined by Borisov and Libgober in [BL03]. We study its equivariant version, and its behavior with respect to fixed point restrictions. In Section 2.4 we define a version “twisted by a line bundle and its section.” This latter version will be used in the rest of the paper.

2.1. Theta functions. For general reference on theta functions see e.g. [We98, Cha85]. We will use the following version

\[ \vartheta(x) = \vartheta(x, q) = x^{1/2} (1 - x^{-1}) \prod_{n \geq 1} (1 - q^n x) (1 - q^n / x). \]

For \( |q| < 1 \) the series is convergent and defines a holomorphic function on a double cover on \( \mathbb{C}^* \). Throughout the paper we will use the function

\[ (1) \quad \delta(a, b) = \frac{\vartheta(ab) \vartheta'(1)}{\vartheta(a) \vartheta(b)}, \]

which is meromorphic on \( \mathbb{C}^* \times \mathbb{C}^* \) with poles whenever \( a \) or \( b \) is a power of \( q \). We will also use theta functions in additive variables, namely, let

\[ \theta(u) = \theta(u, \tau) = \vartheta(e^{2\pi i u}, q) \]

where \( q = e^{2\pi i \tau}, \tau \in \mathbb{C}, \text{im}(\tau) > 0 \). Our function \( \theta \) differs from the classical Jacobi theta-function only by a factor depending on \( \tau \). Namely, according to Jacobi’s product formula [Cha85, Ch V.6],

\[ \theta_{\text{Jacobi}}(u, \tau) = 2 e^{\pi i \tau / 4} \sin(\pi u) \prod_{\ell = 1}^{\infty} (1 - e^{2\pi i \ell \tau}) (1 - e^{2\pi i (\ell u + \tau)}) (1 - e^{2\pi i (\ell - u)}), \]

and hence

\[ \theta_{\text{Jacobi}}(u, \tau) = \frac{1}{q^{1/8}} \prod_{\ell = 1}^{\infty} (1 - q^{\ell}) \cdot \theta(u, \tau). \]

The theta function satisfies the quasi-periodicity identities

\[ (2) \quad \theta(u + 1, \tau) = -\theta(u, \tau), \]

\[ \theta(u + \tau, \tau) = -e^{-2\pi i u - \pi i \tau} \cdot \theta(u, \tau). \]

We will take a closer look at transformation properties of several variable functions built out of theta functions in Section 8.1.

2.2. The Borisov-Libgober elliptic class \( \tilde{E}_\ell \). First we define the elliptic class of a smooth variety \( Z \), cf. [BL00, BL03, Lib17]. For a rank \( n \) bundle \( \mathcal{T} \) over \( Z \) with Grothendieck roots \( t_i \)
Another change is that we introduce the letter $h$ in Hirzebruch
we reserve the letter $\alpha$.

Remark 2.3 (About notation). Borisov and Libgober use the notation $D = -\sum_{k=1}^{\ell} \alpha_k D_k$, but we reserve the letter $\alpha$ for a root of a Lie algebra, and we want to get rid of the minus sign. Another change is that we introduce the letter $h = y^{-1}$. This way we get rid of a collision with $y$ in Hirzebruch $\chi_y$-genus and the variable $h$ agrees with the notation of [RTV17].
Observe that if the divisor $D$ is empty then 

$$\widetilde{\mathcal{E}}(Z, \emptyset) = \left( \frac{\varphi(1)}{\varphi(h)} \right)^{\dim Z} \mathcal{E}(Z).$$

The two definitions above generalize without change to the torus equivariant case: the case when $T = (\mathbb{C}^*)^n$ acts on $Z$ or $(Z, D)$, and $t_i$ are the $T$ equivariant Grothendieck roots. One advantage of torus equivariant $K$ theory is the tool of fixed point restrictions. Let $x$ be a $T$ fixed point on $Z$, and consider the restriction of $\widetilde{\mathcal{E}}(Z, D)$ to $x$. Here each $d_k$ is necessarily a Grothendieck root. By introducing artificial components of $D$ with coefficient $a_i = 0$ if necessary, we may now assume that $d_i = t_i$ for all $i = 1, \ldots, \dim Z$, and we obtain

$$\widetilde{\mathcal{E}}(Z, D)|_x = e^K(TZ) \prod_{i=1}^{\dim Z} \frac{\vartheta(t_i h^{1-a_i})\vartheta'(1)}{\vartheta(t_i)\vartheta(h^{1-a_i})}$$

$$= e^K(TZ) \prod_{i=1}^{\dim Z} \delta(t_i, h^{1-a_i}) \in K_T(x)(h) = R(T)(h).$$

Remark 2.4. In fact, formally, there are complex powers of $h$ in the expression above. By this, here and in the whole paper we mean the following: for a formal variable $z$ satisfying $h = e^{-2\pi iz}$, by $\vartheta(h^a)$ we mean

$$\vartheta(h^a) = \vartheta((e^{-2\pi iz})^a) = \vartheta(e^{-2\pi i z a}) = \theta(-za).$$

To keep expositions simple we will not explicitly work with the $z$ variable anymore, and this kind of dependence on formal powers of $h$ we indicate by $K_T(x)(h)$.

Now we are ready to define the elliptic class for certain pairs of singular varieties. Let $X$ be a possibly singular subvariety of a smooth variety $M$, and $\Delta$ a divisor on $X$. We say that $(X, \Delta)$ is a KLT pair, if

- $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor;
- there exists a map $f : Z \to M$ which is a log-resolution $(Z, D) \to (X, \Delta)$ (i.e. $Z$ is smooth, $D$ is a normal crossing divisor on $Z$, $f$ is proper and is an isomorphism away from $D$) such that
  1. $D = \sum_k a_k D_k$, $a_k < 1$,
  2. $K_Z + D = f^*(K_X + \Delta)$.

Definition 2.5. The elliptic class of a KLT pair $(X, \Delta)$ is defined by

$$\widetilde{\mathcal{E}}(X, \Delta; M) := f_*(\widetilde{\mathcal{E}}(Z, D)) \in K(M)(h).$$

The definition does not depend on the resolution — this is the main result of [BL03]. If we allow complex coefficients $a_k$, the proof of independence on the resolution goes through provided that the coefficients $a_k$ do not belong to $\mathbb{N}_{\geq 1}$. However, to make sense of the definition we have to ensure that some $\mathbb{C}$-multiple of $K_X + \Delta$ is a Cartier divisor.

The torus equivariant elliptic class is defined formally the same way, see details in [Wae08, DBWe16]. We will not indicate the torus action in the notation.

1In the classical definition it is required that $\Delta = \sum d_i \Delta_i$ with $d_i \in [0, 1)$, but we do not need positivity of $a_i$'s.
Remark 2.6. The elliptic class is the “class version” of the elliptic genus studied in detail in the literature. Namely, let \( h = e^{-2\pi iz} \), and let \( Z \) be Calabi-Yau. Then the elliptic genus \( \eta_*(\mathcal{E}(Z)) \) \((\eta : Z \to \text{point})\) is a holomorphic function on \( \mathbb{C}_z \times \mathbb{C}_{i\tau} > 0 \), and it is a quasi-modular form of weight 0 and index \( \dim Z \). Now let us assume that \( n \) times the multiplicities of \( \Delta \) are integers, and that \((X, \Delta)\) is a Calabi-Yau pair (i.e. \( K_X + \Delta = 0 \)). Then the “genus”

\[
\left( \theta_{(-z, \tau)} \right)^{\dim X} \eta_*(\mathcal{E}(X, \Delta; M))
\]

has transformation properties of Jacobi forms of weight \( \dim X \) and index 0 \([BL03, \text{Prop. 3.10}]\), with respect to the subgroup of the full Jacobi group generated by the transformations

\[
(z, \tau) \mapsto (z + n, \tau), \quad (z, \tau) \mapsto (z + n\tau, \tau), \quad (z, \tau) \mapsto (z, \tau + 1), \quad (z, \tau) \mapsto (z/\tau, -1/\tau).
\]

2.3. Calculation of the elliptic class via torus fixed points. The class \( \mathcal{E}(X, \Delta; M) \) is defined via the push-forward map \( f_* \). From the well known localization formulas for such push-forward maps in torus equivariant K theory, see e.g. \([ChGi97, \text{Th. 5.11.7}]\), we obtain the following proposition.

Proposition 2.7. Assume that in the \( T \) equivariant situation of Definition 2.5 the fix point sets \( M^T \) and \( Z^T \) are finite. Then for \( x \in M^T \) in the fraction field of \( K^T(\mathcal{O}) \) we have

\[
\mathcal{E}(X, \Delta; M)|_x \frac{1}{e^K(T_x M)} = \sum_{x' \in f^{-1}(x)} \mathcal{E}(Z, D)|_{x'} \frac{1}{e^K(T_{x'} Z)}.
\]

This formula motivates the definition of the local elliptic class

\[
\mathcal{E}_x(X, \Delta; M) = \frac{\mathcal{E}(X, \Delta; M)|_x}{e^K(T_x M)}
\]

for a fixed point \( x \) on \( X \). In fact the division by \( e^K(T_x M) \) gets rid of the dependence on \( M \), so we set \( \mathcal{E}_x(X, \Delta) = \mathcal{E}_x(X, \Delta; M) \) for some \( M \). Using this definition, from statement (5) we obtain

\[
\mathcal{E}_x(X, \Delta) = \sum_{x' \in f^{-1}(x)} \mathcal{E}_{x'}(Z, D)
\]

\[
= \sum_{x' \in f^{-1}(x)} \prod_{k=1}^{\dim Z} \delta(t_k(x'), h^{1-a_k(x')}),
\]

where \( t_k(x') \) and \( a_k(x') \) denote the tangent weights and multiplicities of the divisors at the torus fixed point \( x' \).

Remark 2.8. In the whole paper we use equivariant K theory as the home of the elliptic characteristic classes. We could have decided differently by setting up equivariant elliptic cohomology (as in e.g. \([FRV07, RTV17]\)) and defining the classes in there. Following that path, in (6) we would divide by the equivariant elliptic Euler class \( \prod \theta(t_i) \) instead of the equivariant K theoretic Euler class \( \prod(1 - t_i^{-1}) \). Remarkably, the thus defined local class would be the ‘same’ in both
theories, due to a version of the Grothendieck-Riemann-Roch theorem. The class \( E_x(X, \Delta) \) is not only independent of the ambient manifold \( M \) but essentially also of the cohomology theory used.

**Example 2.9.** Consider the standard action of \( T = (\mathbb{C}^*)^2 \) on \( M = X = \mathbb{C}^2 \). Denote \( C_x = \{ x = 0 \}, \ C_y = \{ y = 0 \}, \) and let them represent the classes \( t_1 \) and \( t_2 \) in \( K_T(\mathbb{C}^2) \). Consider the divisor \( \Delta = a_1 C_x + a_2 C_y \). Taking the identity map as resolution, using (4) we obtain

\[
\tilde{\ell}(X, \Delta; M) = e^{K(T_0X)} \cdot \delta(t_1, h^{1-a_1})\delta(t_2, h^{1-a_2}) = (1 - t_1^{-1})(1 - t_2^{-1}) \cdot \delta(t_1, h^{1-a_1})\delta(t_2, h^{1-a_2}).
\]

Now we calculate \( \tilde{\ell}(X, \Delta; M) \) in another way. Consider the blow-up \( Z = Bl_0X, f : Z \to X \) with exceptional divisor \( E \), and let the strict transforms of \( C_x, C_y \) be \( \tilde{C}_x, \tilde{C}_y \). Define the divisor \( D = a_1 \tilde{C}_x + a_2 \tilde{C}_y + (a_1 + a_2 - 1)E \). Since \( f^*K_X = K_Z - E \), and \( f^*(\Delta) = a_1 \tilde{C}_x + a_2 \tilde{C}_y + (a_1 + a_2)E \) (from calculations in coordinates) we have \( f^*(K_X + \Delta) = K_Z + D \). Hence, by (5) and using (4) at the two \( T \) fixed points in \( f^{-1}(0) \) we obtain

\[
\frac{\tilde{\ell}(X, \Delta; M)}{e^{K(T_0M)}} = \delta(t_1, h^{2-a_1-a_2})\delta(t_2/t_1, h^{1-a_2}) + \delta(t_2, h^{2-a_1-a_2})\delta(t_1/t_2, h^{1-a_1}).
\]

Observe that the comparison of the two formulas above boils down to the identity

\[
\delta(t_1, h^{1-a_1})\delta(t_2, h^{1-a_2}) = \delta(t_1, h^{2-a_1-a_2})\delta(t_2/t_1, h^{1-a_2}) + \delta(t_2, h^{2-a_1-a_2})\delta(t_1/t_2, h^{1-a_1}),
\]

which is a rewriting of the well known Fay’s trisecant identity

\[
(8) \quad \theta(a + c)\theta(a - c)\theta(b + d)\theta(b - d) = \theta(a + b)\theta(a - b)\theta(c + d)\theta(c - d) + \theta(a + d)\theta(a - d)\theta(b + c)\theta(b - c),
\]

see e.g. [F73], [FRV07, Thm. 5.3], [GaT-L17].

2.4. The elliptic class \( \tilde{\ell} \) twisted by a line bundle. For the main application of the present paper (elliptic classes of Schubert varieties) we need a modified version of the notion \( \tilde{\ell}(X, \Delta; M) \). The following example explains why.

**Example 2.10.** Let \( X_\omega \subset G/B \) be a Schubert variety, and let \( f_\omega : Z_\omega \to X_\omega \) be a Bott-Samelson resolution (as defined in Section 3). Set

\[
\Delta = \partial X_\omega = \sum_{X_{\omega'} \subset X_\omega, \dim(X_{\omega'}) = \dim(X_\omega) - 1} [X_{\omega'}].
\]

It turns out that \( f_\omega^*(K_{X_\omega} + \Delta) = K_{Z_\omega} + D \), where \( D = \partial Z_\omega \) has all coefficients equal to 1, see Theorem 3.2 below. Hence a crucial condition in the definition of \( \tilde{\ell} \) is not satisfied for \( (X_\omega, \partial X_\omega) \).

Let \( X \) be a possibly singular subvariety of a smooth variety \( M \), and \( \Delta \) a divisor on \( X \). Assume that \( K_X + \Delta \) is a \( \mathbb{Q} \)-Cartier divisor; and that there exists a map \( f : Z \to M \) which is a log-resolution \( (Z, D) \to (X, \Delta) \) such that \( K_X + D = f^*(K_X + \Delta) \). Observe that we have not assumed anything about the coefficients of the divisor \( D \) (cf. the definition of KLT pair in Section 2.2).
Let $L$ be an ample line bundle on $X$ and $\xi \in H^0(X; L)$ a section such that $\xi$ does not vanish on $X \setminus \text{supp}(\Delta)$. Assuming $\text{supp}(\Delta) = \text{supp}(\text{zeros}(\xi))$ we have $\text{supp}(f^*D) = \text{supp}(\text{zeros}(f^*\xi))$. Denote

$$\Delta(L, \xi) = \Delta - \text{zeros}(\xi).$$

If $L$ is sufficiently positive then the pair $(X, \Delta(L, \xi))$ is a KLT pair. Therefore we can define the twisted elliptic class of $(X, \Delta)$ by $\mathcal{O}(X, \Delta(L, \xi); M) \in K(M)(h)$.

**Remark 2.11.** In practice, see Section 3.4 below, the line bundle $L$ will be associated with certain integer points $\lambda$ of a vector space, and the choice of the section $\xi$ will be unique. Moreover, the dependence of the twisted elliptic class on $\lambda$ will be meromorphic; so we can extend the definition to a meromorphic function on the vector space: the twisted elliptic class of $(X, \Delta)$ will be a class in $K(M)(h)$ depending on $\lambda$.

3. Bott-Samelson resolution and the elliptic classes of Schubert varieties

Let us consider the usual settings of Schubert calculus: $G$ a semisimple group with Borel subgroup $B$, maximal torus $T$, and Weyl group $W$. Simple roots will be denoted by $\alpha_1, \alpha_2, \ldots$, and the corresponding reflections in $W$ by $s_1, s_2, \ldots$. We consider reduced words in the letters $s_k$, denoted by $\omega$. A word $\omega$ represents an element $\omega \in W$. The length $\ell(\omega)$ of $\omega$ is the length of the shortest reduced word representing it.

We will study the homogeneous space $G/B$. For $\omega \in W$ let $\tilde{\omega} \in N(T) \subset G$ be a representative of $\omega \in W = N(T)/T$, and let $x_\omega = \tilde{\omega}B \in G/B$. The point $x_\omega$ is fixed under the $T$ action. The $B$ orbit $X^\omega = Bx_\omega = B\tilde{\omega}B$ of $x_\omega$ will be called a Schubert cell, and its closure $X_\omega$ the Schubert variety. In this choice of conventions we have $\dim(X_\omega) = \ell(\omega)$.

3.1. The Bott-Samelson resolution of Schubert varieties. Let the reduced word $\omega$ represent $\omega \in W$. The Bott-Samelson variety $Z_\omega$, together with a resolution map $f_\omega : Z_\omega \to X_\omega$ of the Schubert variety $X_\omega$ is constructed inductively as follows. Suppose $\omega = \omega^1k^1$ is a reduced word. Let $P_k \supset B$ be the minimal parabolic containing $B$, such that $W_{P_k} = \langle s_k \rangle$. The map $\pi_k : G/B \to G/P_k$ is a $\mathbb{P}^1$ fibration. It maps the open cell $X^\omega_k$ isomorphically to its image. We have $X_\omega = \pi_k^{-1}\pi_k(X_\omega_k)$ and $\pi_k$ restricted to $X^\omega_k$ is an $\mathbb{A}^1$ fibration. The variety $Z_\omega$ fibers over $Z_\omega_k$ with the fiber $\mathbb{P}^1$. We have a pull-back diagram

$$
\begin{array}{ccc}
Z_\omega & \xrightarrow{f_\omega} & X_\omega \\
\downarrow_{\tilde{\iota}} & & \downarrow_{\pi_k} \\
Z_{\omega'} & \xrightarrow{\pi_k \circ f_\omega} & \pi_k(X_\omega')
\end{array}
\xrightarrow{G/B}
\begin{array}{c}
G/B \\
\downarrow_{\pi_k} \\
G/P_k
\end{array}
$$

The projection $\pi_\omega : Z_\omega \to Z_{\omega'}$ has a section $\iota$, such that $f_\omega \circ \iota = f_{\omega'}$. The relative tangent bundle for $\pi_k$ is denoted by $L_k$. It is associated with the $T$ representation of weight $-\alpha_k$, see [Ram85], [OSWW17, §3]. According to our notation $L_k = \mathcal{L}_{\alpha_k}$.
3.2. **Fixed points of the Bott-Samelson varieties.** Let \( \omega = s_{k_1}s_{k_2} \ldots s_{k_\ell} \) be a reduced word representing \( \omega \in W \) and let \( f_\omega : Z_\omega \to X_\omega \) be the Bott-Samelson resolution of the Schubert variety \( X_\omega \). The \( T \) fixed points of \( X_\omega \) and \( Z_\omega \) are discrete, namely:

- The fixed points \( (X_\omega)^T \) are \( x_\omega' \) where \( \omega' \leq \omega \) in the Bruhat order.
- The fixed points \( (Z_\omega)^T \) are indexed by subwords of \( \omega \) (which are words obtained by leaving out some of the letters from \( \omega \)). We identify subwords with 01-sequences (where 0's mark the positions of the letters to be omitted). We will identify a fixed point with its subword and with its 01-sequence.

The map \( f_\omega \) sends the sequence \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell) \) to \( x_\sigma \) where \( \sigma = s_{k_1}^\epsilon_1 s_{k_2}^\epsilon_2 \ldots s_{k_\ell}^\epsilon_\ell \in W \).

**Example 3.1.** Let \( G = GL_3 \) and \( \omega = s_1s_2s_1 \).

<table>
<thead>
<tr>
<th>sequence</th>
<th>subword</th>
<th>image</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>$\emptyset \emptyset \emptyset$</td>
<td>id = 123</td>
</tr>
<tr>
<td>001</td>
<td>$\emptyset \emptyset s_1$</td>
<td>$s_1 = 213$</td>
</tr>
<tr>
<td>010</td>
<td>$\emptyset s_2 \emptyset$</td>
<td>$s_2 = 132$</td>
</tr>
<tr>
<td>011</td>
<td>$\emptyset s_2s_1$</td>
<td>$s_2s_1 = 312$</td>
</tr>
<tr>
<td>100</td>
<td>$s_1 \emptyset s_1$</td>
<td>$s_1 = 213$</td>
</tr>
<tr>
<td>101</td>
<td>$s_1s_2 \emptyset$</td>
<td>id = 123</td>
</tr>
<tr>
<td>110</td>
<td>$s_1s_2s_1$</td>
<td>$s_1s_2s_1 = 321$</td>
</tr>
<tr>
<td>111</td>
<td>$s_1s_2s_1$</td>
<td>$s_1s_2s_1 = 321$</td>
</tr>
</tbody>
</table>

From the recursive definition of \( Z_\omega \) above we find the recursive description of the tangent weights of \( Z_\omega \) at the fixed point \( x \in \{0, 1\}^\ell \):

\[
\text{weights}(Z_\omega, x) = \begin{cases} 
\text{weights}(Z_\omega', x') \cup \{(L_k)f_\omega(x')\} & \text{if } x = (x', 0) \\
\text{weights}(Z_\omega', x') \cup \{(L_k^{-1})f_\omega(x')\} & \text{if } x = (x', 1).
\end{cases}
\]

Note that \( L_k \) or \( L_k^{-1} \) at a fixed point is just a line with \( T \) action, i.e. a character, so it can indeed be interpreted as a weight. In theory weights form a multiset, so above the \( \cup \) should mean union of multisets, but in fact no repetition of weights occurs, hence there is no need for multisets.

3.3. **Canonical divisors of Schubert and Bott-Samelson varieties.** The starting point for the computation of the elliptic classes of Schubert varieties is the following fact.

**Theorem 3.2.** The divisor \( K_{X_\omega} + \partial X_\omega \) is a \( \mathbb{Q} \)-Cartier divisor, and we have

\[
f_\omega^*(K_{X_\omega} + \partial X_\omega) = K_{Z_\omega} + \partial Z_\omega.
\]

**Proof.** Let \( \rho \in t^* \) be half of the sum of positive roots. Let \( C_{-\rho} \) the trivial bundle with the \( T \) action of weight \(-\rho\) and let \( L_\rho = G \times_B \mathbb{C}_{-\rho} \) be the line bundle associated with weight \( \rho \). We denote ideal sheaves by \( I(\cdot) \), and canonical sheaves by \( \omega_\bullet \). We have the following identities on equivariant sheaves:

1. \( \omega_{X_\omega} = I(\partial X_\omega) \otimes L_\rho^{-1} \otimes \mathbb{C}_{-\rho} \)
2. \( \omega_{Z_\omega} = I(\partial Z_\omega) \otimes f_\omega^*(L_\rho^{-1}) \otimes \mathbb{C}_{-\rho} \).
The first one is proved in [GrKu08, Prop. 2.2] (cf. the non-equivariant version [Ram87, Th 4.2]), and the second one is proved in [BrKu05, Prop. 2.2.2] (cf. the non-equivariant version [Ram85, Prop. 2]).

The boundary $\tilde{D}$ of the opposite open Schubert cell (that is $\tilde{D} = \cup X_s$) intersects $X_\omega$ transversally, hence, from the sheaf identities above we obtain the divisor identities

$$K_{X_\omega} = -\partial X_\omega - \tilde{D} \cap X_\omega, \quad K_{Z_\omega} = -\partial Z_\omega - f^{-1}(\tilde{D} \cap X_\omega).$$

Hence, $K_{X_\omega} + \partial X_\omega$ is Cartier and, by rearrangement we obtain $K_{Z_\omega} + \partial Z_\omega = f^*(K_{X_\omega} + \partial X_\omega)$.

Note, that all the involved Weil divisors are $T$ invariant.

\[\square\]

### 3.4. The $\lambda$-twisted elliptic class of Schubert varieties

Let $\lambda \in \mathfrak{t}^*$ be a dominant weight and let $L_\lambda = G \times_B \mathbb{C}_{-\lambda}$ be the associated (globally generated) line bundle over $G/B$. Then $H^0(G/B; L_\lambda)$ is the irreducible representation of $G$ with highest weight $\lambda$. There exists a unique (up to a constant) section $\xi_\lambda$ of $L_\lambda$, which is invariant with respect to the nilpotent group $N^- \subset B^-$ and on which $T$ acts via the character $\lambda$. Therefore $\xi_\lambda$ does not vanish at the points of the open Schubert cell $X_{w_0}$ and its zero divisor is supported on the union of codimension one Schubert varieties. The translation $\xi_{\omega}^{\lambda} := \tilde{\omega} \omega_0^{-1}(\xi_\lambda)$ of this section by $\tilde{w} \omega_0^{-1} \in N(T)$ is an eigenvector of $B^-$ of weight $\omega \lambda$. The zero divisor of $\xi_{\omega}^{\lambda}|_{X_\omega}$ is supported on $\partial X_\omega$. The multiplicities of this zero divisor are given by the Chevalley formula. Namely, if $\omega = \omega' s_\alpha$, $\ell(\omega) = \ell(\omega') + 1$ and $\alpha$ is a positive root, then the multiplicity of $X_{\omega'}$ is equal to $\langle \lambda, \alpha^\vee \rangle$, where $\alpha^\vee$ is the dual root, [Che58] (or see [Br05, Prop. 1.4.5] for the case of $GL_n$).

**Example 3.3.** Let $G = GL_n$,

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$,
- $\alpha = (0, \ldots, 1, \ldots, -1, \ldots, 0)$, $\alpha^\vee = \lambda_i^* - \lambda_j^*$ for $i < j$,

then $\langle \lambda, \alpha^\vee \rangle = \lambda_i - \lambda_j$.

Our main object for the rest of the paper is the $\lambda$-twisted elliptic class

$$E(X_\omega, \lambda) = \text{Ell}(X_\omega, \partial X_\omega - \text{zeros}(\xi_\lambda); G/B),$$

of the Schubert variety $X_\omega$, cf. Section 3.4. The definition makes sense for “sufficiently large” $\lambda$, i.e. we need to assume that the coefficients of each boundary component in $\text{zeros}(\xi_\lambda)$ is positive. It will be clear from the next section, that it is enough to assume that $\lambda$ is dominant.

### 4. Recursive calculation of local elliptic classes

After using Kempf’s lemma to calculate the multiplicities of $f_\omega^*(\xi_\lambda)$ in Section 4.1, we will give a recursive formula for the local elliptic classes of the Bott-Samelson and Schubert varieties in Sections 4.3, 4.4.
4.1. Multiplicities of the canonical section. As before, let $\omega$ be a reduced word representing $\omega$, and consider the corresponding Bott-Samelson resolution. We have $f^*_\omega(K_{X_\omega} + \partial X_\omega) = K_{Z_\omega} + \partial Z_\omega$ (Theorem 3.2), and hence

$$E(X_\omega, \lambda) = f^*_{\omega'} E_{\ell}(Z_\omega, \partial Z_\omega - \text{zeros}(f^*_{\omega}(\xi_\lambda))).$$

Thus, to calculate $E(X_\omega, \lambda)$ we need to know the multiplicities of $f^*_{\omega}(\xi_\lambda)$ along the components of $\partial Z_\omega$. Recall that the components of $\partial Z_\omega$ correspond to omitting a letter in the word $\omega$. Let $\partial_j Z_\omega$ denote the component corresponding to omitting the $j$'th letter of $\omega$. For our argument in the next section we need the following corollary:

**Proposition 4.1.** Suppose $\lambda \in \mathfrak{t}'$ (not necessarily dominant), then

1. $f^*_{\omega}(L_\lambda) \simeq \pi^*_\omega(L_{s_k \lambda}) \otimes \mathcal{O}_{Z_\omega}(\langle \alpha_k, \lambda \rangle \iota Z_\omega)$,
2. If $\lambda$ is dominant, then the multiplicity of zeros of $f^*_{\omega}(\xi_\lambda)$ along the divisor $\iota Z_\omega$ is equal to $\langle \alpha_k, \lambda \rangle$,
3. the remaining multiplicities of $f^*_{\omega}(\xi_\lambda)$ along the components of $\partial Z_\omega$ are equal to the corresponding multiplicities $f^*_{\omega'}(\xi_{s_k \lambda})$.

**Proof.** Suppose $\omega = \omega' s_k$ is a reduced word. Let $Y = X_{\omega'} \times_{G/P} G/B$. We have the commutative diagram

$$\begin{array}{ccc}
Z_\omega & \xrightarrow{f} & Y \\
\downarrow{\pi_\omega} & & \downarrow{\pi_Y} \\
Z_{\omega'} & \xrightarrow{f_{\omega'}} & X_{\omega'} \xrightarrow{\pi_k} G/P,
\end{array}$$

together with a section $\iota' : X_{\omega'} \rightarrow Y$ which agrees with the section $\iota : Z_{\omega'} \rightarrow Z_\omega$. Recall the following lemma of Kempf (originating from the Chevalley’s work).

**Lemma 4.2** ([Kem76] Lemma 3). Suppose $\lambda \in \mathfrak{t}'$ (not necessarily dominant), then

(i) $f^*_\omega(L_\lambda) \simeq \pi^*_\omega(L_{s_k \lambda}) \otimes \mathcal{O}_Y(\langle \alpha_k, \lambda \rangle \iota' X_{\omega'})$.

(ii) If $f^*_\omega(L_\lambda)$ has a non-zero section, then so does $\pi^*_\omega(L_{s_k \lambda} | X_{\omega'})$. Furthermore, there exists a non-zero section, which is invariant with respect to $B^-$. 

To continue the proof note that (1) follows directly from (i). The bundle $f^*_\omega(L_\lambda)$ is isomorphic to $\mathcal{O}_{Z_\omega}(\sum_{k=1}^{s(\omega)} a_k \partial Z_\omega)$. The section in (ii) is unique and it is equal to $f^*_\omega(\xi_\lambda)$. The multiplicity corresponding to the last component $\partial_{\ell(\omega)} Z_\omega = \iota(Z_{\omega'})$ is equal to $\langle \alpha_k, \lambda \rangle$, the remaining components are pulled back from $Z_{\omega'}$. This proves (2) and (3). ∎

Inductively it is immediate to show the extension to Bott-Samelson variety of Chevalley formula for multiplicities.

**Corollary 4.3.** Let $\omega = \omega_1 s_k \omega_2$ where $s_k$ is the $j$'th letter in the word $\omega$. Write $\omega = \omega_1 \omega_2 s_\alpha$, where $s_\alpha = \omega_1^{-1} s_k \omega_2$ and $\alpha$ is a positive root. Then the multiplicity of $f^*_{\omega}(\xi_\lambda)$ along the boundary component $\partial_j Z_\omega$ is $\langle \lambda, \alpha' \rangle$. ∎
4.2. **Local elliptic classes of Bott-Samelson and Schubert varieties.** Recall from Section 3.2 that the $T$ fixed points of $G/B$ are identified with elements of $W$, and the $T$ fixed points of $Z_{\tilde{\omega}}$ are parameterized by subwords of $\tilde{\omega}$ or equivalently by 01-sequences. For $\sigma \in (G/B)^T$, $x \in \tilde{Z}_{\tilde{\omega}}$ define the local classes

$$E_{\sigma}(X_{\tilde{\omega}}, \lambda) = \frac{E(X_{\tilde{\omega}}, \lambda)}{e^K(T_\sigma(G/B))} \neq \frac{\tilde{\eta}(X_{\tilde{\omega}}, \partial X_{\tilde{\omega}} - \text{zeros}(\xi^w))}{e^K(T_\sigma(G/B))} \in \text{Frac}(R(T)(h)),$$

$$E_x(Z_{\tilde{\omega}}, \lambda) = \frac{\tilde{\eta}(Z_{\tilde{\omega}}, \partial Z_{\tilde{\omega}} - \text{zeros}(f^*_w(\xi^w)))}{x} \in \text{Frac}(R(T)(h))$$

in the fraction field of the representation ring $R(T)$ extended by the formal parameter $h$. If $\omega = \text{id} \in W$ then $\omega = \emptyset$. The Bott-Samelson variety $Z_{\emptyset}$ is one point, a fixed point indexed by the sequence of length 0. Hence we have $f_{\emptyset}(Z_{\emptyset}) = [\text{id}] \in G/B$ and

$$E_{\emptyset}(Z_{\emptyset}, \lambda) = 1, \quad E_{[\text{id}]}(X_{\text{id}}, \lambda) = 1. \quad (9)$$

In the next two subsections we show how the geometry described in Section 3 implies recursions of the local classes, that together with the base step (9) determine them.

4.3. **Recursion for local elliptic classes of Bott-Samelson varieties.** From the description of the fixed points of $Z_{\tilde{\omega}}$ and their tangent weights in Section 3.2 we obtain the following recursion for the local classes on $Z_{\tilde{\omega}}$. Let $\omega = \omega s_k$ be a reduced word.

If $x = (x', 0) \in Z_{\tilde{\omega}}$ then

$$E_x(Z_{\tilde{\omega}}, \lambda) = E_x'(Z_{\tilde{\omega}'}, s_k^x \lambda) \cdot \delta \left( (L_k)_{f_{\omega}(x')}, h^{(\lambda, \alpha_k^\vee)} \right).$$

If $x = (x', 1) \in Z_{\tilde{\omega}}$ then

$$E_x(Z_{\tilde{\omega}}, \lambda) = E_x'(Z_{\tilde{\omega}'}, s_k^x \lambda) \cdot \delta \left( (L_k^{-1})_{f_{\omega}(x')}, h \right).$$

As before, in our notation we identify a bundle restricted to a fixed point with the weight of the obtained $T$ representation on $\mathbb{C}$. Note that in the formula for $x = (x', 1)$ the weight $(L_k^{-1})_{f_{\omega}(x')}$ is equal to $(L_k)_{f_{\omega}(x)}$.

Recall that the classes $E_x(Z_{\tilde{\omega}}, \lambda)$ are defined only for dominant weights $\lambda$. However, the formulas above are meromorphic functions in $\lambda$, so we formally define $E_x(Z_{\tilde{\omega}}, \lambda)$ for all $\lambda \in \mathfrak{t}^*$ by the meromorphic function it satisfies for dominant $\lambda$.

**Example 4.4.** For $G = \text{GL}_3$ we use the notation $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ as before, and let $\mu_k = \mu_k(\lambda) = h^{-\lambda_k}$.

- For $\omega = \emptyset$ we have $E_{\emptyset}(Z_{\tilde{\omega}}, \lambda) = 1$.
- For $\omega = s_1$ we have

\[
E_{(0)}(Z_{s_1}, \lambda) = 1_{|\mu_1 + \mu_2} \cdot \delta(z_2 - z_1, \mu_2 - \mu_1) = \delta(z_2 - z_1, \mu_2 - \mu_1)
\]

\[
E_{(1)}(Z_{s_1}, \lambda) = 1_{|\mu_1 + \mu_2} \cdot \delta(z_1 - z_2, h) = \delta(z_1 - z_2, h).
\]
• For \( \omega = s_2s_1 \) we have
\[
E_{(0,0)}(Z_{s_1s_2}, \lambda) = E_{(0)}(Z_{s_1}, \lambda) |_{\mu_2+\mu_3} \cdot \delta(z_3 - z_2, \mu_3 - \mu_2)
= \delta(z_2 - z_1, \mu_3 - \mu_1)\delta(z_3 - z_2, \mu_3 - \mu_2),
\]
\[
E_{(0,1)}(Z_{s_1s_2}, \lambda) = E_{(0)}(Z_{s_1}, \lambda) |_{\mu_2+\mu_3} \cdot \delta(z_2 - z_3, h)
= \delta(z_2 - z_1, \mu_3 - \mu_1)\delta(z_2 - z_3, h),
\]
\[
E_{(1,0)}(Z_{s_1s_2}, \lambda) = E_{(1)}(Z_{s_1}, \lambda) |_{\mu_2+\mu_3} \cdot \delta(z_3 - z_1, \mu_3 - \mu_2)
= \delta(z_1 - z_2, h)\delta(z_3 - z_1, \mu_3 - \mu_2),
\]
\[
E_{(1,1)}(Z_{s_1s_2}, \lambda) = E_{(1)}(Z_{s_1}, \lambda) |_{\mu_2+\mu_3} \cdot \delta(z_1 - z_3, h)
= \delta(z_1 - z_2, h)\delta(z_1 - z_3, h).
\]

4.4. Recursion for local elliptic classes of Schubert varieties. Let \( \omega \) represent \( \omega \in W \). According to Proposition 2.7 we have
\[
(10) \quad E_\sigma(X_\omega, \lambda) = \sum_x E_x(Z_\omega, \lambda),
\]
where the summation runs for fixed points \( x \) with \( f_\omega(x) = \sigma \).

For example, for \( G = \text{GL}_3 \) let \( \omega = s_1s_2s_1 \) and let \( x \) be the fixed point corresponding to the identity permutation. Then the summation has two summands, corresponding to the 01 sequences (fixed points) \((0,0,0)\) and \((1,0,1)\). In fact for this \( \omega \) only two fixed points (id and \( s_1 \)) are such that the summation has two terms; in the remaining cases there is only one term, see the table in Example 3.1.

The recursion of Section 4.3 for the terms of the right hand side of (10) implies a recursion for the \( E_\sigma(X_\omega, \lambda) \) classes: the initial step is
\[
E_\sigma(X_{\text{id}}, \lambda) = \begin{cases} 1 & \text{if } \sigma = \text{id} \\ 0 & \text{if } \sigma \neq \text{id}, \end{cases}
\]
and for \( \omega = \omega's_k, \ell(\omega) = \ell(\omega') + 1 \) we have
\[
(11) \quad E_\sigma(X_\omega, \lambda) = E_\sigma(X_{\omega'}, s_k(\lambda)) \cdot \delta(L_{k,\sigma}, h^{(\lambda, \alpha^*_k)}) + E_{s_k}(X_{\omega'}, s_k(\lambda)) \cdot \delta(L_{k,\sigma}, h).
\]

Example 4.5. Let \( G = \text{GL}_3 \) and assume we already calculated the fixed point restrictions of \( E(X_{312}, \lambda) \). By applying the recursion above for \( \omega = \omega's_2 = (s_2s_1)s_2 \) we obtain
\[
E_{123}(X_{321}, \lambda) = E_{123}(X_{312}, \lambda) |_{\mu_2+\mu_3} \delta(z_3/z_2, \mu_3/\mu_2) + E_{213}(X_{312}, \lambda) |_{\mu_2+\mu_3} \delta(z_3/z_2, h)
= \delta(z_3/z_2, \mu_2/\mu_1)\delta(z_2/z_1, \mu_3/\mu_1)\delta(z_3/z_2, \mu_3/\mu_2)
+ \delta(z_2/z_3, h)\delta(z_3/z_1, \mu_3/\mu_1)\delta(z_3/z_2, h).
\]

We may calculate the same local class using the recursion for \( \omega = \omega's_1 = (s_1s_2)s_1 \), and we obtain
\[
E_{123}(X_{321}, \lambda) = \delta(z_2/z_1, \mu_3/\mu_2)\delta(z_3/z_2, \mu_3/\mu_1)\delta(z_2/z_1, \mu_2/\mu_1)
+ \delta(z_1/z_2, h)\delta(z_3/z_1, \mu_3/\mu_1)\delta(z_2/z_1, h).
\]
The equality of the expressions (12) and (13) is a non-trivial four term identity for theta functions, for more details see Section 9.

Now we are going to rephrase the recursion (11) in a different language. According to K-theoretic equivariant localization theory, the map

\[ K_T(G/B)(h) \to \bigoplus_{\sigma \in W} K_T(x_\sigma)(h) = \bigoplus_{\sigma \in W} R(T)(h), \]

whose coordinates are the restriction maps to \( x_\sigma \), is injective. As before (cf. Sections 2.3, 4.2) we divide the restriction by the K-theoretic Euler class, and consider the map

\[ \text{res} : K_T(G/B)(h) \to \bigoplus_{\sigma \in W} \text{Frac}(R(T)(h)), \quad \text{res}(\beta) = \left\{ \frac{\beta_{x_\sigma}}{e^K(T_{x_\sigma}(G/B))} \right\}_{\sigma \in W}, \]

where \( \text{Frac}(R(T)(h)) \) is the fraction field of \( R(T)(h) \). Since the map is injective, we may identify an element of \( K_T(G/B)(h) \) with its res-image, i.e. with a tuple of elements from \( \text{Frac}(R(T)(h)) \).

**Theorem 4.6 (Main Theorem).** Regarding the classes \( E_\bullet(X_{\omega}, \lambda) \) as elements of \( \bigoplus_{\sigma \in W} \text{Frac}(R(T)(h)) \), the following recursion holds:

\[ E_\sigma(X_{\text{id}}, \lambda) = \begin{cases} 1 & \sigma = \text{id} \\ 0 & \sigma \neq \text{id} \end{cases}, \]

and for \( \omega = \omega s_k \) with \( \ell(\omega) = \ell(\omega') + 1 \) we have

\[ E_\bullet(X_{\omega}, \lambda) = (\delta_{k}^{\text{bd}} \text{id} + \delta_{k}^{\text{int}} s_k^\gamma) (E_\bullet(X_{\omega'}, s_k \lambda)). \]

Here

- \( s_k^\gamma \) for \( s_k \in W \) acts on the fixed points by right translation \( \sigma \mapsto \sigma s_k \),
- \( s_k \) acts as well on \( \lambda \in t^* \), later this action will be denoted by \( s_k^\mu \),
- \( \delta_{k}^{\text{bd}} \) — multiplication by the element \( \delta(L_k, h^{(\lambda, a\gamma)}) \) (the “boundary factor”)
- \( \delta_{k}^{\text{int}} \) — multiplication by the element \( \delta(L_k, h) \) (the “internal factor”).

Note that boundary and internal factors indeed make sense: restricted to a fixed point \( x_\sigma \) the line bundle \( L_k \) is a \( T \) character depending on \( \sigma \); that is, multiplication by one of these factors means multiplication by a diagonal matrix, not by a constant matrix.

**Proof.** The statement is the rewriting of the recursion (11). \qed

5. **Hecke algebras**

In this section we review various Hecke-type actions on cohomology or K-theory of \( G/B \) giving rise to inductive formulas for various invariants of the Schubert cells. Sections 5.1–5.3—exploring the relation between our elliptic classes and other characteristic classes of singular varieties—is not necessary for the rest of the paper. A reader not familiar with Chern-Schwartz-MacPherson or motivic Chern classes can jump to Section 5.4.
5.1. **Fundamental classes—the nil-Hecke algebra.** Consider the notion of *equivariant fundamental class in cohomology*, denoted by \([\cdot]\). According to [BGG73, Dem74] if \(\omega = \omega's_k\), \(\ell(w) = \ell(\omega') + 1\) then the operation \(D_k = \pi_k^* \circ \pi_{ks}\) in cohomology satisfies
\[
D_k([X_{\omega'}]) = [X_\omega], \quad D_k \circ D_k = 0.
\]
The algebra generated by the operations \(D_k\) is called the nil-Hecke algebra. As before, let us identify elements of \(H^*_T(G/B)\) with their res-image, where

\[
\text{res} : H^*_T(G/B) \to \bigoplus_{\sigma \in W} \mathbb{Q}(t), \quad \beta \mapsto \left\{ \beta_{|x_\sigma} \right\}_{\sigma \in W}.
\]
Here \(\mathbb{Q}(t)\) is the field of rational functions on \(t\), and \(e^H(\cdot)\) is the equivariant cohomological Euler class.

The action of the Demazure operations on the right hand side is given by the formula
\[
D_k = \frac{1}{c_1(L_k)} (\text{id} + s_k^\gamma), \quad \text{that is} \quad D_k(\{f\})_\sigma = \frac{1}{c_1(L_k)_\sigma} (f_\sigma + f_{\sigma s_k}).
\]
The operators \(D_k\) satisfy the braid relations and \(D_k \circ D_k = 0\).

For \(G = GL_n\) we have \(c_1(L_k)_\sigma = z_\sigma(k+1) - z_\sigma(k)\) (where \(z_1, z_2, \ldots, z_n\) are the basic weights of \(T \leq GL_n\)) and we recover the divided difference operators from algebraic combinatorics.

5.2. **CSM-classes and the group ring \(\mathbb{Z}[W]\).** An important one-parameter deformation of the notion of cohomological (equivariant) fundamental class is the equivariant Chern-Schwartz-MacPherson (CSM, in notation \(c^{sm}(\cdot)\)) class. For introduction to this cohomological characteristic class see, e.g. [Oh06, We12, AlMi16, FR18, AMSS17].

It is shown in [AlMi16, AMSS17] that the CSM classes of Schubert cells satisfy the recursion: if \(\omega = \omega's_k\), \(\ell(w) = \ell(\omega') + 1\) then
\[
A_k(c^{sm}(X_{\omega'})) = c^{sm}(X_\omega)
\]
where
\[
A_k = (1 + c_1(L_k))D_k - \text{id}.
\]
In terms of res-images
\[
A_k(\{f\})_\sigma = \frac{1}{c_1(L_k)_\sigma} f_\sigma + \frac{1+c_1(L_k)_\sigma}{c_1(L_k)_\sigma} f_{\sigma s_k}.
\]
By [AlMi16] or by straightforward calculation we find that \(A_k \circ A_k = \text{id}\) and the operators \(A_k\) satisfy the braid relations.

5.3. **Motivic Chern classes—the Hecke algebra.** The K theoretic counterpart of the notion of CSM class is the *motivic Chern class* (in notation \(mC_y(\cdot)\)), see [BSY10, FRW18, AMSS19]. The operators
\[
B_k = (1 + yL_k^{-1})s_k^\pi \pi_{ks} - \text{id},
\]
(see Section 3.1) reproduce the motivic Chern classes \(mC_y\) of the Schubert cells: if \(\omega = \omega's_k\), \(\ell(w) = \ell(\omega') + 1\) then \(B_k(mC_y(X_{\omega'})) = mC_y(X_\omega)\), see [AMSS19], c.f. [SZZ17]. In the local
presentation, i.e. after restriction to the fixed points and division by the K-theoretic Euler class, the operator $B_k$ takes form

$$B_k(\{f_\bullet\})_\sigma = \frac{(1+y)(L_k^{-1})_\sigma}{1-(L_k^{-1})_\sigma} f_\sigma + \frac{1+y(L_k^{-1})_\sigma}{1-(L_k^{-1})_\sigma} f_{\sigma s_k}.$$ 

For example, for $G = \text{GL}_n$ we have $(L_k)^{-1} = z_{\sigma(k)}/z_{\sigma(k+1)}$. The squares of the operators satisfy

$$B_k \circ B_k = -(y+1)B_k - y \text{id},$$

and the operators $B_k$ satisfy the braid relations, compare [Lu85].

5.4. **Elliptic Hecke algebra.** Consider the operator

$$C_k = (\delta_k^{bd} \text{id} + \delta_k^{int} s_k^\mu) s_k^\mu$$

acting on the direct sum of the spaces of meromorphic functions

$$\bigoplus_{\sigma \in W} \mathcal{M}(v^*, \text{Frac}(R(T)(h)),$$

or in coordinates

$$C_k(\{f_\bullet\})_\sigma(\lambda) = (\delta_k^{bd})_\sigma f_\sigma(s_k \lambda) + (\delta_k^{int})_\sigma f_{\sigma s_k}(s_k \lambda).$$

In Section 4.4 we have shown that if $\omega = \omega' s_k$, $\ell(w) = \ell(\omega') + 1$ then

$$E_\bullet(X_{\omega}, \lambda) = C_k(E_\bullet(X_{\omega'}, \lambda)).$$

**Theorem 5.1.** The square of the operator $C_k$ is multiplication by a function depending only on $\lambda$ and $h$:

$$C_k \circ C_k = \kappa_k(\lambda) \text{id},$$

where

$$\kappa_k(\lambda) = \delta(h, \nu_k(\lambda))\delta(h, 1/\nu_k(\lambda)),$$

where $\nu_k(\lambda) = h^{(\lambda, \alpha_k^\vee)}$.

**Proof.** It is enough to check the identity for $G = \text{GL}_2$, $\sigma = \text{id}$:

$$C_1 \circ C_1(\{f_\bullet\})_{\text{id}} = \left( \delta \left( \frac{z_1}{z_2}, h \right) \delta \left( \frac{z_2}{z_1}, h \right) + \delta \left( \frac{z_2}{z_1}, \frac{1}{\nu_1} \right) \delta \left( \frac{z_2}{z_1}, \nu_1 \right) \right) f_\text{id} + \delta \left( \frac{z_2}{z_1}, h \right) \left( \delta \left( \frac{z_1}{z_2}, \frac{1}{\nu_1} \right) + \delta \left( \frac{z_1}{z_2}, \nu_1 \right) \right) f_{s_1},$$

Since the function $\theta$ is antisymmetric ($\theta(1/x) = -\theta(x)$ and hence $\delta(x, y) = -\delta(1/x, 1/y)$) we have

$$C_1 \circ C_1(\{f_\bullet\})_{\text{id}} = \left( \delta \left( \frac{z_1}{z_2}, h \right) \delta \left( \frac{z_2}{z_1}, h \right) + \delta \left( \frac{z_2}{z_1}, \frac{1}{\nu_1} \right) \delta \left( \frac{z_2}{z_1}, \nu_1 \right) \right) f_\text{id}$$

$$= -\frac{\theta(\nu_1)^2}{\theta(\nu_1)^2} \theta \left( \frac{z_1}{z_2}, \frac{1}{\nu_1} \right) \delta \left( \frac{z_2}{z_1}, \nu_1 \right) \delta \left( \frac{z_2}{z_1}, \nu_1 \right) f_\text{id}$$

Setting

$$a = h, \quad b = \frac{z_2}{z_1}, \quad c = \nu_1, \quad d = 1$$

in the Fay’s trisecant identity (8) we obtain the claim. $\square$
Proof of Formula 1.1 relating global nonrestricted classes, we conjugate the operation $C_k$ with the multiplication by the elliptic Euler class

$$e^E(TG/B) = \prod_{\alpha \in \text{positive roots}} \vartheta(\mathcal{L}_\alpha)$$

(according to our convention $\mathcal{L}_\alpha = G \times_B C_{-\alpha}$). Since

$$\vartheta(\mathcal{L}_\alpha|\sigma)/\vartheta(\mathcal{L}_\alpha|\sigma s_k) = -1$$

we obtain the minus sign in (1.1).

If $s_k s_\ell = s_\ell s_k$, then $C_k \circ C_\ell = C_\ell \circ C_k$. Moreover for $GL_n$

$$C_k \circ C_{k+1} \circ C_k = C_{k+1} \circ C_k \circ C_{k+1}.$$  

Hence for $G = GL_n$ the operators $C_k$ define a representation of the braid group. For general $G$ the corresponding braid relation of the Coxeter group are satisfied. This is a immediate consequence of the fact, that the elliptic class does not depend on the resolution.

**Remark 5.2.** Note that the operations $C_\ell$ do not commute with $\kappa_k(\lambda)$ but they satisfy $\kappa_k(\lambda) \circ C_\ell = C_\ell \circ \kappa_k(s_\ell \lambda)$.

This table summarizes the various forms of the Hecke algebras whose operators produce more and more general characteristic classes of Schubert varieties.

<table>
<thead>
<tr>
<th>invariant</th>
<th>operation</th>
<th>square</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-]$</td>
<td>$\frac{1}{c_1(L_k)}(\text{id} + s_k^\gamma)$</td>
<td>$D_k^2 = 0$</td>
</tr>
<tr>
<td>$c^{sm}$</td>
<td>$\frac{1}{c_1(L_k)} \text{id} + \frac{1 + c_1(L_k)}{c_1(L_k)} s_k^\gamma$</td>
<td>$A_k^2 = \text{id}$</td>
</tr>
<tr>
<td>$mC_y$</td>
<td>$\frac{(1+y)L_k^{-1}}{1-L_k^{-1}} \text{id} + \frac{1+y L_k^{-1}}{1-L_k^{-1}} s_k^\gamma$</td>
<td>$B_k^2 = -(y+1)B_k - y \text{id}$</td>
</tr>
<tr>
<td>$E(-, \lambda)$</td>
<td>$\delta(L_k, h^{(\lambda, \alpha_k^\vee)}) s_k^{\mu} + \delta(L_k, h)s_k^\gamma s_k^{\mu}$</td>
<td>$C_k^2 = \kappa_k(\lambda)$</td>
</tr>
</tbody>
</table>

5.5. **Modifying the degeneration of $E(-, \lambda)$ and $C_k$.** The characteristic classes, $[-]$, $c^{sm}$, $mC_y$, $E(-, \lambda)$ are of increasing generality: an earlier in the list can be obtained from a latter in the list by formal manipulations. However, the limiting procedure of getting $mC_y$ from $E(-, \lambda)$ itself is not obvious. We describe this procedure below. As a result we obtain a family of $mC_y$-like classes, with only one of them the $mC_y$-class, as well as a family of corresponding Hecke-type algebras.

The theta function has the limit property

$$\lim_{q \to 0} \vartheta(x) = x^{1/2} - x^{-1/2}.$$
It follows that

$$\lim_{q \to 0} \delta(x, h) = \frac{1 - 1/(xh)}{(1 - 1/x)(1 - 1/h)}.$$  

The motivic Chern class is the limit of the elliptic class when $q = e^{2\pi i r} \to 0$. The limit of the elliptic class of a pair is not what we would expect: the limit of the boundary factor is equal to

$$\lim_{q \to 0} \delta(x, \nu_k(\lambda)) = \frac{1 - L_k^{-1}/\nu_k(\lambda)}{(1 - L_k^{-1})(1 - 1/\nu_k(\lambda))}$$

where $\nu_k(\lambda) = h^{(\lambda, \alpha_\gamma^\vee)}$. The limit Hecke algebra differs from the Hecke algebra computing $mC_y$'s of Schubert cells. The limit classes depend on the parameter $\lambda \in T^\ast$. Calculation shows that in the limit when $q \to 0$ we obtain the operators

$$C_k^{q \to 0} = \left( \frac{1 - L_k^{-1}/\nu_k(\lambda)}{(1 - L_k^{-1})(1 - 1/\nu_k(\lambda))} \right) \frac{1 - 1/(\nu_k(\lambda)h)}{(1 - 1/\nu_k(\lambda))(1 - 1/h)} \cdot s_k^\mu$$

satisfying

$$C_k^{q \to 0} \circ C_k^{q \to 0} = \frac{1 - \nu_k(\lambda)/h}{(1 - \nu_k(\lambda))(1 - 1/h)} \cdot \frac{1 - 1/(\nu_k(\lambda)h)}{(1 - 1/\nu_k(\lambda))(1 - 1/h)} \cdot \id.$$  

Another method of passing to the limit, as in [BL05], is when we first rescale $\lambda$ by $\log(q)/\log(h) = \tau/\log(h)$ and then pass to the limit. For $0 < \Re(z) < 1$, and $m \in \mathbb{Z}$ we have

$$\lim_{q \to 0} \frac{\partial (aq^{m+z})}{\partial (bq^{m+z})} = (a/b)^{-m-1/2}.$$  

We obtain different operations:

$$\lim_{q \to 0} \delta(a, h^{\tau \lambda \alpha_\gamma^\vee}) = \lim_{q \to 0} \delta(a, q^{\lambda \alpha_\gamma^\vee}) = \lim_{q \to 0} \frac{\partial (1)}{\partial (a)} \frac{\partial (aq^{\lambda \alpha_\gamma^\vee})}{\partial (q^{\lambda \alpha_\gamma^\vee})} = \frac{a^{-b_k(\lambda) - 1/2}}{a^{1/2} - a^{-1/2}} = \frac{a^{-b_k(\lambda) - 1}}{1 - a^{-1}},$$

where $b_k(\lambda)$ is the integral part of $\Re(\lambda, \alpha_\gamma^\vee)$, provided that $\langle \lambda, \alpha_\gamma^\vee \rangle \notin \mathbb{Z}$. The limit operation now takes form

$$\tilde{C}_k^{q \to 0} = \left( \frac{L_k^{-b_k(\lambda) - 1}}{1 - L_k^{-1}} \right) \frac{1 - 1/(\nu_k(\lambda)h)}{(1 - 1/\nu_k(\lambda))(1 - 1/h)} \cdot s_k^\mu.$$  

Setting $y = -h^{-1}$ we obtain a form resembling the operation $B_k$:

$$\frac{1}{1 + y} \left( \frac{(1 + y)L_k^{-b_k(\lambda) - 1}}{1 - L_k^{-1}} \right) \frac{1 + y L_k^{-1}}{(1 - L_k^{-1})} \cdot s_k^\mu.$$  

For weights $\lambda$ belonging to the dominant Weyl chamber, which are sufficiently close to 0 we obtain the operation $B_k$. But note that here still the limit operation is composed with the action of $s_k$ on $\lambda \in T^\ast$. In general we obtain “motivic stringy invariant” mentioned in [SchY07, §11.2].
Remark 5.3. The operators $\tilde{C}_k^{q_i=0}$ map the so-called trigonometric weight functions of [RTV17, Section 3.2] into each other. These functions also depend on an extra slope or alcove parameter, where a region in a subset of $t^i$ where the functions $b_k$ are constant. The resulting multiplier for $\tilde{C}_k^{q_i=0} \circ \tilde{C}_k^{q_i=0}$ equals
\[
\lim_{q_i \to 0} \delta(h, q^{(j, \lambda, \nu)}) \delta(h, q^{-(j, \lambda, \nu)}) = \frac{h^{-b_k(\lambda)-1}}{(1-h^{-1})} \cdot \frac{h^{-b_k(-\lambda)-1}}{(1-h^{-1})} = -\frac{h^{-1}}{(1-h^{-1})^2} = -\frac{y}{(1+y)^2}
\]
(since $b_k(\lambda) + b_k(-\lambda) = -1$), which, remarkably, does not depend of $\lambda$.

6. Weight functions

In this section we focus on type-A, and give a formula for the elliptic class of a Schubert variety in terms of natural generators in the K theory of $\mathcal{F}(n) = G/B$. This formula will coincide with the weight function defined in [RTV17] (based on earlier weight function definitions of Tarasov-Varchenko [TV97], Felder-Rimányi-Varchenko [FRV18]). Weight functions play an important role in representation theory, quantum groups, KZ differential equations, and recently (in some situations) they were identified with stable envelopes in Okounkov’s theory.

For a non-negative integer $n$ let $\mathcal{F}(n)$ be the full flag variety parametrizing chains of subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n$ with dim $V_k = k$. We will consider the natural action of $T = (\mathbb{C}^*)^n$ on $\mathcal{F}(n)$. The $T$-equivariant tautological rank $k$ bundle (i.e. the one whose fiber is $V_k$) will be denoted by $T^{(k)}$, and let $G_k$ be the line bundle $T^{(k)}/T^{(k-1)}$.

Let $\gamma_k$ be the class of $G_k$ in $K_T(\mathcal{F}(n))$ and let $t_1^{(k)}, \ldots, t_k^{(k)}$ be the Grothendieck roots of $T^{(k)}$ (i.e. $[T^{(k)}] = t_1^{(k)} + \ldots + t_k^{(k)}$) for $k = 1, \ldots, n$. Let us rename $t_j^{(n)} = z_j$.

It is well known that the $T$-equivariant K ring of $\mathcal{F}(n)$ can be presented as
\[
K_T(\mathcal{F}(n)) = \mathbb{Z}[(t_1^{(k)})^\pm 1, z_j^{\pm 1}]_{k=1,\ldots,n-1, a=1,\ldots,k, j=1,\ldots,n}/(relations)
\]
\[
= \mathbb{Z}[(\gamma_k)^\pm 1, z_j^{\pm 1}]_{k=1,\ldots,n, j=1,\ldots,n}/(relations),
\]
with certain relations. The first presentation is a result of presenting the flag variety as a quotient of the quiver variety
\[
\mathcal{F}(n) = V//G, \quad V = \prod_{k=1}^{n-1} \text{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1}), \quad G = \prod_{k=1}^{n-1} \text{GL}_k,
\]
see [FRW19, §6]. Then
\[
K_{G \times T}(V) \xrightarrow{\sim} K_{G \times T}(U) \simeq K_T(\mathcal{F}(n))
\]
where $U$ is the open subset in $V$ consisting of the family of monomorphisms. The variables $t_a^{(k)}$ are just the characters of the factor $\text{GL}_k$. The second presentation comes from a geometric picture as well: $\mathcal{F}(n) = \text{GL}_n/B_n$ is homotopy equivalent to $\text{GL}_n/T$ and
\[
K_{T \times T}(\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)) \xrightarrow{\sim} K_{T \times T}(\text{GL}_n) \simeq K_T(\text{GL}_n/T).
\]
The variables $\gamma_k$ appearing in the presentation (17) are the characters of the second copy of $T$ acting from the right on $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.
Elliptic classes of Schubert cells via Bott-Samelson resolution

Explicit generators of the ideal of relations could be named in both lines (16), (17), but it is more useful to understand the description of the ideals via “equivariant localization” (a.k.a. “GKM description”, or “moment map description”), as follows.

The \( T \) fixed points \( x_\sigma \) of \( \mathcal{F}(n) \) are parameterized by permutations \( \sigma \in S_n \). The restriction map from \( K_T(\mathcal{F}(n)) \) to \( K_T(x_\sigma) = \mathbb{Z}[z_{j_1}^{\pm 1}, z_{j_2}^{\pm 1}, \ldots, z_{j_n}^{\pm 1}] \) is given by the substitutions

\[
t_a^{(k)} \mapsto z_{\sigma(a)}, \quad \gamma_k \mapsto z_{\sigma(k)}.
\]

Symmetric functions in \( t_a^{(k)}, z_j \), and functions in \( \gamma_k, z_j \) belong to the respective ideals of relations if and only if their substitutions (18) vanish for all \( \sigma \in S_n \) [KR03, Appendix], [ChGi97, Ch. 5-6].

Our main objects of study, the classes \( E(X_\omega) \) live in the completion of \( K_T(\mathcal{F}(n)) \) adjoined with variables \( h, \mu_k \), that is, in the ring

\[
\mathbb{Z}[(t_a^{(k)})^{\pm 1}, z_j^{\pm 1}, h, \mu_j^{\pm 1}]_{k=1, \ldots, n-1, \ a=1, \ldots, n, \ j=1, \ldots, n} / \text{(relations)}
\]

where the same localization description holds for the two ideals of relations. Our goal in this section is to define representatives—that is, functions in \( t_a^{(k)}, z_j, h, \mu_j \) and functions in \( \gamma_k, z_j, h, \mu_j \)—that represent the elliptic classes of \( E(X_\omega) \) of Schubert varieties. This goal will be achieved in Theorem 7.1 below.

6.1. Elliptic weight functions. First we recall some special functions called elliptic weight functions, from [RTV17].

For \( \omega \in S_n, \ k = 1, \ldots, n-1, \ a = 1, \ldots, k \) define the integers

- \( \omega_a^{(k)} \) by \( \{\omega(1), \ldots, \omega(k)\} = \{\omega_1^{(k)} < \ldots < \omega_k^{(k)}\} \),
- \( j_\omega(k, a) \) by \( \omega_a^{(k)} = \omega(j_\omega(k, a)) \),

\[
c_\omega(k, a) = \begin{cases} 0 & \text{if } \omega(k+1) \geq \omega_a^{(k)} \\
1 & \text{if } \omega(k+1) < \omega_a^{(k)} \end{cases}
\]

**Definition 6.1.** [RTV17] For \( \omega \in S_n \) define the elliptic weight function by

\[
w_\omega = \left( \frac{\varphi(h)}{\varphi(1)} \right)^{\dim G} \frac{\text{Sym}_{t^{(1)}} \cdots \text{Sym}_{t^{(n-1)}} U_\omega}{\prod_{k=1}^{n-1} \prod_{i,j=1}^{k} \psi(h t_i^{(k)} / t_j^{(k)})},
\]

where

\[
G = \prod_{k=1}^{n-1} \text{GL}_k, \quad \dim G = \frac{(n-1)n(2n-1)}{6},
\]

\[
\text{Sym}_{t^{(k)}} f(t_1^{(k)}, \ldots, t_k^{(k)}) = \sum_{\sigma \in S_k} f(t_{\sigma(1)}^{(k)}, \ldots, t_{\sigma(k)}^{(k)}),
\]

\[
U_\omega = \prod_{k=1}^{n-1} \prod_{a=1}^{k} \left( \prod_{c=1}^{k+1} \psi_{\omega, k, a, c}(t_{c}^{(k+1)} / t_{a}^{(k)}, h) \prod_{b=a+1}^{k} \delta(t_{b}^{(k)} / t_{a}^{(k)}, h) \right).
\]
The usual names of the variables of the elliptic weight function are:

\[ t^{(k)}_a \quad \text{for} \quad k = 1, \ldots, n - 1, \quad a = 1, \ldots, k \quad \text{the Grothendieck root variables}, \]

\[ z_a := t^{(n)}_a \quad \text{for} \quad a = 1, \ldots, n \quad \text{the equivariant variables}, \]

\[ h \quad \text{the “Planck variable”}, \]

\[ \mu_k = h^{-\lambda_k} \quad \text{for} \quad k = 1, \ldots, n \quad \text{the dynamical (or Kähler) variables}. \]

The function \( w_\omega \) is symmetric in the \( t^{(k)}_a \) variables (for each \( k = 1, \ldots, n - 1 \) separately), but not symmetric in the equivariant variables.

Consider the new variables \( \gamma_1, \ldots, \gamma_n \), and define the modified weight function (of the variables \( \gamma = (\gamma_1, \ldots, \gamma_n), z = (z_1, \ldots, z_n), \mu = (\mu_1, \ldots, \mu_n), \) and \( h \))

\begin{equation}
\hat{w}_\omega(\gamma, z, \mu, h) = w_\omega\left(t^{(k)}_a = \gamma_a \quad \text{for} \quad k = 1, \ldots, n - 1; t^{(n)}_a = z_a\right),
\end{equation}

that is, we substitute \( \gamma_a \) for \( t^{(k)}_a \) for \( k = 1, \ldots, n - 1 \), and rename \( t^{(n)}_a \) to \( z_a \). This substitution corresponds to going from the presentation (16) to the presentation (17).

**Example 6.2.** We have

\[ w_{12} = \frac{1}{\vartheta^1(1)} \vartheta\left(\frac{t^{(2)}_1}{t^{(1)}_1}\right) \vartheta\left(\frac{t^{(2)}_1}{t^{(1)}_1}, \frac{h \mu_2}{\mu_1}\right) \vartheta\left(\frac{h \mu_2 t^{(2)}_1}{\mu_1 t^{(1)}_1}\right), \]

\[ w_{21} = \frac{1}{\vartheta^1(1)} \vartheta\left(\frac{t^{(2)}_1}{t^{(1)}_1}\right) \vartheta\left(\frac{t^{(2)}_1}{t^{(1)}_1}, h\right) \vartheta\left(\frac{t^{(2)}_1}{t^{(1)}_1}, \frac{\mu_2}{\mu_1}\right) \vartheta\left(\frac{h t^{(2)}_1}{t^{(1)}_1}\right) \vartheta\left(\frac{\mu_2 t^{(2)}_1}{\mu_1 t^{(1)}_1}\right) \vartheta\left(\frac{h \mu_2}{\mu_1}\right), \]

and hence

\begin{equation}
\hat{w}_{12} = \frac{1}{\vartheta^1(1)} \vartheta\left(\frac{z_1}{\gamma_1}\right) \vartheta\left(\frac{z_2}{\gamma_1}\right) \delta\left(\frac{z_1}{\gamma_1}, \frac{h \mu_2}{\mu_1}\right), \quad \hat{w}_{21} = \frac{1}{\vartheta^1(1)} \vartheta\left(\frac{z_1}{\gamma_1}\right) \vartheta\left(\frac{z_2}{\gamma_1}\right) \vartheta\left(\frac{z_1 h \mu_3}{\gamma_1 \mu_4}\right) \vartheta\left(\frac{z_2 h \mu_3}{\gamma_1 \mu_4}\right) \delta\left(\frac{z_1}{\gamma_1}, \frac{h \mu_2}{\mu_1}\right).
\end{equation}

For \( n = 3 \) for example we have

\[ \hat{w}_{123} = \frac{\vartheta\left(\frac{z_2}{\gamma_1}\right) \vartheta\left(\frac{z_1}{\gamma_2}\right) \vartheta\left(\frac{z_2}{\gamma_2}\right) \vartheta\left(\frac{z_1 h \mu_3}{\gamma_1 \mu_4}\right) \vartheta\left(\frac{z_2 h \mu_3}{\gamma_1 \mu_4}\right) \vartheta\left(\frac{z_2 h \mu_2}{\gamma_2 \mu_4}\right)}{\vartheta\left(\frac{h \mu_3}{\mu_1}\right) \vartheta\left(\frac{h \mu_3}{\mu_2}\right) \vartheta\left(\frac{h \mu_2}{\gamma_2}\right) \vartheta\left(\frac{h \mu_2}{\gamma_2}\right)}, \]

The key properties of weight functions are the R-matrix recursion property, substitution properties, transformation properties, orthogonality relations, as well as their axiomatic characterizations, see details in [RTV17]. First we recall some obvious substitution properties, and the R-matrix recursion property.
6.2. Substitution properties. Keeping in mind that fixed point restrictions in geometry are obtained by the substitutions (18), for permutations \( \omega, \sigma \) define

\[
\mathbf{w}_{\omega, \sigma} = \mathbf{w}_\omega \big|_{z_k = z_{\sigma(k)}} = \mathbf{w}_\omega \big|_{z_k^{(k)} = z_{\sigma(i)}}.
\]

From the definition of weight functions (or by citing [RTV17, Lemmas 2.4, 2.5]) it follows that \( \mathbf{w}_{\omega, \sigma} = 0 \) unless \( \sigma \leq \omega \) in the Bruhat order, and

\[
\mathbf{w}_{\omega, \omega} = \prod_{i < j} \frac{\partial (z_{\omega(j)}/z_{\omega(i)})}{\prod_{i < j, \omega(i) > \omega(j)} \delta (z_{\omega(j)}/z_{\omega(i)}, h)}.
\]

In particular, we have

\[
\mathbf{w}_{\text{id}, \sigma} = \begin{cases} 
\prod_{i < j} \partial (z_j/z_i) & \text{if } \sigma = \text{id} \\
0 & \text{if } \sigma \neq \text{id}.
\end{cases}
\]

6.3. R-matrix recursion. In [RTV17] (Theorem 2.2 and notation (2.8)) the following identity is proved for weight functions:

\[
w_{\omega, \omega} = \begin{cases} 
\mathbf{w}_\omega \cdot \frac{\delta (\mu_{\omega^{-1}(k)} h)}{\delta (z_{k+1}, h)} - \mathbf{w}_{s_k \omega} \cdot \frac{\delta (z_{k+1}, \mu_{\omega^{-1}(k)})}{\delta (z_{k+1}, h)} & \text{if } \ell (s_k \omega) > \ell (\omega) \\
\mathbf{w}_\omega \cdot \frac{1}{\delta (z_{k+1}, h)} - \mathbf{w}_{s_k \omega} \cdot \frac{\delta (z_{k+1}, \mu_{\omega^{-1}(k+1)} h)}{\delta (z_{k+1}, h)} & \text{if } \ell (s_k \omega) < \ell (\omega),
\end{cases}
\]

where \( s_k \) operates by replacing the \( z_k \) and \( z_{k+1} \) variables. Of course, then the same formula holds for \( \tilde{w} \)-functions (replace \( w \) with \( w \) everywhere in (24)).

Corollary 6.3. If \( \ell (s_k \omega) = \ell (\omega) + 1 \) then

\[
w_{s_k \omega} = \delta \left( \frac{z_{k+1}}{z_k}, \frac{\mu_{\omega^{-1}(k+1)}}{\mu_{\omega^{-1}(k)}} \right) \cdot \mathbf{w}_\omega + \delta \left( z_k, \frac{z_{k+1}}{z_{k+1}}, h \right) \cdot s_k \mathbf{w}_\omega.
\]

and the same holds if \( w \) is replaced with \( \tilde{w} \).

Proof. The statement follows from the second line of (24), after we rename \( \omega \) to \( s_k \omega \).

A key observation is that the recursion in Corollary 6.3, together with the initial condition (23) completely determine the classes \( \mathbf{w}_{\omega, \sigma} \).

7. Weight functions are representatives of elliptic classes

For a rank \( n \) bundle \( \mathcal{T} \) with Grothendieck roots \( t_i \) we defined its K theoretic Euler class in (3). Now we will also need its “elliptic cohomological Euler class”: \( e^E (\mathcal{T}) = \prod_{i=1}^n \partial (t_i) \).

We will be concerned with \( e^E (T \mathcal{F}(n))|_{\sigma} = e^E (T_\sigma \mathcal{F}(n)) = \prod_{i < j} \partial (z_{\sigma(j)}/z_{\sigma(i)}) \) for a permutation \( \sigma \). Using this Euler class, the recursion we obtained in the last section, (23), reads:

\[
e^{E} (T_\sigma \mathcal{F}(n)) = \begin{cases} 
1 & \text{if } \sigma = \text{id} \\
0 & \text{if } \sigma \neq \text{id},
\end{cases}
\]
and for $\ell(s_k\omega) = \ell(\omega) + 1$

$$
\frac{w_{s_k\omega,\sigma}}{e^E(T_0F(n))} = \delta \left( \frac{z_{k+1}}{z_k}, \frac{\mu_{\omega^{-1}(k+1)}}{\mu_{\omega^{-1}(k)}} \right) \cdot \frac{w_{\omega,\sigma}}{e^E(T_0F(n))} + \delta \left( \frac{z_k}{z_{k+1}}, h \right) \cdot s_k^2 \frac{w_{\omega,s_k\sigma}}{e^E(T_{s_k\sigma}F(n))}.
$$

Now we are ready to state the theorem that weight functions represent the elliptic classes of Schubert varieties.

**Theorem 7.1.** Set $\mu_i = h^{-\lambda_i}$. With this identification in presentation (19) we have

$$E(X_\omega, \lambda) = \frac{e^K(TF(n))}{e^E(TF(n))} \cdot [w_\omega],$$

and in presentation (20) we have

$$E(X_\omega, \lambda) = \frac{e^K(TF(n))}{e^E(TF(n))} \cdot [\tilde{w}_\omega].$$

**Remark 7.2.** Continuing Remark 2.8 let us note that if we had set up the elliptic class of varieties not in equivariant K theory but in equivariant elliptic cohomology, then the class would be multiplied by $e^K(TM)/e^K(TM)$ (where $M$ is the ambient space). That is, Theorem 7.1 claims that the functions $w_\omega$, $\tilde{w}_\omega$ represent the elliptic class of Schubert varieties in equivariant elliptic cohomology.

### 7.1. Proof of Theorem 7.1.

Let us fix a notation:

**Convention.** We will skip $\lambda$ in the notation of $E_\sigma(X_\omega, \lambda)$ and we treat $E_\sigma(X_\omega)$ as a function on $\lambda \in t^* \simeq \mathbb{C}^n$ expressed by the basic functions $\mu_k = \mu_k(\lambda) = h^{-\lambda_k}$. The action of $\omega \in W$ on the space of functions generated by $\mu_\bullet$ will be denoted by $s^\mu_k$. We will write $L_{k,\sigma}$ to denote the character of the line bundle $L_k$ at the point $x_\sigma$.

We need to prove that for all $\omega, \sigma$

$$E_\sigma(X_\omega) = \frac{w_{\omega,\sigma}}{e^E(T_0F(n))}.$$

This will be achieved by showing that the recursive characterization (25), (26) of the right hand side holds for the left hand side too. The basic step (25) holds for $E_\sigma(X_{id})$ because of (14).

**Proposition 7.3.** Suppose $G = GL_n$. If $\ell(s_k\omega) = \ell(\omega) + 1$ then the functions $E_\sigma(X_\omega)$ satisfy the recursion

$$E_\sigma(X_{s_k\omega}) = \delta \left( \frac{z_{k+1}}{z_k}, \frac{\mu_{\omega^{-1}(k+1)}}{\mu_{\omega^{-1}(k)}} \right) \cdot E_\sigma(X_\omega) + \delta \left( \frac{z_k}{z_{k+1}}, h \right) \cdot s_k^2 E_{s_k\sigma}(X_\omega).$$

More generally for an arbitrary reductive group

$$E_\sigma(X_{s_k\omega}) = \delta \left( L_{k, id}^{-1}(v_k), \omega^{-1}(v_k) \right) \cdot E_\sigma(X_\omega) + \delta \left( L_{k, id}^{-1}, h \right) \cdot s_k^2 E_{s_k\sigma}(X_\omega).$$

Here $s_k^2$ means the action of $s_k$ on $z$-variables and $\omega^\mu$ acts on the $\mu$-variables, $\nu = h^{-\alpha_\nu}$.

The reader may find it useful to verify the statement for $n = 2$ using the local classes below.
We prove the proposition by induction with respect to the length of $\omega$. We assume that the formula of Proposition 7.3 holds for $\omega_1$ with $\ell(\omega_1) < \ell(\omega)$. We introduce the notation for a group element inverse: $\overline{\omega} = \omega^{-1}$. Note that $\overline{s_k} = s_k$. Let’s assume that $\omega s_\ell$ is a reduced expression of $\omega$ and $\ell(s_\ell \omega_1) = \ell(\omega) + 1$. By Theorem 4.6

$$E_\sigma(X_{s_\ell \omega_1}) = \delta(L_{\ell,\sigma}, \nu_\ell) s_\ell^\mu E_\sigma(X_{s_\ell \omega_1}) + \delta(L_{\ell,\sigma}, h) s_\ell^\mu E_{\sigma s_\ell}(X_{s_\ell \omega_1})$$

By inductive assumption this expression is equal to

$$\delta(L_{\ell,\sigma}, \nu_\ell) s_\ell^\mu \left( \delta(L_{k,\id}, \varpi_1^\mu(\nu_k)) E_\sigma(X_{\omega_1}) + \delta(L_{k,\id}^{-1}, h) s_k^z E_{s_k \sigma}(X_{\omega_1}) \right) + \delta(L_{\ell,\sigma}, h) s_\ell^\mu \left( \delta(L_{k,\id}, \varpi_1^\mu(\nu_k)) E_{\sigma s_\ell}(X_{\omega_1}) + \delta(L_{k,\id}^{-1}, h) s_k^z E_{s_k \sigma s_\ell}(X_{\omega_1}) \right).$$

Note that

$$s_k^z \delta(L_{\ell,\sigma}, \nu_\ell) = \delta(L_{s_k \ell,\sigma}, \nu_\ell), \quad s_k^z \delta(L_{\ell,\sigma}, h) = \delta(L_{s_k \ell,\sigma}, h),$$

$$s_\ell^\mu \delta(L_{k,\sigma}, \varpi_1^\mu(\nu_k)) = \delta(L_{k,\sigma}, s_\ell^\mu \varpi_1^\mu(\nu_k)), \quad s_\ell^\mu \delta(L_{k,\id}^{-1}, h) = \delta(L_{k,\id}^{-1}, h),$$

hence rearranging the expression we obtain

$$\delta(L_{k,\id}, s_\ell^\mu \varpi_1^\mu(\nu_k)) \left( \delta(L_{\ell,\sigma}, \nu_\ell) s_\ell^\mu E_\sigma(X_{\omega_1}) + \delta(L_{\ell,\sigma}, h) s_\ell^\mu E_{\sigma s_\ell}(X_{\omega_1}) \right) + \delta(L_{k,\id}^{-1}, h) s_k^z \left( \delta(L_{s_k \ell,\sigma}, \nu_\ell) s_\ell^\mu E_{s_k \sigma}(X_{\omega_1}) + \delta(L_{s_k \ell,\sigma}, h) s_\ell^\mu E_{s_k \sigma s_\ell}(X_{\omega_1}) \right) =$$

$$= \delta(L_{k,\id}, s_\ell^\mu \varpi_1^\mu(\nu_k)) E_\sigma(X_{\omega_1 s_\ell}) + \delta(L_{k,\id}^{-1}, h) s_k^z E_{s_k \sigma}(X_{\omega_1 s_\ell}) =$$

$$= \delta(L_{k,\id}, \varpi^\mu(\nu_k)) E_\sigma(X_{\omega_1}) + \delta(L_{k,\id}^{-1}, h) s_k^z E_{s_k \sigma}(X_{\omega}).$$

This completes the proof of Theorem 7.1.

**Proposition 7.4.** If $\ell(s_\ell \omega) = \ell(\omega) - 1$, then

$$\delta \left( \frac{z_{k+1}}{z_k}, \frac{\mu^{-1}(k)}{\mu^{-1}(k+1)} \right) \cdot E_\sigma(X_{\omega}) + \delta \left( \frac{z_k}{z_{k+1}}, h \right) \cdot E_{s_k \sigma}(X_{\omega}) = \delta \left( h, \frac{\mu^{-1}(k+1)}{\mu^{-1}(k)} \right) \cdot \delta \left( h, \frac{\mu^{-1}(k)}{\mu^{-1}(k+1)} \right) \cdot E_\sigma(X_{s_k \omega}).$$
Proof. This relation follows from the first line of R matrix relation (24) using identities among of theta functions, not just combinatorics of multiplication and Weyl group actions. For general $G$ this statement follows from a direct calculation when $\omega_1 = s_k \omega_2$, $\ell(\omega_2) < \ell(\omega_1)$, which is exactly the same as the proof of Theorem 5.1. \hfill \Box

To obtain the Formula 1.2 given in the Introduction we multiply and divide by the elliptic Euler classes (15). Here, comparing with the proof of Formula 1.1, the minus sign does not appear because additionally we have the action of $s_k$ compensating the sign.

8. Transformation properties of $E(X_\omega)$

Having proved that for $\text{GL}_n$ the elliptic classes of Schubert varieties are represented by weight functions, we can conclude that all proven properties of weight functions hold for those elliptic classes. One key property of weight functions is a strong constraint on their transformation properties. Hence, such constraint holds for $E(X_\omega)$ in the $\text{GL}_n$ case. Motivated by this fact we will prove an analogous theorem on the transformation properties of elliptic classes of Schubert varieties for any reductive group $G$.

In subsection 8.1 we recall how to encode transformation properties of theta-functions by quadratic forms and recall the known transformation properties of weight functions. In subsection 8.2 we put that statement in context by recalling a whole set of other properties such that together they characterize weight functions. In subsection 8.3 we generalize the transformation property statement to arbitrary $G$. So the new results is only in subsection 8.3, the preceding subsections are only motivations for that.

8.1. Transformation properties of the weight function. Consider functions $\mathbb{C}^p \times \mathbb{H} \to \mathbb{C}$, where $\mathbb{H}$ is the upper half space, and the variable in $\mathbb{H}$ is called $\tau$. Let $M$ be a $p \times p$ symmetric integer matrix, which we identify with the quadratic form $x \mapsto x^T M x$. We say that the function $f : \mathbb{C}^p \times \mathbb{H} \to \mathbb{C}$ has transformation property $M$, if

$$f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_p) = (-1)^{M_{jj}} f(x),$$

$$f(x_1, \ldots, x_{j-1}, x_j + \tau, x_{j+1}, \ldots, x_p) = (-1)^{M_{jj}} e^{-2\pi i (\sum_k M_{jk} x_k) - \pi i M_{jj}} f(x).$$

For a quadratic form $M$ there one may define a line bundle $L(M, 0)$ over the $p$th power of the elliptic curve $\mathbb{C}/\langle 1, \tau \rangle$ such that the sections of $L(M, 0)$ are identified with functions with transformation properties $M$, see [RTV17, Section 6].

Recall from Section 2.1 that we set up theta function in two ways, $\vartheta(\ )$ in “multiplicative variables”, and $\theta(\ )$ in “additive variables”. The transformation property being the quadratic form $M$ is always meant in the additive variables, but naming the quadratic function we use the variable names most convenient for the situation. For example, the function $\vartheta(x)$ has transformation property $M = (1)$ (or equivalently, the quadratic form $x^2$), because of (2).

Iterating this fact one obtains that for integers $r$, the function $\vartheta(\prod_{i=1}^p x_i^{r_i})$ has transformation property $(\sum_{i=1}^p r_i x_i)^2$. For products of functions the quadratic form of their transformation properties add. Hence, for example, the function $\delta(a, b)$ of (1) has transformation property $(a + b)^2 - a^2 - b^2 = 2ab$. Through careful analysis of the combinatorics of the weight functions
defined above (or from [RTV17, Lemmas 6.3, 6.4] by carrying out the necessary convention changes) we obtain

**Proposition 8.1.** The weight function \( \tilde{w}_\omega \) has transformation property

\[
Q(\omega) = \sum_{1 \leq i, j \leq n-1} 2h(\gamma_j - \gamma_i) + \sum_{i=1}^{n-1} \sum_{j=1}^{\omega(i)-1} 2h(z_j - \gamma_i) + \sum_{i=1}^{n-1} 2(z_{\omega(i)} - \gamma_i)(P_{\omega,i}h + \mu_n - \mu_i) + \sum_{i=1}^{n-1} \sum_{j=1}^{n} (z_j - \gamma_i)^2 - \sum_{1 \leq i, j \leq n-1}^{i<j} (\gamma_i - \gamma_j)^2,
\]

(27)

for \( P_{\omega,i} = 1 \) if \( \omega(i) < \omega(n) \) and \( P_{\omega,i} = 0 \) otherwise.

**Example 8.2.** We have

\[
Q(12) = 2(z_1 - \gamma_1)(h + \mu_2 - \mu_1) + (z_1 - \gamma_1)^2 + (z_2 - \gamma_1)^2,
\]

\[
Q(21) = 2h(z_1 - \gamma_1) + 2(z_2 - \gamma_1)(\mu_2 - \mu_1) + (z_1 - \gamma_1)^2 + (z_2 - \gamma_1)^2,
\]

in accordance with formulas (22).

**Corollary 8.3.** We have

\[
Q(\omega) - Q(\omega s_k) = 2(z_{\omega(k)} - z_{\omega(k+1)})(\mu_{k+1} - \mu_k),
\]

\[
Q(\omega) - Q(s_k \omega) = 2(z_k - z_{k+1})(\mu_{\omega^{-1}(k+1)} - \mu_{\omega^{-1}(k)}).
\]

Note that neither line depends on the variables \( \gamma_i \).

**Proof.** Straightforward calculation based on formula (27), carried out separately in the few cases depending on whether \( k < n-1 \) or \( k = n-1 \), whether \( \omega(k) < \omega(n) \) or not. Let us note that a more conceptual proof for (only) the second line can be obtained using the R-matrix relation (24).

8.2. **Axiomatic characterization.** Let us say that a function \( f \) is divisible by another \( g \) if \( f/g \) is holomorphic. Formally, since in theta functions \( \vartheta(x) \) the square root of \( x \) appears, thus formally the classes \( w_{\omega,\sigma} \) depending on \( h, z, \mu \)-variables are defined on a suitable cover of the product of elliptic curves. On the other hand the localized classes \( E_{\sigma}(X_\omega) \) are expressed in terms of \( \delta \) functions. One can check, that in that case no square root appears. Note that the original axioms are about the weight function \( \tilde{w}_\omega \). We translate them to the statements about \( w_{\omega,\sigma} \), not \( w_{\omega,\sigma}/Euler \). The presence of the factor forces us to use theta function \( \vartheta \) not only \( \delta \). Therefore the restricted classes \( \tilde{w}_\omega \) define section of a line bundle over a cover of the product of elliptic curves.
The “constant” (not depending on \( z \) variables)

\[
\psi_\omega = \psi_\omega(\mu, h) = \vartheta(h)^{n(n-1)(n-2)/3} \prod_{i<j} \prod_{\omega(i) < \omega(j)} \vartheta(h\mu_j/\mu_i) \prod_{\omega(i) > \omega(j)} \vartheta(h) \vartheta(\mu_j/\mu_i)
\]

plays a role below.

The consequence of the axiomatic characterizations in [RTV17, Theorem 7.3] is the following axiomatic characterization of \( w_{\omega, \sigma} \) classes.

**Theorem 8.4** ([RTV17] Theorem 7.3, see also [FRV07] Theorem A.1).

(I) The functions \( w_{\omega, \sigma} \) satisfy the properties:

(1.1) (holomorphicity) We have

\[
w_{\omega, \sigma} = \frac{1}{\psi_\omega} \cdot \text{holomorphic function}.
\]

(1.2) (GKM relations) We have

\[
w_{\omega, \sigma} z_{\sigma(k)} = z_{\sigma(k+1)} = w_{\omega, \sigma} z_{\sigma(k)} = z_{\sigma(k+1)}.
\]

(1.3) (transformations) The transformation properties of \( w_{\omega, \sigma} \) are described by the quadratic form \( Q_\gamma(\omega) |_{\gamma = z_\omega(i)} \).

(2) (normalization)

\[
w_{\omega, \sigma} = \prod_{i<j} \vartheta(z_\omega(j)/z_\omega(i)) \prod_{i<j} \delta(z_\omega(j)/z_\omega(i), h).
\]

(3.1) (triangularity) if \( \sigma \not< \omega \) in the Bruhat order then \( w_{\omega, \sigma} = 0 \).

(3.2) (support) if \( \sigma \preceq \omega \) in the Bruhat order then \( w_{\omega, \sigma} \) is of the form

\[
\frac{1}{\psi_\omega} \cdot \prod_{i<j} \vartheta(z_\sigma(j)/z_\sigma(i)) \cdot \text{holomorphic function}
\]

(II) These properties uniquely determine the functions \( w_{\omega, \sigma} \). \( \square \)

The axiom (1.3) may be replaced by the inductive property:

(1.3') If \( \omega = \omega's_k \) with \( \ell(\omega) = \ell(\omega') + 1 \) then the difference of the quadratic forms describing the transformation properties of \( w_{\omega', \sigma} \) and that of \( w_{\omega, \sigma} \) is equal to

\[
2(z_{\omega(k+1)} - z_{\omega(k)})(\mu_{k+1} - \mu_k).
\]

8.3. Transformation properties for general \( G \). A key axiom for the weight functions is (1.3'). Through Theorem 7.1 it implies that the difference of the quadratic forms of \( E_\sigma(X_{\omega'}, \lambda) \) and \( E_\sigma(X_\omega, \lambda) \) is also \( 2(z_{\omega(k+1)} - z_{\omega(k)})(\mu_{k+1} - \mu_k) \). Below, in Theorem 8.9 we will prove that the generalization of this surprising property holds for general \( G \). We will also see that this general argument (using the language of a general Coxeter groups) fits better the transformation properties than the combinatorics of weight functions.
Let $\alpha$ and $\beta \in t^*$ be two roots. The reflection of $\alpha$ about $\beta$ will be denoted by $\alpha^\beta = s_\beta(\alpha)$. Define the functional $\mu_\alpha$ acting on $t^*$

$$\mu_\alpha \in (t^*)^*, \quad \mu_\alpha(\beta) = -\langle \alpha^\vee, \beta \rangle.$$ 

Note that we artificially introduce the minus sign to agree later with the previous conventions (the definition of $L_\lambda$ and $\mu_k = h^{\lambda_k}$ in the weight function). The reflection $s_\alpha \in W$ acting on polynomial functions on $t^*$ is denoted by $s^\mu_\alpha$

$$s^\mu_\alpha(f) = f \circ s_\alpha.$$ 

Define the generalized divided difference operation

$$d_\alpha(f)(x) = \frac{f(x) - f(s_\alpha(x))}{\mu_\alpha(x)}.$$ 

It satisfies the properties

$$s^\mu_\alpha = s^\mu_{-\alpha}, \quad d_\alpha = -d_{-\alpha}, \quad d_\beta \circ s^\mu_\alpha = s^\mu_\alpha \circ d_{\beta_\alpha}, \quad d_\beta(\mu_\alpha) = \langle \beta^\vee, \alpha \rangle.$$ 

In particular, we have $d_\alpha \mu_\alpha = \langle \alpha^\vee, \alpha \rangle = 2$.

Consider the vector space $t \times t^* \times \mathbb{C}$. For a root $\alpha \subset t^*$ the linear functional $z_\alpha \in (t \times t^* \times \mathbb{C})^*$ acts on the first coordinate by $\alpha$. The functional $\mu_\alpha$ acts on the second coordinate, while $h$ acts on the third coordinate. For $\sigma \in W$ and $\alpha \in t^*$ by $\sigma(\alpha)$ we understand the usual action of $\sigma$ on $t^*$.

As usual, we keep fixed the positive roots and the simple roots. To each pair $\omega, \sigma \in W$ such that $\sigma \preceq \omega$ we associate a quadratic form $M(\omega, \sigma)$ such that $M(id, id) = 0$ and inductively: If $\omega = \omega's_\alpha$ with $\ell(\omega) = \ell(\omega') + 1$, then

$$M(\omega, \sigma) = \begin{cases} s^\mu_\alpha(M(\omega', \sigma)) + \mu_\alpha z_{\sigma(\alpha)} & \text{if } \sigma \preceq \omega' \\ s^\mu_\alpha(M(\omega', \sigma s_\alpha)) - h z_{\sigma(\alpha)} & \text{if } \sigma s_\alpha \preceq \omega'. \end{cases}$$ 

If $\sigma \preceq \omega'$ and $\sigma s_\alpha \preceq \omega'$, then the cases of the definitions give the same quadratic form (this follows from the proof of Proposition 8.8 below). Also, $\sigma \preceq \omega$ implies that one of the above conditions holds.

**Example 8.5.** Let $G = GL_3$. We illustrate two ways of computing $M(s_1s_2s_1, id)$:

$$M(id, id) = 0$$

$$M(s_1, id) = \begin{bmatrix} 0 \\ s_1 \downarrow \end{bmatrix}$$

$$M(s_1s_2, id) = \begin{bmatrix} (z_2 - z_1)(\mu_2 - \mu_1) \\ s_2 \downarrow \end{bmatrix}$$

$$M(s_1s_2s_1, id) = \begin{bmatrix} (z_2 - z_1)(\mu_3 - \mu_2) + (z_3 - z_2)(\mu_3 - \mu_2) \\ s_1 \downarrow \end{bmatrix}$$

$$M(s_1s_2s_1, id) = \begin{bmatrix} (z_2 - z_1)(\mu_3 - \mu_2) + (z_3 - z_2)(\mu_3 - \mu_1) + (z_2 - z_1)(\mu_2 - \mu_1) \\ s_1 \downarrow \end{bmatrix}.$$
But also

\[ M(\text{id}, \text{id}) = 0 \]

\[ M(s_1, s_1) = h(z_1 - z_2) \]

\[ M(s_1 s_2, s_1) = h(z_1 - z_2) + (z_3 - z_1)(\mu_3 - \mu_2) \]

\[ M(s_1 s_2 s_1, \text{id}) = h(z_1 - z_2) + (z_3 - z_1)(\mu_3 - \mu_1) + h(z_2 - z_1). \]

In both cases we obtain \( M(s_1 s_2 s_1, \text{id}) = (z_3 - z_1)(\mu_3 - \mu_1) \). One can check that presenting this permutation as \( s_2 s_1 s_2 \) and performing analogous computations we obtain the same result.

The inductive procedure of constructing the elliptic classes can be translated to a description of the associated quadratic form.

**Proposition 8.6.** The quadratic form \( 2M(\omega, \sigma) \) describes the transformation properties of \( E_\sigma(X_\omega, \lambda) \).

**Proof.** If \( \omega = \text{id} \) then

\[ E_\sigma(X_\text{id}, \lambda) = \begin{cases} 1 & \text{if } \text{id} = \sigma \\ 0 & \text{if } \text{id} \neq \sigma, \end{cases} \]

hence the associated form at \((\text{id}, \text{id})\) is 0. By Theorem 4.6, when passing from \( \omega' \) to \( \omega \) the elliptic class changes by the factor \( \delta_k^{\text{in}} = \delta(L_k, h) \), which is at the point \( \sigma \) has the transformation properties \( 2z_{-\sigma(\alpha_k)}h = -2z_{\sigma(\alpha_k)}h \) (since \( L_k = L_{\alpha_k} \)) or by the factor \( \delta_k^{\text{bd}} = \delta(L_k, h(\lambda, \alpha_k^\vee)) \) which at the point \( \sigma \) has the transformation properties \( 2z_{-\alpha_k}\mu_{-\alpha_k} = 2z_{\sigma(\alpha_k)}\mu_{\alpha_k} \). \( \square \)

**Proposition 8.7.** The quadratic form at the smooth point of the cell is equal to

\[ M(\omega, \omega) = h \sum_{\alpha \in \Sigma_+ \cap \omega(\Sigma_-)} z_\alpha, \]

where \( \Sigma_\pm \) denotes the set of positive/negative roots. The roots appearing in the summation are the tangent weights of \( X_\omega \) at \( x_\omega \).

**Proof.** At the smooth point \( x_\omega \) the localized elliptic class \( E_\omega(X_\omega, \lambda) \) is given by the product of \( \delta \) functions and the transformation matrix for \( \delta(x, h) \) is equal to \( 2xh \) by (7). \( \square \)

**Proposition 8.8.** Suppose \( \sigma \preceq \omega \), then

\[ d_\beta(M(\omega, \sigma)) = z_{\sigma(\beta)} - z_{\omega(\beta)}. \]

**Proof.** Obviously, the statement holds of \( \omega = \text{id} \).

Consider the first case of the inductive definition. We have

\[ d_\beta(M(\omega, \sigma)) = d_\beta(s_\alpha^\mu(M(\omega, \sigma)) + \mu_\alpha z_{\sigma(\alpha)}) \]

\[ = s_\alpha^\mu d_\beta s_\alpha^\mu(M(\omega, \sigma)) + (\beta^\vee, \alpha)z_{\sigma(\alpha)} \]

\[ = s_\alpha^{\mu_\beta} (z_{\sigma(\beta^\mu)} - z_{\omega(\beta^\mu)}) + (\beta^\vee, \alpha)z_{\sigma(\alpha)} \]

\[ = z_{\sigma s_\alpha(\beta)} - z_{\omega(\beta)} + (\beta^\vee, \alpha)z_{\sigma(\alpha)}. \]
Since
\[ s_\alpha(\beta) = \beta - \langle \beta^\vee, \alpha \rangle \alpha, \quad s_\alpha(\beta) + \langle \beta^\vee, \alpha \rangle \alpha = \beta, \quad z_{\sigma s_\alpha(\beta)} + \langle \beta^\vee, \alpha \rangle z_{\sigma(\alpha)} = z_{\sigma(\beta)} \]
the conclusion follows.

Consider the second case of the inductive definition. We have
\[ d_\beta(M(\omega, \sigma)) = d_\beta(s_\alpha^\mu(M_{\omega'}, \sigma s_\alpha) - h z_{\sigma(\alpha)}) \]
\[ = s_\alpha^\mu d_\beta'(M(\omega', \sigma s_\alpha)) \]
\[ = s_\alpha^\mu(z_{\sigma s_\alpha(\beta^\vee)} - z_{\omega'(\beta^\vee)}) \]
\[ = s_\alpha^\mu(z_{\sigma(\beta)} - z_{\omega(\beta)}) \]
\[ = z_{\sigma(\beta)} - z_{\omega(\beta)}. \]
\[ \square \]

Note that both cases of inductive definition (28) are linear in \( \mu_\alpha \) variables. If both cases are applicable in the above proof, then we get the same result of the differential \( d_\beta \) for any root \( \beta \):
\[ d_\beta(s_\alpha^\mu(M(\omega', \sigma)) + \mu_\alpha z_{\sigma(\alpha)}) = d_\beta(s_\alpha^\mu(M(\omega', \sigma s_\alpha)) + h z_{\sigma(\alpha)}) \]
It follows that quadratic forms are equal:
\[ s_\alpha^\mu(M(\omega', \sigma)) + \mu_\alpha z_{\sigma(\alpha)} = s_\alpha^\mu(M(\omega', \sigma s_\alpha)) + h z_{\sigma(\alpha)} \]
Therefore two cases of the formula (28) do not create a contradiction.

**Theorem 8.9.** Let \( \alpha \) be a simple root. Suppose \( \sigma \preceq \omega \preceq \omega s_\alpha \). Then
\[ M(\omega s_\alpha, \sigma) = M(\omega, \sigma) + \mu_\alpha z_{\omega(\alpha)} \]

**Proof.** From the inductive definition of \( M \) (case 1) and Proposition 8.8 it follows:
\[ M(\omega, \sigma) - M(\omega s_\alpha, \sigma) = M(\omega, \sigma) - (s_\alpha^\mu(M(\omega, \sigma)) + \mu_\alpha z_{\sigma(\alpha)}) \]
\[ = \mu_\alpha d_\alpha(M(\omega, \sigma)) - \mu_\alpha z_{\sigma(\alpha)} \]
\[ = \mu_\alpha(z_{\sigma(\alpha)} - z_{\omega(\alpha)} - z_{\sigma(\alpha)}) \]
\[ = -\mu_\alpha z_{\omega(\alpha)}. \]
\[ \square \]

The Theorem 8.9 is an extension of the property \((1.3')\) of the axiomatic characterization of the weight function to the case of general \( G \). Of course, this inductive step together with the diagonal data determines all transformation properties \( M(\omega, \sigma) \):

**Proposition 8.10.** If two families of quadratic forms \( M_1(\omega, \sigma) \) and \( M_2(\omega, \sigma) \) are defined for \( \sigma \preceq \omega \) and satisfy the formula of Theorem 8.9. Moreover suppose \( M_1(\sigma, \sigma) = M_2(\sigma, \sigma) \) for all \( \sigma \in W \), then \( M_1(\omega, \sigma) = M_2(\omega, \sigma) \) for all \( \sigma \preceq \omega \).
Proof. We show equality of forms inductively, keeping the second variable of \(M_i(\omega, \sigma)\) fixed. We can only change \(\omega\) by an elementary reflection \(s_i\). The starting point for the induction is \(M_1(\sigma, \sigma) = M_2(\sigma, \sigma)\). Increasing the length of \(\omega\) by 1 we can arrive to \(M_1(\omega_0, \sigma) = M_2(\omega_0, \sigma)\). Now decreasing the length \(\omega\) for we can go down to any \(\omega\) satisfying \(\sigma \prec \omega \prec \omega_0\).

9. Weight function vs lexicographically smallest reduced word

Let us revisit Example 4.5, and study the underlying geometry and its relation with the weight functions.

The class \(E_{123}(X_{321})\) is calculated in Example 4.5 in two ways, one corresponding to the reduced word \(s_1 s_2 s_1\), the other corresponding to the reduced word \(s_2 s_1 s_2\) of the permutation 321. The two obtained expressions are, respectively,

\[
E_{id}(X_{s_1 s_2 s_1}) = \delta \left( \frac{z_2}{z_1}, \frac{\mu_2}{\mu_1} \right) \delta \left( \frac{z_3}{z_2}, \frac{\mu_3}{\mu_2} \right) \delta \left( \frac{z_2}{z_1}, \frac{\mu_2}{\mu_1} \right) + \delta \left( \frac{z_1}{z_2}, h \right) \delta \left( \frac{z_3}{z_1}, \frac{\mu_3}{\mu_1} \right) \delta \left( \frac{z_2}{z_1}, h \right)
\]

\[
E_{id}(X_{s_2 s_1 s_2}) = \delta \left( \frac{z_3}{z_2}, \frac{\mu_3}{\mu_1} \right) \delta \left( \frac{z_3}{z_1}, \frac{\mu_3}{\mu_1} \right) \delta \left( \frac{z_3}{z_2}, h \right) \delta \left( \frac{z_3}{z_1}, \frac{\mu_3}{\mu_1} \right) \delta \left( \frac{z_3}{z_2}, h \right).
\]

As we mentioned, the equality of these two expressions follows from general theory of Borisov and Libgober, or can be shown to be equivalent to the four term identity [RTV17, eq. (2.7)].

To explore the underlying geometry, let \(u_{21}, u_{31}\) and \(u_{32}\) be the coordinates in the standard affine neighborhood of the fixed point \(x_{123}\), that is, \(u_{ij}\) are the entries of the lower-triangular 3 \(\times\) 3 matrices. The weight of the coordinate \(u_{ij}\) is equal to \(z_i/z_j\). The open cell \(X^{\circ}_{s_1 s_2 s_1}\) intersected with this affine neighborhood is the complement of the sum of divisors

\[
\{u_{31} - u_{21} u_{32} = 0\} \quad \text{and} \quad \{u_{31} = 0\}.
\]

The intersections of the divisors is singular. The resolution corresponding to the word \(s_1 s_2 s_1\) coincides with the blow-up of the \(u_{21}\) axis. The fiber above 0 contains two fixed points which give the two contributions in (29). Analogously, the expression obtained by blowing up the axis \(u_{32}\) is (30).

The reader is invited to verify that the (long) calculation for

\[
\frac{W_{321,123}}{e^E(T_{id} F(3))} = \frac{W_{321,123}}{\vartheta(z_2/z_1) \vartheta(z_3/z_1) \vartheta(z_3/z_2)}
\]

results exactly the expression (29). This and many other examples calculated by us suggest the following conjecture: the weight function formula for \(w_{\omega, \sigma}/e^E(T_{\sigma} F(n))\) coincides (without using any \(\vartheta\)-function identities) with the expression obtained for \(E_\sigma(X_\omega)\) using the lexicographically smallest reduced word for \(\omega\).
10. Action of $C$-operations on weight functions.

We still consider the case of $G = \text{GL}_n$. Let $\mathfrak{c}_k$ be a family of operators on the space of meromorphic functions no $t \times t^* \times \mathbb{C}$ indexed by the simple roots:

\begin{equation}
(31) \quad \mathfrak{c}_k(f)(z, \gamma, \lambda, h) = \delta(L_k^\gamma, \nu_k) \cdot f(z, \gamma, s_k(\lambda), h) + \delta(L_k^\gamma, h) \cdot f(s_k(z), \gamma, s_k(\lambda), h).
\end{equation}

Here

\[
L_k^\gamma = \frac{\gamma_{k+1}}{\gamma_k}
\]

denotes the weight of the relative tangent bundle $G/B \to G/P_k$ (equal to $-\alpha_k$ at the point $x_{id}$), but living in the $\gamma$-copy of variables and written multiplicatively. We also recall that

\[
\nu_k = \frac{\mu_{k+1}}{\mu_k} = h^{-\langle \alpha_k, - \rangle}.
\]

The operators are constructed in such a way that they descend to the operators $C_k$ acting on the K theory of $\mathcal{F}(n)$. One may think about them as acting on $K_{T \times T}(\text{Hom}(\mathbb{C}^n, \mathbb{C}^n))(\mu, h)$.

**Theorem 10.1.** The operators $\mathfrak{c}_k$ satisfy

- braid relations,
- $\mathfrak{c}_k^2 = \kappa(\alpha_k)$.

**Proof.** This is straightforward checking and a repetition of the proof of Theorem 5.1. \hfill $\square$

Let us set $w_{id} = \hat{\omega}_{id}/e_\gamma^E$, where $e_\gamma^E$ is the elliptic Euler class written in $\gamma$ variables. The operators $\mathfrak{c}_k$ recursively define the functions $w_\omega$ living in the same space as the weight function $\hat{w}(\omega)$. They have the restrictions to the fixed points of $\mathcal{F}(n)$ equal to the restrictions of $\hat{w}_\omega$ divided by the elliptic Euler class. Nevertheless one can check that they are essentially different from the weights function. The difference lies in the ideal defining K theory of $\mathcal{F}(n)$.

**Example 10.2.** Let $G = \text{GL}_2$. Setting

\[
\frac{\hat{w}_{id}}{e_\gamma^E} = \frac{\vartheta \left( \frac{z_1}{\gamma_1} \right) \vartheta \left( \frac{z_2}{\gamma_1} \right) \vartheta \left( \frac{h_m z_1}{\gamma_1 \mu_1} \right)}{\vartheta \left( \frac{z_1}{\gamma_1} \right) \vartheta \left( \frac{h_m z_2}{\gamma_1 \mu_1} \right)} = \frac{\vartheta \left( \frac{z_1}{\gamma_1} \right) \vartheta \left( \frac{h_m z_1}{\gamma_1 \mu_1} \right)}{\vartheta \left( \frac{z_1}{\gamma_1} \right) \vartheta \left( \frac{h_m z_2}{\gamma_1 \mu_1} \right)}
\]

we obtain

\[
\frac{w_{s_1}}{\vartheta \left( \frac{z_1}{\gamma_1} \right) \cdot \vartheta \left( \frac{h_m z_1}{\gamma_1 \mu_1} \right)} = \frac{\vartheta \left( \frac{z_1}{\gamma_1} \right) \cdot \vartheta \left( \frac{h_m z_1}{\gamma_1 \mu_1} \right)}{\vartheta \left( \frac{z_1}{\gamma_1} \right) \cdot \vartheta \left( \frac{h_m z_1}{\gamma_1 \mu_1} \right)}
\]

Note that the operations $\mathfrak{c}_i$ (as well as $C_i$) do not preserve the initial transformation form. The summands of (31) might have different transformation properties. The equality holds in the quotient ring $K_T(\text{GL}_2/B)$.

**Remark 10.3.** Similarly when we define the operation

\[
\hat{B}_k(f)(z, \gamma, y) = \frac{1 + y/L_k^\gamma}{1 - e^{-\alpha_i}} f(z, \gamma, y) + \frac{1 + y/L_k^\gamma}{1 - 1/L_k^\gamma} f(z, s_i(\gamma), z).
\]

It satisfies braid relations and Hecke relations.
11. A Tale of Two Recursions for Weight Functions

11.1. Bott-Samelson recursion for weight functions. The main achievement in Sections 6–9 was the identification of the geometrically defined $E(X_\omega)$ classes with the weight functions whose origin is in representation theory. The way our identification went was through recursions. The elliptic classes satisfied the Bott-Samelson recursion of Theorem 4.6, and the weight functions satisfied the R-matrix recursion of (24). In Proposition 7.3 we showed that the two recursions are consistent, and hence both recursions hold for both objects.

One important consequence is that (the fixed point restrictions of) weight functions satisfy the Bott-Samelson recursion, as follows:

**Theorem 11.1.** We have

\[
[\tilde{\omega}_{s^k}] = \begin{cases} 
  s^\mu_k [\tilde{\omega}] \cdot \delta(\frac{\gamma_{k+1}}{\gamma_k}, \frac{\mu_{k+1}}{\mu_k}) - s^\mu_k s^\gamma_k [\tilde{\omega}] \cdot \delta(\frac{\gamma_{k+1}}{\gamma_k}, h) & \text{if } \ell(\omega s_k) > \ell(\omega) \\
  s^\mu_k [\tilde{\omega}] \cdot \delta(\frac{\mu_k}{\mu_{k+1}},h)\delta(\frac{\gamma_{k+1}}{\gamma_k},\frac{\mu_{k+1}}{\mu_k}) - s^\mu_k s^\gamma_k [\tilde{\omega}] \cdot \delta(\frac{\gamma_{k+1}}{\gamma_k},h) & \text{if } \ell(\omega s_k) < \ell(\omega), 
\end{cases}
\]

or equivalently, using the normalization

\[
\tilde{\omega}' = \tilde{\omega} \cdot \frac{1}{\prod_{i<j, \omega(i) > \omega(j)} \delta(\frac{\mu_i}{\mu_j}, h)}.
\]

we have the unified

\[
[\tilde{\omega}'_{s^k}] = s^\mu_k [\tilde{\omega}] \cdot \delta(\frac{\gamma_{k+1}}{\gamma_k}, \frac{\mu_{k+1}}{\mu_k}) - s^\mu_k s^\gamma_k [\tilde{\omega}] \cdot \delta(\frac{\gamma_{k+1}}{\gamma_k}, h).
\]

\[
\square
\]

This new property of weight functions will play an important role in a followup paper [RSVZ] in connection with elliptic stable envelopes. It is worth pointing out that the normalization (32) also makes the R-matrix property of (24) unified:

\[
\tilde{\omega}'_{s^k} = \tilde{\omega}' \cdot \frac{\delta(\frac{\mu_{w^{-1}(k+1)}}{\mu_{w^{-1}(k)}}, \frac{\gamma_{k+1}}{\gamma_k})}{\delta(\frac{\mu_{w^{-1}(k)}}{\mu_{w^{-1}(k+1)}}, h)} + s^\gamma_k \tilde{\omega}' \cdot \frac{\delta(\frac{\gamma_{k+1}}{\gamma_k}, h)}{\delta(\frac{\mu_{w^{-1}(k+1)}}{\mu_{w^{-1}(k)}}, h)}.
\]

There is, however, an essential difference between (33) and (34): the latter holds for the weight functions themselves, while the former only holds for the cosets $[\tilde{\omega}']$ of $\tilde{\omega}'$ functions. Already for $n = 2, \omega = \text{id}, k = 1$ the two sides of (33) only hold for the fixed point restrictions, not for the $\tilde{\omega}'$ functions themselves.

In essence, the remarkable geometric object $E(X_\omega)$ satisfies two different recursions: the Bott-Samelson recursion and the R-matrix recursion. The weight functions are the lifts of $E(X_\omega)$ classes satisfying only one of these recursions.
11.2. **Two recursions for the local elliptic classes.** Although we already stated and proved that both Bott-Samelson and R-matrix recursions hold for the elliptic classes, let us rephrase these statements in a convenient normalization.

- Let $\zeta_k \in \text{Hom}(T, C^*)$ be the inverse of a root written multiplicatively, e.g. $\zeta_k = \frac{\bar{z}_k^{k+1}}{z_k}$ for $G = \text{GL}_n$.
- Let $\nu_k \in \text{Hom}(C^*, T)$ be the inverse of a coroot written multiplicatively, e.g. $\nu_k = \frac{\mu_{k+1}}{\mu_k}$ for $G = \text{GL}_n$.

For $\nu \in \text{Hom}(C^*, T)$ define

$$\kappa(\nu) = \vartheta(h, \nu) \vartheta(h, \nu^{-1}).$$

The Bott-Samelson recursion for local elliptic classes is:

$$\delta (\sigma^z(\zeta_k), \mu_k) \cdot s_k^\mu E_\sigma(X_{\omega}) + \delta (\sigma^z(\zeta_k), h) \cdot s_k^\mu E_{\sigma s_k}(X_{\omega}) = \begin{cases} E_\sigma(X_{\omega, s_k}) & \text{if } \ell(\omega s_k) = \ell(\omega) + 1 \\ \kappa(\nu_k) E_{\sigma s_k} & \text{if } \ell(\omega s_k) = \ell(\omega) - 1, \end{cases}$$

and the R-matrix recursion for local elliptic classes is

$$\delta (\zeta_k, \varpi(\nu_k)) \cdot E_\sigma(X_{\omega}) + s_k^z (\delta (\zeta_k, h) \cdot E_{\sigma s_k}(X_{\omega})) = \begin{cases} E_\sigma(X_{s_k \omega}) & \text{if } \ell(s_k \omega) = \ell(\omega) + 1 \\ \kappa(\varpi(\nu_k)) E_{s_k \omega} & \text{if } \ell(s_k \omega) = \ell(\omega) - 1. \end{cases}$$

The $G = \text{GL}_n$ special case of the above two formulas is

$$\delta \left( \frac{z_{\sigma(k+1)}}{z_{\sigma(k)}}, \frac{\mu_{k+1}}{\mu_k} \right) \cdot s_k^\mu E_\sigma(X_{\omega}) + \delta \left( \frac{z_{\sigma(k+1)}}{z_{\sigma(k)}}, h \right) \cdot s_k^\mu E_{\sigma s_k}(X_{\omega}) = \begin{cases} E_\sigma(X_{\omega, s_k}) & \text{if } \ell(\omega s_k) = \ell(\omega) + 1 \\ \delta \left( h, \frac{\mu_{k+1}}{\mu_k} \right) \delta \left( h, \frac{\mu_k}{\mu_{k+1}} \right) E_{\sigma}(X_{\omega s_k}) & \text{if } \ell(\omega s_k) = \ell(\omega) - 1, \end{cases}$$

and

$$\delta \left( \frac{z_{k+1}}{z_k}, \frac{\mu_{\omega^{-1}(k+1)}}{\mu_{\omega^{-1}(k)}} \right) \cdot E_\sigma(X_{\omega}) + \delta \left( \frac{z_k}{z_{k+1}}, h \right) \cdot s_k^z E_{\sigma s_k}(X_{\omega}) = \begin{cases} E_\sigma(X_{s_k \omega}) & \text{if } \ell(s_k \omega) = \ell(\omega) + 1 \\ \delta \left( h, \frac{\mu_{\omega^{-1}(k+1)}}{\mu_{\omega^{-1}(k)}} \right) \delta \left( h, \frac{\mu_{\omega^{-1}(k)}}{\mu_{\omega^{-1}(k+1)}} \right) E_\sigma(X_{s_k \omega}) & \text{if } \ell(s_k \omega) = \ell(\omega) - 1. \end{cases}$$
References


ELLIPITIC CLASSES OF SCHUBERT CELLS VIA BOTT-SAMELSON RESOLUTION


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