A THEOREM ON REMOVING $\Sigma^r$ SINGULARITIES

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Abstract. As an application of the generalized Pontrjagin-Thom construction ([2]) and a theorem of Golubjatnikov [1] here we prove a result on removing $\Sigma^r$ singularities in a certain cobordism class of smooth mappings of positive codimension.

The integer $k > 0$ will be fixed throughout the paper. Let $\eta : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{n+k}, 0)$ be a smooth map germ. By a suspension of $\eta$ we mean a germ $\Sigma^r \eta : (\mathbb{R}^{n+v}, 0) \rightarrow (\mathbb{R}^{(n+k)+v}, 0)$ defined by $(x, u) \mapsto (\eta(x), u)$ — otherwise we will use the standard notions and notations of singularity theory, see e.g. [3]. Now consider stable smooth maps between smooth manifolds of codimension $k$. For such a map $f : N^m \rightarrow P^{m+k}$ we define the submanifolds

$$\eta(f) = \{ y \in P \mid f^{-1}(y) \text{ has only one element and the germ of } f \text{ at } f^{-1}(y) \text{ is } A \text{-equivalent to a suspension of } \eta \},$$

$$\Sigma^r(f) = \{ x \in N \mid \text{the germ of } f \text{ at } x \text{ is of Thom-Boardman type } \Sigma^r. \}.$$ 

Let $\eta_r : (\mathbb{R}^{r^2+rk}, 0) \rightarrow (\mathbb{R}^{r^2+rk+k}, 0)$ denote the miniversal unfolding of the germ $\zeta_r : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^{r+k}, 0)$ defined by

$$(x_1, \ldots, x_r) \mapsto (x_2^2, \ldots, x_r^2, x_1x_2, x_1x_3, \ldots, x_{r-1}x_r, 0, \ldots, 0),$$

where there are $t := k - \binom{r}{2}$ 0's at the end. The Thom-Boardman type of $\eta_r$ is $\Sigma^{r,0}$, and, in fact, this is the “simplest” among the germs of codimension $k$ and of Thom-Boardman type $\Sigma^r$. That is, if $f : N^m \rightarrow P^{m+k}$ is a map, then the closure of the submanifold $f^{-1}(\eta_r(f)) \subset N$ contains $\Sigma^r(f)$.

Definition 1. If $g : N^m \rightarrow P^{m+k} \times \mathbb{R}^r$ is an immersion then the composition $f = pr_P \circ g : N^m \rightarrow P \times \mathbb{R}^r \rightarrow P$ will be called a prim-$\Sigma^r$ map. If, in addition, $f$ does not have other $\Sigma^r$ singularities than $\eta_r$, then we will call it a prim-$\eta_r$ map. If $f$ does not have $\Sigma^r$ points at all, then we call it prim-$\emptyset$.

The word prim stands for projected immersion, and $\Sigma^r$ and $\eta_r$ refers to types of the most difficult singularities such a map may have. Now let us fix $\mathbb{R}^{m+k}$ and consider prim-$\eta_r$ (prim-$\Sigma^r$, prim-$\emptyset$) maps of $m$-manifolds into it. We call two such map $f_1 : N_1^m \rightarrow \mathbb{R}^{m+k}$ and $f_2 : N_2^m \rightarrow \mathbb{R}^{m+k}$ cobordant if there is an abstract manifold $W^{m+1}$ with boundary $N_1 \cup N_2$ and a prim-$\eta_r$ (prim-$\Sigma^r$, prim-$\emptyset$) map

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Theorem 2. Let \( f \) is given by \( \text{a class of immersions with a double point} \), etc. Let \( k \) be even, \( m < (r + 1)k + (r^2 - 1) \), and let \( f : N^m \rightarrow \mathbb{R}^{m+k} \) be a stable prim-\( \eta_r \) map. Then the following conditions are equivalent:

1. there is a prim-\( \eta_r \) map \( g : M^n \rightarrow \mathbb{R}^{n+k} \) cobordant to \( f \) with \( \eta_r(g) = 0 \);

2. the abstract manifold \( \eta_r(f) \) is nullcobordant.

Proof. The implication \((1) \Rightarrow (2)\) is clear. Indeed, if the cobordism between \( f \) and \( g \) is given by \( F : W \rightarrow \mathbb{R}^{(n+k)+r} \) then the manifold \( \eta_r(F) \) is a cobordism between \( \eta_r(f) \) and the emptyset.

For the converse implication we need some notions and results from [2].

Consider the set of stable map germs \((\mathbb{R}^*, \text{finite set}) \rightarrow (\mathbb{R}^{*+k}, 0)\). The set of equivalence classes of this set under the equivalence relation generated by \( \mathcal{A} \)-equivalence and suspension is called \( T \). There is an obvious hierarchy on \( T \), whose top element is the class of \((k\text{-codimensional)}\) embeddings, and right under this is the class of immersions with a double point, etc. Let \( \tau \) be an ascending subset of \( T \). A map \( f : N^m \rightarrow P^{m+k} \) is called a \( \tau \)-map if for all \( y \in f(N) \) the germ of \( f \) at \( f^{-1}(y) \) is from \( \tau \). For more details and examples see [2]. If for two \( \tau \)-maps \( f_i : N_i \rightarrow P^{m+k} \) (\( i = 0, 1 \)) an abstract cobordism \( W \) is given between \( N_0 \) and \( N_1 \), as well as a \( \tau \)-map \( F : W \rightarrow P^{m+k} \times [0, 1] \) with \( F|_{N_i} = f_i \times \{i\} \), then we call \( f_0 \) and \( f_1 \) cobordant. The set of cobordism classes is denoted by: \( \text{Cob}_m(P^{m+k}; \tau) \).

Definition 3. The space \( X \) is called a classifying space for \( \tau \)-maps, if for any closed manifold \( P^{m+k} \) there is a bijection between

\[ \text{Cob}_m(P^{m+k}, \tau) \quad \text{and} \quad [P, X] = \text{homotopy classes of maps } P \rightarrow X. \]

Now let \( \tau = \tau' \cup [\eta] \), where \( \tau \) and \( \tau' \) are ascending subsets of \( T \). Suppose also that \( \eta \) is the “simplest” in its equivalence class, that is suppose that \( \eta : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k} \) is not the suspension of any other germ — germs having this property are called isolated. Let \( G \) be the maximal compact subgroup of the symmetry group

\[ \text{Aut}_{\mathcal{A} \eta} = \{ (\varphi, \phi) \in \text{Diff}(\mathbb{R}^n, 0) \times \text{Diff}(\mathbb{R}^{n+k}, 0) \mid \phi \circ \eta \circ \varphi^{-1} = \eta \}, \]

with the representations \( \lambda_1 \) and \( \lambda_2 \) on \( \mathbb{R}^n \) and \( \mathbb{R}^{n+k} \) (both of which can be supposed to be linear). The vector bundle associated to the universal principal \( G \)-bundle using the representation \( \lambda_i \) will be denoted by \( E\lambda_i \rightarrow BG \) and its disc bundle by \( D\lambda_i \). The following theorem is proved in [2].

Theorem 4. If \( X\tau' \) is a classifying space for \( \tau' \)-maps then the space \( X\tau = X\tau' \cup_{\rho} D\lambda_2 \) is a classifying space for \( \tau \)-maps (for some \( \rho : \partial D\lambda_2 \rightarrow X\tau' \)). \( \square \)
Since the one-point-space is a classifying space for $\emptyset$-maps, and any other $\tau$ can be build up from $\emptyset$ by consecutively adding new $[\eta]$‘s, the above theorem can be considered as a construction of a classifying space for $\tau$-maps for all $\tau$. Well, almost... In fact, to carry out this procedure we need some knowledge of the group $\text{Aut}_A\eta$, its maximal compact subgroup and its representations $\lambda_1, \lambda_2$. This problem is also essentially solved in [2], we will come back to these results in the concrete examples where we need them.

Now turn back to prim-$\eta_r$ maps to $\mathbb{R}^{m+k}$. Their cobordism group (defined above) is not $\text{Cob}_m(\mathbb{R}^{m+k}, \tau)$ for any $\tau$, but one can evidently extend the notion of classifying space for prim-$\eta_r$ (as well as prim-$\Sigma^r$ and prim-$\emptyset$) maps. And the theorem quoted above remains true with a minor modification. For this we need some notation. The germ $\eta_r : \mathbb{R}^{r^2+kr} \to \mathbb{R}^{r^2+kr+k}$ is of Thom-Boardman type $\Sigma^r$, so the kernel $K$ of its differential is $r$-dimensional. Now let $G$ be the subgroup of $G = \text{MC Aut}_\eta$, whose induced action on $K$ is trivial. The restriction of $\lambda_i$ to $G$ will be called $\lambda_i$, and $E\lambda_i \to B\lambda_i (D\lambda_i \to B\lambda_i)$ denotes the vector bundle (disc bundle) associated to the universal principal $G$-bundle using the representation $\lambda_i$. The following theorem is analogous to Theorem 4. We will not give a proof for it, since it goes the same way.

**Theorem 5.** Let $X'$ be a classifying space for prim-$\emptyset$ maps. Then $X = X' \cup_{\rho} D\lambda_2$ is a classifying space for prim-$\eta_r$ maps (for some $\rho : D\lambda_2 \to X'$).

Now let us consider a portion of the homotopy exact sequence of the pair $(X, X')$:

$$
\text{Cob}_m(\mathbb{R}^{m+k}, \text{prim-}\emptyset) \longrightarrow \text{Cob}_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r) \longrightarrow \pi_{m+k}(X') \longrightarrow i_* \longrightarrow \pi_{m+k}(X) \longrightarrow \delta \longrightarrow \pi_{m+k}(X, X').
$$

The statement (2)$\Rightarrow$(1) in this terms is the following: if $[f] \in \text{Cob}_m(\mathbb{R}^{m+k}, \text{prim-}\eta_r)$ is such that $\eta_r(f)$ is null-cobordant, then $[f]$ is in the image of $i_*$ — or, what is the same, $\delta([f]) = 0$. Now let us study the group $\pi_{m+k}(X, X')$. First observe that

a) $X'$ is $(k - 1)$-connected, since $\pi_i(X') = \text{Cob}_{i-k}(S^i, \text{prim-}\emptyset) = 0$ if $i - k < 0$; and

b) the pair $(X, X')$ is $r^2 + rk + k - 1$-connected, since $H^i(X, X') = H^i(T\lambda_2) = 0$ for $i = 1, \ldots, r^2 + rk + k - 1$, because $rk \lambda_2 = r^2 + rk + k$.

Now, due to the homotopy excision theorem and our dimension restrictions we have the natural isomorphism $\pi_{m+k}(X, X') = \pi_{m+k}(X/X')$ which latter is $\pi_{m+k}(T\lambda_2)$. Now we have to analyse the group $\bar{G}$ and the representation $\bar{\lambda}_2$.

**Lemma 6.**

$$\bar{G} = O(t) \quad \bar{\lambda}_2 = r \cdot \rho_t \oplus (r^2 - r + r\left(\frac{r + 1}{2}\right)) \cdot 1,$$

where $t = k - \binom{k}{2}$, $\rho_t$ is the standard $t$-dimensional representation of $O(t)$, and $1$ is the trivial $1$-dimensional representation.

**Proof of Lemma 6.** First we recall from [2] some results about the maximal compact automorphism group of $\eta_r$:

$$\text{MC Aut}_A\eta_r = \text{MC Aut}_K\zeta_r \leq \text{Aut} Q_{\zeta_r} \times O(k - d) =$$
then

$$\lambda$$

so the subgroup $\bar{\lambda}$ the representations $m$-morphic to the cobordism group of embeddings of closed $R$-R. RIMÁNYI, A. SZÜCS

where $Q_{\zeta_r}$ is the local algebra of $\zeta_r$ and $d$ is its defect. In fact $O(r) \times O(t)$ acts as an $A$-equivalence group (and therefore as an $K$-equivalence group) of $\zeta_r$, so there is equation instead of $\leq$ in the formula. So $G = O(r) \times O(t)$.

To determine $\bar{G} \leq G$ and the representation $\bar{\lambda}_2$ we recall some more notions and results from [2] and [4]. Since the germ $\eta_r$ is a miniversal unfolding of $\zeta_r$ with $d\zeta_r(0) = 0$, therefore $\eta_r$ is $A$-equivalent to

$$\mathbb{R}^r \times V \longrightarrow \mathbb{R}^{r+k} \times V$$

$$(x, \phi) \mapsto (x + \phi(x), \phi),$$

where $V$ is a complement of the subspace $t\zeta_r(\theta_r) + \zeta^*_r(m(r+k))\theta_{\zeta_r}$ in the vector space $\theta_{\zeta_r}$. Since $G$ actually acts as an $A$ automorphism, so it has representations $\alpha$ and $\beta$ on $\mathbb{R}^r$ and $\mathbb{R}^{r+k}$ respectively. The group $G$ also acts on $\theta_{\zeta_r}$ by $(\alpha, \beta) \cdot \phi = \beta \circ \phi \circ \alpha^{-1}$ — leaving $t\zeta_r(\theta_a) + \zeta^*_r(m(a+k))\theta_{\zeta_r}$ invariant. If $V$ is chosen to be $G$-invariant ($G$ compact, so it is possible) then $G$ also acts on $V$. Let this action be $\gamma$. A theorem in [2] proves that the maximal compact subgroup of $Aut_A \eta_r$ is $G$ with the representations $\lambda_1 := \alpha \oplus \gamma$, $\lambda_2 := \beta \oplus \gamma$ on the source $(\mathbb{R}^r \times V)$ and target $(\mathbb{R}^{r+k} \times V)$ spaces, respectively.

Now observe that $\alpha$ is $\rho_r \circ pr_{O(r)}$, where $\rho_r$ is the standard $r$-dimensional representation of $O(r)$. Observe also that the kernel of $d\eta_r$ is (the tangent space to) $\mathbb{R}^r$, so the subgroup $\bar{G} \leq G$ must be $O(t)$. If we choose $V$ to be spanned by

$$(x_1, \ldots, x_r) \mapsto (0, \ldots, 0, x_i, 0, \ldots, 0)$$

the coordinate is $j = 1, \ldots, r, i \neq j$

$$(x_1, \ldots, x_r) \mapsto (0, \ldots, 0, x_i, 0, \ldots, 0)$$

the coordinate is $j = r + 1, \ldots, r + k$,

then $V$ will be $O(r) \times O(t)$-invariant, and using the definition of $\alpha$, $\beta$, $\gamma$ above we can compute

$$\alpha|_{\bar{G}} = r \cdot 1$$

$$\beta|_{\bar{G}} = \rho_t \oplus \left( \binom{r}{2} + r \right) \cdot 1$$

$$\gamma|_{\bar{G}} = r \cdot \rho_t \oplus (r(r-1) + r(\binom{r}{2})) \cdot 1,$$

which proves the lemma. □

Now, according to the original Thom-construction, the group $\pi_{m+k}(T\bar{\lambda}_2)$ is isomorphic to the cobordism group of embeddings of closed $m - r^2 - rk$-manifolds into $\mathbb{R}^{m+k}$ with a fixed splitting of the normal bundle to the direct sum of $r + 1$ isomorphic $t$-dimensional bundles and a trivial bundle. If the dimension $n - r^2 - rk$ of the embedded manifold is smaller than $t$ (which holds in the dimension range of the Theorem) then these bundles are already stable normal bundles, so the group $\pi_{m+k}(T\bar{\lambda}_2)$ is isomorphic to the group $\Omega^{(r+1)\gamma}_{m-r^2-rk}$ defined by Golubjatnikov

\footnote{the notation $\Omega^{(r+1)\gamma}_{m-r^2-rk}$ would be perhaps better}
in [1]. Golubjatnikov also proves that in case \( r + 1 \) is odd then the forgetful map
\[ \Omega_{m-r^2-rk}^{(r+1)\gamma} \rightarrow \mathcal{N}_{m-r^2-rk} \]
to the abstract cobordism group is an isomorphism.

Putting all these together we see that \( \pi_{m+k}(X, X') \cong \mathcal{N}_{m-r^2-rk} \), and it is easy
to see that the image of \( \delta([f]) \) in \( \mathcal{N}_{m-r^2-rk} \) is the abstract cobordism class of \( \eta_r(f) \).
Since \( \eta_r(f) \) is null-cobordant, we have proved the theorem. \( \square \)

Remark. In fact, the dimension restriction \( m < (r+1)k + (r^2-1) \) in the theorem
implies that a stable map \( N^m \rightarrow \mathbb{R}^{m+k} \) does not have any other singularities of
type \( \Sigma^r \) than \( \eta_r \). Therefore the condition “\( f: N^m \rightarrow \mathbb{R}^{m+k} \) is a stable prim-
\( \eta_r \) map” can be weakened as “\( f: N^m \rightarrow \mathbb{R}^{m+k} \) is a stable prim-\( \Sigma^r \) map”; so
Theorem 2 can actually be considered as a theorem on removing \( \Sigma^r \) singularities.

References

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